



# Extremal Kähler metrics and Bach–Merkulov equations



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## ABSTRACT

In this paper, we study a coupled system of equations on oriented compact 4-manifolds which we call the Bach–Merkulov equations. These equations can be thought of as the conformally invariant version of the classical Einstein–Maxwell equations. Inspired by the work of C. LeBrun on Einstein–Maxwell equations on compact Kähler surfaces, we give a variational characterization of solutions to Bach–Merkulov equations as critical points of the Weyl functional. We also show that extremal Kähler metrics are solutions to these equations, although, contrary to the Einstein–Maxwell analogue, they are not necessarily minimizers of the Weyl functional. We illustrate this phenomenon by studying the Calabi action on Hirzebruch surfaces.

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## 1. Introduction

Let  $M$  be a smooth oriented  $n$ -manifold. A Riemannian metric  $g$  on  $M$  is said to satisfy the *Einstein–Maxwell equations* if

$$\begin{aligned} [r + F \circ F]_{\circ} &= 0 \\ dF &= 0, \quad d * F = 0 \end{aligned} \quad (1)$$

for some 2-form  $F$  on  $M$ . Here,  $r$  is the Ricci tensor of  $g$ ;  $(F \circ F)_{ij} = F_i^s F_{sj}$  is the composition of  $F$  with itself as an endomorphism of the tangent bundle  $TM$ ;  $[\cdot]_{\circ}$  denotes the trace-free part of a  $(2, 0)$ -tensor, and  $*$  is the Hodge operator with respect to the metric  $g$ . When  $M$  is compact, the second line of (1), which is called *Maxwell equations*, is equivalent to saying that  $F$  is harmonic with respect to  $g$ , i.e.  $\Delta F = 0$ .

By Hodge theory we know that any harmonic form  $F$  minimizes the  $L^2$  norm  $F \mapsto \int_M |F|_g^2 d\mu_g$  among the forms cohomologous to  $F$ , namely on  $[F] \in H_{dR}^2(M, \mathbb{R})$ . If, in addition,  $M$  has dimension 4, the integral  $\int_M |F|_g^2 d\mu_g$  is unchanged if  $g$  is replaced by any conformally related metric  $\tilde{g} := ug$ , for a positive smooth function  $u$  on  $M$ . Therefore, if  $F$  is harmonic with respect to  $g$ , it will be harmonic with respect to  $\tilde{g}$ . By contrast, the first line of (1) is certainly not conformally invariant in any dimension. There is, however, an interesting conformally invariant counterpart of these equations introduced by Merkulov in [1]:

$$\begin{aligned} B + [F \circ F]_{\circ} &= 0 \\ dF &= 0, \quad d * F = 0 \end{aligned} \quad (2)$$

where  $B_{ij} = (\nabla^s \nabla^t + \frac{1}{2} r^{st}) W_{ijst}$  is the *Bach tensor* [2]. When  $M$  is compact, this tensor arises from the Euler–Lagrange equations for the *Weyl energy functional*  $g \mapsto \int_M |W|^2 d\mu_g$  over the space of all metrics. That is, if we vary the metric  $g_t = g_0 + th + o(t^2)$ , then [3]

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{W}(g_t) = \int_M \langle h, B \rangle d\mu = \int_M g^{is} g^{tj} h_{st} B_{ij} d\mu. \quad (3)$$

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Note that in 4 dimensions,  $\mathcal{W}$  is indeed conformally invariant since the conformal change  $\tilde{g} = ug$  of the metric implies

$$d\tilde{\mu} = u^2 d\mu \quad \text{and} \quad \tilde{W}_{ijk}^l = W_{ijk}^l.$$

Bach tensor, too, behaves well under conformal change:  $\tilde{B}_{ij} = \frac{1}{u} B_{ij}$ . To see this note that for the rescaled variation  $\tilde{g}_t = ug_0 + tuh + o(t^2)$  we have

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{W}(\tilde{g}_t) = \int_M \langle uh, \tilde{B} \rangle d\tilde{\mu} = \int_M \tilde{g}^{is} \tilde{g}^{tj} u h_{st} \tilde{B}_{ij} d\tilde{\mu}$$

and comparing it to (3) we deduce that  $\tilde{B}_{ij} = \frac{1}{u} B_{ij}$ . Also,  $B$  is symmetric, trace-free and divergence-free. Note also that  $[F \circ F]_\circ$ , the other term in (2), rescales similarly to  $B_{ij}$  under conformal rescaling. Clearing out the  $\frac{1}{u}$  factors, we see that, when  $M$  is a compact manifold of dimension 4, the coupled system of Eqs. (2) is conformally invariant in the sense that if  $(g, F)$  is a solution, so is  $(ug, F)$  for any positive smooth function  $u$ .

Both Einstein–Maxwell and Bach–Merkulov equations stem from a variational origin. For any given de Rham class  $\Omega \in H_{dR}^2(M, \mathbb{R})$ , solutions  $(g, F)$  of Einstein–Maxwell equations with  $[F] = \Omega$  are in fact the critical points of the coupled action

$$\begin{aligned} \mathcal{G}_1 \times \Omega &\longrightarrow \mathbb{R} \\ (g, F) &\longmapsto \int_M s_g + |F|_g^2 d\mu_g \end{aligned}$$

where  $\mathcal{G}_1$  stands for the space of unit volume metrics [4]. Similarly [2,1], Bach–Merkulov solutions are the critical point of the action

$$\begin{aligned} \mathcal{G}_1 \times \Omega &\longrightarrow \mathbb{R} \\ (g, F) &\longmapsto \int_M |W|_g^2 + |F|_g^2 d\mu_g. \end{aligned}$$

In [4], C. LeBrun studied Einstein–Maxwell equations (1) on compact smooth 4-manifolds, and discovered some fascinating properties of these equations in relation to Kähler geometry. He showed that *constant scalar curvature Kähler* metrics satisfy (1); all solutions to (1) are critical points of  $L^2$ -norm of scalar curvature on  $\mathcal{G}_\Omega$ , the space of metrics for which a fixed cohomology class  $\Omega$  is represented by a self-dual harmonic form  $\Omega_g$ ; and on complex surfaces constant scalar curvature Kähler metrics are *global minimizers* of that action if  $c_1 \cdot \Omega \leq 0$ . These results are summarized in Section 2. The aim of this paper is to state and prove the relevant properties of the Bach–Merkulov equations. The first main results is:

**Theorem 1.** *Let  $M$  be a compact complex surface, and let  $g$  be a metric conformal to an extremal Kähler metric on  $M$ . Then  $g$  satisfies the Bach–Merkulov equations with some  $F$ . As a consequence, on any compact complex surface of Kähler type we can solve (2).*

In other words, extremal Kähler metrics are standard solutions of Bach–Merkulov equations on a compact complex surface. On a more general compact oriented 4-manifold the Bach–Merkulov solutions are in a natural bijective correspondence with the critical points of the Weyl energy functional  $g \mapsto \int_M |W|^2 d\mu$ :

**Theorem 2.** *Let  $M$  be a smooth compact oriented 4-manifold, and  $\Omega \in H_{dR}^2(M, \mathbb{R})$  be any de Rham class. A metric  $g \in \mathcal{G}_\Omega$  is a critical point of the restriction of the Weyl functional to  $\mathcal{G}_\Omega$  if and only if  $g$  is a solution of Bach–Merkulov equations in conjunction with a unique harmonic form  $F$  whose self-dual part is  $F^+ = \Omega_g$ .*

On a compact Kähler surface, one could therefore ask analogously if extremal Kähler metrics are *absolute minimizers* of the Weyl functional on  $\mathcal{G}_\Omega$  where  $\Omega$  is the Kähler class represented by the extremal Kähler metric. It turns out that this is not the case:

**Theorem 3.** *For any given  $\Omega \in H_{dR}^2(M, \mathbb{R})$  on Kähler-type smooth 4-manifolds  $\mathbb{C}P_2 \# \overline{\mathbb{C}P_2}$  or  $\mathbb{C}P_1 \times \mathbb{C}P_1$  the extremal Kähler metrics in  $\mathcal{G}_\Omega$  (with respect to some complex structure) are not necessarily minimizers of the Weyl functional restricted to  $\mathcal{G}_\Omega$ .*

Theorems 1 and 2 follow from Propositions 3–5, proved in Section 3. Theorem 3 is a consequence of the discussion in Section 4.

Recall that, given a compact complex manifold  $(M, J)$  with a Kähler class  $\Omega$  (i.e.  $\Omega$  is represented by a Kähler form), an *extremal Kähler metric* is, by definition, a critical point of the action

$$\begin{aligned} \Omega^+ &\longrightarrow \mathbb{R} \\ \omega &\longmapsto \int_M s_\omega^2 d\mu_\omega \end{aligned} \tag{4}$$

where  $\Omega^+$  stands for the space of Kähler forms in the de Rham class  $\Omega$ . This notion of extremal metrics was introduced by Calabi [5] in an attempt to show the existence of constant scalar curvature Kähler metrics on compact complex manifolds. The Euler–Lagrange equations of this action are given by  $\frac{\partial^2 s}{\partial \bar{z}_i \partial z_j} = 0$ . In particular, every constant scalar curvature Kähler metric is extremal. The converse, however, is not true: For any given Kähler class on Hirzebruch surfaces  $\mathbb{F}_k = \mathbb{P}(\mathcal{O}(-k) \oplus \mathcal{O})$ , Calabi constructed explicit extremal Kähler metrics in that class. However, the first Hirzebruch surface  $\mathbb{F}_1 \approx \mathbb{C}P_2 \# \mathbb{C}P_2$  cannot admit constant scalar curvature Kähler metric by the Matsushima–Lichnerowicz theorem because the maximal compact subgroup Lie group of automorphisms is not reductive [5].

By computing the second variation of this action at a critical metric, Calabi was able to show that extremal Kähler metrics are local minimizers [5]. Indeed, they turn out to be global minimizers as proven recently by Donaldson and Chen [6,7]. As we will discuss in Section 2, on compact complex surfaces, LeBrun showed that the constant scalar curvature Kähler metrics remain global minimizers of the action (4), even if we extend the domain from  $\Omega^+$  to  $\mathcal{G}_\Omega$ , provided  $c_1 \cdot \Omega \leq 0$ . However, we will show in Section 4 that the extremal Kähler metrics are not necessarily global minimizers of the Weyl energy functional.

## 2. Einstein–Maxwell equations

This section summarizes some of the results in [4].

Recall that the Euler–Lagrange equations of the action  $g \mapsto \int_M s_g d\mu_g$ , where  $g$  is allowed to vary over all unit volume metrics are precisely  $\hat{r} = 0$  (characterizing Einstein metrics). Also, from Hodge theory, the Euler–Lagrange equations of the action  $F \mapsto \int_M |F|_g^2 d\mu_g$ , where  $g$  is fixed but  $F$  varies over all closed 2-forms in a fixed de Rham class  $[F] \in H_{dR}^2(M, \mathbb{R})$  are the Laplace equation  $\Delta F = 0$ . Therefore, the Einstein–Maxwell equations are precisely the Euler–Lagrange equations of the joint action  $(g, F) \mapsto \int_M s_g + |F|_g^2 d\mu_g$  where  $g$  varies over unit volume Riemannian metrics and  $F$  varies over a fixed de Rham class.

If we restrict the first action to the conformal class of a critical metric, we get the Einstein–Hilbert action whose critical points are well known to have constant scalar curvature [8]. Thus, any Einstein–Maxwell metric is of constant scalar curvature. Conversely, C. LeBrun observed the following remarkable fact:

**Proposition 1 (LeBrun).** *Suppose that  $(M^4, g, J)$  is a Kähler surface with Kähler form  $\omega = g(J \cdot, \cdot)$  and Ricci form  $\rho = r(J \cdot, \cdot)$ . If  $g$  has constant scalar curvature, then  $g$  satisfies Einstein–Maxwell equations with  $F = \omega + \frac{1}{2} \hat{\rho}$ , where  $\hat{\rho} = \hat{r}(J \cdot, \cdot)$  is the primitive part of the Ricci form  $\rho$  of  $g$ .*

Recall that constant scalar curvature Kähler metrics are in particular extremal Kähler metrics, that is, critical points of the  $L^2$ -norm of scalar curvature  $g \mapsto \int_M s_g^2 d\mu_g$ , where  $g$  varies over Kähler metrics in a fixed Kähler class  $\Omega \in H_{dR}^2(M, \mathbb{R})$ . C. LeBrun generalized the notion of a Kähler class and Calabi problem for a Kähler surface to the Riemannian setting, where, a priori, there may not be a complex structure at all. The generalization is as follows:

Let  $M$  be a smooth 4-manifold; and let  $\Omega$  be a de Rham class as above. By Hodge theory, we know that any Riemannian metric  $g$  gives a unique harmonic representative  $\Omega_g$  of  $\Omega$ . If  $\Omega_g$  is self-dual,  $g$  is called an  $\Omega$ -adapted metric. The space of all  $\Omega$ -adapted metrics is denoted by  $\mathcal{G}_\Omega$ ; i.e.  $\mathcal{G}_\Omega = \{g : *\Omega_g = \Omega_g\}$ .

Observe that if  $M$  is a complex surface and  $\Omega$  is a Kähler class, then  $\mathcal{G}_\Omega$  contains all Kähler metrics in  $\Omega$ , because any Kähler form is self-dual. In this sense,  $\mathcal{G}_\Omega$  is a Riemannian generalization of a Kähler class. Also note that if  $g \in \mathcal{G}_\Omega$ , so is  $\tilde{g} = ug \in \mathcal{G}_\Omega$  since Hodge  $*$ -operator is unchanged under conformal changes of the metric.

Now, as in the Calabi problem, C. LeBrun considers the action

$$g \mapsto \int_M s_g^2 d\mu_g \tag{*}$$

on  $\mathcal{G}_\Omega$ , and sees which metrics are critical points of this action:

**Proposition 2 (LeBrun).** *Critical points of (\*) are either*

- (1) scalar-flat metrics (i.e.  $s \equiv 0$ ), or
- (2) Einstein–Maxwell metrics  $g$  with  $F^+ = \Omega_g$ .

Thus, in particular, constant scalar curvature Kähler metrics are critical points of (\*). Moreover, they are actually minimizers if  $c_1 \cdot \Omega \leq 0$ .

**Theorem 4 (LeBrun).** *Let  $(M^4, J)$  be a compact complex surface and  $\Omega$  a Kähler class with  $c_1 \cdot \Omega \leq 0$ . Then any metric  $g$  in  $\mathcal{G}_\Omega$  satisfies  $\int_M s^2 d\mu \geq 32\pi^2 \frac{(c_1 \cdot \Omega)^2}{\Omega \cdot \Omega}$ , and equality holds if and only if  $g$  is a constant scalar curvature Kähler metric.*

Another observation of C. LeBrun is that any compact smooth 4-manifold of Kähler type admits a solution of (1). This follows from Shu’s result [9], which says that such 4-manifolds admit a constant scalar curvature Kähler metrics unless they are diffeomorphic to  $\mathbb{C}P_2 \# \mathbb{C}P_2$  or  $\mathbb{C}P_2 \# 2\mathbb{C}P_2$ . However, both of these manifolds admit Einstein metrics (Page metric [10] and Chen–LeBrun–Weber metric [11]) which are automatically Einstein–Maxwell with  $F = 0$ .

### 3. Bach–Merkulov equations

In this section we will state and prove analogues of LeBrun’s results stated in Section 2 for Bach–Merkulov equations.

First we start by observing the following proposition which shows that Bach–Merkulov equations possess an interesting family of solutions.

**Proposition 3.** *Let  $g$  be an extremal Kähler metric on a compact complex surface  $(M, J)$ . Then  $(g, F)$  satisfies the Bach–Merkulov equations with  $F = \omega + \frac{1}{2}\psi$  where  $\psi = B(J \cdot, \cdot)$ . Hence any metric conformal to an extremal Kähler metric is a solution of (2).*

**Proof.** The proof is similar to the one of Proposition 1. First, observe that  $[F \circ F]_{\circ} = 2F^+ \circ F^-$  where  $F^+$  and  $F^-$  are the self-dual and anti-self-dual parts of  $F$ . Since  $g$  is Kähler,  $\omega$  is a self-dual harmonic 2-form. Moreover, since  $g$  is extremal,  $\psi = B(J \cdot, \cdot)$  is an anti-self-dual harmonic 2-form (see [11]). Thus, setting  $F^+ = \omega$  and  $F^- = \frac{\psi}{2}$ , we see that  $2F^+ \circ F^- = \omega_i^s \psi_{sj} = \psi(J \cdot, \cdot) = -B$ . Thus we get  $B + [F \circ F]_{\circ} = 0$ . Moreover,  $F$  is harmonic since both  $F^+$  and  $F^-$  are so. Therefore,  $(g, F)$  is a solution of Bach–Merkulov equations.  $\square$

More explicitly, if  $g$  is an extremal Kähler metric, then the Bach tensor can be re-written in the form

$$B = \frac{1}{12}(s\ddot{r} + 2\text{Hess}_{\circ}(s))$$

and therefore  $\psi = \frac{1}{12}[s\rho + i\partial\bar{\partial}s]_{\circ}$  where  $[\cdot]_{\circ}$  stands for the primitive part of a  $(1, 1)$ -form (see [11]). In particular, if the extremal Kähler metric turns out to have non-zero constant scalar curvature, then  $\psi$  simplifies to  $\frac{s}{12}\rho$ . So we see that the solution of Proposition 3 becomes  $(g, F = \omega + \frac{s}{24}\rho)$  which is quite similar to LeBrun’s solution to Einstein–Maxwell equations  $(g, F = \omega + \frac{\rho}{2})$ .

Proposition 3 together with Shu’s result [9] implies the following:

**Proposition 4.** *Let  $M$  be the underlying 4-manifold of any compact complex surface of Kähler type. Then  $M$  admits a solution  $(g, F)$  of Bach–Merkulov equations.*

Next, we will prove the analogue of Proposition 2 for Bach–Merkulov equations:

**Proposition 5.** *An  $\Omega$ -adapted metric  $g$  is a critical point of the restriction of Weyl functional to  $\mathcal{G}_{\Omega}$  iff  $g$  is a solution of Bach–Merkulov equations in conjunction with a unique harmonic form  $F$  with  $F^+ = \Omega_g$ .*

**Proof.** The proof is similar to the one of Proposition 2. Let  $g_t = g + th + O(t^2)$  be a variation of a metric  $g$  in  $\mathcal{G}_{\Omega}$ . Donaldson showed [12] that the tangent space  $T_g \mathcal{G}_{\Omega}$  is precisely the  $L^2$ -orthogonal complement of  $\{\Omega_g \circ \varphi : \varphi \in \mathcal{H}_g^-\}$  in  $\Gamma(\odot^2 T^*M)$ . Thus, in our case,  $h$  can be taken such that  $\int_M \langle h, \Omega_g \circ \varphi \rangle d\mu_g = 0$  for all  $\varphi \in \mathcal{H}_g^-$ .

The first variation of the Weyl functional is given by [2,3]

$$\frac{d}{dt} \int_M \|W\|^2 d\mu_{g_t} \Big|_{t=0} = \int_M h^{ij} B_{ij} d\mu_g = \int_M \langle h, B \rangle d\mu_g.$$

Thus,  $g$  is a critical point if and only if  $h$  is  $L^2$ -orthogonal to  $B$ . By Donaldson’s result [12], this implies that  $B = \Omega_g \circ \varphi$  for some  $\varphi \in \mathcal{H}_g^-$ . So, taking  $F^+ = \Omega_g$  and  $F^- = -\frac{\varphi}{2}$ , we see that  $g$  satisfies (2).

Conversely, if  $g$  is an  $\Omega$ -adapted solution of (2) with  $F^+ = \Omega_g$ , then  $\int \langle h, B \rangle d\mu = 2 \int \langle h, F^+ \circ F^- \rangle d\mu = 0$  for any variation  $h$  as above. Thus, again by Donaldson’s result [12],  $g$  is a critical point.  $\square$

In particular, extremal Kähler metrics are also critical points of this functional. The natural question to ask is whether they are global minimizers in  $\mathcal{G}_{\Omega}$ . In the next section, we will show that the answer to this question is negative: the analogue of Theorem 4 does not hold for Bach–Merkulov equations.

### 4. Example: Hirzebruch surfaces

In this section we will show that extremal Kähler metrics adapted to a fixed cohomology class  $\Omega$  do not necessarily have the same Weyl energy. We will illustrate this fact on Hirzebruch surfaces, by showing the existence of two closed forms in  $\Omega$  which are extremal Kähler with respect to different complex structures. In view of the formula in [13] it will turn out that the Weyl energy of the corresponding extremal Kähler metrics are different.

Recall that the  $k$ -th Hirzebruch surface  $\mathbb{F}_k$  is defined as the projectivization of the rank-2 complex vector bundle  $\mathcal{O}(-k) \oplus \mathcal{O}$  over  $\mathbb{C}\mathbb{P}_1$  (see [14,15] for details).  $\mathbb{F}_k$  is diffeomorphic to  $S^2 \times S^2$  if  $k$  is even, and to  $\mathbb{C}\mathbb{P}_2 \# \overline{\mathbb{C}\mathbb{P}_2}$  if  $k$  is odd [16]. They are, however, all biholomorphically distinct as complex surfaces (see [16]). They are simply connected; they have the second Betti number  $b_2(\mathbb{F}_k) = 2$  and Euler characteristic  $\chi(\mathbb{F}_k) = 4$ .

For the generators of the homology of  $\mathbb{F}_k$  we will take the fiber  $F$  and the image of the section  $\{z \mapsto [0 : z]\} : \mathbb{C}\mathbb{P}_1 \rightarrow \mathbb{P}(\mathcal{O}(-k) \oplus \mathcal{O}) = \mathbb{F}_k$ , which we will denote by  $C_k$ . Note that  $F \cdot F = 0$ ,  $F \cdot C_k = 1$  and  $C_k \cdot C_k = -k$  so that in this basis the intersection pairing becomes

$$\begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}.$$

Let the Poincaré dual of  $C_k$  and  $F$  be  $c_k$  and  $f$  respectively. Then any de Rham class  $\Omega \in H_{dR}^2(\mathbb{F}_k, \mathbb{R})$  can be written as  $\Omega = pc_k + qf$  for some  $p, q \in \mathbb{R}$ . If  $k$  and  $n$  are two positive integers of the same parity, then  $\mathbb{F}_k$  and  $\mathbb{F}_n$  are diffeomorphic; so we can represent  $\Omega$  as a combination of the basis  $\{c_n, f\}$ . The following lemma gives the change of basis formula:

**Lemma 1.** *We have*

$$c_k = c_n + \frac{n-k}{2}f.$$

Therefore,

$$\Omega = pc_k + qf = pc_n + \tilde{q}f$$

where  $\tilde{q} = p\frac{n-k}{2} + q$ .

**Proof.** Let

$$C_k = sC_n + tF \tag{5}$$

for some constants  $s, t$ . Take the intersection of both sides with  $F$ :

$$C_k \cdot F = sC_n \cdot F + tF \cdot F.$$

Since  $C_k \cdot F = C_n \cdot F = 1$  and  $F \cdot F = 0$ , we have  $s = 1$ . On the other hand, take the self intersection of both sides:

$$\begin{aligned} C_k \cdot C_k &= (C_n + tF) \cdot (C_n + tF) \\ -k &= -n + 2t. \end{aligned}$$

Therefore  $t = \frac{n-k}{2}$ . Taking the Poincaré dual of (5) proves the first equality. The second equality follows immediately from this.  $\square$

Note also that  $\Omega = pc_k + qf \in H_{dR}^2(\mathbb{F}_k, \mathbb{R})$  is a Kähler class iff  $\Omega \cdot C_k > 0$  and  $\Omega \cdot F > 0$ , that is, iff  $p > 0$  and  $q > kp$ . Now we can deduce when the same de Rham class  $\Omega$  is a Kähler class in  $\mathbb{F}_n$ , where  $n$  and  $k$  have the same parity. Let  $J_k$  denote the complex structure of the complex surface  $\mathbb{F}_k$ .

**Lemma 2.** *A Kähler class  $\Omega = pc_k + qf$  in  $\mathbb{F}_k$  is a Kähler class in  $\mathbb{F}_n$  if and only if  $n < 2\frac{q}{p} - k$ . In particular,  $\Omega$  is Kähler with respect to only finitely many  $J_n$ 's.*

**Proof.**  $\Omega = pc_k + qf$  is Kähler with respect to  $J_k$  iff  $p > 0$  and  $q > kp$ . By Lemma 1,  $\Omega = pc_n + (p\frac{n-k}{2} + q)$ . Now, by the previous paragraph, this class is Kähler with respect to  $J_n$  iff  $p > 0$  and  $p\frac{n-k}{2} + q > np$ . The second inequality is the same as  $n < 2\frac{q}{p} - k$ . Notice that  $2\frac{q}{p} - k$  is positive since  $\Omega = pc_k + qf$  is assumed to be Kähler since  $q > kp$ . Hence there are only finitely many possibilities for  $n$  so that  $\Omega$  remains Kähler with respect to  $J_n$ .  $\square$

So there are de Rham classes on smooth 4-manifolds  $S^2 \times S^2$  or  $\mathbb{C}\mathbb{P}_2 \# \overline{\mathbb{C}\mathbb{P}_2}$  which are Kähler with respect to different complex structures. However, Calabi [5] showed that every Kähler class on a Hirzebruch surface is represented by an extremal Kähler metric.

So, with our previous notation all Riemannian metrics  $g$  whose Kähler form  $\omega = g(J_k \cdot, \cdot)$  with respect to any of the complex structures  $J_k$  are in  $\mathcal{G}_\Omega$ . Thus, we have essentially distinct extremal Kähler metrics in  $\mathcal{G}_\Omega$ . Each of those metrics are critical points of the restriction of the Weyl functional to  $\mathcal{G}_\Omega$ .

Next, we will show that this the Weyl energy levels of those metrics are different. First note that [17] for Kähler metrics we have  $|W^+|^2 = \frac{\tau^2}{24}$ . By signature formula,  $\int |W^+|^2 d\mu = \frac{1}{2} \int |W|^2 d\mu + 6\pi^2 \tau$ . Since the signature  $\tau$  of the Hirzebruch surfaces is 0, we see that the Weyl energy of a Kähler metric is equal to its Calabi energy up to an overall multiplicative constant. Hwang and Simanca [13] gave the following formula for the Calabi energy of an extremal metric in a Kähler class on the Hirzebruch surface  $\mathbb{F}_k$ .

**Proposition 6** (Hwang and Simanca). *The Calabi energy of the extremal Kähler metric in the class  $\Omega = 4\pi c_k + 2\pi(a+k)f$  in  $\mathbb{F}_k$  is given as:*

$$\tilde{\mathcal{C}}(a, k) := 12\pi \frac{a^3 + 4a^2 + (4+k^2)a - 4k^2}{3a^2 - k^2}. \tag{6}$$

Note that the Calabi energy and the Weyl energy are scale-invariant in dimension four. Therefore by appropriate scaling we see that the Calabi energy of the extremal Kähler metric in  $\Omega = pc_k + qf$  is given by

$$\mathcal{C}(p, q, \mathbb{F}_k) := \tilde{\mathcal{C}}\left(2\frac{q}{p} - k, k\right) = 12\pi \frac{\left(2\frac{q}{p} - k\right)^3 + 4\left(2\frac{q}{p} - k\right)^2 + (4 + k^2)\left(2\frac{q}{p} - k\right) - 4k^2}{3\left(2\frac{q}{p} - k\right)^2 - k^2}. \quad (7)$$

We therefore see that the extremal Kähler metrics with respect to different complex structures in  $\Omega$  have different energy, i.e.  $\mathcal{C}(p, q, \mathbb{F}_k) \neq \mathcal{C}(p, p\frac{n-k}{2} + q, \mathbb{F}_n)$  in general. Thus, they also have different Weyl energy.

This shows that the analogue of [Theorem 4](#) cannot hold for the Bach–Merkulov equations.

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