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Deformations of generalized holomorphic structures

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ABSTRACT

A deformation theory of generalized holomorphic structures in the setting of (generalized) principal fibre bundles is developed. It allows the underlying generalized complex structure to vary together with the generalized holomorphic structure. We study the related differential graded Lie algebra, which controls the deformation problem via the Maurer–Cartan equation. As examples, we check the content of the Maurer–Cartan equation in detail in the special cases where the underlying generalized complex structure is symplectic or complex. A deformation theorem, together with some non-obstructed examples, is also included.

1. Introduction

In generalized complex geometry, the notion of generalized holomorphic structures is the analogue of holomorphic structures in classical complex geometry, including flat bundles over symplectic manifolds, co-Higgs bundles and holomorphic Poisson modules as extreme examples. These examples are also the most studied cases to date: the flat case is the most trivial one; N. Hitchin has studied certain aspects of co-Higgs bundles and also provided some interesting examples [1,2], while [3] contains a detailed investigation of stable co-Higgs bundles over $\mathbb{P}^1$ and $\mathbb{P}^2$; the case of holomorphic Poisson modules is less touched, in particular in the setting of generalized complex geometry—[4,5] contain some topics concerning this. The construction of more general generalized holomorphic structures often involves more effort; for some progress in this direction see [5,6].

In [6,7] the author has explored some local features of generalized holomorphic structures. In the formalism of reduction theory of Courant algebroids and Dirac structures developed in [8], the author has also extended the notion of generalized holomorphic structures to the context of (generalized) principal bundles [6]. This paper is then a continuation of that work, motivated by the attempt to find more examples of generalized holomorphic structures.

One possible way to obtain more examples of generalized holomorphic structures is by deforming a given one. Recall that in classical theory of deformations of holomorphic structures [9], one fixes a compact complex manifold (M, J) together with a holomorphic vector bundle V and tries to find nearby holomorphic structures, *but all these holomorphic structures are w.r.t. the same complex structure* J. Then infinitesimal deformations are contained in $H^1(M, \mathcal{O}(\mathrm{End}(V)))$ while the obstructions for an infinitesimal deformation to be integrable live in $H^2(M, \mathcal{O}(\mathrm{End}(V)))$.

One can certainly routinely apply a similar method to the generalized case, *where the underlying generalized complex structure should be fixed, and in this sense we call the resulting theory traditional.*. However, there are some drawbacks. First,

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when deforming a usual holomorphic structure (viewed as a generalized holomorphic structure), one cannot go too far and at most gets co-Higgs bundles. Second, the relevant cohomology groups (e.g. those associated to a Poisson module) are, to some extent, the starting point to find nearby generalized holomorphic structures, but generally there still lacks any effective way to compute them. Thus, for practical purposes, e.g. to probe more general generalized holomorphic vector bundles, this way of deformation is not that useful.

Therefore, in this paper, we will develop a more general deformation theory, which, more or less, can overcome the above drawbacks. As many existing examples of generalized complex manifolds are obtained by deforming simple ones, our choice is that, we *no longer* restrict ourselves to a *fixed* underlying generalized complex structure—the generalized holomorphic structure varies more freely, because the underlying generalized complex structure is also allowed to vary. In this direction, the formalism of [6] is rather suitable for our purpose, so we start in the context of principal bundles and then work out its vector-bundle counterpart.

The paper is organized as follows. In Section 2, we collect the necessary basics of generalized complex geometry. Section 3 is devoted to finding the correct differential graded Lie algebra (DGLA for short) governing the deformation problem. We show how to define the differential and bracket at the level of equivariant objects over the principal bundle **P** and then descend to the base manifold M. It turns out that the resulting DGLA is an extension of the DGLA controlling deformations of the underlying generalized complex structure by the DGLA controlling traditional deformations of the generalized holomorphic structure. This results in the infinitesimal deformation theory being described by a long exact sequence of cohomology groups (cf. Theorem 3.6). In Section 4 we present the Maurer–Cartan equation and investigate it in detail in the cases where the underlying generalized complex structure is actually symplectic or complex. Some new possibilities occur, which are missing in the existing literature. Section 5 is devoted to proving the deformation theorem (cf. Theorem 5.3). As the procedure is rather standard, we only outline the proof. Examples are presented, in which the obstruction vanishes.

2. Some preliminaries

We collect the basic material concerning generalized complex structures and generalized holomorphic structures. The most relevant references are [4,6,8,10]. In this paper, M will always be a connected orientable smooth $2m$-manifold.

Generalized geometry is the geometry related to the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$, or more generally, a so-called exact Courant algebroid E.

Definition 2.1. A Courant algebroid over M is a real vector bundle $E \to M$ with a bracket $[\cdot,\cdot]_c$ (Courant bracket) on $\Gamma(E)$, a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$, and an anchor map $\pi : E \to TM$, satisfying the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

- $\pi([e_1, e_2]_c) = [\pi(e_1), \pi(e_2)]$,
- $[e_1, [e_2, e_3]_c]_c = [[e_1, e_2]_c, e_3]_c + [e_2, [e_1, e_3]_c]_c$,
- $[e_1, fe_2]_c = f[e_1, e_2]_c + (\pi(e_1)f)e_2$,
- $\pi(e_1)\langle e_2, e_3\rangle = \langle [e_1, e_2]_c, e_3\rangle + \langle e_2, [e_1, e_3]_c\rangle$,
- $[e_1, e_1]_c = \frac{1}{2}\mathcal{D}\langle e_1, e_1\rangle$,

where $\mathcal{D} = \pi^* \circ d : C^\infty(M) \to \Gamma(E)$ (E and E^* are identified using $\langle\cdot,\cdot\rangle$).

Definition 2.2. E is called exact if it is an extension of TM by T^*M, i.e. the sequence

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0$$

is exact.

Courant algebroids encountered in this paper are all exact and called Courant algebroids for short. Given E, one can always find an isotropic right splitting $s : TM \to E$, which has a curvature form $H \in \Omega^3_{cl}(M)$ defined by

$$H(X, Y, Z) = \langle [s(X), s(Y)]_c, s(Z)\rangle, \quad X, Y, Z \in \Gamma(TM).$$

By the bundle isomorphism $s + \pi^* : TM \oplus T^*M \to E$, the Courant algebroid structure can be transported onto $\mathbb{T}M$. Then the pairing $\langle\cdot,\cdot\rangle$ is the natural one, i.e. $\langle X + \xi, Y + \eta\rangle = \xi(Y) + \eta(X)$, and the Courant bracket is

$$[X + \xi, Y + \eta]_H = [X, Y] + L_X\eta - \iota_Y d\xi + \iota_Y\iota_X H,$$

called the H-twisted Courant bracket. Different splittings are related by B-field transforms, i.e. $e^B(X + \xi) = X + \xi + \iota_X B$, where B is a 2-form.

A Courant algebroid E has more symmetries than the tangent bundle; in particular, the left adjoint action by a section of E gives rise to an infinitesimal inner automorphism of E.

An isotropic subbundle $A \subset E$ is called a generalized distribution and called integrable if it is involutive w.r.t. the Courant bracket. An integrable maximal generalized distribution L is called a Dirac structure. These notions can be complexified and what interests us here is the following complex Dirac structure[1]:

1 We use $V_{\mathbb{C}}$ to denote the complexification of a real vector space or bundle V.

Definition 2.3. A generalized complex structure in E is an orthogonal complex structure $\mathbb{J}$ of E, such that the i-eigenbundle $L \subset E_{\mathbb{C}}$ of $\mathbb{J}$ is integrable. We call $(M, \mathbb{J})$ a generalized complex manifold.

Two extreme generalized complex structures are symplectic and complex structures. Let $E = \mathbb{T}M$ with $H = 0$. If ω is a symplectic structure, then $L = \{X - i\omega(X) | X \in T_{\mathbb{C}}M\}$; if J is a complex structure, then $L = T_{0,1} \oplus T^*_{1,0}$. A more complicated example is a holomorphic Poisson manifold (M, J, β). In this case, $L = \{X + \xi + \beta(\xi) | X + \xi \in T_{0,1} \oplus T^*_{1,0}\}$.

We will use $\mathbb{J}$ and L interchangeably to label the generalized complex structure under consideration. By the pairing we can identify L^* with $\bar{L}$, and a differential $d_L : \Gamma(\wedge^k \bar{L}) \to \Gamma(\wedge^{k+1} \bar{L})$ can be defined: for $\sigma \in \Gamma(\wedge^k \bar{L})$, $a_i \in \Gamma(L)$,

$$d_L\sigma(a_0, \ldots, a_k) = \sum_i (-1)^i \pi(a_i)\sigma(a_0, \ldots, \widehat{a_i}, \ldots, a_k) + \sum_{i<j} (-1)^{i+j} \sigma([a_i, a_j]_c, a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_k), \tag{2.1}$$

where $\widehat{a_i}$ means a_i is omitted. Since L is involutive, $d_L^2 = 0$ and we get the following elliptic complex

$$0 \longrightarrow C^\infty(M) \xrightarrow{d_L} \Gamma(\bar{L}) \xrightarrow{d_L} \Gamma(\wedge^2 \bar{L}) \xrightarrow{d_L} \cdots \xrightarrow{d_L} \Gamma(\wedge^{2m} \bar{L}) \longrightarrow 0. \tag{$\mathcal{C}_1$}$$

This is just the deformation complex of the underlying generalized complex structure. The corresponding cohomology groups are denoted by $H^k(\bar{L}, d_L)$, but for our purpose, we decrease the degree of each term in the complex by 1 and denote the cohomology groups by $\tilde{H}^k(\bar{L}, d_L)$, i.e.

$$\tilde{H}^k(\bar{L}, d_L) = H^{k+1}(\bar{L}, d_L).$$

The analogue of holomorphic structures is defined as follows:

Definition 2.4 ([4]). Let L be a generalized complex structure in E and V a complex vector bundle over M. An L-connection D in V is a differential operator $\mathrm{D} : \Gamma(V) \longrightarrow \Gamma(\bar{L} \otimes V)$ satisfying

$$\mathrm{D}(fs) = d_L f \otimes s + f\mathrm{D}s, \quad s \in \Gamma(V), f \in C^\infty(M).$$

If D is flat, i.e. $\mathrm{D}^2 = 0$, it is called a generalized holomorphic structure and (V, D) is called a generalized holomorphic vector bundle.

As was implied in Section 1, for a symplectic manifold, a generalized holomorphic vector bundle is a flat one, while for a complex manifold, a generalized holomorphic vector bundle is a co-Higgs bundle, i.e. a holomorphic vector bundle V together with a holomorphic section ϕ of $T_{1,0} \otimes \mathrm{Hom}(V)$ such that $[\phi, \phi] = 0$. If M is a holomorphic Poisson manifold, then a generalized holomorphic vector bundle is precisely a holomorphic Poisson module [4].

For a generalized holomorphic vector bundle (V, D), since $\mathrm{D}^2 = 0$, we have the following elliptic complex

$$0 \longrightarrow \Gamma(V) \xrightarrow{\mathrm{D}} \Gamma(\bar{L} \otimes V) \xrightarrow{\mathrm{D}} \Gamma(\wedge^2 \bar{L} \otimes V) \xrightarrow{\mathrm{D}} \cdots \xrightarrow{\mathrm{D}} \Gamma(\wedge^{2m} \bar{L} \otimes V) \longrightarrow 0. \tag{$\mathcal{C}_2$}$$

This complex controls the deformation of D in the traditional sense. We denote the corresponding cohomology groups by $H^k(V)$ if the underlying D is clear.

We also recall a modest knowledge of extended group actions. A detailed account can be found in [8].

Let E be a Courant algebroid over M. Let G be a connected real Lie group and $\mathfrak{g}$ its Lie algebra. Assume that G acts freely and properly on M on the right, infinitesimally described by $\tilde{\psi} : \mathfrak{g} \to \Gamma(TM)$. An isotropic trivially extended action of $\tilde{\psi}$ is a linear map $\psi : \mathfrak{g} \to \Gamma(E)$ covering $\tilde{\psi}$ such that

$$[\psi(a), \psi(b)]_c = -\psi([a, b]), \quad a, b \in \mathfrak{g}$$

and the subbundle $K \subset E$ generated by $\psi(\mathfrak{g})$ is isotropic. It is also required that ψ integrate to a G-action on E.

Let $K^\perp$ be the orthogonal complement of K in E w.r.t. $\langle \cdot, \cdot \rangle$. Then the Courant algebroid structure on E descends to M/G, and the resulting Courant algebroid is $\frac{K^\perp}{K}/G$. Under some mild conditions, a complex Dirac structure $L \subset E_{\mathbb{C}}$ also descends to a complex Dirac structure $L_r \subset \frac{K_{\mathbb{C}}^\perp}{K_{\mathbb{C}}}/G$:

$$L_r := \frac{L \cap K_{\mathbb{C}}^\perp + K_{\mathbb{C}}}{K_{\mathbb{C}}}.$$

This procedure actually also applies to generalized distributions which are not necessarily maximal, and this is what happens in this paper.

Definition 2.5. A generalized principal G-bundle over M is a triple $(\mathbf{P}, \mathbf{E}, \psi)$ such that

(i) $p : \mathbf{P} \to M$ is a usual principal G-bundle,
(ii) $\mathbf{E}$ is a Courant algebroid over $\mathbf{P}$ and ψ is an isotropic trivially extended action on $\mathbf{E}$ of the natural G-action on $\mathbf{P}$.

As was mentioned, $\mathbf{E}$ descends to a Courant algebroid E over M. By π_1 and π_2 denote the anchor maps of $\mathbf{E}$ and E respectively. To distinguish the two Courant brackets, we suppress the subscript c of the Courant bracket on $\Gamma(\mathbf{E})$. If the underlying $\mathbf{E}$ and ψ are clear, we shall just refer to $\mathbf{P}$ as a generalized principal G-bundle.

Example 2.6. Let $p : \mathbf{P} \to M$ be a usual principal G-bundle over M and $H \in \Omega^3_{cl}(M)$. Let $\mathbf{E}$ be $\mathbb{T}\mathbf{P}$ equipped with the $p^*(H)$-twisted Courant bracket. G acts on $\mathbf{E}$ in the ordinary manner, i.e. $g \cdot (X, \xi) = (g_*(X), g^{*-1}(\xi))$. This way, $\mathbf{P}$ becomes a generalized principal G-bundle. Note that the reduced Courant algebroid is just $\mathbb{T}M$ with the H-twisted Courant bracket. In this paper, when viewing a usual principal G-bundle as a generalized one, for convenience, we always follow this procedure with $H = 0$.

For a generalized principal G-bundle $(\mathbf{P}, \mathbf{E}, \psi)$, a right splitting s is called admissible if $\psi(\mathfrak{g}) \subset \Gamma(s(T\mathbf{P}))$. Use such a splitting to identify $\mathbf{E}$ with $\mathbb{T}\mathrm{P}$. Then for $a \in \mathfrak{g}$, $\psi(a) = X_a$, i.e. the vector field generated by a. For later use, we give the following lemma.

Lemma 2.7. *For sufficiently small open set $W \subset M$, there are G-invariant admissible isotropic splittings for $(\mathbf{P}, \mathbf{E}, \psi)$ restricted on W. The B-field relating two such splittings is a basic 2-form.*

Proof. Assume $\mathbf{P}|_W$ to be of the form $W \times G$. It is not hard to find that admissible isotropic splittings do exist and the B-field relating two such splittings is a horizontal 2-form. Let s be an admissible splitting and $\mathbf{H}$ be the curvature. Then $\iota_{X_a}\iota_{X_b}\mathbf{H} = 0$ for $a, b \in \mathfrak{g}$.

Let $\{y^i\}$ be the coordinates of W. ∂_{y^i} can be viewed as a section of $\mathbb{T}\mathrm{P}|_W$ over the set $W \times \{e\} \subset \mathrm{P}|_W$, and by G-action it extends to the whole of $\mathrm{P}|_W$. The result Y_i is of the form $\partial_{y^i} + \xi_i$, where ξ_i is a 1-form over $\mathrm{P}|_W$.[2] By construction, Y_i is G-invariant and therefore

$$L_{X_a}\xi_i + \iota_{\partial_{y^i}}\iota_{X_a}\mathbf{H} = 0, \quad a \in \mathfrak{g}.$$

This equation, together with $\iota_{X_a}\iota_{X_b}\mathbf{H} = 0$ and the fact that $\xi_i|_{W\times\{0\}} = 0$, implies that ξ_i is a horizontal 1-form.

Let $\{Z_\alpha\}$ be a G-invariant frame of K. Then $\{\partial_{y^i}; Z_\alpha\}$ is a frame of $T\mathbf{P}|_W$ and $\{Y_i; Z_\alpha\}$ gives rise to a G-invariant splitting. Let $\{dy^i; Z^*_\alpha\}$ be the dual frame of $T^*\mathbf{P}|_W$. Then it is easy to see that the new splitting relates to the old by the B-field $\frac{1}{2}\xi_i(\partial_{y_j})dy^i dy^j$, which is horizontal. Therefore the new splitting is both admissible isotropic and G-invariant.

If s and s' are both G-invariant admissible isotropic splittings, relating to each other by a B-field B, then the corresponding curvatures satisfy $\mathbf{H}' = \mathbf{H} - dB$. One must have $\iota_{y^i}\iota_{X_a}dB = 0$ and $\iota_{X_a}\iota_{X_b}dB = 0$. As B is horizontal, these imply that $L_{X_a}B = 0$, i.e. B is basic. □

Remark. The proof of the lemma implies that for a general generalized principal G-bundle, at least locally, it is safe to assume that G acts on $\mathbf{E}$ in the manner described in Example 2.6: For a G-invariant admissible isotropic splitting, ∂_{y^i} as a section of $\mathbf{E}$ is G-invariant. Then

$$0 = [X_a, \partial_{y^i}]_{\mathbf{H}} = [X_a, \partial_{y^i}] + \iota_{\partial_{y^i}}\iota_{X_a}\mathbf{H} = \iota_{\partial_{y^i}}\iota_{X_a}\mathbf{H}.$$

This, together with $\iota_{X_a}\iota_{X_b}\mathbf{H} = 0$, implies $\iota_{X_a}\mathbf{H} = 0$. Since $\mathbf{H}$ is closed, then it is also basic, i.e. of the form $p^*(H)$ for a 2-form H on W. Then for a vector field X over $\mathbf{P}|_W$, $[X_a, X]_{\mathbf{H}} = [X_a, X]$.

Now let G be a connected complex Lie group, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_h \oplus \mathfrak{g}_a$, i.e. the sum of holomorphic and anti-holomorphic components. If $(\mathbf{P}, \mathbf{E}, \psi)$ is a generalized principal G-bundle, then $\psi(\mathfrak{g}_a)$ and $\psi(\mathfrak{g}_h)$ generate K_a and K_h respectively, both subbundles of $K_{\mathbb{C}}$.

Definition 2.8 (*[6]*)**.** Let $(\mathbf{P}, \mathbf{E}, \psi)$ be a generalized principal G-bundle over M and $A \subset \mathbf{E}_{\mathbb{C}}$ a G-invariant generalized distribution such that

(i) $K_a \subset A \subset K^{\perp}_{\mathbb{C}}$,
(ii) $A \oplus \overline{A} = K^{\perp}_{\mathbb{C}}$,
(iii) A descends to a generalized complex structure $L \subset E_{\mathbb{C}}$.

Then A is called an almost generalized holomorphic structure w.r.t. L. If moreover A is integrable, it is called a generalized holomorphic structure.

A is essentially a complex structure in $K^{\perp}$, which restricted to K is minus the canonical complex structure induced from the complex group G. Condition (iii) means $L := \frac{A \cap K^{\perp}_{\mathbb{C}} + K_{\mathbb{C}}}{K_{\mathbb{C}}}/G$ is a generalized complex structure in E. It is worth mentioning that there is a natural G-isomorphism

$$\widetilde{\Pi} : A/K_a \to (A \cap K^{\perp}_{\mathbb{C}} + K_{\mathbb{C}})/K_{\mathbb{C}}. \tag{2.2}$$

[2] Since the G-action preserves the bilinear form, one must have $\xi_i(\partial_{y^j}) + \xi_j(\partial_{y^i}) = 0$.

For some examples of generalized holomorphic principal bundles, see [6]. A remarkable observation of [6] is that a G-invariant generalized distribution A, satisfying conditions (i), (ii), always descends to a generalized distribution A_r such that $A_r \oplus \overline{A}_r = E_{\mathbb{C}}$. If further A is integrable, so is A_r. Therefore if an integrable G-invariant generalized distribution A satisfies (i) and (ii), then (iii) follows automatically. This suggests that, if we deform A in such a way that conditions (i), (ii) are preserved and the resulting generalized distribution A' is still integrable, then A' is a generalized holomorphic structure w.r.t. an underlying generalized complex structure $\mathbb{J}'$, possibly differing from the original $\mathbb{J}$. This will be the very starting point of our deformation theory in the coming sections.

Given an almost generalized holomorphic structure A and a finite dimensional holomorphic representation (ρ, U), a canonical L-connection in $V := \mathrm{P} \times_\rho U$ can be defined. Recall that $\Gamma(V)$ can be identified with $\Gamma(\mathbf{P}, U)^G$, the space of G-equivariant U-valued functions on $\mathbf{P}$. For $s \in \Gamma(V)$, by $\tilde{s}$ denote the corresponding element in $\Gamma(\mathbf{P}, U)^G$. One can define the L-connection D as follows: let $x \in M$ and W be a small neighbourhood of x. For $a \in \Gamma(L)$, choose $\hat{a} \in \Gamma(A|_{p^{-1}(W)})^G$, which descends to $a|_W$. Then

$$(\mathrm{D}_a s)(x) := [(q, (\pi_1(\hat{a})\tilde{s})(q))], \tag{2.3}$$

where q is any point in $p^{-1}(x)$. If further A is integrable, then D is a generalized holomorphic structure in V. In particular, if ρ is the adjoint representation of G on $\mathfrak{g}_h$, then we get the canonical generalized holomorphic vector bundle $(\mathfrak{g}^A, \mathrm{D})$.

3. The underlying DGLA

As can be expected from general deformation theory [11], deformations of a generalized holomorphic structure are governed by a DGLA, through a Maurer–Cartan type equation. The purpose of this section is then to explore the underlying DGLA in detail, also preparing for the next sections.

Definition 3.1. A DGLA $(S, [\cdot,\cdot], d)$ is the data of a $\mathbb{Z}$-graded complex vector space $S = \bigoplus_{i\in\mathbb{Z}} S^i$ with a bilinear bracket $[\cdot,\cdot] : S \times S \to S$ and a linear map $d \in \mathrm{Hom}(S, S)$ satisfying the following conditions:

(i) $[\cdot,\cdot]$ is graded skew-symmetric, i.e. $[S^i, S^j] \subset S^{i+j}$ and

$$[a, b] + (-1)^{|a||b|}[b, a] = 0$$

for every a, b homogeneous ($|a|$ is the degree of a).

(ii) Each triple of homogeneous elements a, b, c satisfies the graded Jacobi identity

$$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]].$$

(iii) $d(S^i) \subset S^{i+1}$, $d^2 = 0$ and $d[a, b] = [da, b] + (-1)^{|a|}[a, db]$.

$\mathcal{A}_1 = \bigoplus_{k=0}^{2m} \Gamma(\wedge^k \overline{L})$ is an example of DGLA, where the degree of $\Gamma(\wedge^k \overline{L})$ is $k-1$, the bracket is the Schouten bracket[3] (still denoted by $[\cdot,\cdot]_c$) associated to the Lie algebroid $\overline{L}$, and the differential is just d_L; $\mathcal{A}_2 = \bigoplus_{k=0}^{2m} \Gamma(\wedge^k \overline{L} \otimes \mathfrak{g}^A)$ is another example, where for $a_i \in \Gamma(\wedge^{k_i} \overline{L})$, $F_i \in \Gamma(\mathfrak{g}^A)$, $i = 1, 2$, the bracket is characterized by

$$[a_1 \otimes F_1, a_2 \otimes F_2] = (a_1 \wedge a_2) \otimes [F_1, F_2], \tag{3.1}$$

and the differential is D. In this section, we will find a new DGLA, which, to some extent, is a combination of the above two.

Let $(M, \mathbb{J})$ be a compact generalized complex manifold, and A_0 a generalized holomorphic structure w.r.t $\mathbb{J}$ in a generalized principal G-bundle $\mathbf{P}$ over M. We continue to use the notation of the previous section.

Definition 3.2. A deformation of A_0 is a G-equivariant bundle homomorphism $\varepsilon : A_0 \to \overline{A}_0$ which vanishes on K_a and is skew, i.e. $\langle a, \varepsilon(b)\rangle + \langle \varepsilon(a), b\rangle = 0$ for $a, b \in A_0$.

ε can also be viewed as a homomorphism from $\mathfrak{L} := A_0/K_a$ to $\overline{A}_0$, i.e. $\varepsilon \in \Gamma(\mathfrak{L}^* \otimes \overline{A}_0)^G$. For ε small enough, the graph A_ε of ε satisfies conditions (i), (ii) in Definition 2.8. ε is called integrable if A_ε is involutive.

By abuse of notation, we will label an element $a \in \mathfrak{L}$ and its representative in A_0 by the same letter. No ambiguity would arise from the context. We call an element $s \in \wedge^k \mathfrak{L}^* \otimes \overline{A}_0$ skew if for $a_i \in A_0$, $i = 1, \ldots, k+1$

$$\langle s(a_1, \ldots, a_k), a_{k+1}\rangle + \langle a_1, s(a_{k+1}, a_2, \ldots, a_k)\rangle = 0.$$

Let $\overline{A}_k \subset \wedge^k \mathfrak{L}^* \otimes \overline{A}_0$ ($k = 1, \ldots, 2m$) be the subbundle consisting of skew elements. We also denote the trivial complex line bundle over $\mathbf{P}$ by $\overline{A}_{-1}$.

Consider $\mathfrak{A} = \bigoplus_{k=-1}^{2m} \Gamma(\overline{A}_k)^G$ and set the degree of $\Gamma(\overline{A}_k)$ to be k. We show there is a differential and a bracket on $\mathfrak{A}$, which are ingredients to define a DGLA.

[3] We refer the reader to [10] for all properties of this Schouten algebra, which will be used without mention.

Since for $f \in \Gamma(\overline{A}_{-1})^G$, $df \in \Gamma(K_{\mathbb{C}}^{\perp})$, by $d^{\overline{A}}f$ we denote the $\overline{A}$-component of df. Also for $a, b \in \Gamma(K_{\mathbb{C}}^{\perp})^G$, $[a, b] \in \Gamma(K_{\mathbb{C}}^{\perp})^G$ and the $\overline{A}$-component $[a, b]^{\overline{A}}$ makes sense.

Define $\delta : \Gamma(\overline{A}_{-1})^G \to \Gamma(\overline{A}_0)^G$ by $\delta f = d^{\overline{A}}f$. Define $\delta : \Gamma(\overline{A}_0)^G \to \Gamma(\overline{A}_1)^G$ by

$$\delta s(a) = -[s, a]^{\overline{A}}, \quad a \in \Gamma(A)^G, \tag{3.2}$$

or generally, define $\delta : \Gamma(\overline{A}_k)^G \to \Gamma(\overline{A}_{k+1})^G$ by

$$\begin{aligned}\delta s(a_0, \ldots, a_k) &= \sum_i (-1)^i [a_i, s(a_1, \ldots, \widehat{a_i}, \ldots, a_k)]^{\overline{A}} \\ &\quad + \sum_{i<j} (-1)^{i+j} s([a_i, a_j], a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_k) + (-1)^{k+1} d^{\overline{A}} \langle s(a_0, \ldots, a_{k-1}), a_k \rangle.\end{aligned}$$

Lemma 3.3. *δ is well-defined, and the sequence*

$$0 \longrightarrow \Gamma(\overline{A}_{-1})^G \xrightarrow{\delta} \Gamma(\overline{A}_0)^G \xrightarrow{\delta} \Gamma(\overline{A}_1)^G \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Gamma(\overline{A}_{2m})^G \longrightarrow 0$$

is a complex.

Proof. It is easy to see that if s is G-invariant, so is δs. By definition, $\delta s(a_0, \ldots, a_k)$ is skew-symmetric w.r.t. its arguments. We prove that it is $C^\infty(\mathbf{P})^G$-linear. It suffices to prove $\delta s(fa_0, \ldots, a_k) = f\delta s(a_0, \ldots, a_k)$:

$$\begin{aligned}\delta s(fa_0, \ldots, a_k) &= f\delta s(a_0, \ldots, a_k) + d^{\bar{A}}f \langle a_0, s(a_1, \ldots, a_k)\rangle + \sum_{i\neq 0} (-1)^i \pi_1(a_i) f s(a_0, a_1, \ldots, \widehat{a_i}, \ldots, a_k) \\ &\quad + \sum_{i\neq 0} (-1)^{i+1} \pi_1(a_i) f s(a_0, a_1, \ldots, \widehat{a_i}, \ldots, a_k) + (-1)^{k+1} d^{\bar{A}} f \langle s(a_0, \ldots, a_{k-1}), a_k\rangle \\ &= f\delta s(a_0, \ldots, a_k).\end{aligned}$$

To see δs is skew, we only prove the case in which s is of degree 0, since the general case only involves more complicated notation:

$$\begin{aligned}\langle \delta s(a_0), a_1\rangle &= -\langle [s, a_0]^{\overline{A}}, a_1\rangle \\ &= -\pi_1(s)\langle a_0, a_1\rangle + \langle a_0, [s, a_1]\rangle \\ &= -\langle a_0, \delta s(a_1)\rangle.\end{aligned}$$

We also will not prove the formula $\delta^2 = 0$ completely, and this shall be done at the level of complexes over M (Lemma 3.5). However, to show how the Jacobi identity plays its role, we prove $\delta^2 s = 0$ for s of degree 0[4]:

$$\begin{aligned}\delta^2 s(a_0, a_1) &= [a_0, \delta s(a_1)]^{\overline{A}} + [\delta s(a_0), a_1]^{\overline{A}} - \delta s([a_0, a_1]) \\ &= -[a_0, [s, a_1]^{\overline{A}}]^{\overline{A}} - [[s, a_0]^{\overline{A}}, a_1]^{\overline{A}} + [s, [a_0, a_1]]^{\overline{A}} \\ &= 0,\end{aligned}$$

where the last equality holds due to the Jacobi identity. □

To define the bracket on $\bigoplus_k \Gamma(\overline{A}_k)^G$, we first remark that $\bigoplus_k \overline{A}_k$ has another characterization:

Lemma 3.4. *Let $I(\wedge^2 K_h)$ be the ideal generated by $\wedge^2 K_h$ in the exterior algebra $\wedge^\bullet \overline{A}_0$. Then*

$$\bigoplus_k \overline{A}_k \cong \wedge^\bullet \overline{A}_0 / I(\wedge^2 K_h),$$

and $\overline{A}_k$ is precisely identified with $\wedge^{k+1} \overline{A}_0 / I(\wedge^2 K_h)$.

Proof. At any point $q \in \mathbf{P}$, by the natural pairing on $\mathbf{E}$, each element in the fibre at q of $\wedge^{k+1} \overline{A}_0 / I(\wedge^2 K_h)$ defines a skew element in $\wedge^k \mathfrak{L}_q^* \otimes \overline{A}_{0q}$. It is easy to see this correspondence is injective. Counting dimensions then leads to the claimed identification. □

4 For degree -1, this holds due to the general fact that the left adjoint action by an exact 1-form on $\Gamma(E)$ is trivial.

Since $\overline{A}_0$ is a G-equivariant Lie algebroid over $\mathbf{P}$, the Schouten bracket makes sense on G-invariant sections of $\wedge^\bullet \overline{A}_0$. Note that

$$[\Gamma(\wedge^\bullet \overline{A}_0)^G, \Gamma(I(\wedge^2 K_h))^G] \subset \Gamma(I(\wedge^2 K_h))^G.$$

The Schouten bracket then descends to $\Gamma(\wedge^\bullet \overline{A}_0/I(\wedge^2 K_h))^G$, or $\bigoplus_k \Gamma(\overline{A}_k)^G$. This way, we get the necessary bracket, which we denote by $[\cdot,\cdot]_A$.

What is most relevant to us in this paper is the case of degree 1. Let $s, t \in \Gamma(\overline{A}_1)^G$. Then it can be checked that

$$[s,t]_A(a,b) = [s(a), t(b)] + [t(a), s(b)] - s([a, t(b)]^A + [t(a), b]^A) - t([a, s(b)]^A + [s(a), b]^A).$$

In particular,

$$[s,s]_A(a,b) = 2[s(a), s(b)] - 2s([a, s(b)]^A + [s(a), b]^A).$$

By taking the quotient, all G-equivariant vector bundles over $\mathbf{P}$ descend to M. Let $\mathfrak{g}^A$, $\overline{\mathrm{A}}_k$ and L^A be obtained this way from K_h, $\overline{A}_k$ and $\mathfrak{L}$ respectively. Due to the G-isomorphism $\widetilde{\Pi}$, L^A can be canonically identified with L, and $\overline{\mathrm{A}}_k$ with a subbundle of $\wedge^k \overline{L} \otimes \overline{\mathrm{A}}_0$. Note that $\overline{\mathrm{A}}_0$ is an extension lying in $K_{\mathbb{C}}^{\perp}/G$ of $\overline{L}$ by the adjoint bundle $\mathfrak{g}^A$:

$$0 \longrightarrow \mathfrak{g}^A \xrightarrow{i} \overline{\mathrm{A}}_0 \xrightarrow{\Pi} \overline{L} \longrightarrow 0. \tag{3.3}$$

Generally $\overline{\mathrm{A}}_k$, $k = 0, 1, \ldots, 2m$, is an extension of $\wedge^{k+1}\overline{L}$ by $\wedge^k \overline{L} \otimes \mathfrak{g}^A$:

$$0 \longrightarrow \wedge^k \overline{L} \otimes \mathfrak{g}^A \xrightarrow{i} \overline{\mathrm{A}}_k \xrightarrow{\Pi} \wedge^{k+1}\overline{L} \longrightarrow 0. \tag{3.4}$$

Remark. Note that when $k = 2m$, $\wedge^{2m+1}\overline{L} = \{0\}$. For our purpose, we also set $\wedge^{-1}\overline{L} \otimes \mathfrak{g}^A = \{0\}$ and then the above sequence also exists for this case.

Before proceeding further, we describe $\overline{\mathrm{A}}_k$ in terms of local data.

Let $\{e_\alpha\}$ be a basis of $\mathfrak{g}_h$, and $\widetilde{e}_\alpha$ (resp. $\widetilde{\bar{e}}_\alpha$) be the vector fields generated by e_α (resp. $\bar{e}_\alpha$). Due to the remark below Lemma 2.7, over a sufficiently small open set $W \subset M$, we can choose a local section ϕ of $\mathbf{P}$ and a G-invariant admissible isotropic splitting such that we are in the situation of Example 2.6. Then $\mathbf{P}|_W$ is of the form $W \times G$ and an element $s \in \Gamma(\overline{A}_0)^G$ is locally of the form

$$b - \overline{\theta^\alpha}(b)\widetilde{\bar{e}}_\alpha + F^\alpha \widetilde{e}_\alpha, \quad b \in \Gamma(\overline{L}|_W).$$

Here, b is viewed as a section of $p^*(\bar{L})|_{\mathbf{P}|_W}$ by pull-back, $\theta^\alpha \in \Gamma(p^*(\bar{L})|_{\mathbf{P}|_W})$, and $\overline{\theta^\alpha}$ are the complex conjugate of θ^α; F^α are functions over $\mathbf{P}|_W$. G-invariance implies that $\theta^\alpha(\bar{b})$, F^α are holomorphic in the G-orbit directions, and form G-equivariant $\mathfrak{g}_h$-valued functions $\theta(\bar{b}) = \theta^\alpha(\bar{b})e_\alpha$, $F = F^\alpha e_\alpha$. θ restricted to ϕ is precisely the connection form of $(\mathfrak{g}^A, \mathrm{D})$ in the frame induced by ϕ.

If one chooses a new G-invariant admissible splitting related to the old by $p^*(B)$ for a 2-form B on W, then the new $(b', F') = (b - \iota_{\pi_2(b)}B, F)$. Since b and $b - \iota_{\pi_2(b)}B$ represent the same local section of L w.r.t. the corresponding local splittings of E related by B, ϕ actually provides a local splitting of the sequence (3.3). Thus, with the help of θ, a section $s \in \Gamma(\overline{\mathrm{A}}_0)$ is locally of the form (b, F), where b and F are local sections of $\overline{L}$ and $\mathfrak{g}^A$ respectively. It can be checked that, if $\phi' = \phi g$ is another local section of $\mathbf{P}$, then in the new frame induced by ϕ',

$$s = (b, \mathrm{Ad}_{g^{-1}}(F) - \iota_{\pi_2(b)}(g^{-1}dg)),$$

where $g^{-1}dg$ is the pull-back of the Maurer–Cartan form on G.

The above argument holds generally. ϕ provides a local splitting of the sequence (3.4). If $\Gamma(\overline{\mathrm{A}}_k) \ni s = (b, F)$ in the frame induced by ϕ, then in the frame induced by ϕ',

$$s = (b, \mathrm{Ad}_{g^{-1}}(F) + (-1)^{k+1}b(g^{-1}d_{\bar{L}}g)),$$

where $g^{-1}d_{\bar{L}}g$ is the L-component of $g^{-1}dg$.

Let $\{W_\alpha\}$ be an open cover of M such that $\mathbf{P}$ is trivialized on each W_α by a local section ϕ_α. If $g_{\alpha\beta}$ are the transition functions, then the identification on the overlaps $W_\alpha \cap W_\beta$

$$(b, F) \sim (b, \mathrm{Ad}_{g_{\alpha\beta}^{-1}}(F) + (-1)^{k+1}b(g_{\alpha\beta}^{-1}d_{\bar{L}}g_{\alpha\beta}))$$

gives rise to a vector bundle R_k isomorphic to $\overline{\mathrm{A}}_k$, which is, however, independent of A.

The differential δ and the bracket $[\cdot,\cdot]_A$ all have their counterparts on M. Without ambiguity we use the same symbols to denote them. This way the sequence in Lemma 3.3 descends to a complex over M:

$$0 \longrightarrow C^\infty(M) \xrightarrow{\delta} \Gamma(R_0) \xrightarrow{\delta} \Gamma(R_1) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Gamma(R_{2m}) \longrightarrow 0. \tag{$\mathcal{C}$}$$

Lemma 3.5. *The sequence ($\mathcal{C}$) is an elliptic complex.*

Proof. As the problem is local in essence, we can choose a local section ϕ of **P**, which provides special splittings for the sequences (3.4). Let θ be the connection form of $(\mathfrak{g}^A, \mathrm{D})$ associated to ϕ. Then for a function f,

$$\delta f = (d_L f, \pi_2(\theta) f).$$

For a local section (b, F) of R_0,

$$\delta(b, F) = (d_L b, d_L F + [\theta, F] + [b, \theta]_c).$$

It can be checked that generally for a local section (b, F) of R_k,

$$\delta(b, F) = (d_L b, d_L F + [\theta, F] + (-1)^k [b, \theta]_c),$$

where $[b, \theta]_c$ is the Schouten bracket of b and θ. Therefore,

$$\begin{aligned} \delta^2 s &= (d_L^2 b, \mathrm{D}(\mathrm{D}F + (-1)^k [b, \theta]_c) + (-1)^{k+1}[d_L b, \theta]) \\ &= (0, (-1)^k \mathrm{D}[b, \theta]_c + (-1)^{k+1}[d_L b, \theta]), \end{aligned}$$

where the formula $\mathrm{D}^2 = 0$ is used. Noting that

$$\begin{aligned} \mathrm{D}[b, \theta]_c &= d_L[b, \theta]_c + [\theta, [b, \theta]_c] \\ &= [d_L b, \theta]_c + (-1)^k [b, d_L \theta]_c + [\theta, [b, \theta]_c], \end{aligned}$$

and that

$$[b, [\theta, \theta]]_c = [[b, \theta]_c, \theta] + (-1)^k [\theta, [b, \theta]_c] = (-1)^k 2[\theta, [b, \theta]_c],$$

we have

$$\delta^2 s = \left(0, \left[b, d_L \theta + \frac{1}{2}[\theta, \theta]\right]_c\right) = 0,$$

where the Maurer–Cartan equation for θ is used.

Denote $\delta : \Gamma(R_k) \to \Gamma(R_{k+1})$ by δ_k , $k = -1, \ldots, 2m - 1$. Then the first two symbol maps are (R_k should be pulled back to T^*M)

$$\sigma_\xi(\delta_{-1})(f, 0) = (f \xi^{\bar{L}}, \langle \xi^L, \theta \rangle f),$$

and

$$\sigma_\xi(\delta_0)(b, F) = (\xi^{\bar{L}} \wedge b, \xi^{\bar{L}} \otimes F - \langle \xi^L, \theta \rangle b),$$

where $\xi \in T^*M$ and $\xi^{L/\bar{L}}$ are its $L/\bar{L}$-components. For general k,

$$\sigma_\xi(\delta_k)(b, F) = (\xi^{\bar{L}} \wedge b, \xi^{\bar{L}} \wedge F + (-1)^{k+1} \langle \xi^L, \theta \rangle b).$$

From this local description, it is easy to find out that the complex is elliptic. $\square$

The bracket $[\cdot, \cdot]_A$ can also be described explicitly in local terms. Let $s_i = (b_i, E_i)$ be of degree d_i, $i = 1, 2$ respectively. Then it is easy to find

$$[s_1, s_2]_A = ([b_1, b_2]_c, [b_1, E_2]_c + (-1)^{d_1 d_2 + 1}[b_2, E_1]_c - [E_1, E_2]). \tag{3.5}$$

Note that unlike δ, this bracket is actually independent of A_0.

Combining the argument in this section and the elliptic complexes ($\mathcal{C}_1$) and ($\mathcal{C}_2$), we thus obtain

Theorem 3.6. *There exists a short exact sequence of elliptic complexes:*

$$0 \longrightarrow \mathcal{C}_2 \xrightarrow{i} \mathcal{C} \xrightarrow{\Pi} \mathcal{C}_1 \longrightarrow 0.$$

In particular, we have the following long exact sequence of cohomology groups:

$$\begin{aligned} 0 &\to H^0(\mathfrak{g}^A) \to H^0(A) \to \tilde{H}^0(\bar{L}, d_L) \to H^1(\mathfrak{g}^A) \\ &\to H^1(A) \to \tilde{H}^1(\bar{L}, d_L) \to H^2(\mathfrak{g}^A) \to H^2(A) \to \tilde{H}^2(\bar{L}, d_L) \\ &\to \cdots \to H^{2m}(A) \to 0. \end{aligned}$$

Proof. Note that $H^{-1}(A) = \tilde{H}^{-1}(\bar{L}, d_L) = \mathbb{C}$ and hence that the connecting homomorphism $\tilde{H}^{-1}(\bar{L}, d_L) \to H^0(\mathfrak{g}^A)$ is zero. The long exact sequence then follows. $\square$

Proposition 3.7. *$\mathcal{A} = \bigoplus_{k=-1}^{2m} \Gamma(R_k)$ equipped with the differential δ and the bracket $[\cdot,\cdot]_A$ is a DGLA, and the short exact sequence in Theorem 3.6 is a sequence of DGLAs.*

Proof. It is convenient to use the local description. It holds obviously that the bracket is graded skew-symmetric. Let $s_i = (b_i, F_i)$ be of degree d_i, $i = 1, 2, 3$.

For the graded Jacobi identity, due to linearity and graded skew-symmetry, we need to consider 6 cases according to whether $b_i = 0$ or $F_i = 0$, but here we only treat the case where $F_1 = 0, F_2 = 0$ and $b_3 = 0$ and leave others to the interested reader. Noting that

$$[s_1, [s_2, s_3]_A]_A = (0, [b_1, [b_2, F_3]_c]_c), \qquad [s_2, [s_1, s_3]_A]_A = (0, [b_2, [b_1, F_3]_c]_c),$$

and

$$[[s_1, s_2]_A, s_3]_A = (0, [[b_1, b_2]_c, F_3]_c),$$

we need to prove that

$$[b_1, [b_2, F_3]_c]_c = [[b_1, b_2]_c, F_3]_c + (-1)^{d_1 d_2}[b_2, [b_1, F_3]_c]_c,$$

which holds because of the graded Jacobi identity in $\mathcal{A}_1$.

We next prove that δ and $[\cdot,\cdot]_A$ are compatible, i.e.

$$\delta[s_1, s_2]_A = [\delta s_1, s_2]_A + (-1)^{d_1}[s_1, \delta s_2]_A.$$

It is enough to consider the case where $F_1 = 0$ and $b_2 = 0$. Since

$$[s_1, s_2]_A = (0, [b_1, F_2]_c),$$

then

$$\delta[s_1, s_2]_A = (0, d_L[b_1, F_2]_c + [\theta, [b_1, F_2]_c]).$$

On the other side, we have

$$[\delta s_1, s_2]_A = (0, [d_L b_1, F_2]_c + (-1)^{d_1+1}[[b_1, \theta]_c, F_2]),$$

and

$$[s_1, \delta s_2]_A = (0, [b_1, d_L F_2 + [\theta, F_2]]_c).$$

We have to prove

$$[\theta, [b_1, F_2]_c] = (-1)^{d_1+1}[[b_1, \theta]_c, F_2] + (-1)^{d_1}[b_1, [\theta, F_2]]_c,$$

which certainly holds due to the compatibility of adjoint actions with wedge products in $\mathcal{A}_1$.

It is easy to find that

$$\delta i = i\mathrm{D}, \quad i([\cdot,\cdot]) = [i(\cdot), i(\cdot)]_A$$

and

$$\Pi\delta = d_L\Pi, \qquad \Pi([\cdot,\cdot]_A) = [\Pi(\cdot), \Pi(\cdot)]_c,$$

i.e. i and Π are all homomorphisms of DGLAs. Thus, the short exact sequence in Theorem 3.6 is actually a sequence of DGLAs. □

4. The Maurer–Cartan equation and some simple examples

We continue to use the notation of the previous sections. Let ε be a small deformation of a generalized holomorphic structure A. Integrability of ε means for arbitrary $a, b \in \Gamma(A)^G$

$$\varepsilon([a, \varepsilon(b)]^A + [\varepsilon(a), b]^A) = -[a, \varepsilon(b)]^{\bar{A}} - [\varepsilon(a), b]^{\bar{A}} + [\varepsilon(a), \varepsilon(b)] + \varepsilon([a, b]). \tag{4.1}$$

Denote the element in $\Gamma(R_1)$ corresponding to ε by the same label. In terms of δ and $[\cdot,\cdot]_A$, Eq. (4.1) can be reformulated as

$$\delta\varepsilon - \frac{1}{2}[\varepsilon, \varepsilon]_A = 0. \tag{4.2}$$

If $\Pi(\varepsilon) = b$, as Π is a homomorphism between DGLAs, we have

$$d_L b - \frac{1}{2}[b, b]_c = 0,$$

which is precisely the Maurer–Cartan equation for b as a deformation of the underlying generalized complex structure $\mathbb{J}$.[5]

[5] Compared with the common form, the extra minus sign is only a matter of convention.

In local terms, if $\varepsilon = (b, F)$, then Eq. (4.2) splits into two: one is the Maurer–Cartan equation for b, and the other is

$$d_L F + [\theta, F] - [b, \theta]_c - [b, F]_c + \frac{1}{2}[F, F] = 0. \tag{4.3}$$

When $b = 0$, Eq. (4.3) is the traditional Maurer–Cartan equation for deformations of a generalized holomorphic structure:

$$\mathrm{D}F + \frac{1}{2}[F, F] = 0. \tag{4.4}$$

In the following, we interpret the content of Eq. (4.3) in the context of symplectic or complex manifolds. As there are many examples of flat vector bundles, co-Higgs bundles and Poisson modules in the existing literature, we will not provide any further concrete examples of these here, but will pay more attention to other possibilities.

Example 4.1. Let (M, ω) be a symplectic manifold. By projection, we can identify L with $T_{\mathbb{C}}M$, and then $\wedge^{\cdot} \bar{L}$ with $\wedge^{\cdot} T^*_{\mathbb{C}}M$. This way, $d_L = d$, and the Schouten bracket on forms is obtained from the Schouten bracket on multi-vector fields via ω, e.g. for 1-forms ξ, η,

$$[\xi, \eta]_c = -\frac{i}{2}\omega([\omega^{-1}(\xi), \omega^{-1}(\eta)]).$$

Thus an integrable deformation of ω is described by a 2-form b, satisfying the equation $db - \frac{1}{2}[b, b]_c = 0$. Let $\mathbf{P}$ be a flat (hence generalized holomorphic, see [6]) principal G-bundle over M and $\Pi(\varepsilon) = b$. In a flat local frame, $\varepsilon = (b, F)$, and Eq. (4.3) turns to

$$dF - [b, F]_c + \frac{1}{2}[F, F] = 0.$$

Note that $d - [b,]_c$ is a twisted de Rham operator d_b. Then F can be said to determine a d_b-flat structure in $\mathbf{P}$. Another interpretation is that F determines a connection, whose curvature is $[b, F]_c$. Note that the original flat structure ($F = 0$) is also d_b-flat.

Concretely, we provide the following simple example.

Example 4.2. Let (M, ω) be a symplectic manifold and l a line bundle over M. Then the frame bundle $\mathbf{P}$ of l is a flat principal $\mathbb{C}^*$-bundle and $\mathfrak{g}^A$ is a trivial line bundle. If b is a multiple of ω and F is a closed 1-form, then $dF - [b, F]_c = 0$ always holds, because in this case for any form ξ, $[b, \xi]_c$ is a multiple of $d\xi$.

Example 4.3. Let (M, J) be a complex manifold, and $\mathbf{P}$ a holomorphic principal G-bundle over it. When viewed as a generalized holomorphic structure, locally a traditional deformation decomposes into $F = F_1 + F_2$ w.r.t. the splitting $\bar{L} = T^*_{0,1} \oplus T_{1,0}$. Eq. (4.4) then decomposes:

$$\bar{\partial}F_1 + \frac{1}{2}[F_1, F_1] = 0, \qquad \bar{\partial}F_2 + [F_1, F_2] = 0, \qquad [F_2, F_2] = 0.$$

The first equation determines a new holomorphic structure, the second means F_2 is a $\mathfrak{g}^A$-valued holomorphic vector field, and then the third means F_2 is a co-Higgs field.

A complex structure, as a generalized one, may have three primary kinds of integrable deformations, described respectively by holomorphic Poisson structures, $\bar{\partial}$-closed $(0, 2)$-forms and $T_{1,0}$-valued $(0, 1)$-forms satisfying the Maurer–Cartan equation [4]. Accordingly, if we deform the underlying generalized complex structure at the same time, deformations of a holomorphic structure are combinations of three primary cases. In the following several examples, we continue to use the notation in Example 4.3.

Example 4.4. Let ε be an integrable deformation such that $\Pi(\varepsilon) = \beta$ is a holomorphic Poisson structure. In a holomorphic frame, $\varepsilon = (\beta, F_1 + F_2)$ and Eq. (4.3) splits:

$$\bar{\partial}F_1 + \frac{1}{2}[F_1, F_1] = 0, \qquad \bar{\partial}F_2 + [F_1, F_2] - [\beta, F_1]_c = 0, \qquad \frac{1}{2}[F_2, F_2] - [\beta, F_2]_c = 0.$$

Since in another holomorphic frame related to the old by g, the new $F_1' = \mathrm{Ad}_{g^{-1}}(F_1)$, such a deformation still gives rise to a new holomorphic structure D'. It is easy to check that the second equation means, in a local frame holomorphic w.r.t. D', F_2 is a $\mathfrak{g}_h$-valued holomorphic vector field. The third equation then precisely means that $\mathfrak{g}^A$ is a holomorphic Poisson module. In particular, if $F_1 = 0$, then F_2 is holomorphic w.r.t. the original holomorphic structure; however, F_2 is not a global section of $\mathfrak{g}^A$, since in the frame related to the old by g, the new

$$F_2' = \mathrm{Ad}_{g^{-1}}(F_2) + \beta(g^{-1}dg).$$

Note that if $s = (\beta, F_1 + F_2)$ is an integrable deformation, so is $s_c = (c\beta, F_1 + cF_2)$ for any complex constant c.

Example 4.5. Let ε be an integrable deformation such that $\Pi(\varepsilon) = B$ is a $\overline{\partial}$-closed (0, 2)-form. In a holomorphic frame, $\varepsilon = (B, F_1 + F_2)$ and Eq. (4.3) splits:

$$\overline{\partial}F_1 - [B, F_2]_c + \frac{1}{2}[F_1, F_1] = 0, \qquad \overline{\partial}F_2 + [F_1, F_2] = 0, \qquad [F_2, F_2] = 0.$$

F_1 and F_2 are actually global sections of $T^*_{0,1} \otimes \mathfrak{g}^A$ and $T_{1,0} \otimes \mathfrak{g}^A$ respectively. F_1 determines a semi-connection $\overline{\partial'}$, whose curvature is $[B, F_2]_c$. The second equation means F_2 is $\overline{\partial'}$-flat, while the third is again an algebraic commutative condition. We call such a triple $(\mathfrak{g}^A, \overline{\partial'}, F_2)$ a B-twisted co-Higgs bundle. Note that when $F_2 = 0$ or B is also d-closed, B plays no essential role and we obtain usual holomorphic structures or co-Higgs bundles.

As B-twisted co-Higgs bundles are scarcely touched in the literature, we provide the following example.

Example 4.6. If M is a Kähler manifold, any $\overline{\partial}$-closed (0, 2)-form B has the form $B_h + \overline{\partial}\xi$, where B_h is the harmonic part of B and ξ is a (0, 1)-form. Therefore, for a B-twisted co-Higgs bundle, the essential part of B is $\overline{\partial}\xi$. The simplest B-twisted co-Higgs bundles are constructed from genuine co-Higgs bundles: let $(V, \overline{\partial}, \Phi)$ be a co-Higgs bundle over M, and define $\overline{\partial'} = \overline{\partial} + [\xi, \Phi]_c$. Then it is not hard to check that $(V, \overline{\partial'}, \Phi)$ is a B-twisted co-Higgs bundle. It is natural to ask whether all B-twisted co-Higgs bundles are obtained in this way, but we will not tackle this problem in this paper.

Example 4.7. Let ε be an integrable deformation such that $\Pi(\varepsilon) = \varphi$ is a deformation of the underlying complex structure in the classical sense. In a holomorphic frame, $\varepsilon = (\varphi, F_1 + F_2)$ and Eq. (4.3) splits:

$$\overline{\partial}F_1 - [\varphi, F_1]_c + \frac{1}{2}[F_1, F_1] = 0, \qquad \overline{\partial}F_2 - [\varphi, F_2] + [F_1, F_2] = 0, \qquad [F_2, F_2] = 0.$$

$\overline{\partial} - [\varphi,]_c$ is a new Dolbeault operator $\overline{\partial}_\varphi$ associated to the complex structure J_φ determined by φ. Note that in another holomorphic frame related to the old by g, since g is holomorphic w.r.t. J,

$$F_1' = \mathrm{Ad}_{g^{-1}}(F_1) + \varphi(g^{-1}\partial g) = \mathrm{Ad}_{g^{-1}}(F_1) + g^{-1}\overline{\partial}_\varphi g,$$

i.e. the first equation gives rise to a holomorphic structure D_φ w.r.t. J_φ. Then as one may expect, the remainder equations only mean that F_2 is actually a co-Higgs field w.r.t. D_φ.

5. The deformation theorem

We come back to the general theory. To obtain a finite dimensional moduli space, certain deformations should be identified. It is convenient to deal with the problem at the level of G-equivariant bundles over **P**. If $v \in \Gamma(K^\perp)^G$, v would generate a family of inner automorphisms of **E** with one parameter, which also preserves $K^\perp$ because $[\Gamma(K^\perp)^G, \Gamma(K^\perp)^G] \subset \Gamma(K^\perp)^G$. We denote this family by F_{tv}. Let A be a generalized holomorphic structure. We identify A with $F_{tv}(A) \subset K^\perp_{\mathbb{C}}$. This way, F_{tv} acts on deformations of A.

Proposition 5.1. *Let A be a generalized holomorphic structure in a generalized principal G-bundle* **P** *over $(M, \mathbb{J})$, and ε be a small deformation of A. If $v \in \Gamma(K^\perp)^G$, then for sufficiently small $t \in \mathbb{R}$, the action of F_{tv} on ε has the following form:*

$$F_{tv}(\varepsilon) = \varepsilon + t\delta v^{\overline{A}} + R(\varepsilon, tv), \tag{5.1}$$

where R satisfies

$$R(t\varepsilon, tv) = t^2\tilde{R}(\varepsilon, v, t),$$

and $\tilde{R}$ is smooth.

Proof. We follow closely the treatment of [10]. If ε is a deformation, then A_ε can be described by the following endomorphism on $K^\perp_{\mathbb{C}} = A \oplus \overline{A}$:

$$A_\varepsilon = \begin{pmatrix} 1 & -\bar{\varepsilon} \\ -\varepsilon & 1 \end{pmatrix}.$$

For $s \in \mathbb{R}$, the combined action of $s\varepsilon$ and F_{tv} on $K^\perp_{\mathbb{C}}$ is of the following form:

$$F_{tv}A_{s\varepsilon} = \begin{pmatrix} \mu & -\bar{\nu} \\ -\nu & \bar{\mu} \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix}\begin{pmatrix} 1 & -\bar{\tau} \\ -\tau & 1 \end{pmatrix} = C_\mu A_\tau.$$

Then $F_{tv}(s\varepsilon) = \tau$. Differentiating $A_\tau = C_\mu^{-1}F_{tv}A_{s\varepsilon}$ and evaluated at $(s, t) = (0, 0)$, we get

$$\dot{A}_\tau|_{(0,0)} = -\dot{C}_\mu|_{(0,0)} + \dot{F}_{tv}|_{(0,0)} + \dot{A}_{s\varepsilon}|_{(0,0)}.$$

Let $a \in \Gamma(A)^G$. We need to compute the $\overline{A}$-component of $\dot{A}_\tau|_{(0,0)}(a)$. Since C_μ preserves the splitting $K_{\mathbb{C}}^\perp = A \oplus \overline{A}$, the term $\dot{C}_\mu|_{(0,0)}$ has no contribution. By definition, the term

$$\dot{F}_{tv}|_{(0,0)}(a) = [v, a]^{\overline{A}} = -\delta v^{\overline{A}}(a),$$

and

$$\dot{A}_{s\varepsilon}|_{(0,0)}(a) = -\varepsilon(a).$$

Therefore, by Taylor's theorem,

$$\tau = s\varepsilon + t\delta v^{\overline{A}} + R(s\varepsilon, tv), \tag{5.2}$$

where the remainder $R(s\varepsilon, tv)$ is of order s^2, st and t^2. Setting $s = t$ in Eq. (5.2), we get $R(t\varepsilon, tv) = t^2\tilde{R}(\varepsilon, v, t)$; setting $s = 1$ in Eq. (5.2), our first claim then follows. $\square$

Remark. Though Proposition 5.1 is presented in terms of G-invariant objects over **P**, we will use it later in its reduced form over M without mention.

Example 5.2. The B-twisted co-Higgs bundles in Example 4.6 are just obtained by carrying out the transformation $F_{\xi+\bar{\xi}}$ on the co-Higgs bundle $(V, \overline{\partial}, \Phi)$.

We discuss briefly the corresponding Hodge theory. Choose a Hermitian structure on R_0. Then $\mathfrak{g}^A$ as a subbundle acquires a natural Hermitian structure and $\overline{L}$ can be identified with the orthogonal complement of $\mathfrak{g}^A$ in R_0. Hence Π is just the orthogonal projection onto $\overline{L}$. This way, $\wedge^k \overline{L} \otimes \mathfrak{g}^A$, $\wedge^{k+1} \overline{L}$ and R_k are all equipped with Hermitian structures.

Let δ^* be the formal adjoint of δ and $|\cdot|_l$ be the L_l^2-Sobolev norm on $\Gamma(R_k)$ induced from the Hermitian structure (l is large enough). By $\mathbb{H}^k(A)$ denote the harmonic part of $\Gamma(R_k)$, which is isomorphic to $H^k(A)$. Let $\mathcal{G}$ be the Green smoothing operator quasi-inverse to $\Delta_\delta = \delta\delta^* + \delta^*\delta$, i.e.

$$\mathcal{G} : L_l^2(R_k) \to L_{l+2}^2(R_k)$$

so that we have the following orthogonal decomposition:

$$\mathrm{Id} = \mathcal{H} + \mathcal{G}\Delta_\delta = \mathcal{H} + \Delta_\delta\mathcal{G},$$

where $\mathcal{H}$ is the orthogonal projection from $\Gamma(R_k)$ onto $\mathbb{H}^k(A)$. As in [10], we introduce the once-smoothing operator

$$\mathcal{Q} = \delta^*\mathcal{G} : L_l^2(R_k) \to L_{l+1}^2(R_k).$$

We then have $\mathcal{Q}^2 = 0$ and

$$\mathrm{Id} = \mathcal{H} + \delta\mathcal{Q} + \mathcal{Q}\delta. \tag{5.3}$$

Note that Δ_δ does not necessarily commute with Π. Hence that ε is harmonic does not necessarily imply that $\Pi(\varepsilon)$ is.

We now turn to the deformation theorem. Let A be a generalized holomorphic structure in **P** w.r.t. $\mathbb{J}$. We follow closely the lines of [12,13] concerning the deformation theorem of (generalized) complex structures.

Theorem 5.3 (*Deformation Theorem*). *There exists an open neighbourhood $W \subset H^1(A)$ containing 0, a smooth family $\widetilde{\mathfrak{M}} = \{\varepsilon_u | u \in W, \varepsilon_0 = 0\}$ of almost generalized holomorphic structures deforming A, and an analytic obstruction map $\Phi : W \to H^2(A)$ with $\Phi(0) = 0$ and $d\Phi(0) = 0$, such that the deformations in the sub-family $\mathfrak{M} = \{\varepsilon_z : z \in \mathfrak{Z} = \Phi^{-1}(0)\}$ are precisely the integrable ones. Moreover, any sufficiently small deformation ε of A is equivalent to at least one member of the family $\mathfrak{M}$. In particular, when the obstruction map vanishes, $\mathfrak{M}$ is a smooth locally complete family.*

Proof. As the proof is quite standard, for completeness, we just outline it here. For the details, one can refer to [12].

First, we construct the family $\widetilde{\mathfrak{M}}$ and show it contains the family $\mathfrak{M}$ of integrable deformations defined by a map Φ.

Define a map $\Psi : \varepsilon \to \varepsilon + \frac{1}{2}\mathcal{Q}[\varepsilon, \varepsilon]_A$ and extend it to a map

$$\Psi : L_l^2(R_1) \to L_l^2(R_1).$$

Ψ is a smooth map of the Hilbert space into itself and whose derivative at the origin is the identity. By the inverse function theorem in Banach space, Ψ^{-1} maps a neighbourhood of the origin in $L_l^2(R_1)$ smoothly and bijectively to another neighbourhood of the origin. Then for a sufficiently small $\epsilon > 0$, Ψ^{-1} takes the finite-dimensional subset of harmonic sections

$$W = \{u \in \mathbb{H}^1(A) \subset L_l^2(R_1) | |u|_l < \epsilon\}$$

into another finite dimensional set, which is precisely the family $\widetilde{\mathfrak{M}}$ (elliptic regularity implies that these elements are smooth).

Integrable elements in $\widetilde{\mathfrak{M}}$ are singled out by the zero locus of the map $\Phi : \varepsilon(u) \mapsto \mathcal{H}[\varepsilon(u), \varepsilon(u)]_A$, i.e. $\mathfrak{M} = \widetilde{\mathfrak{M}} \cap \Phi^{-1}(0)$. In fact, since

$$\varepsilon(u) + \frac{1}{2}\mathcal{Q}[\varepsilon(u), \varepsilon(u)]_A = u,$$

we have

$$\begin{aligned}\delta\varepsilon(u) + \frac{1}{2}[\varepsilon(u), \varepsilon(u)]_A &= -\frac{1}{2}\delta\mathcal{Q}[\varepsilon(u), \varepsilon(u)]_A + \frac{1}{2}[\varepsilon(u), \varepsilon(u)]_A \\ &= \frac{1}{2}(\mathcal{Q}\delta + \mathcal{H})[\varepsilon(u), \varepsilon(u)]_A,\end{aligned}$$

where Eq. (5.3) is used. Since the two terms in the last line are orthogonal, we only need to prove that $\mathcal{H}[\varepsilon(u), \varepsilon(u)]_A = 0$ implies $\mathcal{Q}\delta[\varepsilon(u), \varepsilon(u)]_A = 0$. This is correct, because due to our analysis of the underlying DGLA in the previous section, all the essential identities which lead to the corresponding conclusion concerning deformations of a generalized complex structure [10] have their counterparts here. We omit the details.

The local model $\mathfrak{M}$ has another characterization: $\mathfrak{M}$ is actually a neighbourhood of zero in the set

$$\mathfrak{M}' = \left\{\varepsilon \in \Gamma(R_1) | \delta\varepsilon + \frac{1}{2}[\varepsilon, \varepsilon]_A = \delta^*\varepsilon = 0\right\}.$$

Since no original ideas are needed to prove this claim, we omit the details and the interested reader can refer to [12].

Second, we need to prove the miniversality property of $\mathfrak{M}$. Noting that $K^{\perp}_{\mathbb{C}}/G = R_0 + \overline{R_0}$, by v^{R_0} we denote the R_0-component of $v \in \Gamma(K^{\perp}/G)$. Let P be the image of δ^* on $\Gamma(R_1)$. In the following, we prove that there are neighbourhoods of the origin $U \subset \Gamma(R_1)$ and $V \subset P$ such that for any element $\varepsilon \in U$, there exists a unique $v \in \Gamma(K^{\perp}/G)$ such that $v^{R_0} \in V$ and the gauge-fixing condition $\delta^*F_v(\varepsilon) = 0$ is satisfied. This conclusion shall lead to that every sufficiently small integrable deformation is equivalent to another in $\mathfrak{M}$.

Restricting to sufficiently small neighbourhood of the origin in $\Gamma(K^{\perp}/G)$ so that we can set $t = 1$ in Eq. (5.1). We see that $\delta^*F_v(\varepsilon) = 0$ iff

$$\delta^*\varepsilon + \delta^*\delta v^{R_0} + \delta^*R(\varepsilon, v) = 0.$$

Assuming $v^{R_0} \in P$, we obtain $\delta^*v^{R_0} = \mathcal{H}v^{R_0} = 0$, and

$$\delta^*\varepsilon + \Delta_\delta v^{R_0} + \delta^*R(\varepsilon, v) = 0.$$

Applying $\mathcal{G}$, we get

$$v^{R_0} + \mathcal{Q}\varepsilon + \mathcal{Q}R(\varepsilon, v) = 0.$$

The map

$$\Upsilon : (\varepsilon, v) \mapsto v^{R_0} + \mathcal{Q}\varepsilon + \mathcal{Q}R(\varepsilon, v)$$

is continuous from a neighbourhood of the origin in $\Gamma(R_1) \times P$ to P (all spaces are equipped with suitable L^2_l-norms), and thus can be extended to a continuous map from the completion of the domain. Note that the derivative of Υ w.r.t. v at 0 is invertible. By the implicit function theorem, for sufficiently small ε, the equation $\Upsilon(\varepsilon, v) = 0$ has a unique solution v, depending on ε smoothly. Elliptic regularity implies that this solution is smooth.

The above argument is also enough to imply that when the obstruction vanishes, $\mathfrak{M}$ is a smooth locally complete family. □

Example 5.4. Let (M, ω) be a symplectic surface and G be a reductive complex group. If **P** is a flat principal G-bundle such that $H^0(\mathfrak{g}^A) = 0$, then the obstruction vanishes: since $\tilde{H}^2(\bar{L}, d_L) \cong H^3(M, \mathbb{C})$ and we are in real dimension 2, $\tilde{H}^2(\bar{L}, d_L) = 0$; by Poincaré duality, $H^2(\mathfrak{g}^A) = 0$.

Example 5.5. Let M be a Riemann surface and **P** be a holomorphic principal G-bundle. Then $\mathfrak{g}^A$ is a holomorphic vector bundle. Since we are in complex dimension 1, $H^2(\mathfrak{g}^A) = \tilde{H}^2(\bar{L}, d_L) = 0$. Therefore all infinitesimal deformations are integrable, and since the only possible deformations of the underlying complex structure are classical ones, all possible deformations are actually co-Higgs bundles.

Example 5.6. Let V be a holomorphic vector bundle over $\mathbb{P}^2$ (with the Fubini-Study metric), viewed as a generalized holomorphic bundle. Let **P** be the frame bundle of V. In this case $\mathfrak{g}^A = \mathrm{Hom(V)}$, and $\tilde{H}^2(\bar{L}, d_L) = \bigoplus_{p+q=3} H^q(\mathbb{P}^2, \mathcal{O}(\wedge^p T_{1,0}))$. Since we are in complex dimension 2 and $\wedge^2 T_{1,0} = \mathcal{O}(3)$, we have

$$H^3(\mathbb{P}^2, \mathcal{O}) = H^0(\mathbb{P}^2, \wedge^3 T_{1,0}) = H^1(\mathbb{P}^2, \mathcal{O}(\wedge^2 T_{1,0})) = 0.$$

By the Euler sequence, we also have $H^2(\mathbb{P}^2, \mathcal{O}(T_{1,0})) = 0$. Thus, the vanishing of the obstruction space requires $H^2(\mathbb{P}^2, \mathcal{O}(\mathrm{Hom(V)})) = 0$. By Serre duality, this is equivalent to $H^0(\mathbb{P}^2, \mathcal{O}(K \otimes \mathrm{Hom(V)})) = 0$, where K is the canonical line bundle.

If the holomorphic structure of V is obtained from a Hermitian–Einstein (H–E for short) connection and the H–E constant γ is negative, then no nontrivial global holomorphic section exists [14, Chap. 2]. In our case, if V is stable, then one can find an H–E metric in V; in particular this also holds for Hom(V) with $\gamma = 0$. Since $K = \mathcal{O}(-3)$, K has a negative H–E constant, implying that $K \otimes \mathrm{Hom(V)}$ has a negative H–E constant and $H^0(\mathbb{P}^2, \mathcal{O}(K \otimes \mathrm{Hom(V)})) = 0$.

Combining these facts together, we conclude that any infinitesimal deformation of a stable vector bundle over $\mathbb{P}^2$ is integrable.

Remark. The above example is also good to illustrate how the second drawback mentioned in Section 1 may be overcome: There are fairly many holomorphic Poisson structures over $\mathbb{P}^2$, which provide nontrivial deformations of the complex structure. Then our example shows around the holomorphic structure of a stable vector bundle, there are many Poisson modules. However, this claim only depends on the cohomological information of the original holomorphic vector bundle, which is often more computable than the cohomology groups involved in the traditional deformation theory when one tries to get Poisson modules near a given one.

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