Journal of Geometry and Physics 149 (2020) 103576

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Geometric crystals and cluster ensembles in Kac–Moody setting

Yuki Kanakubo *, Toshiki Nakashima

Division of Mathematics, Sophia University, Kioicho 7-1, Chiyoda-ku, Tokyo 102-8554, Japan

ARTICLE INFO

Article history:
Received 18 May 2019
Received in revised form 13 October 2019
Accepted 5 December 2019
Available online 16 December 2019

Keywords:
Cluster ensembles
Double Bruhat cells
Geometric crystals
Crystals

ABSTRACT

For a Kac–Moody group G, double Bruhat cells $G^{u,e}$ (u is a Weyl group element) have positive geometric crystal structures. In Williams (2013), it is shown that there exist birational maps between 'cluster tori' $\mathcal{X}_\Sigma$ (resp. $\mathcal{A}_\Sigma$) and $G^{u,e}_{\mathrm{Ad}}$ (resp. $G^{u,e}$), and they are extended to regular maps from cluster $\mathcal{X}$ (resp. $\mathcal{A}$) -varieties to $G^{u,e}_{\mathrm{Ad}}$ (resp. $G^{u,e}$). The aim of this article is to compute explicit formulae for the induced geometric crystal structures on these cluster tori. As a corollary, the sets of $\mathbb{Z}^T$-valued points of these cluster varieties have crystal structures.

1. Introduction

In [4], V.V.Fock and A.B.Goncharov have introduced the "cluster ensemble", which is a pair of schemes "$\mathcal{A}$-variety $\mathcal{A}_{|\Sigma|}$" and "$\mathcal{X}$-variety $\mathcal{X}_{|\Sigma|}$" with a morphism $\mathcal{A}_{|\Sigma|} \to \mathcal{X}_{|\Sigma|}$ called an *ensemble map*. For the construction of these schemes, *cluster* $\mathcal{A}$ (resp. $\mathcal{X}$)*-tori* $\mathcal{A}_\Sigma = \{(A_i) \mid i \in I,\ A_i \in \mathbb{C}^\times\}$ (resp. $\mathcal{X}_\Sigma = \{(X_i) \mid i \in I,\ X_i \in \mathbb{C}^\times\}$) associated with *seeds* $\Sigma = (I, I_0, B, d)$ (see Section 3) are defined. The schemes are defined as unions of cluster tori which are glued by the following *mutations* $\mu_k : \mathcal{A}_\Sigma \to \mathcal{A}_{\Sigma'}, \mathcal{X}_\Sigma \to \mathcal{X}_{\Sigma'}$

$$\mu_k^*(A'_{M_k(i)}) = \begin{cases} A_i & \text{if } i \neq k, \\ \frac{\prod_{b_{k,j}>0} A_j^{b_{k,j}} + \prod_{b_{k,j}<0} A_j^{-b_{k,j}}}{A_k} & \text{if } i = k, \end{cases}$$

$$\mu_k^*(X'_{M_k(i)}) = \begin{cases} X_i X_k^{[b_{i,k}]_+}(1+X_k)^{-b_{i,k}} & \text{if } i \neq k, \\ X_k^{-1} & \text{if } i = k, \end{cases}$$

where $\Sigma' = (I', I_0, B', d')$ is a new seed, and the map $M_k : I \to I'$ is as in Section 3.

Furthermore, in [4] Fock and Goncharov presented a conjecture on "tropical duality" between these two cluster varieties. To be more precise, the conjecture claimed that the "universal positive Laurent polynomial ring" on $\mathcal{A}_{|\Sigma|}$(resp. $\mathcal{X}_{|\Sigma|}$) is described by the positive summation of points in the set of $\mathbb{Z}$-valued points (see 3.2) $\mathcal{X}_{|\Sigma^\vee|}(\mathbb{Z}^T)$ (resp. $\mathcal{A}_{|\Sigma^\vee|}(\mathbb{Z}^T)$), where $\Sigma^\vee$ is the Langlands dual seed of Σ [4]. Though in [6], unfortunately, counterexamples for the conjecture have been found, and in [7], the conjecture has been refined to be more valid, which is now called "full Fock–Goncharov conjecture". It seems to be still generally open except for several special cases.

* Corresponding author.
E-mail addresses: j_chi_sen_you_ky@eagle.sophia.ac.jp (Y. Kanakubo), toshiki@sophia.ac.jp (T. Nakashima).

https://doi.org/10.1016/j.geomphys.2019.103576
0393-0440/

As topics related to cluster varieties, in [1], Berenstein, Fomin and Zelevinsky proved that the coordinate ring of the double Bruhat cell $G^{u,v}$ holds an upper cluster algebra structure in the case G is a simply connected, connected, semisimple complex algebraic group, where a certain family of generalized minors plays a role of an initial cluster. In [5], Fock and Goncharov show that double Bruhat cells of the adjoint form of a semisimple algebraic group have cluster $\mathcal{X}$-variety structures. H. Williams generalized these results, and considered the relations between cluster ensembles associated with Weyl group elements and double Bruhat cells in Kac–Moody setting [16]. Therein, he also constructed the regular map p_M from $\mathcal{A}_{|\Sigma|}$ to $\mathcal{X}_{|\Sigma|}$ compatible with all cluster mutations (Proposition 3.4), which plays a role of the ensemble map in the context of [16].

In [3], A.Berenstein and D.Kazhdan have initiated the theory of "geometric crystals", which is aimed at constructing a geometric analogue of the Kashiwara's crystal base theory [9] on a variety birationally isomorphic to a split torus. In [14], the second author extended this notion to the Kac–Moody setting and gave some explicit forms of geometric crystals on Schubert/Bruhat cells. Geometric crystals have a bunch of remarkable properties, in particular, the fact that positive geometric crystals can be transferred to the Langlands dual Kashiwara's crystal bases by the "tropicalization" procedure is one of the most crucial features. More precisely, there exists a functor *Trop* (denoted by $\mathcal{UD}$ in [14,15]) from a category of split tori to the category of sets, and via this functor a geometric crystal structure on a torus induces a crystal structure on its set of co-characters. It is an interesting problem to find morphisms corresponding to operators (e.g. the star operator $*$) on crystals.

The aim of this article is to compute explicit formulae for the induced geometric crystal structures on the cluster tori $\mathcal{X}_\Sigma$ and $\mathcal{A}_\Sigma$ of [16]. These main results are presented as Theorems 5.1, 6.1 and 6.3. Due to this, we can integrate the geometric crystal theory with cluster theory. As a corollary of them, we see that the sets of $\mathbb{Z}^T$-valued points $\mathcal{X}_{|\Sigma|}(\mathbb{Z}^T)$, $\mathcal{A}_{|\Sigma|}(\mathbb{Z}^T)$ have crystal structures. We expect this approach will be a guide to construct the geometric analogue of operators on crystals in terms of cluster theory (the mutations, ensemble maps, and so on).

To achieve the aim, first we define geometric crystal structures on the cluster tori $\mathcal{X}_\Sigma$, $\mathcal{A}_\Sigma$. We will see that double Bruhat cells $G^{u,e}$ and their quotients $G^{u,e}_{\mathrm{Ad}}$ have geometric crystal structures in Proposition 2.9 and Definition 4.1. Using birational maps from $\mathcal{A}_\Sigma$, $\mathcal{X}_\Sigma$ to $G^{u,e}$, $G^{u,e}_{\mathrm{Ad}}$ given by H. Williams, we obtain geometric crystal structures on $\mathcal{A}_\Sigma$, $\mathcal{X}_\Sigma$ (Definition 4.3(1)). By using "twist map" $\zeta^{u,e}$, we can also construct another geometric crystal structures on them (Definition 4.3(2)). Second, we will verify compatibilities between these structures in Proposition 4.4. We mainly treat the geometric crystal structures of Definition 4.3(1) on $\mathcal{X}_\Sigma$ and those of Definition 4.3(2) on $\mathcal{A}_\Sigma$ in this article. Third, we will present explicit formulae for geometric crystal structures on the tori $\mathcal{A}_\Sigma$, $\mathcal{X}_\Sigma$. These explicit formulae imply the tori have positive geometric crystal structures. Since the set of $\mathbb{Z}^T$-valued points $\mathcal{A}_{|\Sigma|}(\mathbb{Z}^T)$ (resp. $\mathcal{X}_{|\Sigma|}(\mathbb{Z}^T)$) is a union of the sets of co-characters of $\mathcal{A}_\Sigma$ (resp. $\mathcal{X}_\Sigma$), we see that it has a crystal structure.

The organization of this article is as follows. In Section 2, we review the theory of geometric crystals and some explicit formulae for a geometric crystal on the double Bruhat cell $G^{u,e}$ which will be needed in the rest of the article. In Section 3, the cluster ensembles will be introduced and we will see the cluster ensembles associated with arbitrary Weyl group elements. Cluster tori $\mathcal{A}_{\Sigma_{\mathbf{i}}}$, $\mathcal{X}_{\Sigma_{\mathbf{i}}}$ birationally isomorphic to $G^{u,e}$, $G^{u,e}_{\mathrm{Ad}}$ are defined in this section. In Section 4, we define geometric crystal structures on the cluster tori $\mathcal{A}_{\Sigma_{\mathbf{i}}}$, $\mathcal{X}_{\Sigma_{\mathbf{i}}}$. We will also present a compatibility between these structures. In Section 5, we give explicit formulae for geometric crystal structures on $\mathcal{X}_{\Sigma_{\mathbf{i}}}$. In Section 6, we also give explicit formulae for geometric crystal structures on $\mathcal{A}_{\Sigma_{\mathbf{i}}}$ by using the compatibility shown in Section 4. In Section 7, we present explicit formulae for geometric crystal structures on $\mathcal{A}_{\Sigma_{\mathbf{i}}}$ in the case $G = SL_{r+1}(\mathbb{C})$ and u is the longest element of W in a different way from Section 6.

2. Geometric crystals

In this section, we will review notion of geometric crystal following [2,3,14,15].

2.1. Notation and definitions

Following [16], we define several notions. For a positive integer l, we set $[1,l] := \{1,2,\ldots,l\}$. Let $A = (a_{i,j})_{i,j\in[1,r]}$ be a symmetrizable generalized Cartan matrix with a symmetrizer $\mathrm{diag}(d_1,\ldots,d_r)$ ($d_i \in \mathbb{Z}_{>0}$), and $\mathfrak{g} = \mathfrak{g}(A) = \langle e_i, f_i, \mathfrak{h}\rangle$ the Kac–Moody Lie algebra associated with A over $\mathbb{C}$. The Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ contains simple coroots $\alpha_1^\vee,\ldots,\alpha_r^\vee$ and $\mathfrak{h}^*$ contains simple roots $\alpha_1,\ldots,\alpha_r$, which satisfy $\alpha_j(\alpha_i^\vee) = a_{i,j}$. Let $\tilde{r} = \dim \mathfrak{h} = 2r - \mathrm{rank}\, A$. The simple reflections $s_i \in \mathrm{Aut}(\mathfrak{h}^*)$ ($i \in [1,r]$) are defined as $s_i(\beta) := \beta - \beta(\alpha_i^\vee)\alpha_i$, which generate the Weyl group W. Let $P := \{\lambda \in \mathfrak{h}^* | \lambda(\alpha_i^\vee) \in \mathbb{Z} \text{ for all } i \in [1,r]\}$ and $\{\Lambda_i\}_{i\in[1,\tilde{r}]} \subset P$ be a basis which satisfies $\Lambda_i(\alpha_j^\vee) = \delta_{i,j}$ for $i \in [1,\tilde{r}]$, $j \in [1,r]$. Let G be the Kac–Moody group associated with $(\mathfrak{g}, P)$ and $H \subset G$ a maximal torus. The set P is identified with the set $\mathrm{Hom}(H, \mathbb{C}^\times)$ of characters. We call Λ_i ($i \in [1,\tilde{r}]$) *fundamental weights*. By fixing $\{\Lambda_i\}_{i\in[1,\tilde{r}]}$, we obtain a corresponding dual basis of $\mathrm{Hom}(\mathbb{C}^\times, H)$ and denote its elements $\alpha_1^\vee,\ldots,\alpha_{\tilde{r}}^\vee$. If $i \in [1,r]$ then $\alpha_i^\vee$ is just the ith coroot of $\mathfrak{g}$. For $t \in \mathbb{C}^\times$ and $h \in \mathrm{Hom}(\mathbb{C}^\times, H)$, let t^h denote the element $h(t) \in H$. The equation

$$\alpha_j = \sum_{1\le i\le \tilde{r}} a_{ij}\Lambda_i \tag{2.1}$$

defines numbers a_{ij} for $i \in \{r+1, \ldots, \tilde{r}\}$ and $j \in \{1, 2, \ldots, r\}$. We can also define elements $\{\alpha_i\}_{i\in\{r+1,\ldots,\tilde{r}\}}$ of P by

$$\alpha_i = D \sum_{j=1}^{r} d_j^{-1} a_{ij} \Lambda_j,$$

where D is the least common integer multiple of $d_1, \ldots, d_r$.

For each real root α, there exists a one-parameter subgroup $\{x_\alpha(t) | t \in \mathbb{C}\} \subset G$, and G is generated by all one-parameter subgroups and H [8,13]. Let N, N^- be the subgroups of G generated by $\{x_\alpha(t) | t \in \mathbb{C},\ \alpha : \text{positive root}\}$, $\{x_\alpha(t) | t \in \mathbb{C},\ \alpha : \text{negative root}\}$. Let $B = HN$, $B^- = HN^-$ be Borel subgroups. For $T \in H$, let $\alpha_i(T)$ or T^{α_i} denote the value of α_i at T (as the character).

2.2. Double Bruhat cells

We set $x_i(c) := \exp(ce_i)$, $y_i(c) := \exp(cf_i) \in G$ for $c \in \mathbb{C}$. We also set $\overline{s_i} := x_i(-1)y_i(1)x_i(-1)$ for $i \in [1, r]$, and for a reduced expression $w = s_{j_1} \cdots s_{j_n} \in W$, set $\overline{w} := \overline{s_{j_1}} \cdots \overline{s_{j_n}}$. The following two kinds of Bruhat decompositions of G are known [13]:

$$G = \coprod_{u\in W} B\overline{u}B = \coprod_{u\in W} B^-\overline{u}B^-.$$

Then, for $u, v \in W$, the *double Bruhat cell* $G^{u,v}$ is defined as follows:

$$G^{u,v} := B\overline{u}B \cap B^-\overline{v}B^-.$$

Proposition 2.1 ([16]). *For $u, v \in W$, the double Bruhat cell $G^{u,v}$ is a rational affine variety and* $\dim G^{u,v} = l(u) + l(v) + \tilde{r}$.

2.3. Crystals

Let us recall the definition of *crystals* [10]. We use the notation in 2.1.

Definition 2.2. A *crystal* is a set $\mathcal{B}$ together with the maps $\mathrm{wt}_i : \mathcal{B} \to \mathbb{Z}$, $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$ ($i \in [1, r]$) satisfying the following: For $b, b' \in \mathcal{B}$, $i, j \in [1, r]$,

(1) $\varphi_i(b) = \varepsilon_i(b) + \mathrm{wt}_i(b)$,
(2) $\mathrm{wt}_j(\tilde{e}_i b) = \mathrm{wt}_j(b) + a_{j,i}$ if $\tilde{e}_i(b) \in \mathcal{B}$, $\quad \mathrm{wt}_j(\tilde{f}_i b) = \mathrm{wt}_j(b) - a_{j,i}$ if $\tilde{f}_i(b) \in \mathcal{B}$,
(3) $\varepsilon_i(\tilde{e}_i(b)) = \varepsilon_i(b) - 1$, $\quad \varphi_i(\tilde{e}_i(b)) = \varphi_i(b) + 1$ if $\tilde{e}_i(b) \in \mathcal{B}$,
(4) $\varepsilon_i(\tilde{f}_i(b)) = \varepsilon_i(b) + 1$, $\quad \varphi_i(\tilde{f}_i(b)) = \varphi_i(b) - 1$ if $\tilde{f}_i(b) \in \mathcal{B}$,
(5) $\tilde{f}_i(b) = b'$ if and only if $b = \tilde{e}_i(b')$,
(6) if $\varphi_i(b) = -\infty$ then $\tilde{e}_i(b) = \tilde{f}_i(b) = 0$.

We call $\tilde{e}_i$, $\tilde{f}_i$ *Kashiwara operators*, and wt_i *weight functions*. A crystal $\mathcal{B}$ is said to be *free* if the Kashiwara operators $\tilde{e}_i$ ($i \in [1, r]$) are bijections $\tilde{e}_i : \mathcal{B} \to \mathcal{B}$.

Note that the above definition of crystals is slightly weaker than the original one in [10]. In the case the generalized Cartan matrix $(a_{i,j})_{i,j\in[1,r]}$ has rank r then the above definition is equivalent to the one in [10].

For two crystals $(\mathcal{B}, \{\tilde{e}_i\}, \{\tilde{f}_i\}, \{\varepsilon_i\}, \{\varphi_i\}, \{\mathrm{wt}_i\})$, $(\mathcal{B}', \{\tilde{e}'_j\}, \{\tilde{f}'_i\}, \{\varepsilon'_i\}, \{\varphi'_i\}, \{\mathrm{wt}'_i\})$, a bijection $f : \mathcal{B} \to \mathcal{B}'$ is called a *crystal isomorphism* if it satisfies $f(\tilde{e}_i(b)) = \tilde{e}'_i(f(b))$, $f(\tilde{f}_i(b)) = \tilde{f}'_i(f(b))$, $\varepsilon'_i(f(b)) = \varepsilon_i(b)$, $\varphi'_i(f(b)) = \varphi_i(b)$ and $\mathrm{wt}'_i(f(b)) = \mathrm{wt}_i(b)$ for $b \in \mathcal{B}$ and $i \in [1, r]$. Here we understand $f(0) = 0$.

2.4. Geometric crystals

For algebraic varieties X, Y and a rational function $f : X \to Y$, let $\mathrm{dom}(f)$ denote the maximal open subset of X on which f is defined.

Definition 2.3. For a symmetrizable generalized Cartan matrix $A = (a_{i,j})_{i,j\in[1,r]}$ and an irreducible algebraic variety X over $\mathbb{C}$, let γ_i, ε_i ($i \in [1, r]$) be rational functions on X, and $e_i : \mathbb{C}^\times \times X \to X$ a rational $\mathbb{C}^\times$-action ($i \in [1, r]$) (to be denoted by $(c, x) \mapsto e_i^c(x)$). A quintuple $(X, \{e_i\}_{i\in[1,r]}, \{\gamma_i\}_{i\in[1,r]}, \{\varepsilon_i\}_{i\in[1,r]})$ is called a *geometric crystal* if

(*i*) For $i \in [1, r]$, $(\{1\} \times X) \cap \mathrm{dom}(e_i)$ is open dense in $\{1\} \times X$.
(*ii*) For any $i, j \in [1, r]$, the rational functions $\{\gamma_i\}_{i\in[1,r]}$ satisfy $\gamma_j(e_i^c(x)) = c^{a_{ij}} \gamma_j(x)$.
(*iii*) For any $t \in H$, $w \in W$ and its two reduced words $\mathbf{i}$, $\mathbf{i}'$, the relation $e_{\mathbf{i}}(t) = e_{\mathbf{i}'}(t)$ holds, where for a reduced word $\mathbf{i} = (i_1, \ldots, i_n)$ of w, we define $e_{\mathbf{i}}(t) = e_{i_1}^{(\alpha^{(1)}(t))} e_{i_2}^{(\alpha^{(2)}(t))} \cdots e_{i_n}^{(\alpha^{(n)}(t))}$, $\alpha^{(j)} := s_{i_n} \cdots s_{i_{j+1}}(\alpha_{i_j})$.
(*iv*) The rational functions $\{\varepsilon_i\}_{i\in[1,r]}$ satisfy $\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$ and $\varepsilon_i(e_j^c(x)) = \varepsilon_i(x)$ if $a_{i,j} = a_{j,i} = 0$.

Let $X^*(T) := \mathrm{Hom}(T, \mathbb{C}^\times)$ be the set of characters for a split algebraic torus T.

Definition 2.4. Let T, T' be split algebraic tori over $\mathbb{C}$.

(i) A regular function $f = \sum_{\mu \in X^*(T)} c_\mu \cdot \mu$ on T is positive if all coefficients c_μ are non-negative numbers. A rational function on T is said to be positive if there exist positive regular functions g, h such that $f = \frac{g}{h}$ $(h \neq 0)$.

(ii) Let $f : T \to T'$ be a rational map between T and T'. Then f is called *positive* if for any $\xi \in X^*(T')$, the rational function $\xi \circ f$ is positive in the sense of (i).

Let $\mathcal{T}_+$ be a category whose objects are algebraic tori over $\mathbb{C}$ and morphisms are positive rational maps. In [2,14], a functor $Trop : \mathcal{T}_+ \to \mathfrak{Set}$ is introduced, where $\mathfrak{Set}$ is the category of all sets. Each torus T in $\mathcal{T}_+$ corresponds to the set of co-characters $X_*(T) = \mathrm{Hom}(\mathbb{C}^\times, T)$ under the functor *Trop*.

Definition 2.5. Let $\chi = (X, \{e_i\}_{i\in[1,r]}, \{\gamma_i\}_{i\in[1,r]}, \{\varepsilon_i\}_{i\in[1,r]})$ be a geometric crystal, T an algebraic torus, $\theta : T \to X$ a birational map. The map θ is called *positive structure* on χ if it satisfies the following:

(i) For $i \in [1, r]$, the rational functions $\gamma_i \circ \theta$, $\varepsilon_i \circ \theta$ are positive.

(ii) For $i \in [1, r]$, the rational map $e_{i,\theta} : \mathbb{C}^\times \times T \to T$, defined by $(c, t) \mapsto \theta^{-1} \circ e_i^c \circ \theta(t)$ is positive.

We say (χ, θ) is a *positive geometric crystal.*

Applying the functor *Trop* to $e_{i,\theta}$, $\gamma_i \circ \theta$ and $\varepsilon_i \circ \theta$, we get

$$\tilde{e}_i = Trop(e_{i,\theta}) : \mathbb{Z} \times X_*(T) \to X_*(T),$$

$$\tilde{\gamma}_i = Trop(\gamma_i \circ \theta) : X_*(T) \to \mathbb{Z}, \quad \tilde{\varepsilon}_i = Trop(\varepsilon_i \circ \theta) : X_*(T) \to \mathbb{Z},$$

Theorem 2.6 ([2,14]). *Let $\chi = (X, \{e_i\}_{i\in[1,r]}, \{\gamma_i\}_{i\in[1,r]}, \{\varepsilon_i\}_{i\in[1,r]})$ be a geometric crystal, T an algebraic torus, $\theta : T \to X$ its positive structure. Then $(X_*(T), \{\tilde{e}_i\}_{i\in[1,r]}, \{\tilde{\gamma}_i\}_{i\in[1,r]}, \{\tilde{\varepsilon}_i\}_{i\in[1,r]})$ has a free crystal structure.*

In the above notation, for $x \in X_*(T)$, $x \mapsto \tilde{e}_i(1, x)$ and $x \mapsto \tilde{e}_i(-1, x)$ give actions of Kashiwara operators on $X_*(T)$, and $\tilde{\gamma}_i$ define the weight functions $x \mapsto \tilde{\gamma}_i(x)$ on $X_*(T)$. The maps φ_i are defined by $\varphi_i(x) = \tilde{\varepsilon}_i(x) + \tilde{\gamma}_i(x)$ $(i \in [1, r])$.

2.5. Geometric crystal actions on $G^{u,e}$

For a Weyl group element $u \in W$, let $\gamma_i : G^{u,e} \to \mathbb{C}^\times$ be the rational function defined by

$$\gamma_i : G^{u,e} \hookrightarrow B^- \xrightarrow{\sim} H \times N^- \xrightarrow{\mathrm{proj}} H \xrightarrow{\alpha_i} \mathbb{C}^\times.$$

For $\alpha \in \Delta^{\mathrm{re}}$, let $\mathfrak{g}_\alpha$ be the root space, and $N_\alpha := \exp(\mathfrak{g}_\alpha)$. For $i \in [1, r]$, we set $N_i^\pm := N^\pm \cap \bar{s}_i N^\mp \bar{s}_i^{-1}$ and $N_\pm^i := N^\pm \cap \bar{s}_i N^\pm \bar{s}_i^{-1}$, where $N^+ = N$. Indeed, we have $N_i^\pm = N_{\pm\alpha_i}$. We also set

$$Y_{\pm\alpha_i} := \langle x_{\pm i}(t) N_\alpha x_{\pm i}(-t) | t \in \mathbb{C}, \alpha \in \Delta_\pm^{\mathrm{re}} \setminus \{\pm\alpha_i\}\rangle,$$

where $x_{-i}(t) := y_i(t)$.

Lemma 2.7 ([13]). *For a simple root α_i $(i \in [1, r])$, we have*
(i) $Y_{\pm\alpha_i} = N_\pm^i$,
(ii) $N^\pm = N_i^\pm \cdot Y_{\pm\alpha_i}$ (semi-direct product).

By this lemma, we have the unique decomposition:

$$N^- = N_i^- \cdot Y_{-\alpha_i} = N_{-\alpha_i} \cdot N_-^i,$$

and we get the canonical projection $\xi_i : N^- \to N_{-\alpha_i}$. Let χ_i be the function on N^- defined as $\chi_i := y_i^{-1} \circ \xi_i : N^- \to \mathbb{C}$, where $y_i : \mathbb{C} \to N_{-\alpha_i}$ is defined as $c \mapsto y_i(c)$. We extend this to the function on B^- by $\chi_i(u \cdot t) := \chi_i(u)$ for $u \in N^-$ and $t \in H$. We set

$$\varphi_i := (\chi_i|_{G^{u,e}})^{-1} : G^{u,e} \to \mathbb{C}^\times, \quad \varepsilon_i := \frac{\varphi_i}{\gamma_i} : G^{u,e} \to \mathbb{C}^\times.$$

For a reduced expression $u = s_{i_1} \cdots s_{i_n}$, we suppose that $\{i_1, \ldots, i_n\} = \{1, 2, \ldots, r\}$. Then $\chi_i|_{G^{u,e}}$ is not identically zero [14]. Thus, in this case, we can define the rational functions φ_i and ε_i $(i \in [1, r])$.

For each $i \in [1, r]$, let us define a rational $\mathbb{C}^\times$-action e_i on $G^{u,e}$ as follows: First, let $g : G \to B^- \times N$ be the inverse birational map of the multiplication map $B^- \times N \to G$. We set a rational morphism $\pi^- : G \to B^-$ as $\pi^- := \mathrm{proj}_{B^-} \circ g$, and define rational N-action on B^- by $\alpha_{B^-} := \pi^- \circ m : N \times B^- \to B^-$, where m is the multiplication map $N \times B^- \to G$. Then we define a rational $\mathbb{C}^\times$-action as $(c \in \mathbb{C}^\times, x \in G^{u,e})$

$$e_i^c(x) := \alpha_{B^-}(x_i((c-1)\varphi_i(x)), x) = x_i((c-1)\varphi_i(x)) x x_i((c^{-1}-1)\varepsilon_i(x)),$$

if $\chi_i|_{G^{u,e}} \not\equiv 0$ and $e_i^c(x) = x$ if $\chi_i|_{G^{u,e}} \equiv 0$. In the above definition, the second equality is shown in [3]. Let $\alpha_i^\vee(T) := T^{\alpha_i^\vee} \in H$ for $T \in \mathbb{C}^\times$.

We define the set $\overline{\mathbb{B}}_u^-$ as

$$\overline{\mathbb{B}}_u^- := \{ty_{i_1}(c_1)\cdots y_{i_n}(c_n) | t \in H,\ c_1, \ldots, c_n \in \mathbb{C}^\times, \} \tag{2.2}$$

which is an open subset of $G^{u,e}$ [16].

Proposition 2.8. *For $u \in W$ and its reduced expression $u = s_{i_1}\cdots s_{i_n}$, we define the set $\mathbb{B}_u^-$ as*

$$\mathbb{B}_u^- := \{ty_{i_1}(c_1)\alpha_{i_1}^\vee(c_1^{-1})\cdots y_{i_n}(c_n)\alpha_{i_n}^\vee(c_n^{-1}) | t \in H,\ c_1, \ldots, c_n \in \mathbb{C}^\times\}.$$

Then we have $\overline{\mathbb{B}}_u^- = \mathbb{B}_u^-$.

Proof. Note that $y_i(S)\alpha_j^\vee(T^{-1}) = \alpha_j^\vee(T^{-1})y_i(ST^{-a_{j,i}})$ for $S, T \in \mathbb{C}^\times$ and $i, j \in \{1, 2, \ldots, r\}$. We have

$$\begin{aligned}
&\mathbb{B}_u^- \ni ty_{i_1}(c_1)\alpha_{i_1}^\vee(c_1^{-1})\cdots y_{i_n}(c_n)\alpha_{i_n}^\vee(c_n^{-1})\\
&= t\alpha_{i_1}^\vee(c_1^{-1})\cdots\alpha_{i_n}^\vee(c_n^{-1})y_{i_1}(\frac{1}{c_1}\prod_{j=2}^n c_j^{-a_{j,1}})\cdots y_{i_s}(\frac{1}{c_s}\prod_{j=s+1}^n c_j^{-a_{j,s}})\cdots y_{i_n}(\frac{1}{c_n}) \in \overline{\mathbb{B}}_u^-,
\end{aligned}$$

which means $\mathbb{B}_u^- \subset \overline{\mathbb{B}}_u^-$. Next, for $c_1, \ldots, c_n \in \mathbb{C}^\times$, we put $\zeta_n := c_n$ and $\zeta_s := \zeta_n^{-a_{i_n,i_s}}\zeta_{n-1}^{-a_{i_{n-1},i_s}}\cdots\zeta_{s+1}^{-a_{i_{s+1},i_s}}c_s$ for $s = 1, \ldots, n-1$. It is clear that $\zeta_j \in \mathbb{C}^\times$ for $j = 1, 2, \ldots, n$. We obtain

$$\begin{aligned}
&\overline{\mathbb{B}}_u^- \ni ty_{i_1}(c_1)\cdots y_{i_n}(c_n)\\
&= t\alpha_{i_1}^\vee(\zeta_1^{-1})\cdots\alpha_{i_n}^\vee(\zeta_n^{-1})\alpha_{i_1}^\vee(\zeta_1)\cdots\alpha_{i_n}^\vee(\zeta_n)y_{i_1}(c_1)\cdots y_{i_n}(c_n)\\
&= t\alpha_{i_1}^\vee(\zeta_1^{-1})\cdots\alpha_{i_n}^\vee(\zeta_n^{-1})y_{i_1}(\frac{c_1}{\zeta_1^2}\prod_{j=2}^n \zeta_j^{-a_{i_j,i_1}})\alpha_{i_1}^\vee(\zeta_1)\\
&\quad\cdots y_{i_s}(\frac{c_s}{\zeta_s^2}\prod_{j=s+1}^n \zeta_j^{-a_{i_j,i_s}})\alpha_{i_s}^\vee(\zeta_s)\cdots y_{i_n}(\zeta_n^{-1})\alpha_{i_n}^\vee(\zeta_n)\\
&= t\alpha_{i_1}^\vee(\zeta_1^{-1})\cdots\alpha_{i_n}^\vee(\zeta_n^{-1})y_{i_1}(\zeta_1^{-1})\alpha_{i_1}^\vee(\zeta_1)\cdots y_{i_s}(\zeta_s^{-1})\alpha_{i_s}^\vee(\zeta_s)\cdots y_{i_n}(\zeta_n^{-1})\alpha_{i_n}^\vee(\zeta_n) \in \mathbb{B}_u^-,
\end{aligned}$$

which means $\overline{\mathbb{B}}_u^- \subset \mathbb{B}_u^-$. □

Proposition 2.9 ([14,15]). *For $u \in W$ and its reduced expression $u = s_{i_1}\cdots s_{i_n}$, we suppose that $\{i_1, \ldots, i_n\} = \{1, 2, \ldots, r\}$. Then the quintuple $(G^{u,e}, \{e_i\}_{i\in[1,r]}, \{\gamma_i\}_{i\in[1,r]}, \{\varepsilon_i\}_{i\in[1,r]})$ is a geometric crystal. The map $H \times (\mathbb{C}^\times)^n \to G^{u,e}$, $(t, c_1, \ldots, c_n) \mapsto ty_{i_1}(c_1)\alpha_{i_1}^\vee(c_1^{-1})\cdots y_{i_n}(c_n)\alpha_{i_n}^\vee(c_n^{-1})$ is a positive structure on this geometric crystal.*

Proposition 2.10 ([14,15]). *Let $u = s_{i_1}\cdots s_{i_n}$ be a reduced expression of $u \in W$ such that $\{i_1, \ldots, i_n\} = \{1, 2, \ldots, r\}$. The action of e_j^c on the open subset*

$$\mathbb{B}_u^- = \{ty_{i_1}(t_1)\alpha_{i_1}^\vee(t_1^{-1})y_{i_2}(t_2)\alpha_{i_2}^\vee(t_2^{-1})\cdots y_{i_n}(t_n)\alpha_{i_n}^\vee(t_n^{-1}) | t \in H,\ t_1, \ldots, t_n \in \mathbb{C}^\times\} \subset G^{u,e}$$

is given by

$$\begin{aligned}
&e_j^c(ty_{i_1}(t_1)\alpha_{i_1}^\vee(t_1^{-1})y_{i_2}(t_2)\alpha_{i_2}^\vee(t_2^{-1})\cdots y_{i_n}(t_n)\alpha_{i_n}^\vee(t_n^{-1}))\\
&= ty_{i_1}(t'_1)\alpha_{i_1}^\vee(t_1'^{-1})y_{i_2}(t'_2)\alpha_{i_2}^\vee(t_2'^{-1})\cdots y_{i_n}(t'_n)\alpha_{i_n}^\vee(t_n'^{-1}),
\end{aligned}$$

where

$$t'_k = t_k\frac{c\sum_{1\le m<k, i_m=j} t_1^{a_{i_1,j}}\cdots t_{m-1}^{a_{i_{m-1},j}}t_m + \sum_{k\le m\le n, i_m=j} t_1^{a_{i_1,j}}\cdots t_{m-1}^{a_{i_{m-1},j}}t_m}{c\sum_{1\le m\le k, i_m=j} t_1^{a_{i_1,j}}\cdots t_{m-1}^{a_{i_{m-1},j}}t_m + \sum_{k<m\le n, i_m=j} t_1^{a_{i_1,j}}\cdots t_{m-1}^{a_{i_{m-1},j}}t_m}.$$

Furthermore,

$$\varepsilon_j((ty_{i_1}(t_1)\alpha_{i_1}^\vee(t_1^{-1})\cdots y_{i_n}(t_n)\alpha_{i_n}^\vee(t_n^{-1}))) = \left(\sum_{1\le m\le n,\ i_m=j}\frac{1}{t_m t_{m+1}^{a_{i_{m+1},j}}\cdots t_n^{a_{i_n,j}}}\right)^{-1},$$

$$\gamma_j((ty_{i_1}(t_1)\alpha_{i_1}^\vee(t_1^{-1})\cdots y_{i_n}(t_n)\alpha_{i_n}^\vee(t_n^{-1}))) = \frac{\alpha_j(t)}{t_1^{a_{i_1,j}}\cdots t_n^{a_{i_n,j}}}.$$

2.6. Generalized minors and a bilinear form

We set $G_0 := N^-HN$, and let $x = [x]_-[x]_0[x]_+$ with $[x]_- \in N^-$, $[x]_0 \in H$, $[x]_+ \in N$ be the corresponding decomposition.

Definition 2.11. For $i \in \{1,\dots,\tilde{r}\}$ and $w,\ w' \in W$, the *generalized minor* $\Delta_{w'\Lambda_i,w\Lambda_i}$ is a regular function on G whose restriction to the open set $\overline{w'}G_0\overline{w}^{-1}$ is given by $\Delta_{w'\Lambda_i,w\Lambda_i}(x) = ([\overline{w'}^{-1}x\overline{w}]_0)^{\Lambda_i}$. Here, Λ_i is the ith fundamental weight and for $a = T^h \in H$ $(h \in \oplus_{j\in\{1,\dots,\tilde{r}\}}\mathbb{Z}\alpha_j^\vee,\ T \in \mathbb{C}^\times)$, we set $a^{\Lambda_i} := T^{\Lambda_i(h)}$.

Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be the anti-involution

$$\omega(e_i) = f_i, \quad \omega(f_i) = e_i, \quad \omega(h) = h,$$

and extend it to G by setting $\omega(x_i(c)) = y_i(c)$, $\omega(y_i(c)) = x_i(c)$ and $\omega(t) = t$ $(t \in H)$. One can calculate the generalized minors as follows. Let $V(\lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with a fixed highest weight vector v_λ $(\lambda \in \bigoplus_{i\in[1,\tilde{r}]}\mathbb{Z}_{\geq 0}\Lambda_i)$. Since $V(\lambda)$ is an integrable module, it is also a G-module. There exists a bilinear form on $V(\lambda)$ such that $\langle v_\lambda, v_\lambda\rangle = 1$ and

$$\langle au, v\rangle = \langle u, \omega(a)v\rangle, \qquad (u, v \in V(\lambda),\ a \in \mathfrak{g} \text{ (or } G)).$$

For $g \in G$, we have the following simple fact:

$$\Delta_{\Lambda_i,\Lambda_i}(g) = \langle g v_{\Lambda_i}, v_{\Lambda_i}\rangle,$$

where v_{Λ_i} is a fixed highest weight vector in $V(\Lambda_i)$. Hence, for $w,\ w' \in W$, we have

$$\Delta_{w'\Lambda_i,w\Lambda_i}(g) = \Delta_{\Lambda_i}(\overline{w'}^{-1}g\overline{w}) = \langle g\overline{w}\cdot v_{\Lambda_i}, \overline{w'}\cdot v_{\Lambda_i}\rangle. \tag{2.3}$$

Using generalized minors, the rational functions γ_i and φ_i in 2.5 are written as

$$\gamma_i = \prod_{j=1}^{\tilde{r}} \Delta_{\Lambda_j,\Lambda_j}^{a_{j,i}}, \quad \varphi_i = \frac{\Delta_{\Lambda_i,\Lambda_i}}{\Delta_{s_i\Lambda_i,\Lambda_i}}, \tag{2.4}$$

where we use (2.1) in the first relation.

3. Cluster ensembles

Following [16], let us recall the notions of cluster $\mathcal{A}$-variety and $\mathcal{X}$-variety.

3.1. Definitions of cluster $\mathcal{A}$-variety and $\mathcal{X}$-variety

Definition 3.1. A *seed* $\Sigma = (I, I_0, B, d)$ is a quintuple of the following data:

(i) I is a finite index set and I_0 is a subset of I. The elements of I_0 are called frozen.
(ii) $B = (b_{i,j})$ is an $I \times I$-matrix called *exchange matrix* which satisfies $b_{i,j} \in \mathbb{Z}$ unless both i and j are frozen.
(iii) $d = (d_i)_{i\in I}$ is a set of positive integers such that $b_{i,j}d_j = -b_{j,i}d_i$.

For $k \in I\setminus I_0$, we say a seed $\Sigma' = (I', I_0, B', d')$ is obtained from Σ by *mutation* at k if there exists a bijection $M_k : I \to I'$ which satisfies $M_k(i) = i$ for $i \in I_0$, $d'_{M_k(j)} = d_j$ for $j \in I$ and

$$b'_{M_k(i),M_k(j)} := \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

Then we write $\Sigma' = \mu_k(\Sigma)$.

To a seed $\Sigma = (I, I_0, B, d)$, we associate a set of variables $\{A_i\}_{i\in I}$ and a split algebraic torus $\mathcal{A}_\Sigma = \mathrm{Spec}\ \mathbb{C}[A_i^{\pm 1}|i \in I]$ called *cluster $\mathcal{A}$-torus* . We call $\{A_i\}_{i\in I}$ *cluster $\mathcal{A}$-coordinate* on $\mathcal{A}_\Sigma$. If $\Sigma' = (I', I_0, B', d') = \mu_k(\Sigma)$ then there exists a birational map (called *mutation*) $\mu_k : \mathcal{A}_\Sigma \to \mathcal{A}_{\Sigma'}$ defined as

$$\mu_k^*(A'_{M_k(i)}) = \begin{cases} A_i & \text{if } i \neq k, \\ \frac{\prod_{b_{k,j}>0} A_j^{b_{k,j}} + \prod_{b_{k,j}<0} A_j^{-b_{k,j}}}{A_k} & \text{if } i = k, \end{cases}$$

where $\{A'_i\}_{i\in I'}$ is the cluster $\mathcal{A}$-coordinate associated to Σ'.

Definition 3.2. The cluster $\mathcal{A}$-variety $\mathcal{A}_{|\Sigma|}$ is the scheme obtained by gluing together all tori $\mathcal{A}_{\Sigma'}$ of seeds Σ' which are obtained from Σ by an iteration of mutations.

To a seed $\Sigma = (I, I_0, B, d)$, we also associate an algebraic torus $\mathcal{X}_\Sigma := \mathrm{Spec}\ \mathbb{C}[X_i^{\pm 1} | i \in I]$ called *cluster $\mathcal{X}$-torus* with variables $\{X_i\}_{i\in I}$. We call $\{X_i\}_{i\in I}$ *cluster $\mathcal{X}$-coordinate* on $\mathcal{X}_\Sigma$. If $\Sigma' = \mu_k(\Sigma)$ then there exists a birational map (called *mutation*) $\mu_k : \mathcal{X}_\Sigma \to \mathcal{X}_{\Sigma'}$ defined as

$$\mu_k^*(X'_{M_k(i)}) = \begin{cases} X_i X_k^{[b_{i,k}]_+}(1+X_k)^{-b_{i,k}} & \text{if } i \neq k, \\ X_k^{-1} & \text{if } i = k, \end{cases}$$

where $\{X'_i\}_{i\in I'}$ is the cluster $\mathcal{X}$-coordinate associated to Σ' and $[b_{i,k}]_+ := \max(b_{i,k}, 0)$.

Definition 3.3. The cluster $\mathcal{X}$-variety $\mathcal{X}_{|\Sigma|}$ is the scheme obtained by gluing together all tori $\mathcal{X}_{\Sigma'}$ of seeds Σ' which are obtained from Σ by an iteration of mutations.

In what follows, we identify I with I' by M_k, and write $b'_{M_k(i),M_k(j)} = b'_{i,j}$, $A'_{M_k(i)} = A'_i$ and $X'_{M_k(i)} = X'_i$.

Proposition 3.4 (*[16]*). *Let $M = (M_{i,j})$ be an $I \times I$-matrix such that $M_{i,j} = 0$ unless both i and j are frozen. For a seed $\Sigma = (I, I_0, B, d)$ such that $\tilde{B} = B + M$ is an integer matrix, we define a map $p_M : \mathcal{A}_\Sigma \to \mathcal{X}_\Sigma$ as*

$$p_M^*(X_i) = \prod_{j\in I} A_j^{\tilde{B}_{i,j}}.$$

Then p_M extends to a regular map $p_M : \mathcal{A}_{|\Sigma|} \to \mathcal{X}_{|\Sigma|}$.

If $\Sigma' = \mu_k(\Sigma)$ and B' is the exchange matrix of Σ' then $\tilde{B'} = B' + M'$ is an integer matrix. Thus, we can define $p'_M : \mathcal{A}_{\Sigma'} \to \mathcal{X}_{\Sigma'}$ by $p'^*_M(X'_i) = \prod_{j\in I} A_j'^{\tilde{B}'_{i,j}}$. This proposition means the following diagram is commutative:

$$\begin{tikzcd} \mathcal{A}_\Sigma \arrow[r, "\mu_k"] \arrow[d, "p_M"'] & \mathcal{A}_{\Sigma'} \arrow[d, "p'_M"] \\ \mathcal{X}_\Sigma \arrow[r, "\mu_k"'] & \mathcal{X}_{\Sigma'} \end{tikzcd} \tag{3.1}$$

The map $p_M : \mathcal{A}_{|\Sigma|} \to \mathcal{X}_{|\Sigma|}$ is called an *ensemble map* in the context of [16].

3.2. The set of $\mathbb{Z}^T$-valued points

Let $\mathbb{Z}^T$ be the tropical semi-field of integers, that is, it is equal to $\mathbb{Z}$ as sets, and product and sum are defined as $+$ and max respectively. For a split torus H, we set

$$H(\mathbb{Z}^T) := X_*(H),$$

where $X_*(H)$ is the group of co-characters of H. We can verify that $H(\mathbb{Z}^T) \cong (\mathbb{Z}^T)^{\dim H}$. Note that a positive rational map $f : H \to H'$ induces a map $Trop(f) : H(\mathbb{Z}^T) \to H'(\mathbb{Z}^T)$, where *Trop* is the functor in 2.4.

Definition 3.5 (*[4]*). For a seed Σ and the cluster $\mathcal{A}$-variety $\mathcal{A}_{|\Sigma|}$ and $\mathcal{X}$-variety $\mathcal{X}_{|\Sigma|}$, we set

$$\mathcal{A}_{|\Sigma|}(\mathbb{Z}^T) := \coprod \mathcal{A}_{\Sigma'}(\mathbb{Z}^T)/\{\text{identifications } Trop(\mu),\ \mu : \text{mutation}\},$$

$$\mathcal{X}_{|\Sigma|}(\mathbb{Z}^T) := \coprod \mathcal{X}_{\Sigma'}(\mathbb{Z}^T)/\{\text{identifications } Trop(\mu),\ \mu : \text{mutation}\},$$

where Σ' runs over the set of all seeds which are obtained from Σ by an iteration of mutations. We call them the sets of *$\mathbb{Z}^T$-valued points* of cluster $\mathcal{A}(\mathcal{X})$-variety.

The sets of $\mathbb{Z}^T$-valued points of cluster varieties appear in the context of Fock–Goncharov conjecture in [4,7].

3.3. Seeds associated with reduced words

We will use the notation in 2.1. For $u \in W$ and its reduced word $\mathbf{i} = (i_1, \ldots, i_n)$, one associate a seed $\Sigma_{\mathbf{i}}$ as follows. We set $i_{-j} = -j$ for $j = 1, 2, \ldots, \tilde{r}$.

Definition 3.6 (*[16]*). We define the index set as $I := \{-\tilde{r}, \ldots, -2, -1\} \cup \{1, 2, \ldots, n\}$. For $k \in I$, we set $k^+ := \min\{l \in I | l > k, |i_l| = |i_k|\} \cup \{n+1\}$ and $I_0 := \{k \in I | k < 0, \text{ or } k^+ > n\}$. The exchange matrix $B_{\mathbf{i}} = (b_{j,k})$ is defined by

$$\begin{aligned} b_{j,k} = \frac{a_{|i_k|,|i_j|}}{2}(&-[j = k^+] + [j^+ = k] - [k < j < k^+][j > 0] + [k < j^+ < k^+][j^+ \leq n] \\ &+ [j < k < j^+][k > 0] - [j < k^+ < j^+][k^+ \leq n]), \end{aligned}$$

where for a proposition P,

$$[P] = \begin{cases} 1 & \text{if } P : \text{true}, \\ 0 & \text{if } P : \text{false}. \end{cases}$$

Let $d_k = d_{|i_k|}$ for $k \in I$, where the right-hand side means the symmetrizer of the Cartan matrix $(a_{i,j})_{i,j \in I}$. Then we define a seed $\Sigma_{\mathbf{i}} := (I, I_0, B_{\mathbf{i}}, d)$.

The lattice $\bigoplus_{1\leq i\leq \tilde{r}} \mathbb{Z}\alpha_i$ is a sublattice of P, and we see that its kernel $\{t \in H | t^{\alpha_i} = 1,\ 1 \leq i \leq \tilde{r}\}$ is a discrete subgroup of the center of G. Let G_{Ad} denote the quotient of G by the discrete subgroup $\{t \in H | t^{\alpha_i} = 1,\ 1 \leq i \leq \tilde{r}\}$ and H_{Ad} denote the image of H in G_{Ad}. The character lattice $\mathrm{Hom}(H_{\mathrm{Ad}}, \mathbb{C}^\times)$ is canonically isomorphic to $\bigoplus_{1\leq i\leq \tilde{r}} \mathbb{Z}\alpha_i$. Thus, the co-character lattice $\mathrm{Hom}(\mathbb{C}^\times, H_{\mathrm{Ad}})$ has a dual basis $\Lambda_1^\vee, \ldots, \Lambda_{\tilde{r}}^\vee$ of *fundamental coweights* such that $\alpha_i(\Lambda_j^\vee) = \delta_{i,j}$ for $i, j \in [1, \tilde{r}]$. Let $T^{\Lambda_i^\vee}$ be an element of H_{Ad} such that $\alpha_j(T^{\Lambda_i^\vee}) = T^{\delta_{i,j}}$ for $i, j \in [1, \tilde{r}]$ and $T \in \mathbb{C}^\times$. Now we define numbers a_{ij} $(i, j \in [1, \tilde{r}])$ as $a_{i,j} := \alpha_j(\alpha_i^\vee)$.

Proposition 3.7 ([16]). *The $\tilde{r} \times \tilde{r}$ integer matrix $(a_{i,j})_{1\leq i,j\leq \tilde{r}}$ is nondegenerate and symmetrizable. We also get $\alpha_i^\vee = \sum_{j=1}^{\tilde{r}} a_{ij}\Lambda_j^\vee$.*

Definition 3.8 ([16]). Let $\mathcal{X}_{\Sigma_{\mathbf{i}}}$ be the cluster $\mathcal{X}$-torus which associates to the seed $\Sigma_{\mathbf{i}}$. An open immersion $x_{\Sigma_i} : \mathcal{X}_{\Sigma_{\mathbf{i}}} \to G^{u,e}_{\mathrm{Ad}}$ is defined for an element $u \in W$ and its reduced word $\mathbf{i} = (i_1, i_2, \ldots, i_n)$:

$$x_{\Sigma_i} : (X_{-\tilde{r}}, \ldots, X_{-1}, X_1, \ldots, X_n) \mapsto X_{-\tilde{r}}^{\Lambda_{\tilde{r}}^\vee} \cdots X_{-1}^{\Lambda_1^\vee} y_{i_1}(1) X_1^{\Lambda_{i_1}^\vee} y_{i_2}(1) X_2^{\Lambda_{i_2}^\vee} \cdots y_{i_n}(1) X_n^{\Lambda_{i_n}^\vee}.$$

Let $u_{\leq k} := s_{i_1} \cdots s_{i_k}$ for $k \in [1, n]$ and $u_{\leq k} = e$ for $k \in \{-\tilde{r}, \ldots, -2, -1\}$.

Lemma 3.9 ([16]). *Let $\mathcal{A}_{\Sigma_i}$ be the cluster $\mathcal{A}$-torus which associates to the seed $\Sigma_{\mathbf{i}}$. There exists an open immersion $a_{\Sigma_i} : \mathcal{A}_{\Sigma_i} \to G^{u,e}$ such that the pull-back $a^*_{\Sigma_i}$ identifies each coordinate function A_k with a generalized minor $\Delta_{u_{\leq k}\Lambda_{|i_k|}, \Lambda_{|i_k|}}$ for $k \in I$.*

Definition 3.10 ([16]). For $x \in G_0 := N^- H N$, we write $x = [x]_-[x]_0[x]_+$ with $[x]_- \in N^-$, $[x]_0 \in H$, $[x]_+ \in N_+$. For $u \in W$, the twist map $\zeta^{u,e} : G^{u,e} \to G^{u^{-1},e}$ is defined by

$$x \mapsto \theta([\overline{u}^{-1}x]_-^{-1}\overline{u}^{-1}x),$$

where θ is the automorphism of G such that $\theta(a) = a^{-1}$ $(a \in H)$, $\theta(x_i(T)) = y_i(T)$ and $\theta(y_i(T)) = x_i(T)$ $(T \in \mathbb{C})$. We also define ι as the antiautomorphism of G defined by $a \mapsto a^{-1}$ for $a \in H$ and $x_i(T) \mapsto x_i(T)$, $y_i(T) \mapsto y_i(T)$ for $T \in \mathbb{C}$.

Proposition 3.11 ([16]). *For $x \in G^{u,e}$, we get $\overline{u}^{-1}x \in G_0$. The map $\iota \circ \zeta^{u,e} : G^{u,e} \to G^{u,e}$ is a biregular isomorphism.*

Theorem 3.12 ([16]). *Let $M = (M_{j,k})$ be the following $I \times I$-matrix*

$$M_{j,k} = \frac{a_{|i_k|,|i_j|}}{2}([j^+, k^+ > n] + [j, k < 0]).$$

(1) There is a regular map $a_{|\Sigma_i|} : \mathcal{A}_{|\Sigma_i|} \to G^{u,e}$ which extends $a_{\Sigma_i} : \mathcal{A}_{\Sigma_i} \to G^{u,e}$ in Lemma 3.9. It induces an algebra isomorphism $\mathbb{C}[G^{u,e}] \to \mathbb{C}[\mathcal{A}_{|\Sigma_i|}]$.

(2) There is a regular map $x_{|\Sigma_i|} : \mathcal{X}_{|\Sigma_i|} \to G^{u,e}_{\mathrm{Ad}}$ which extends $x_{\Sigma_i} : \mathcal{X}_{\Sigma_i} \to G^{u,e}_{\mathrm{Ad}}$ in Definition 3.8.

(3) The all entries of $B_{\mathbf{i}} + M$ are integer and the following diagram is commutative:

$$\begin{tikzcd} \mathcal{A}_{|\Sigma_{\mathbf{i}}|} \arrow[r, "a_{|\Sigma_{\mathbf{i}}|}"] \arrow[d, "p_M"'] & G^{u,e} \arrow[d, "p_G"] \\ \mathcal{X}_{|\Sigma_{\mathbf{i}}|} \arrow[r, "x_{|\Sigma_{\mathbf{i}}|}"'] & G^{u,e}_{\mathrm{Ad}} \end{tikzcd}$$

where $p_G : G^{u,e} \to G^{u,e}_{\mathrm{Ad}}$ is the composition of the automorphism $\iota \circ \zeta^{u,e}$ and the quotient map G to G_{Ad}.

4. Compatibility between geometric crystal structures on the cluster tori

In 3.3, we defined tori $\mathcal{A}_\Sigma$ (resp. $\mathcal{X}_\Sigma$) birationally isomorphic to $G^{u,e}$ (resp. $G^{u,e}_{\mathrm{Ad}}$). In this section, we will define two geometric crystal structures on each torus $\mathcal{A}_\Sigma$ (resp. $\mathcal{X}_\Sigma$) (Definition 4.3). Furthermore, we will discuss a compatibility between these geometric crystal structures (Proposition 4.4). In Sections. 5, 6, we will calculate the explicit formulae of them. In the rest of article, for a reduced expression $u = s_{i_1} \cdots s_{i_n}$, we suppose that $\{i_1, \ldots, i_n\} = \{1, 2, \ldots, r\}$. Let $Z = \{t \in H | t^{\alpha_i} = 1,\ 1 \leq i \leq \tilde{r}\}$.

First, we define a geometric crystal structure on $G^{u,e}_{\mathrm{Ad}}$. Recall that $G^{u,e}$ has a geometric crystal structure $(G^{u,e}, \{e_j\}_{j\in[1,r]}, \{\varepsilon_j\}_{j\in[1,r]}, \{\gamma_j\}_{j\in[1,r]})$ (Proposition 2.9). The definitions imply that ε_j and γ_j are Z-invariant, and e_j satisfies $e_j(c, tx) = te_j(c, x)$ for $t \in Z$, $(c, x) \in \mathrm{dom}(e_j)$. Thus, the geometric crystal structure on $G^{u,e}$ induces a geometric crystal structure on $G^{u,e}_{\mathrm{Ad}}$.

Definition 4.1. We write this geometric crystal structure as $(G^{u,e}_{\mathrm{Ad}}, \{e_j\}_{j\in[1,r]}, \{\varepsilon_j\}_{j\in[1,r]}, \{\gamma_j\}_{j\in[1,r]})$, that is, we use the same notation for e_j, ε_j and γ_j on $G^{u,e}_{\mathrm{Ad}}$ as those on $G^{u,e}$.

In the formulae of Proposition 2.10 for e^c_j, ε_j and γ_j, an element $t \in H$ is replaced with $t \in H_{\mathrm{Ad}}$ when we consider the geometric crystal structure on $G^{u,e}_{\mathrm{Ad}}$.

Proposition 4.2. *The biregular map $G^{u,e}_{\mathrm{Ad}} \to G^{u,e}_{\mathrm{Ad}}$ defined as $xZ \mapsto (\iota \circ \zeta^{u,e}(x))Z$ $(x \in G^{u,e})$ is well-defined. Let us denote it by $\overline{\iota \circ \zeta^{u,e}}$.*

Proof. We take $x \in G^{u,e}$ and $z \in Z$. By Proposition 3.11, we have $\overline{u}^{-1}x \in G_0$. Considering the decomposition $\overline{u}^{-1}x = [\overline{u}^{-1}x]_-[\overline{u}^{-1}x]_0[\overline{u}^{-1}x]_+$, we obtain $\overline{u}^{-1}xz = [\overline{u}^{-1}x]_-[\overline{u}^{-1}x]_0z[\overline{u}^{-1}x]_+$ and $[\overline{u}^{-1}x]_0z \in H$, which yields

$$[\overline{u}^{-1}xz]_- = [\overline{u}^{-1}x]_-. \tag{4.1}$$

It follows from (4.1) that

$$\begin{aligned}(\iota \circ \zeta^{u,e})(xz) &= \iota \circ \theta([\overline{u}^{-1}xz]_-^{-1}\overline{u}^{-1}xz)\\ &= \iota \circ \theta([\overline{u}^{-1}x]_-^{-1}\overline{u}^{-1}xz)\\ &= \iota(\theta([\overline{u}^{-1}x]_-^{-1}\overline{u}^{-1}x)z^{-1})\\ &= \iota(\theta([\overline{u}^{-1}x]_-^{-1}\overline{u}^{-1}x))z\\ &= (\iota \circ \zeta^{u,e})(x)z.\end{aligned}$$

Thus, the biregular map $G^{u,e}_{\mathrm{Ad}} \to G^{u,e}_{\mathrm{Ad}}$ is induced. □

Definition 4.3. We use the same notation as in Theorem 3.12. Let Σ be a seed obtained from Σ_i by an iteration of mutations $\overline{\mu}$, and $\overline{\mu}^a : \mathcal{A}_{\Sigma_i} \to \mathcal{A}_\Sigma$, $\overline{\mu}^x : \mathcal{X}_{\Sigma_i} \to \mathcal{X}_\Sigma$ be the corresponding birational maps. Let a_Σ, x_Σ denote the birational maps $a_{\Sigma_i} \circ (\overline{\mu}^a)^{-1} : \mathcal{A}_\Sigma \to G^{u,e}$, $x_{\Sigma_i} \circ (\overline{\mu}^x)^{-1} : \mathcal{X}_\Sigma \to G^{u,e}_{\mathrm{Ad}}$, respectively. We define two geometric crystal structures on the torus $\mathcal{A}_\Sigma$ (resp. $\mathcal{X}_\Sigma$) as follows:

(1) The first one is

$$(\mathcal{A}_\Sigma,\ a_\Sigma^{-1} \circ e^c_j \circ a_\Sigma,\ \varepsilon_j \circ a_\Sigma,\ \gamma_j \circ a_\Sigma), \quad (\text{resp. } (\mathcal{X}_\Sigma,\ x_\Sigma^{-1} \circ e^c_j \circ x_\Sigma,\ \varepsilon_j \circ x_\Sigma,\ \gamma_j \circ x_\Sigma)).$$

(2) The second one is

$$(\mathcal{A}_\Sigma,\ a_\Sigma^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e^c_j \circ (\iota \circ \zeta^{u,e}) \circ a_\Sigma,\ \varepsilon_j \circ (\iota \circ \zeta^{u,e}) \circ a_\Sigma,\ \gamma_j \circ (\iota \circ \zeta^{u,e}) \circ a_\Sigma),$$

$$(\text{resp. } (\mathcal{X}_\Sigma,\ x_\Sigma^{-1} \circ (\overline{\iota \circ \zeta^{u,e}}) \circ e^c_j \circ (\overline{\iota \circ \zeta^{u,e}})^{-1} \circ x_\Sigma,\ \varepsilon_j \circ (\overline{\iota \circ \zeta^{u,e}})^{-1} \circ x_\Sigma,\ \gamma_j \circ (\overline{\iota \circ \zeta^{u,e}})^{-1} \circ x_\Sigma)).$$

Proposition 4.4. *Let $p = p_\Sigma$ be the restriction of the map p_M in Theorem 3.12 to the torus $\mathcal{A}_\Sigma$. The following commutative diagrams hold:*

$$\begin{tikzcd} \mathcal{A}_\Sigma \arrow[r, "a_\Sigma^{-1}\circ(\iota\circ\zeta^{u,e})^{-1}\circ e^c_j\circ\iota\circ\zeta^{u,e}\circ a_\Sigma"] \arrow[d, "p"'] & \mathcal{A}_\Sigma \arrow[d, "p"] \\ \mathcal{X}_\Sigma \arrow[r, "x_\Sigma^{-1}\circ e^c_j\circ x_\Sigma"'] & \mathcal{X}_\Sigma \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_\Sigma \arrow[r, "a_\Sigma^{-1}\circ e^c_j\circ a_\Sigma"] \arrow[d, "p"'] & \mathcal{A}_\Sigma \arrow[d, "p"] \\ \mathcal{X}_\Sigma \arrow[r, "x_\Sigma^{-1}\circ(\overline{\iota\circ\zeta^{u,e}})\circ e^c_j\circ(\overline{\iota\circ\zeta^{u,e}})^{-1}\circ x_\Sigma"'] & \mathcal{X}_\Sigma \end{tikzcd} \tag{4.2}$$

We also get the following commutative diagrams:

$$\begin{tikzcd} \mathcal{A}_\Sigma \arrow[d, "p"'] \arrow[dr, "\varepsilon_j\circ(\iota\circ\zeta^{u,e})\circ a_\Sigma"] & \\ \mathcal{X}_\Sigma \arrow[r, "\varepsilon_j\circ x_\Sigma"'] & \mathbb{C}^\times \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_\Sigma \arrow[d, "p"'] \arrow[dr, "\gamma_j\circ(\iota\circ\zeta^{u,e})\circ a_\Sigma"] & \\ \mathcal{X}_\Sigma \arrow[r, "\gamma_j\circ x_\Sigma"'] & \mathbb{C}^\times \end{tikzcd} \tag{4.3}$$

$$\begin{tikzcd} \mathcal{A}_\Sigma \arrow[d, "p"'] \arrow[dr, "\varepsilon_j\circ a_\Sigma"] & \\ \mathcal{X}_\Sigma \arrow[r, "\varepsilon_j\circ(\overline{\iota\circ\zeta^{u,e}})^{-1}\circ x_\Sigma"'] & \mathbb{C}^\times \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_\Sigma \arrow[d, "p"'] \arrow[dr, "\gamma_j\circ a_\Sigma"] & \\ \mathcal{X}_\Sigma \arrow[r, "\gamma_j\circ(\overline{\iota\circ\zeta^{u,e}})^{-1}\circ x_\Sigma"'] & \mathbb{C}^\times \end{tikzcd}$$

Proof. First, let us prove them for $\Sigma = \Sigma_i$. We set $p := p_{\Sigma_i}$. It follows from Theorem 3.12(3), definitions of p_G and action of e_j^c that

$$\begin{aligned} x_{\Sigma_i}^{-1} \circ e_j^c \circ x_{\Sigma_i} \circ p &= x_{\Sigma_i}^{-1} \circ e_j^c \circ p_G \circ a_{\Sigma_i} \\ &= x_{\Sigma_i}^{-1} \circ e_j^c \circ q \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} \\ &= x_{\Sigma_i}^{-1} \circ q \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} \\ &= x_{\Sigma_i}^{-1} \circ q \circ (\iota \circ \zeta^{u,e}) \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} \\ &= x_{\Sigma_i}^{-1} \circ p_G \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} \\ &= p \circ a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i}, \end{aligned}$$

and

$$\begin{aligned} p \circ a_{\Sigma_i}^{-1} \circ e_j^c \circ a_{\Sigma_i} &= x_{\Sigma_i}^{-1} \circ p_G \circ e_j^c \circ a_{\Sigma_i} \\ &= x_{\Sigma_i}^{-1} \circ (\overline{\iota \circ \zeta^{u,e}}) \circ q \circ e_j^c \circ a_{\Sigma_i} \\ &= x_{\Sigma_i}^{-1} \circ (\overline{\iota \circ \zeta^{u,e}}) \circ e_j^c \circ q \circ a_{\Sigma_i} \\ &= x_{\Sigma_i}^{-1} \circ (\overline{\iota \circ \zeta^{u,e}}) \circ e_j^c \circ (\overline{\iota \circ \zeta^{u,e}})^{-1} \circ (\overline{\iota \circ \zeta^{u,e}}) \circ q \circ a_{\Sigma_i} \\ &= x_{\Sigma_i}^{-1} \circ (\overline{\iota \circ \zeta^{u,e}}) \circ e_j^c \circ (\overline{\iota \circ \zeta^{u,e}})^{-1} \circ p_G \circ a_{\Sigma_i} \\ &= x_{\Sigma_i}^{-1} \circ (\overline{\iota \circ \zeta^{u,e}}) \circ e_j^c \circ (\overline{\iota \circ \zeta^{u,e}})^{-1} \circ x_{\Sigma_i} \circ p. \end{aligned}$$

Thus, we get (4.2) for $\Sigma = \Sigma_i$.

The definitions of γ_j on $G^{u,e}$ and $G^{u,e}_{\mathrm{Ad}}$ mean $\gamma_j \circ q(g) = \gamma_j(g)$ for $g \in G^{u,e}$. Hence, we have

$$\begin{aligned} \gamma_j \circ x_{\Sigma_{\mathbf{i}}} \circ p &= \gamma_j \circ p_G \circ a_{\Sigma_{\mathbf{i}}} \\ &= \gamma_j \circ q \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}}} \\ &= \gamma_j \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}}}. \end{aligned}$$

Thus, we get the first commutative diagram in (4.3), and one can verify other diagrams via similar ways for $\Sigma = \Sigma_i$.

Next, we assume that Σ is obtained from Σ_i by an iteration of mutations $\overline{\mu}$. Let $\overline{\mu}^a : \mathcal{A}_{\Sigma_i} \to \mathcal{A}_\Sigma$ (resp. $\overline{\mu}^x : \mathcal{X}_{\Sigma_i} \to \mathcal{X}_\Sigma$) be the corresponding birational map. By Proposition 3.4, (3.1) and Definition 4.3, we obtain $a_{\Sigma_i} = a_\Sigma \circ \overline{\mu}^a$, $x_{\Sigma_i} = x_\Sigma \circ \overline{\mu}^x$ and $p_{\Sigma_i} = (\overline{\mu}^x)^{-1} \circ p_\Sigma \circ \overline{\mu}^a$. In conjunction with commutative diagrams for Σ_i, we obtain the diagrams (4.2), (4.3) for general seeds Σ. $\square$

Remark 4.5. In the rest of article, we will treat geometric crystals $(\mathcal{X}_\Sigma,\ x_\Sigma^{-1} \circ e_j^c \circ x_\Sigma,\ \varepsilon_j \circ x_\Sigma,\ \gamma_j \circ x_\Sigma)$ and $(\mathcal{A}_\Sigma,\ a_\Sigma^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_\Sigma,\ \varepsilon_j \circ (\iota \circ \zeta^{u,e}) \circ a_\Sigma,\ \gamma_j \circ (\iota \circ \zeta^{u,e}) \circ a_\Sigma)$ only.

Proposition 4.6. *For $i \in [1, \tilde{r}]$ and $w \in W$, we have*

$$(\iota \circ \zeta^{u,e})^* \Delta_{w\Lambda_i,\Lambda_i,} = \Delta_{u\Lambda_i,w\Lambda_i}.$$

Proof. Recall that we defined anti-involution $\omega : G \to G$ in 2.6, One can verify that $\iota \circ \theta = \omega$. For $x \in G^{u,e}$, we get

$$\begin{aligned} \iota \circ \zeta^{u,e}(x) &= \iota \circ \theta([\overline{u}^{-1}x]_-^{-1}\overline{u}^{-1}x) \\ &= \omega([\overline{u}^{-1}x]_0[\overline{u}^{-1}x]_+) \\ &= \omega([\overline{u}^{-1}x]_+) \cdot \omega([\overline{u}^{-1}x]_0). \end{aligned}$$

For $x \in G^{u,e}$, we get $\overline{u}^{-1}x \in G_0$ by Proposition 3.11. Writing $\overline{u}^{-1}x = [\overline{u}^{-1}x]_-[\overline{u}^{-1}x]_0[\overline{u}^{-1}x]_+$, we have $\omega(\overline{u}^{-1}x) = \omega([\overline{u}^{-1}x]_+)\omega([\overline{u}^{-1}x]_0)\omega([\overline{u}^{-1}x]_-)$ and $\omega([\overline{u}^{-1}x]_+) \in N^-$, $\omega([\overline{u}^{-1}x]_0) \in H$, $\omega([\overline{u}^{-1}x]_-) \in N$. Using the bilinear form in 2.6, we get

$$\begin{aligned}(\iota\circ\zeta^{u,e})^*\Delta_{w\Lambda_i,\Lambda_i}(x) &= \langle \overline{w}v_{\Lambda_i}, (\iota\circ\zeta^{u,e})(x)v_{\Lambda_i}\rangle \\ &= \langle \overline{w}v_{\Lambda_i}, \omega([\overline{u}^{-1}x]_+)\cdot\omega([\overline{u}^{-1}x]_0)v_{\Lambda_i}\rangle \\ &= \langle \overline{w}v_{\Lambda_i}, \omega(\overline{u}^{-1}x)v_{\Lambda_i}\rangle \\ &= \langle \overline{u}^{-1}x\overline{w}v_{\Lambda_i}, v_{\Lambda_i}\rangle \\ &= \langle x\overline{w}v_{\Lambda_i}, \overline{u}v_{\Lambda_i}\rangle = \Delta_{u\Lambda_i,w\Lambda_i}(x). \quad \square\end{aligned}$$

A result similar to Proposition 4.6 for unipotent quantum minors of quantum unipotent cells is obtained in [12].

5. Explicit formulae for geometric crystals on cluster $\mathcal{X}$-tori

In this section, we will reveal the explicit formulae for geometric crystal structures $(\mathcal{X}_{\Sigma_i},\ x_{\Sigma_i}^{-1}\circ e_j^c\circ x_{\Sigma_i},\ \varepsilon_j\circ x_{\Sigma_i},\ \gamma_j\circ x_{\Sigma_i})$ in Definition 4.3.

Theorem 5.1. *Let e_j^c $(j\in[1,r],\ c\in\mathbb{C}^\times)$ be the rational $\mathbb{C}^\times$-action on $G^{u,e}_{\mathrm{Ad}}$ (Definition 4.1). We set*

$$(X'_{-\tilde{r}},\dots,X'_{-1},X'_1,\dots,X'_n) := x_{\Sigma_i}^{-1}\circ e_j^c\circ x_{\Sigma_i}(X_{-\tilde{r}},\dots,X_{-1},X_1,\dots,X_n),$$

and $\{K_1,K_2,\dots,K_l\} := \{K \mid 1\le K\le n,\ i_K=j\}$ $(K_1<\cdots<K_l)$. Then

$$X'_{K_p} = X_{K_p}\cdot\frac{c\sum_{m=1}^{p+1}(X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}}) + \sum_{m=p+2}^{l}(X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}})}{c\sum_{m=1}^{p-1}(X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}}) + \sum_{m=p}^{l}(X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}})}. \tag{5.1}$$

For $k\in\{1,2,\dots,n\}$ with $i_k\neq j$, we also set $\{k_1,k_2,\dots,k_s\} := \{K \mid k<K<k^+,\ j=i_K\}$ $(k_1<k_2<\cdots<k_s)$. We can write $k_1=K_\gamma$ with some $\gamma\in\{1,2,\dots,l\}$. Then

$$X'_k = X_k\left(\frac{c\sum_{m=1}^{\gamma+s-1}X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}} + \sum_{m=\gamma+s}^{l}X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}}}{c\sum_{m=1}^{\gamma-1}X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}} + \sum_{m=\gamma}^{l}X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}}}\right)^{a_{j,i_k}}. \tag{5.2}$$

For $i\in[1,\tilde{r}]$,

$$X'_{-i} = X_{-i}c^{a_{ji}}\prod_{1\le s\le n,\, i_s=i}X_sX_s'^{-1}.$$

Furthermore, let γ_j, ε_j be the functions of geometric crystal on $G^{u,e}_{\mathrm{Ad}}$. Then

$$\gamma_j\circ x_{\Sigma_i}(X_{-\tilde{r}},\dots,X_{-1},X_1,\dots,X_n) = X_{-j}X_{K_1}\cdots X_{K_l},$$

$$\varepsilon_j\circ x_{\Sigma_i}(X_{-\tilde{r}},\dots,X_{-1},X_1,\dots,X_n) = \left(\sum_{p=0}^{l-1}X_{K_{p+1}}X_{K_{p+2}}\cdots X_{K_l}\right)^{-1}.$$

First, let us prove the following lemma.

Lemma 5.2.

(1) A map $f:(\mathbb{C}^\times)^n\to(\mathbb{C}^\times)^n$,

$$(t_1,\dots,t_s,\dots,t_n)\mapsto\left(\cdots,\frac{t_{s+1}^{-a_{i_{s+1},i_s}}t_{s+2}^{-a_{i_{s+2},i_s}}\cdots t_n^{-a_{i_n,i_s}}}{t_s},\dots,\frac{1}{t_n}\right)$$

is bijective.

(2) A map $g:(\mathbb{C}^\times)^n\to(\mathbb{C}^\times)^n$,

$$(t_1,\dots,t_j,\dots,t_n)\mapsto\left(\prod_{1\le k\le n,\, i_k=i_1}t_k,\dots,\prod_{j\le k\le n,\, i_k=i_j}t_k,\dots,t_n\right)$$

is bijective.

Proof. (1) For $(\zeta_1,\dots,\zeta_n)\in(\mathbb{C}^\times)^n$, by setting $t_n=\frac{1}{\zeta_n}$, $t_{n-1}=\frac{(t_n)^{a_{i_n,i_{n-1}}}}{\zeta_{n-1}}$, …, $t_s=\frac{t_{s+1}^{-a_{i_{s+1},i_s}}t_{s+2}^{-a_{i_{s+2},i_s}}\cdots t_n^{-a_{i_n,i_s}}}{\zeta_s}$, $\cdots$ inductively, we have

$$(\zeta_1,\dots,\zeta_n) = f(t_1,\dots,t_n),$$

which means f is surjective.

Next, we assume $f(t_1,\ldots,t_n)=f(t'_1\cdots,t'_n)$. Then we get $\frac{1}{t_n}=\frac{1}{t'_n}$, $\frac{t_n^{-a_{i_n,i_{n-1}}}}{t_{n-1}}=\frac{{t'}_n^{-a_{i_n,i_{n-1}}}}{t'_{n-1}}$, …, $\frac{t_{s+1}^{-a_{i_{s+1},i_s}}t_{s+2}^{-a_{i_{s+2},i_s}}\cdots t_n^{-a_{i_n,i_s}}}{t_s}=\frac{{t'}_{s+1}^{-a_{i_{s+1},i_s}}{t'}_{s+2}^{-a_{i_{s+2},i_s}}\cdots {t'}_n^{-a_{i_n,i_s}}}{t'_s}$, ⋯. Hence, we can inductively show that $t_n=t'_n$, $t_{n-1}=t'_{n-1}$, …, $t_s=t'_s$, ⋯. Therefore, f is injective. Similarly, we can prove (2). □

Proof of Theorem 5.1. Note that $y_i(t)X^{\Lambda_l^\vee}=X^{\Lambda_l^\vee}y_i(X^{\delta_{i,l}}t)$ holds. Therefore,

$$\begin{aligned}
&x_{\Sigma_i}(X_{-\tilde{r}},\ldots,X_{-1},X_1,\ldots,X_n)\\
&=X_{-\tilde{r}}^{\Lambda_{\tilde{r}}^\vee}\cdots X_{-1}^{\Lambda_1^\vee}y_{i_1}(1)X_1^{\Lambda_{i_1}^\vee}y_{i_2}(1)X_2^{\Lambda_{i_2}^\vee}\cdots y_{i_n}(1)X_n^{\Lambda_{i_n}^\vee}\\
&=X_{-\tilde{r}}^{\Lambda_{\tilde{r}}^\vee}\cdots X_{-1}^{\Lambda_1^\vee}X_1^{\Lambda_{i_1}^\vee}\cdots X_n^{\Lambda_{i_n}^\vee}\\
&\quad\cdot y_{i_1}\Big(\prod_{1\le k\le n,i_k=i_1}X_k\Big)y_{i_2}\Big(\prod_{2\le k\le n,i_k=i_2}X_k\Big)y_{i_3}\Big(\prod_{3\le k\le n,i_k=i_3}X_k\Big)\cdots y_{i_n}(X_n).
\end{aligned}$$

By Lemma 5.2, there exist $t_1,\ldots,t_n\in\mathbb{C}^\times$ such that

$$\prod_{s\le k\le n,i_k=i_s}X_k=\frac{t_{s+1}^{-a_{i_{s+1},i_s}}t_{s+2}^{-a_{i_{s+2},i_s}}\cdots t_n^{-a_{i_n,i_s}}}{t_s}\tag{5.3}$$

for $s=1,2,\ldots,n$.

Hence,

$$\begin{aligned}
&x_{\Sigma_i}(X_{-\tilde{r}},\ldots,X_{-1},X_1,\ldots,X_n)\\
&=X_{-\tilde{r}}^{\Lambda_{\tilde{r}}^\vee}\cdots X_{-1}^{\Lambda_1^\vee}X_1^{\Lambda_{i_1}^\vee}\cdots X_n^{\Lambda_{i_n}^\vee}\\
&\quad\cdot y_{i_1}\Big(\frac{t_2^{-a_{i_2,i_1}}t_3^{-a_{i_3,i_1}}\cdots t_n^{-a_{i_n,i_1}}}{t_1}\Big)y_{i_2}\Big(\frac{t_3^{-a_{i_3,i_2}}t_4^{-a_{i_4,i_2}}\cdots t_n^{-a_{i_n,i_2}}}{t_2}\Big)\\
&\quad\cdot y_{i_3}\Big(\frac{t_4^{-a_{i_4,i_3}}\cdots t_n^{-a_{i_n,i_3}}}{t_3}\Big)\cdots y_{i_n}\Big(\frac{1}{t_n}\Big)\\
&=X_{-\tilde{r}}^{\Lambda_{\tilde{r}}^\vee}\cdots X_{-1}^{\Lambda_1^\vee}X_1^{\Lambda_{i_1}^\vee}\cdots X_n^{\Lambda_{i_n}^\vee}\alpha_{i_1}^\vee(t_1)\alpha_{i_2}^\vee(t_2)\cdots\alpha_{i_n}^\vee(t_n)\\
&\quad\cdot y_{i_1}(t_1)\alpha_{i_1}^\vee(t_1^{-1})y_{i_2}(t_2)\alpha_{i_2}^\vee(t_2^{-1})\cdots y_{i_n}(t_n)\alpha_{i_n}^\vee(t_n^{-1}),
\end{aligned}\tag{5.4}$$

where we use

$$y_i(S)\alpha_p^\vee(T^{-1})=\alpha_p^\vee(T^{-1})y_i(ST^{-a_{p,i}}),\quad (S,T\in\mathbb{C}^\times,\ i,p\in\{1,2,\ldots,r\})\tag{5.5}$$

in the second equality. Here, we denote the image of $\alpha_p^\vee(T^{-1})\in G^{u,e}$ under the quotient map $G\to G_{\mathrm{Ad}}$ by the same notation $\alpha_p^\vee(T^{-1})$.

By the definition of e_j^c and Proposition 2.10, if $(c,x_{\Sigma_i}(X_{-\tilde{r}},\ldots,X_{-1},X_1,\ldots,X_n))\in\mathrm{dom}(e_j)$ then we have

$$\begin{aligned}
&e_j^c\circ x_{\Sigma_i}(X_{-\tilde{r}},\ldots,X_{-1},X_1,\ldots,X_n)\\
&=X_{-\tilde{r}}^{\Lambda_{\tilde{r}}^\vee}\cdots X_{-1}^{\Lambda_1^\vee}X_1^{\Lambda_{i_1}^\vee}\cdots X_n^{\Lambda_{i_n}^\vee}\alpha_{i_1}^\vee(t_1)\alpha_{i_2}^\vee(t_2)\cdots\alpha_{i_n}^\vee(t_n)\\
&\quad\cdot y_{i_1}(t'_1)\alpha_{i_1}^\vee(t_1'^{-1})y_{i_2}(t'_2)\alpha_{i_2}^\vee(t_2'^{-1})\cdots y_{i_n}(t'_n)\alpha_{i_n}^\vee(t_n'^{-1}),
\end{aligned}$$

with $t'_1,\ldots,t'_n\in\mathbb{C}^\times$ in Proposition 2.10.

Using (5.5) again, we obtain

$$\begin{aligned}
&e_j^c\circ x_{\Sigma_i}(X_{-\tilde{r}},\ldots,X_{-1},X_1,\ldots,X_n)\\
&=X_{-\tilde{r}}^{\Lambda_{\tilde{r}}^\vee}\cdots X_{-1}^{\Lambda_1^\vee}X_1^{\Lambda_{i_1}^\vee}\cdots X_n^{\Lambda_{i_n}^\vee}\alpha_{i_1}^\vee(t_1)\alpha_{i_2}^\vee(t_2)\cdots\alpha_{i_n}^\vee(t_n)\\
&\quad\cdot\alpha_{i_1}^\vee(t_1'^{-1})\alpha_{i_2}^\vee(t_2'^{-1})\cdots\alpha_{i_n}^\vee(t_n'^{-1})y_{i_1}\Big(\frac{{t'}_2^{-a_{i_2,i_1}}{t'}_3^{-a_{i_3,i_1}}\cdots {t'}_n^{-a_{i_n,i_1}}}{t'_1}\Big)\\
&\quad\cdot y_{i_2}\Big(\frac{{t'}_3^{-a_{i_3,i_2}}{t'}_4^{-a_{i_4,i_2}}\cdots {t'}_n^{-a_{i_n,i_2}}}{t'_2}\Big)y_{i_3}\Big(\frac{{t'}_4^{-a_{i_4,i_3}}\cdots {t'}_n^{-a_{i_n,i_3}}}{t'_3}\Big)\cdots y_{i_n}\Big(\frac{1}{t'_n}\Big).
\end{aligned}\tag{5.6}$$

It follows from Lemma 5.2 that there exist $X'_1, \ldots, X'_n \in \mathbb{C}^\times$ such that

$$\frac{(t'_{k+1})^{-a_{i_{k+1},i_k}}(t'_{k+2})^{-a_{i_{k+2},i_k}}\cdots(t'_n)^{-a_{i_n,i_k}}}{t'_k} = \prod_{k\le p\le n, i_p=i_k} X'_p \tag{5.7}$$

for $k = 1, 2, \ldots, n$. Substituting these into (5.6), we get

$$\begin{aligned}
&e_j^c \circ x_{\Sigma_i}(X_{-\tilde{r}}, \ldots, X_{-1}, X_1, \ldots, X_n)\\
&= X_{-\tilde{r}}^{\Lambda_{\tilde{r}}^\vee}\cdots X_{-1}^{\Lambda_1^\vee}X_1^{\Lambda_{i_1}^\vee}\cdots X_n^{\Lambda_{i_n}^\vee}\alpha_{i_1}^\vee(t_1)\alpha_{i_2}^\vee(t_2)\cdots\alpha_{i_n}^\vee(t_n)\\
&\cdot\alpha_{i_1}^\vee(t_1'^{-1})\alpha_{i_2}^\vee(t_2'^{-1})\cdots\alpha_{i_n}^\vee(t_n'^{-1})y_{i_1}(\prod_{1\le l\le n, i_l=i_1} X'_l)\\
&\cdot y_{i_2}(\prod_{2\le l\le n, i_l=i_2} X'_l)y_{i_3}(\prod_{3\le l\le n, i_l=i_3} X'_l)\cdots y_{i_n}(X'_n)\\
&= X_{-\tilde{r}}^{\Lambda_{\tilde{r}}^\vee}\cdots X_{-1}^{\Lambda_1^\vee}X_1^{\Lambda_{i_1}^\vee}\cdots X_n^{\Lambda_{i_n}^\vee}\alpha_{i_1}^\vee(t_1)\alpha_{i_2}^\vee(t_2)\cdots\alpha_{i_n}^\vee(t_n)\\
&\cdot\alpha_{i_1}^\vee(t_1'^{-1})\alpha_{i_2}^\vee(t_2'^{-1})\cdots\alpha_{i_n}^\vee(t_n'^{-1})X_1'^{-\Lambda_{i_1}^\vee}\cdots X_n'^{-\Lambda_{i_n}^\vee}\\
&y_{i_1}(1)X_1'^{\Lambda_{i_1}^\vee}y_{i_2}(1)X_2'^{\Lambda_{i_2}^\vee}\cdots y_{i_n}(1)X_n'^{\Lambda_{i_n}^\vee},
\end{aligned}$$

where we use $y_i(T)X^{\Lambda_l^\vee} = X^{\Lambda_l^\vee}y_i(X^{\delta_{i,l}}T)$ in the second equality. Note that putting $\{K_1, K_2, \ldots, K_l\} := \{1 \le K \le n | i_K = j\}$ $(K_1 < \cdots < K_l)$,

$$t_k t_k'^{-1} = \begin{cases} \dfrac{c\sum_{m=1}^{p} t_1^{a_{i_1,j}}t_2^{a_{i_2,j}}\cdots t_{K_m-1}^{a_{i_{K_m-1},j}}t_{K_m} + \sum_{m=p+1}^{l} t_1^{a_{i_1,j}}t_2^{a_{i_2,j}}\cdots t_{K_m-1}^{a_{i_{K_m-1},j}}t_{K_m}}{c\sum_{m=1}^{p-1} t_1^{a_{i_1,j}}t_2^{a_{i_2,j}}\cdots t_{K_m-1}^{a_{i_{K_m-1},j}}t_{K_m} + \sum_{m=p}^{l} t_1^{a_{i_1,j}}t_2^{a_{i_2,j}}\cdots t_{K_m-1}^{a_{i_{K_m-1},j}}t_{K_m}} & \text{if } k = K_p,\\ 1 & \text{if } i_k \neq j. \end{cases} \tag{5.8}$$

Thus, by Proposition 3.7, we get

$$\begin{aligned}
&\alpha_{i_1}^\vee(t_1)\alpha_{i_2}^\vee(t_2)\cdots\alpha_{i_n}^\vee(t_n)\cdot\alpha_{i_1}^\vee(t_1'^{-1})\alpha_{i_2}^\vee(t_2'^{-1})\cdots\alpha_{i_n}^\vee(t_n'^{-1})\\
&= \alpha_j^\vee(t_{K_1}t_{K_1}'^{-1}t_{K_2}t_{K_2}'^{-1}\cdots t_{K_l}t_{K_l}'^{-1}) = \alpha_j^\vee(c) = c^{\sum_{i=1}^{\tilde{r}} a_{ji}\Lambda_i^\vee}.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
&e_j^c \circ x_{\Sigma_i}(X_{-\tilde{r}}, \ldots, X_{-1}, X_1, \ldots, X_n)\\
&= X_{-\tilde{r}}'^{\Lambda_{\tilde{r}}^\vee}\cdots X_{-1}'^{\Lambda_1^\vee}y_{i_1}(1)X_1'^{\Lambda_{i_1}^\vee}y_{i_2}(1)X_2'^{\Lambda_{i_2}^\vee}\cdots y_{i_n}(1)X_n'^{\Lambda_{i_n}^\vee},
\end{aligned}$$

where for $i \in [1, \tilde{r}]$,

$$X'_{-i} = X_{-i}c^{a_{ji}}\prod_{1\le s\le n, i_s=i} X_s X_s'^{-1}.$$

Finally, let us prove (5.1) and (5.2). Eq. (5.7) means that

$$X'_{K_p} = \frac{\prod_{p\le s\le l} X'_{K_s}}{\prod_{p<s\le l} X'_{K_s}} = \frac{t'^{-a_{i_{K_p+1},j}}_{K_p+1}t'^{-a_{i_{K_p+2},j}}_{K_p+2}\cdots t'^{-a_{i_{K_{p+1}-1},j}}_{K_{p+1}-1}}{t'_{K_p}t'_{K_{p+1}}}, \quad (p = 1, 2, \ldots, l) \tag{5.9}$$

where $T'_{K_{l+1}} := 1$. Substituting $c = 1$, we obtain

$$X_{K_p} = \frac{t_{K_p+1}^{-a_{i_{K_p+1},j}}t_{K_p+2}^{-a_{i_{K_p+2},j}}\cdots t_{K_{p+1}-1}^{-a_{i_{K_{p+1}-1},j}}}{t_{K_p}t_{K_{p+1}}}. \tag{5.10}$$

Therefore,

$$(X_{K_1}X_{K_2}\cdots X_{K_{m-1}})^{-1} = t_{K_1}t_{K_1+1}^{a_{i_{K_1+1},j}}t_{K_1+2}^{a_{i_{K_1+2},j}}t_{K_1+3}^{a_{i_{K_1+3},j}}\cdots t_{K_m-1}^{a_{i_{K_m-1},j}}t_{K_m},$$

where we use $t_{K_p}^{a_{i_{K_p},j}} = t_{K_p}^{a_{j,j}} = t_{K_p}^2$. Thus,

$$t_1^{a_{i_1,j}} t_2^{a_{i_2,j}} \cdots t_{K_1-1}^{a_{i_{K_1-1},j}} t_{K_1} (X_{K_1}X_{K_2}\cdots X_{K_{m-1}})^{-1} = t_1^{a_{i_1,j}} t_2^{a_{i_2,j}} t_3^{a_{i_3,j}} \cdots t_{K_m-1}^{a_{i_{K_m-1},j}} t_{K_m}. \tag{5.11}$$

Using (5.8) and (5.9), we obtain

$$\begin{aligned} X'_{K_p} &= \frac{t_{K_p+1}^{-a_{i_{K_p+1},j}} t_{K_p+2}^{-a_{i_{K_p+2},j}} \cdots t_{K_{p+1}-1}^{-a_{i_{K_{p+1}-1},j}}}{t'_{K_p} t'_{K_{p+1}}} \\ &= \frac{t_{K_p+1}^{-a_{i_{K_p+1},j}} t_{K_p+2}^{-a_{i_{K_p+2},j}} \cdots t_{K_{p+1}-1}^{-a_{i_{K_{p+1}-1},j}}}{t_{K_p} t_{K_{p+1}}} \\ &\quad \cdot \frac{c\sum_{m=1}^{p+1} t_1^{a_{i_1,j}} t_2^{a_{i_2,j}} \cdots t_{K_m-1}^{a_{i_{K_m-1},j}} t_{K_m} + \sum_{m=p+2}^{l} t_1^{a_{i_1,j}} t_2^{a_{i_2,j}} \cdots t_{K_m-1}^{a_{i_{K_m-1},j}} t_{K_m}}{c\sum_{m=1}^{p-1} t_1^{a_{i_1,j}} t_2^{a_{i_2,j}} \cdots t_{K_m-1}^{a_{i_{K_m-1},j}} t_{K_m} + \sum_{m=p}^{l} t_1^{a_{i_1,j}} t_2^{a_{i_2,j}} \cdots t_{K_m-1}^{a_{i_{K_m-1},j}} t_{K_m}}. \end{aligned} \tag{5.12}$$

It follows from (5.10), (5.11) and (5.12) that

$$\begin{aligned} &X'_{K_p} \\ &= X_{K_p} \cdot \frac{c\sum_{m=1}^{p+1}(X_{K_1}X_{K_2}\cdots X_{K_{m-1}})^{-1} + \sum_{m=p+2}^{l}(X_{K_1}X_{K_2}\cdots X_{K_{m-1}})^{-1}}{c\sum_{m=1}^{p-1}(X_{K_1}X_{K_2}\cdots X_{K_{m-1}})^{-1} + \sum_{m=p}^{l}(X_{K_1}X_{K_2}\cdots X_{K_{m-1}})^{-1}} \\ &= X_{K_p} \cdot \frac{c\sum_{m=1}^{p+1}(X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}}) + \sum_{m=p+2}^{l}(X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}})}{c\sum_{m=1}^{p-1}(X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}}) + \sum_{m=p}^{l}(X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}})}. \end{aligned}$$

Thus, we obtain (5.1).

Next, for $k \in \{1,2,\dots,n\}$, let us suppose that $i_k \neq j$. We set $\{k_1,k_2,\dots,k_s\} := \{K | k < K < k^+,\ j = i_K\}$ $(k_1 < k_2 < \cdots < k_s)$. Because of (5.7),

$$\begin{aligned} &X'_k = \frac{\prod_{k\le p\le n, i_p=i_k} X'_p}{\prod_{k^+\le p\le n, i_p=i_k} X'_p} \\ &= \frac{(t'_{k+1})^{-a_{i_{k+1},i_k}} (t'_{k+2})^{-a_{i_{k+2},i_k}} \cdots (t'_{k^+-1})^{-a_{i_{k^+-1},i_k}}}{t'_k t'_{k^+}} \\ &= \frac{1}{t_k t_{k^+}} (t_{k+1})^{-a_{i_{k+1},i_k}} \cdots (t_{k_1-1})^{-a_{i_{k_1-1},i_k}} (t'_{k_1})^{-a_{j,i_k}} (t_{k_1+1})^{-a_{i_{k_1+1},i_k}} \\ &\quad \cdots (t_{k_2-1})^{-a_{i_{k_2-1},i_k}} (t'_{k_2})^{-a_{j,i_k}} (t_{k_2+1})^{-a_{i_{k_2+1},i_k}} \\ &\quad \cdots (t_{k_s-1})^{-a_{i_{k_s-1},i_k}} (t'_{k_s})^{-a_{j,i_k}} (t_{k_s+1})^{-a_{i_{k_s+1},i_k}} \cdots (t_{k^+-1})^{-a_{i_{k^+-1},i_k}}. \end{aligned}$$

Since $i_{k_1} = j$, there exists $\gamma \in \{1,2,\dots,l\}$ such that $k_1 = K_\gamma$. In this case, we have $k_s = K_{\gamma+s-1}$. By (5.8) and (5.10),

$$\begin{aligned} &X'_k \\ &= X_k \left(\frac{c\sum_{m=1}^{\gamma+s-1} t_1^{a_{i_1,j}} t_2^{a_{i_2,j}} \cdots t_{K_m-1}^{a_{i_{K_m-1},j}} t_{K_m} + \sum_{m=\gamma+s}^{l} t_1^{a_{i_1,j}} t_2^{a_{i_2,j}} \cdots t_{K_m-1}^{a_{i_{K_m-1},j}} t_{K_m}}{c\sum_{m=1}^{\gamma-1} t_1^{a_{i_1,j}} t_2^{a_{i_2,j}} \cdots t_{K_m-1}^{a_{i_{K_m-1},j}} t_{K_m} + \sum_{m=\gamma}^{l} t_1^{a_{i_1,j}} t_2^{a_{i_2,j}} \cdots t_{K_m-1}^{a_{i_{K_m-1},j}} t_{K_m}} \right)^{a_{j,i_k}} \\ &= X_k \left(\frac{c\sum_{m=1}^{\gamma+s-1}(X_{K_1}X_{K_2}\cdots X_{K_{m-1}})^{-1} + \sum_{m=\gamma+s}^{l}(X_{K_1}X_{K_2}\cdots X_{K_{m-1}})^{-1}}{c\sum_{m=1}^{\gamma-1}(X_{K_1}X_{K_2}\cdots X_{K_{m-1}})^{-1} + \sum_{m=\gamma}^{l}(X_{K_1}X_{K_2}\cdots X_{K_{m-1}})^{-1}} \right)^{a_{j,i_k}} \\ &= X_k \left(\frac{c\sum_{m=1}^{\gamma+s-1} X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}} + \sum_{m=\gamma+s}^{l} X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}}}{c\sum_{m=1}^{\gamma-1} X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}} + \sum_{m=\gamma}^{l} X_{K_m}X_{K_{m+1}}\cdots X_{K_{l-1}}} \right)^{a_{j,i_k}}. \end{aligned}$$

Thus, we get (5.2).

Taking into account (5.4), we get

$$\begin{aligned} \gamma_j \circ x_{\Sigma_i}(X_{-\tilde{r}},\dots,X_{-1},X_1,\dots,X_n) &= \frac{\alpha_j(X_{-\tilde{r}}^{\Lambda_{\tilde{r}}^\vee} \cdots X_{-1}^{\Lambda_1^\vee} X_1^{\Lambda_{i_1}^\vee} \cdots X_n^{\Lambda_{i_n}^\vee} \alpha_{i_1}^\vee(t_1)\alpha_{i_2}^\vee(t_2)\cdots\alpha_{i_n}^\vee(t_n))}{t_1^{a_{i_1,j}} \cdots t_n^{a_{i_n,j}}} \\ &= \alpha_j(X_{-\tilde{r}}^{\Lambda_{\tilde{r}}^\vee} \cdots X_{-1}^{\Lambda_1^\vee} X_1^{\Lambda_{i_1}^\vee} \cdots X_n^{\Lambda_{i_n}^\vee}) \\ &= X_{-j}X_{K_1}\cdots X_{K_l}. \end{aligned}$$

Using (5.3), we also get

$$\varepsilon_j \circ x_{\Sigma_i}(X_{-\tilde{r}},\ldots,X_{-1},X_1,\ldots,X_n) = \left(\sum_{1\le m\le n,\ i_m=j} \frac{1}{t_m t_{m+1}^{a_{i_{m+1},j}} \cdots t_n^{a_{i_n,j}}}\right)^{-1}$$
$$= \left(\sum_{p=0}^{l-1} X_{K_{p+1}} X_{K_{p+2}} \cdots X_{K_l}\right)^{-1}. \quad \square$$

By Theorem 5.1, we can verify that the map $x_{\Sigma_{\mathbf{i}}} : \mathcal{X}_{\Sigma_{\mathbf{i}}} \to G^{u,e}_{\mathrm{Ad}}$ is a positive structure on the geometric crystal $(G^{u,e}_{\mathrm{Ad}}, \{e_i\}_{i\in[1,r]}, \{\gamma_i\}_{i\in[1,r]}, \{\varepsilon_i\}_{i\in[1,r]})$. Let Σ be a seed obtained from $\Sigma_{\mathbf{i}}$ by an iteration of mutations. Then the corresponding birational map $\mathcal{X}_{\Sigma_{\mathbf{i}}} \to \mathcal{X}_{\Sigma}$ is positive. Hence, we obtain the following:

Corollary 5.3. *The map $x_\Sigma : \mathcal{X}_\Sigma \to G^{u,e}_{\mathrm{Ad}}$ is a positive structure on the geometric crystal $(G^{u,e}_{\mathrm{Ad}}, \{e_i\}_{i\in[1,r]}, \{\gamma_i\}_{i\in[1,r]}, \{\varepsilon_i\}_{i\in[1,r]})$. Applying Theorem 2.6, we obtain a crystal*

$$(X_*(\mathcal{X}_\Sigma), \{\tilde{e}_i\}_{i\in[1,r]}, \{\tilde{\gamma}_i\}_{i\in[1,r]}, \{\tilde{\varepsilon}_i\}_{i\in[1,r]}).$$

It induces a crystal structure on the set of $\mathbb{Z}$-valued points $\mathcal{X}_{|\Sigma_{\mathbf{i}}|}(\mathbb{Z}^T)$.

6. Explicit formulae for geometric crystals on cluster $\mathcal{A}$-tori

In this section, we will present the explicit formulae for geometric crystal structures $(\mathcal{A}_{\Sigma_i},\ a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i},\ \varepsilon_j \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i},\ \gamma_j \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i})$ in Definition 4.3. For a fixed reduced word $\mathbf{i} = (i_1,\ldots,i_n)$ of u and $k \in \{1,2,\ldots,n\}$, we set $k^- := \max\{l \in I | l < k,\ |i_l| = |i_k|\}$. For $j \in [1,r]$, we also set $j_{\max} := \max\{l \in I | i_l = j\}$.

Theorem 6.1. *Let M be the matrix in Theorem 3.12, and $p = p_M$ be the map in Proposition 3.4. We put*

$$(\overline{X}_{-\tilde{r}},\ldots,\overline{X}_{-1},\overline{X}_1,\ldots,\overline{X}_n) = x_{\Sigma_i}^{-1} \circ e_j^c \circ x_{\Sigma_i} \circ p(A_{-\tilde{r}},\ldots,A_{-1},A_1,\ldots,A_n).$$

Then for $j \in \{1,\ldots,r\}$ and $k \in \{1,\ldots,n\}$, we have

$$(a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_{k^-} \tag{6.1}$$
$$= A_{k^-} \cdot c^{D_{\{k,k+1,\ldots,n\},\{(k+1)^-,(k+2)^-,\ldots,n^-,j_{\max}\}}} \left(\frac{p(A)_k}{\overline{X}_k}\right) \left(\frac{p(A)_{k+1}}{\overline{X}_{k+1}}\right)^{D_{k,(k+1)^-}}$$
$$\cdot \left(\frac{p(A)_{k+2}}{\overline{X}_{k+2}}\right)^{D_{\{k,k+1\},\{(k+1)^-,(k+2)^-\}}} \cdots \left(\frac{p(A)_n}{\overline{X}_n}\right)^{D_{\{k,k+1,\ldots,n-1\},\{(k+1)^-,(k+2)^-,\ldots,n^-\}}},$$

where $D_{\{j_1,\ldots,j_l\},\{k_1,\ldots,k_l\}}$ is the minor of $\tilde{B}_{\mathbf{i}} = B_{\mathbf{i}} + M = (\tilde{B}_{i,s})_{i,s\in\{-\tilde{r},\ldots,-1,1,2,\ldots,n\}}$ whose rows (resp. columns) are labelled by $\{j_1,\ldots,j_l\}$ (resp $\{k_1,\ldots,k_l\}$). Furthermore, for $k \in \{1,2,\ldots,r\}$, we obtain

$$(a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_{k_{\max}} = c^{\delta_{j,k}} A_{k_{\max}}. \tag{6.2}$$

If $i \in \{1,2,\ldots,\tilde{r}\} \setminus \{i_1,i_2,\ldots,i_n\}$ then

$$(a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_{-i} = A_{-i}. \tag{6.3}$$

Remark 6.2. In the above theorem, we obtain formulae of $(a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_k$ for all $k \in I$.

Proof of Theorem 6.1. First, let us prove (6.2). By a property of the irreducible highest weight module $V(\Lambda_k)$ with the highest weight vector v_{Λ_k}, we obtain $f_j v_{\Lambda_k} = 0$ $(j \neq k)$ and $f_k v_{\Lambda_k} = \overline{s_k} v_{\Lambda_k}$ ([13], Chap.II). If $j \neq k$, by using the bilinear form in 2.6, we obtain

$$\begin{aligned}
((e_j^c)^* \Delta_{\Lambda_k,\Lambda_k})(x) &= \Delta_{\Lambda_k,\Lambda_k}(x_j((c-1)\varphi_j(x)) x x_j((c^{-1}-1)\varepsilon_j(x))) \\
&= \langle v_{\Lambda_k}, x_j((c-1)\varphi_j(x)) x x_j((c^{-1}-1)\varepsilon_j(x)) v_{\Lambda_k}\rangle \\
&= \langle y_j((c-1)\varphi_j(x)) v_{\Lambda_k}, x v_{\Lambda_k}\rangle \\
&= \langle v_{\Lambda_k}, x v_{\Lambda_k}\rangle \\
&= \Delta_{\Lambda_k,\Lambda_k}(x),
\end{aligned}$$

which means $(e_j^c)^* \Delta_{\Lambda_k,\Lambda_k} = \Delta_{\Lambda_k,\Lambda_k}$.

We also obtain

$$\begin{aligned}((e_j^c)^*\Delta_{\Lambda_j,\Lambda_j})(x) &= \langle y_j((c-1)\varphi_j(x))v_{\Lambda_j}, xv_{\Lambda_j}\rangle \\ &= \langle v_{\Lambda_j}, xv_{\Lambda_j}\rangle + (c-1)\varphi_j(x)\langle f_j v_{\Lambda_j}, xv_{\Lambda_j}\rangle \\ &= \langle v_{\Lambda_j}, xv_{\Lambda_j}\rangle + (c-1)\varphi_j(x)\langle \overline{s}_j v_{\Lambda_j}, xv_{\Lambda_j}\rangle \\ &= \langle v_{\Lambda_j}, xv_{\Lambda_j}\rangle + (c-1)\langle v_{\Lambda_j}, xv_{\Lambda_j}\rangle = c\langle v_{\Lambda_j}, xv_{\Lambda_j}\rangle,\end{aligned}$$

where we use (2.4) in the fourth equality. Thus, we get $(e_j^c)^*\Delta_{\Lambda_j,\Lambda_j} = c\Delta_{\Lambda_j,\Lambda_j}$ and

$$\begin{aligned}&(a_{\Sigma_i}^{-1}\circ(\iota\circ\zeta^{u,e})^{-1}\circ e_j^c\circ(\iota\circ\zeta^{u,e})\circ a_{\Sigma_i})^*A_{k_{\max}} \\ =\ & a_{\Sigma_i}^*\circ(\iota\circ\zeta^{u,e})^*\circ(e_j^c)^*\circ((\iota\circ\zeta^{u,e})^{-1})^*\circ(a_{\Sigma_i}^{-1})^*A_{k_{\max}} \\ =\ & a_{\Sigma_i}^*\circ(\iota\circ\zeta^{u,e})^*\circ(e_j^c)^*\circ((\iota\circ\zeta^{u,e})^{-1})^*\Delta_{u\Lambda_k,\Lambda_k} \\ =\ & a_{\Sigma_i}^*\circ(\iota\circ\zeta^{u,e})^*\circ(e_j^c)^*\Delta_{\Lambda_k,\Lambda_k} \\ =\ & a_{\Sigma_i}^*\circ(\iota\circ\zeta^{u,e})^* c^{\delta_{j,k}}\Delta_{\Lambda_k,\Lambda_k} \\ =\ & c^{\delta_{j,k}}a_{\Sigma_i}^*\Delta_{u\Lambda_k,\Lambda_k} = c^{\delta_{j,k}}A_{k_{\max}},\end{aligned}$$

where we use Proposition 4.6 in the third and fifth equality. Thus, we obtain (6.2).

Next, let us prove (6.3). If $i\in\{1,2,\ldots,\tilde{r}\}\setminus\{i_1,i_2,\ldots,i_n\}$ then $u\Lambda_i=\Lambda_i$. Since we know Proposition 4.6, our claim (6.3) follows from the following calculation:

$$\begin{aligned}&(a_{\Sigma_i}^{-1}\circ(\iota\circ\zeta^{u,e})^{-1}\circ e_j^c\circ(\iota\circ\zeta^{u,e})\circ a_{\Sigma_i})^*A_{-i} \\ =\ & a_{\Sigma_i}^*\circ(\iota\circ\zeta^{u,e})^*\circ(e_j^c)^*\circ((\iota\circ\zeta^{u,e})^{-1})^*\circ(a_{\Sigma_i}^{-1})^*A_{-i} \\ =\ & a_{\Sigma_i}^*\circ(\iota\circ\zeta^{u,e})^*\circ(e_j^c)^*\circ((\iota\circ\zeta^{u,e})^{-1})^*\Delta_{\Lambda_i,\Lambda_i} \\ =\ & a_{\Sigma_i}^*\circ(\iota\circ\zeta^{u,e})^*\circ(e_j^c)^*\circ((\iota\circ\zeta^{u,e})^{-1})^*\Delta_{u\Lambda_i,\Lambda_i} \\ =\ & a_{\Sigma_i}^*\circ(\iota\circ\zeta^{u,e})^*\circ(e_j^c)^*\Delta_{\Lambda_i,\Lambda_i} \\ =\ & a_{\Sigma_i}^*\circ(\iota\circ\zeta^{u,e})^*\Delta_{\Lambda_i,\Lambda_i} \\ =\ & a_{\Sigma_i}^*\Delta_{u\Lambda_i,\Lambda_i} = A_{-i}.\end{aligned}$$

We now turn to (6.1) by induction on $n-k$. First, let us consider the case $n-k=0$ so that $n=k$. By Proposition 4.4(4.2), we have $(p\circ a_{\Sigma_i}^{-1}\circ(\iota\circ\zeta^{u,e})^{-1}\circ e_j^c\circ\iota\circ\zeta^{u,e}\circ a_{\Sigma_i}A)_n=\overline{X}_n$. We set $F:=\{i\in I|i^+=n+1\}$. Note that $\tilde{B}_{n,n^-}=-1$ and if $l\in I\setminus M\setminus\{n^-\}$ then $\tilde{B}_{n,l}=0$ by $l<l^+<n$ and Definition 3.6 and the definition of the matrix M in Theorem 3.12. Therefore, the definition of p and (6.2) imply

$$\begin{aligned}&\overline{X}_n = (p\circ a_{\Sigma_i}^{-1}\circ(\iota\circ\zeta^{u,e})^{-1}\circ e_j^c\circ\iota\circ\zeta^{u,e}\circ a_{\Sigma_i}A)_n \\ =\ & \prod_{l\in I}(a_{\Sigma_i}^{-1}\circ(\iota\circ\zeta^{u,e})^{-1}\circ e_j^c\circ\iota\circ\zeta^{u,e}\circ a_{\Sigma_i}A)_l^{\tilde{B}_{n,l}} \\ =\ & \prod_{l\in I\setminus F}(a_{\Sigma_i}^{-1}\circ(\iota\circ\zeta^{u,e})^{-1}\circ e_j^c\circ\iota\circ\zeta^{u,e}\circ a_{\Sigma_i}A)_l^{\tilde{B}_{n,l}} \\ & \prod_{l\in F}(a_{\Sigma_i}^{-1}\circ(\iota\circ\zeta^{u,e})^{-1}\circ e_j^c\circ\iota\circ\zeta^{u,e}\circ a_{\Sigma_i}A)_l^{\tilde{B}_{n,l}} \\ =\ & (a_{\Sigma_i}^{-1}\circ(\iota\circ\zeta^{u,e})^{-1}\circ e_j^c\circ\iota\circ\zeta^{u,e}\circ a_{\Sigma_i}A)_{n^-}^{-1}c^{\tilde{B}_{n,j_{\max}}}\prod_{l\in F}A_l^{\tilde{B}_{n,l}} \\ =\ & (a_{\Sigma_i}^{-1}\circ(\iota\circ\zeta^{u,e})^{-1}\circ e_j^c\circ\iota\circ\zeta^{u,e}\circ a_{\Sigma_i}A)_{n^-}^{-1}c^{\tilde{B}_{n,j_{\max}}}A_{n^-}p(A)_n,\end{aligned}$$

which yields our claim $(a_{\Sigma_i}^{-1}\circ(\iota\circ\zeta^{u,e})^{-1}\circ e_j^c\circ\iota\circ\zeta^{u,e}\circ a_{\Sigma_i}A)_{n^-}=A_{n^-}c^{\tilde{B}_{n,j_{\max}}}\frac{p(A)_n}{\overline{X}_n}$ for $k=n$.

Next, let us consider the case $n-k>0$. Using Proposition 4.4 (4.2), we have $\overline{X}_k=(p\circ a_{\Sigma_i}^{-1}\circ(\iota\circ\zeta^{u,e})^{-1}\circ e_j^c\circ(\iota\circ\zeta^{u,e})\circ a_{\Sigma_i}A)_k$. Note that $\tilde{B}_{k,k^-}=-1$ and if $l\in\{-\tilde{r},\ldots,-1,1,\ldots,n\}\setminus F\setminus\{n^-,\ldots,(k+1)^-,k^-\}$ then $\tilde{B}_{k,l}=0$ by $l<l^+<k$ and Definition 3.6 and the definition of the matrix M in Theorem 3.12. Hence the submatrix $(\tilde{B_{i,l}})_{k\le i\le n,\ l=(k+1)^-,\ldots,n^-,j_{\max}}$ of $\tilde{B}_{\mathbf{i}}$ is as follows:

$$\begin{pmatrix} \tilde{B}_{k,(k+1)^-} & \tilde{B}_{k,(k+2)^-} & \cdots & \tilde{B}_{k,n^-} & \tilde{B}_{k,j_{\max}} \\ -1 & \tilde{B}_{k+1,(k+2)^-} & \cdots & \tilde{B}_{k+1,n^-} & \tilde{B}_{k+1,j_{\max}} \\ 0 & -1 & \cdots & \tilde{B}_{k+2,n^-} & \tilde{B}_{k+2,j_{\max}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \tilde{B}_{n,j_{\max}} \end{pmatrix}$$

In conjunction with the definition of p, (6.2) and induction hypothesis, we obtain

$$
\begin{aligned}
&\overline{X}_k = (p \circ a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_k \\
&= \prod_{l \in \{-\tilde{r},\dots,-1,1,\dots,n\}} (a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_l^{\tilde{B}_{k,l}} \\
&= (a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_{k^-}^{-1} \\
&\quad \cdot \prod_{l \in \{-\tilde{r},\dots,-1,1,\dots,n\} \setminus F \setminus \{n^-,\dots,(k+1)^-,k^-\}} (a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_l^{\tilde{B}_{k,l}} \\
&\quad \cdot \prod_{l \in \{n^-,\dots,(k+1)^-\}} (a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_l^{\tilde{B}_{k,l}} \\
&\quad \cdot \prod_{l \in F} (a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_l^{\tilde{B}_{k,l}} \\
&= (a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_{k^-}^{-1} \\
&\quad \cdot \prod_{l \in \{n^-,\dots,(k+1)^-\}} (a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_l^{\tilde{B}_{k,l}} \\
&\quad \cdot c^{\tilde{B}_{k,j\mathrm{max}}} \prod_{l \in F} A_l^{\tilde{B}_{k,l}} \\
&= (a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_{k^-}^{-1} \cdot c^{\tilde{B}_{k,j\mathrm{max}}} \prod_{l \in F} A_l^{\tilde{B}_{k,l}} \\
&\quad \cdot \prod_{l \in \{n,\dots,k+1\}} \left(A_{l^-} c^{D_{\{l,l+1,\dots,n\},\{(l+1)^-,(l+2)^-,\dots,n^-,j\mathrm{max}\}}} \left(\frac{p(A)_l}{\overline{X}_l} \right) \left(\frac{p(A)_{l+1}}{\overline{X}_{l+1}} \right)^{D_{l,(l+1)^-}} \right. \\
&\quad \left. \cdot \left(\frac{p(A)_{l+2}}{\overline{X}_{l+2}} \right)^{D_{\{l,l+1\},\{(l+1)^-,(l+2)^-\}}} \cdots \left(\frac{p(A)_n}{\overline{X}_n} \right)^{D_{\{l,l+1,\dots,n-1\},\{(l+1)^-,(l+2)^-,\dots,n^-\}}} \right)^{\tilde{B}_{k,l^-}} \\
&= (a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_{k^-}^{-1} \cdot A_{k^-} p(A)_k \\
&\quad \cdot c^{\tilde{B}_{k,j\mathrm{max}} + \sum_{l=n,n-1,\dots,k+1} D_{\{l,l+1,\dots,n\},\{(l+1)^-,\dots,n^-,j\mathrm{max}\}} \tilde{B}_{k,l^-}} \\
&\quad \cdot \prod_{l \in \{n,\dots,k+1\}} \left(\frac{p(A)_l}{\overline{X}_l} \right)^{\tilde{B}_{k,l^-} + \sum_{s=l-1,l-2,\dots,k+1} \tilde{B}_{k,s^-} D_{\{s,\dots,l-2,l-1\},\{(s+1)^-,\dots,(l-1)^-,l^-\}}} \\
&= (a_{\Sigma_i}^{-1} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e_j^c \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_i} A)_{k^-}^{-1} \cdot A_{k^-} p(A)_k \\
&\quad \cdot c^{D_{\{k,k+1,\dots,n\},\{(k+1)^-,\dots,n^-,j\mathrm{max}\}}} \\
&\quad \cdot \prod_{l \in \{n,\dots,k+1\}} \left(\frac{p(A)_l}{\overline{X}_l} \right)^{D_{\{k,k+1,\dots,l-1\},\{(k+1)^-,\dots,l^-\}}},
\end{aligned}
$$

which yields our claim (6.1). □

Theorem 6.3. *Let* $\mathbf{i} = (i_1, \dots, i_n)$ *be a reduced word of* $u \in W$, $A = (A_{-\tilde{r}}, \dots, A_{-1}, A_1, \dots, A_n) \in \mathcal{A}_{\Sigma_{\mathbf{i}}}$, M *be the matrix in Theorem 3.12, and* $p = p_M : \mathcal{A}_{\Sigma_{\mathbf{i}}} \to \mathcal{X}_{\Sigma_{\mathbf{i}}}$ *be the map in Proposition 3.4. For* $j \in [1, r]$, *we set* $\{K_1, K_2, \dots, K_l\} := \{K | 1 \le K \le n,\ i_K = j\}$ $(K_1 < \dots < K_l)$.

$$\gamma_j \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}}}(A) = p(A)_{-j} p(A)_{K_1} \cdots p(A)_{K_l},$$

$$\varepsilon_j \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}}}(A) = \left(\sum_{s=0}^{l-1} p(A)_{K_{s+1}} p(A)_{K_{s+2}} \cdots p(A)_{K_l} \right)^{-1}.$$

Proof. It follows from Proposition 4.4 (4.3) that $\varepsilon_j \circ x_{\Sigma_{\mathbf{i}}} \circ p = \varepsilon_j \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}}}$ and $\gamma_j \circ x_{\Sigma_{\mathbf{i}}} \circ p = \gamma_j \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}}}$. Our claims follow from Theorem 5.1. □

Theorems 6.1 and 6.3 imply that the map $(\iota \circ \zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}}} : \mathcal{A}_{\Sigma_{\mathbf{i}}} \to G^{u,e}$ is a positive structure on the geometric crystal $(G^{u,e}, \{e_i\}_{i \in [1,r]}, \{\gamma_i\}_{i \in [1,r]}, \{\varepsilon_i\}_{i \in [1,r]})$. Let Σ be a seed obtained from $\Sigma_{\mathbf{i}}$ by an iteration of mutations. Then the corresponding birational map $\mathcal{A}_{\Sigma_{\mathbf{i}}} \to \mathcal{A}_{\Sigma}$ is positive. Hence, we obtain the following corollary.:

Corollary 6.4. *The map $(\iota\circ\zeta^{u,e})\circ a_{\Sigma}:\mathcal{A}_{\Sigma}\to G^{u,e}$ is a positive structure on the geometric crystal $(G^{u,e},\{e_i\}_{i\in[1,r]},\{\gamma_i\}_{i\in[1,r]},\{\varepsilon_i\}_{i\in[1,r]})$. Applying Theorem 2.6, we obtain a crystal $(X_*(\mathcal{A}_{\Sigma}),\{\tilde{e}_i\}_{i\in[1,r]},\{\tilde{\gamma}_i\}_{i\in[1,r]},\{\tilde{\varepsilon}_i\}_{i\in[1,r]})$. It induces a crystal structure on the set of $\mathbb{Z}$-valued points $\mathcal{A}_{|\Sigma_{\mathbf{i}}|}(\mathbb{Z}^T)$.*

7. Type A-case

In the rest of article, we set $G=SL_{r+1}(\mathbb{C})$ and consider the cell $G^{w_0,e}$ with the longest element $w_0\in W$. Let $n:=\frac{r(r+1)}{2}$, $I=\{-r,\ldots,-1,1,2,\ldots,n\}$ and $\mathbf{i}_0$ be the following reduced word of w_0:

$$\mathbf{i}_0=(\underbrace{1,2,\ldots,r-2,r-1,r}_{\text{1st cycle}},\underbrace{1,2,\ldots,r-2,r-1}_{\text{2nd cycle}},\underbrace{1,2,\ldots,r-2}_{\text{3rd cycle}},\ldots,\underbrace{1,2}_{r-1\text{th cycle}},1)$$

and i_k be the kth index of $\mathbf{i}_0$ from the left.

In this section, we shall calculate explicit forms of $a^{-1}_{\Sigma_{\mathbf{i}_0}}\circ(\iota\circ\zeta^{w_0,e})^{-1}\circ e^c_j\circ(\iota\circ\zeta^{w_0,e})\circ a_{\Sigma_{\mathbf{i}_0}}$ by a direct calculation. In the last subsection, we will verify the explicit forms coincide with those of Theorem 6.1.

7.1. Fundamental representation of Type A_r

First, we review the fundamental representations of the complex simple Lie algebras $\mathfrak{g}$ of type A_r [11,15]. Let $\mathfrak{g}=\mathfrak{sl}(r+1,\mathbb{C})$ be the simple Lie algebra of type A_r. The Cartan matrix $A=(a_{i,j})_{i,j\in\{1,2,\ldots,r\}}$ of $\mathfrak{g}$ is as follows:

$$a_{i,j}=\begin{cases}2 & \text{if } i=j,\\ -1 & \text{if } |i-j|=1,\\ 0 & \text{otherwise.}\end{cases}$$

For $\mathfrak{g}=\langle\mathfrak{h},e_i,f_i(i\in\{1,2,\ldots,r\})\rangle$, let us describe the vector representation $V(\Lambda_1)$. Set $\mathbf{B}^{(r)}:=\{v_i|\ i=1,2,\cdots,r+1\}$ and define $V(\Lambda_1):=\bigoplus_{v\in\mathbf{B}^{(r)}}\mathbb{C}v$. The weights of v_i $(i=1,\cdots,r+1)$ are given by $\mathrm{wt}(v_i)=\Lambda_i-\Lambda_{i-1}$, where $\Lambda_0=\Lambda_{r+1}=0$. We define the $\mathfrak{g}$-action on $V(\Lambda_1)$ as follows:

$$hv_j=\langle h,\mathrm{wt}(v_j)\rangle v_j\quad (h\in\oplus_{i\in[1,r]}\mathbb{Z}\alpha_i^{\vee},\ j\in\{1,2,\ldots,r+1\}), \tag{7.1}$$

$$f_iv_i=v_{i+1},\quad e_iv_{i+1}=v_i\quad (1\le i\le r), \tag{7.2}$$

and the other actions are trivial.

Let Λ_i be the ith fundamental weight of type A_r. As is well-known that the fundamental representation $V(\Lambda_i)$ $(1\le i\le r)$ is embedded in $\wedge^iV(\Lambda_1)$ with multiplicity free. The explicit form of the highest (resp. lowest) weight vector v_{Λ_i} (resp. u_{Λ_i}) of $V(\Lambda_i)$ is realized in $\wedge^iV(\Lambda_1)$ as follows:

$$v_{\Lambda_i}=v_1\wedge v_2\wedge\cdots\wedge v_i,\qquad u_{\Lambda_i}=v_{r-i+2}\wedge v_{r-i+3}\wedge\cdots\wedge v_{r+1}. \tag{7.3}$$

It is known that if $1\le j_1<\cdots<j_s\le r+1$, $1\le l_1<\cdots<l_s\le r+1$ and $x\in G=SL_{r+1}(\mathbb{C})$ then the value of bilinear form in 2.6 on x

$$\langle v_{j_1}\wedge v_{j_2}\wedge\cdots\wedge v_{j_s},xv_{l_1}\wedge v_{l_2}\wedge\cdots\wedge v_{l_s}\rangle$$

is equal to an ordinary minor $D_{\{j_1,\ldots,j_s\},\{l_1,\ldots,l_s\}}(x)$.

7.2. Geometric crystal action on cluster $\mathcal{A}$-varieties for type A_r case

Lemma 7.1. *The rational maps $\gamma_j,\varepsilon_j:G^{w_0,e}\to\mathbb{C}^{\times}$ can be described as*

$$\varepsilon_j=\frac{D_{j+1,j+1}}{D_{j+1,j}},\quad \gamma_j=\frac{D_{j,j}}{D_{j+1,j+1}}.$$

Proof. Our claim follows by (2.4) and (7.3). □

Lemma 7.2. *We suppose that $1\le j_1<\cdots<j_s\le r+1$ and $j\in\{1,2,\ldots,r\}$. If there exists $i\in\{1,2,\ldots,s\}$ such that $j=j_i$ and $j_{i+1}>j_i+1$ (we set $j_{s+1}=r+2$) then*

$$(e^c_j)^*D_{\{j_1,\ldots,j_s\},\{1,\ldots,s\}}=D_{\{j_1,\ldots,j_s\},\{1,\ldots,s\}}+(c-1)\frac{D_{\{1,\ldots,j\},\{1,\ldots,j\}}}{D_{\{1,\ldots,j-1,j+1\},\{1,\ldots,j\}}}D_{\{j_1,\ldots,j_{i-1},j_i+1,j_{i+1},\ldots,j_s\},\{1,\ldots,s\}},$$

otherwise,

$$(e^c_j)^*D_{\{j_1,\ldots,j_s\},\{1,\ldots,s\}}=D_{\{j_1,\ldots,j_s\},\{1,\ldots,s\}}.$$

Proof. Because of Lemma 7.1, we obtain

$$\varphi_j = \varepsilon_j\gamma_j = \frac{D_{j,j}}{D_{j+1,j}} = \frac{D_{\{1,\dots,j\},\{1,\dots,j\}}}{D_{\{1,\dots,j-1,j+1\},\{1,\dots,j\}}}.$$

We can calculate

$$\begin{aligned}
&(e_j^c)^* D_{\{j_1,\dots,j_s\},\{1,\dots,s\}}(x)\\
&= \langle v_{j_1}\wedge v_{j_2}\wedge\cdots\wedge v_{j_s}, e_j^c(x)v_1\wedge v_2\wedge\cdots\wedge v_s\rangle\\
&= \langle v_{j_1}\wedge v_{j_2}\wedge\cdots\wedge v_{j_s}, x_j((c-1)\varphi_j(x))xx_j((c^{-1}-1)\varepsilon_i(x))v_1\wedge v_2\wedge\cdots\wedge v_s\rangle\\
&= \langle y_j((c-1)\varphi_j(x))v_{j_1}\wedge v_{j_2}\wedge\cdots\wedge v_{j_s}, xv_1\wedge v_2\wedge\cdots\wedge v_s\rangle.
\end{aligned}$$

If there exists $i\in\{1,2,\dots,s\}$ such that $j=j_i$ and $j_{i+1}>j_i+1$ then

$$\begin{aligned}
&(e_j^c)^* D_{\{j_1,\dots,j_s\},\{1,\dots,s\}}(x)\\
&= \langle (1+(c-1)\varphi_j(x)f_j)v_{j_1}\wedge v_{j_2}\wedge\cdots\wedge v_{j_s}, xv_1\wedge v_2\wedge\cdots\wedge v_s\rangle\\
&= \langle v_{j_1}\wedge v_{j_2}\wedge\cdots\wedge v_{j_s}, xv_1\wedge v_2\wedge\cdots\wedge v_s\rangle\\
&\quad +(c-1)\varphi_j(x)\langle v_{j_1}\wedge\cdots\wedge v_{j_{i-1}}\wedge v_{j_i+1}\wedge\cdots\wedge v_{j_s}, xv_1\wedge v_2\wedge\cdots\wedge v_s\rangle\\
&= D_{\{j_1,\dots,j_s\},\{1,\dots,s\}}(x) + (c-1)\varphi_j(x) D_{\{j_1,\dots,j_{i-1},j_i+1,j_{i+1},\dots,j_s\},\{1,\dots,s\}}(x)\\
&= D_{\{j_1,\dots,j_s\},\{1,\dots,s\}}(x) + (c-1)\frac{D_{\{1,\dots,j\},\{1,\dots,j\}}}{D_{\{1,\dots,j-1,j+1\},\{1,\dots,j\}}} D_{\{j_1,\dots,j_{i-1},j_i+1,j_{i+1},\dots,j_s\},\{1,\dots,s\}}(x).
\end{aligned}$$

Otherwise, we have $(e_j^c)^* D_{\{j_1,\dots,j_s\},\{1,\dots,s\}}(x) = \langle v_{j_1}\wedge v_{j_2}\wedge\cdots\wedge v_{j_s}, xv_1\wedge v_2\wedge\cdots\wedge v_s\rangle = D_{\{j_1,\dots,j_s\},\{1,\dots,s\}}(x)$. □

Recall that a regular map $p_M : \mathcal{A}_{\Sigma_{\mathbf{i}_0}} \to \mathcal{X}_{\Sigma_{\mathbf{i}_0}} = \mathcal{X}_{\mathbf{i}_0}$ is defined as

$$p^*(X_i) = \prod_{k\in I} A_k^{\tilde{B}_{i,k}},$$

where $(X_i)_{i\in I}$ is a coordinate of $\mathcal{X}_{\Sigma_{\mathbf{i}_0}}$. In what follows, we denote p_M by p.

We write a coordinate $A=(A_{-r},\dots,A_{-1},A_1,\dots,A_n)\in\mathcal{A}_{\Sigma_{\mathbf{i}_0}}$ as

$$(A_{-r},\dots,A_{-1},A_{1,1},A_{1,2},\dots,A_{1,r},A_{2,1},A_{2,2},\dots,A_{2,r-1},\dots,A_{r-1,1},A_{r-1,2},A_{r,1}).$$

Note that $(a^*_{\Sigma_{\mathbf{i}_0}})^{-1}A_{m,d} = \langle v_{m+1}\wedge v_{m+2}\wedge\cdots\wedge v_{m+d}, \cdot v_1\wedge v_2\wedge\cdots\wedge v_d\rangle$ for $m\in\{1,2,\dots,r\}$ and $d\in\{1,2,\dots,r-m+1\}$.

We also set

$$(P_{-r},\dots,P_{-1},P_{1,1},P_{1,2},\dots,P_{1,r},P_{2,1},P_{2,2},\dots,P_{2,r-1},\dots,P_{r-1,1},P_{r-1,2},P_{r,1}) = p(A).$$

Theorem 7.3. *For $m\in\{0,1,2,\dots,r\}$ and $d\in\{1,2,\dots,\min(r-m+1,r)\}$, we obtain*

$$\begin{aligned}
&(a^{-1}_{\Sigma_{\mathbf{i}_0}}\circ(\iota\circ\zeta^{w_0,e})^{-1}\circ e_j^c\circ(\iota\circ\zeta^{w_0,e})\circ a_{\Sigma_{\mathbf{i}_0}}A)_{m,d}\\
&= \begin{cases} A_{m,d}\frac{c(P_{1,d}P_{2,d}\cdots P_{r-d,d}+P_{2,d}\cdots P_{r-d,d}+\cdots+P_{m,d}\cdots P_{r-d,d})+P_{m+1,d}\cdots P_{r-d,d}+\cdots+P_{r-d,d}+1}{P_{1,d}P_{2,d}\cdots P_{r-d,d}+P_{2,d}\cdots P_{r-d,d}+\cdots+P_{r-d,d}+1} & \text{if } j=d,\\ A_{m,d} & \text{if } j\neq d,\end{cases}
\end{aligned}$$

where we set $A_{0,d} := A_{-d}$.

Proof. By Proposition 4.6 and Lemma 7.2, we get

$$\begin{aligned}
&A_{m,d} \xrightarrow{(a^{-1}_{\Sigma_{\mathbf{i}_0}})^*} \langle v_{m+1}\wedge v_{m+2}\wedge\cdots\wedge v_{m+d}, \cdot v_1\wedge v_2\wedge\cdots\wedge v_d\rangle\\
&= \frac{\langle v_{m+1}\wedge v_{m+2}\wedge\cdots\wedge v_{r+1}, \cdot v_1\wedge v_2\wedge\cdots\wedge v_d\wedge v_{m+d+1}\wedge\cdots\wedge v_{r+1}\rangle}{\langle v_{m+d+1}\wedge v_{m+d+2}\wedge\cdots\wedge v_{r+1}, \cdot v_{m+d+1}\wedge\cdots\wedge v_{r+1}\rangle}\\
&= \frac{\langle \overline{w_0}v_1\wedge v_2\wedge\cdots\wedge v_{r-m+1}, \cdot v_1\wedge v_2\wedge\cdots\wedge v_d\wedge v_{m+d+1}\wedge\cdots\wedge v_{r+1}\rangle}{\langle \overline{w_0}v_1\wedge v_2\wedge\cdots\wedge v_{r-m-d+1}, \cdot v_{m+d+1}\wedge\cdots\wedge v_{r+1}\rangle}\\
&\xrightarrow{((\iota\circ\zeta^{w_0,e})^{-1})^*} \frac{\langle v_1\wedge v_2\wedge\cdots\wedge v_d\wedge v_{m+d+1}\wedge\cdots\wedge v_{r+1}, \cdot v_1\wedge v_2\wedge\cdots\wedge v_{r-m+1}\rangle}{\langle v_{m+d+1}\wedge\cdots\wedge v_{r+1}, \cdot v_1\wedge v_2\wedge\cdots\wedge v_{r-m-d+1}\rangle}\\
&= \frac{D_{\{1,\dots,d,m+d+1,\dots,r+1\},\{1,2,\dots,r-m+1\}}}{D_{\{m+d+1,\dots,r+1\},\{1,2,\dots,r-m-d+1\}}}\\
&\xrightarrow{(e_d^c)^*} \frac{1}{D_{\{m+d+1,\dots,r+1\},\{1,2,\dots,r-m-d+1\}}}\Big(D_{\{1,\dots,d,m+d+1,\dots,r+1\},\{1,2,\dots,r-m+1\}}
\end{aligned}$$

$$+(c-1)\frac{D_{\{1,\dots,d\},\{1,\dots,d\}}}{D_{\{1,\dots,d-1,d+1\},\{1,\dots,d\}}}D_{\{1,\dots,d-1,d+1,m+d-1,\dots,r+1\},\{1,2,\dots,r-m+1\}}\Bigg)$$

$$\xrightarrow{(\iota\circ\zeta^{w_0,e})^*}\frac{1}{D_{\{m+d+1,\dots,r+1\},\{m+d+1,\dots,r+1\}}}\left(D_{\{m+1,\dots,r+1\},\{1,2,\dots,d,m+d+1,\dots,r+1\}}\right.$$

$$\left.+(c-1)\frac{D_{\{r-d+2,\dots,d\},\{1,\dots,d\}}}{D_{\{r-d+2,\dots,r+1\},\{1,\dots,d-1,d+1\}}}D_{\{m+1,\dots,r+1\},\{1,2,\dots,d-1,d+1,m+d+1,\dots,r-m+1\}}\right).$$

Hence, we need to show that

$$\begin{aligned}&(a_{\Sigma_{\mathbf{i}_0}})^*\frac{1}{D_{\{m+d+1,\dots,r+1\},\{m+d+1,\dots,r+1\}}}\left(D_{\{m+1,\dots,r+1\},\{1,2,\dots,d,m+d+1,\dots,r+1\}}\right.\\&\left.+(c-1)\frac{D_{\{r-d+2,\dots,r+1\},\{1,\dots,d\}}}{D_{\{r-d+2,\dots,r+1\},\{1,\dots,d-1,d+1\}}}D_{\{m+1,\dots,r+1\},\{1,2,\dots,d-1,d+1,m+d+1,\dots,r-m+1\}}\right)\\&=A_{m,d}\frac{c(P_{1,d}P_{2,d}\cdots P_{r-d,d}+P_{2,d}\cdots P_{r-d,d}+\cdots+P_{m,d}\cdots P_{r-d,d})+P_{m+1,d}\cdots P_{r-d,d}+\cdots+P_{r-d,d}+1}{P_{1,d}P_{2,d}\cdots P_{r-d,d}+P_{2,d}\cdots P_{r-d,d}+\cdots+P_{r-d,d}+1}.\end{aligned}\tag{7.4}$$

Let us take an element of the open subset $\overline{\mathbb{B}}^-_{w_0}$ in (2.2)

$$x=ay_r(t_{1,r})\cdots y_1(t_{1,1})y_r(t_{2,r})\cdots y_2(t_{2,2})\cdots y_r(t_{r-1,r})y_{r-1}(t_{r-1,r-1})y_r(t_{r,r})\in\overline{\mathbb{B}}^-_{w_0}\subset G^{w_0,e}$$

with $a=\mathrm{diag}(a_1,\dots,a_r,a_{r+1})\in H$ and $t_{s,i}\in\mathbb{C}^\times$. We get

$$\begin{aligned}&\frac{1}{D_{\{m+d+1,\dots,r+1\},\{m+d+1,\dots,r+1\}}}\left(D_{\{m+1,\dots,r+1\},\{1,2,\dots,d,m+d+1,\dots,r+1\}}\right.\\&\left.+(c-1)\frac{D_{\{r-d+2,\dots,r+1\},\{1,\dots,d\}}}{D_{\{r-d+2,\dots,r+1\},\{1,\dots,d-1,d+1\}}}D_{\{m+1,\dots,r+1\},\{1,2,\dots,d-1,d+1,m+d+1,\dots,r-m+1\}}\right)(x)\\&=a_{m+1}\cdots a_{m+d}\left(\prod_{i=1}^{d}t_{i,i}t_{i,i+1}\cdots t_{i,m+i-1}+(c-1)\prod_{i=1}^{d}t_{i,i}t_{i,i+1}\cdots t_{i,r-d+i}\right.\\&\left(\prod_{i=1}^{d-1}t_{i,i}t_{i,i+1}\cdots t_{i,r-d+i}(\sum_{i=d}^{r}t_{d+1,d+1}t_{d+1,d+2}\cdots t_{d+1,i}t_{d,i+1}\cdots t_{d,r})\right)^{-1}\\&\left.\left(\prod_{i=1}^{d-1}t_{i,i}t_{i,i+1}\cdots t_{i,m+i-1}(\sum_{i=d}^{m+d-1}t_{d+1,d+1}t_{d+1,d+2}\cdots t_{d+1,i}t_{d,i+1}\cdots t_{d,m+d-1})\right)\right)\\&=a_{m+1}\cdots a_{m+d}\prod_{i=1}^{d}t_{i,i}t_{i,i+1}\cdots t_{i,m+i-1}\left(1+(c-1)t_{d,m+d}t_{d,m+d+1}\cdots t_{d,r}\right.\\&\left.\frac{\sum_{i=d}^{m+d-1}t_{d+1,d+1}t_{d+1,d+2}\cdots t_{d+1,i}t_{d,i+1}\cdots t_{d,m+d-1}}{\sum_{i=d}^{r}t_{d+1,d+1}t_{d+1,d+2}\cdots t_{d+1,i}t_{d,i+1}\cdots t_{d,r}}\right)\\&=a_{m+1}\cdots a_{m+d}\prod_{i=1}^{d}t_{i,i}t_{i,i+1}\cdots t_{i,m+i-1}\\&\frac{\sum_{i=m+d}^{r}t_{d+1,d+1}t_{d+1,d+2}\cdots t_{d+1,i}t_{d,i+1}\cdots t_{d,r}+c\sum_{i=d}^{m+d-1}t_{d+1,d+1}\cdots t_{d+1,i}t_{d,i+1}\cdots t_{d,r}}{\sum_{i=d}^{r}t_{d+1,d+1}t_{d+1,d+2}\cdots t_{d+1,i}t_{d,i+1}\cdots t_{d,r}}.\end{aligned}\tag{7.5}$$

Next, let us calculate

$$(a^{-1}_{\Sigma_{\mathbf{i}_0}})^*A_{m,d}\frac{c(P_{1,d}\cdots P_{r-d,d}+\cdots+P_{m,d}\cdots P_{r-d,d})+P_{m+1,d}\cdots P_{r-d,d}+\cdots+P_{r-d,d}+1}{P_{1,d}P_{2,d}\cdots P_{r-d,d}+P_{2,d}\cdots P_{r-d,d}+\cdots+P_{r-d,d}+1}(x).$$

The construction of $\tilde{B}_{\mathbf{i}_0}$ means $P_{s,d}=\frac{A_{s+1,d}A_{s,d-1}A_{s-1,d+1}}{A_{s+1,d-1}A_{s,d+1}A_{s-1,d}}$ $(s=1,2,\dots,r-d)$. We obtain

$$(a^{-1}_{\Sigma_{\mathbf{i}_0}})^*A_{s+1,d}(x)=D_{\{s+2,\dots,s+d+1\},\{1,\dots,d\}}(x)=a_{s+2}\cdots a_{s+d+1}\prod_{i=1}^{d}t_{i,i}t_{i,i+1}\cdots t_{i,s+i},$$

$$(a^{-1}_{\Sigma_{\mathbf{i}_0}})^*A_{s,d-1}(x)=D_{\{s+1,\dots,s+d-1\},\{1,\dots,d-1\}}(x)=a_{s+1}\cdots a_{s+d-1}\prod_{i=1}^{d-1}t_{i,i}t_{i,i+1}\cdots t_{i,s+i-1},$$

$$(a_{\Sigma_{\mathbf{i}_0}}^{-1})^{*}A_{s-1,d+1}(x)=D_{\{s,\ldots,s+d\},\{1,\ldots,d+1\}}(x)=a_s\cdots a_{s+d}\prod_{i=1}^{d+1}t_{i,i}t_{i,i+1}\cdots t_{i,s+i-2},$$

$$(a_{\Sigma_{\mathbf{i}_0}}^{-1})^{*}A_{s+1,d-1}(x)=D_{\{s+2,\ldots,s+d\},\{1,\ldots,d-1\}}(x)=a_{s+2}\cdots a_{s+d}\prod_{i=1}^{d-1}t_{i,i}t_{i,i+1}\cdots t_{i,s+i},$$

$$(a_{\Sigma_{\mathbf{i}_0}}^{-1})^{*}A_{s,d+1}(x)=D_{\{s+1,\ldots,s+d+1\},\{1,\ldots,d+1\}}(x)=a_{s+1}\cdots a_{s+d+1}\prod_{i=1}^{d+1}t_{i,i}t_{i,i+1}\cdots t_{i,s+i-1},$$

$$(a_{\Sigma_{\mathbf{i}_0}}^{-1})^{*}A_{s-1,d}(x)=D_{\{s,\ldots,s+d-1\},\{1,\ldots,d\}}(x)=a_s\cdots a_{s+d-1}\prod_{i=1}^{d}t_{i,i}t_{i,i+1}\cdots t_{i,s+i-2}.$$

Consequently, we have $(a_{\Sigma_{\mathbf{i}_0}}^{-1})^{*}P_{s,d}(x)=\frac{t_{d,s+d}}{t_{d+1,s+d}}$. In conjunction with $(a_{\Sigma_{\mathbf{i}_0}}^{-1})^{*}A_{m,d}(x)=D_{\{m+1,\ldots,m+d\},\{1,\ldots,d\}}(x)=a_{m+1}\cdots a_{m+d}$ $\prod_{i=1}^{d}t_{i,i}t_{i,i+1}\cdots t_{i,s+i-1}$, one obtain

$$\begin{aligned}
&(a_{\Sigma_{\mathbf{i}_0}}^{-1})^{*}A_{m,d}\frac{c(P_{1,d}\cdots P_{r-d,d}+\cdots+P_{m,d}\cdots P_{r-d,d})+P_{m+1,d}\cdots P_{r-d,d}+\cdots+P_{r-d,d}+1}{P_{1,d}P_{2,d}\cdots P_{r-d,d}+P_{2,d}\cdots P_{r-d,d}+\cdots+P_{r-d,d}+1}(x)\\
&=a_{m+1}\cdots a_{m+d}\prod_{i=1}^{d}t_{i,i}t_{i,i+1}\cdots t_{i,s+i-1}\\
&\quad\frac{c(\sum_{s=1}^{m}\frac{t_{d,s+d}t_{d,s+d+1}\cdots t_{d,r}}{t_{d+1,s+d}t_{d+1,s+d+1}\cdots t_{d+1,r}})+\sum_{s=m+1}^{r-d+1}\frac{t_{d,s+d}t_{d,s+d+1}\cdots t_{d,r}}{t_{d+1,s+d}t_{d+1,s+d+1}\cdots t_{d+1,r}}}{\sum_{s=1}^{r-d+1}\frac{t_{d,s+d}t_{d,s+d+1}\cdots t_{d,r}}{t_{d+1,s+d}t_{d+1,s+d+1}\cdots t_{d+1,r}}}\\
&=a_{m+1}\cdots a_{m+d}\prod_{i=1}^{d}t_{i,i}t_{i,i+1}\cdots t_{i,s+i-1}\qquad(7.6)\\
&\quad\frac{c(\sum_{s=1}^{m}t_{d+1,1}\cdots t_{d+1,s-1+d}t_{d,s+d}t_{d,s+d+1}\cdots t_{d,r})+\sum_{s=m+1}^{r-d+1}t_{d+1,1}\cdots t_{d+1,s-1+d}t_{d,s+d}\cdots t_{d,r}}{\sum_{s=1}^{r-d+1}t_{d+1,1}\cdots t_{d+1,s-1+d}t_{d,s+d}t_{d,s+d+1}\cdots t_{d,r}}.
\end{aligned}$$

The relation (7.4) follows from (7.5) and (7.6). □

7.3. An example

Example 7.4. We consider the case $SL_5(\mathbb{C})$, $\mathbf{i}_0=(1,2,3,4,1,2,3,1,2,1)$. The matrix $\tilde{B}_{\mathbf{i}_0}$ is as follows:

$$\tilde{B}_{\mathbf{i}_0}=\begin{array}{c}\\ -4\\ -3\\ -2\\ -1\\ 1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ 8\\ 9\\ 10\end{array}\begin{array}{c}\begin{array}{cccccccccccccc}-4&-3&-2&-1&1&2&3&4&5&6&7&8&9&10\end{array}\\
\begin{pmatrix}
1&0&0&0&0&0&-1&1&0&0&0&0&0&0\\
-1&1&0&0&0&-1&1&0&0&0&0&0&0&0\\
0&-1&1&0&-1&1&0&0&0&0&0&0&0&0\\
0&0&-1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&1&-1&0&-1&0&0&1&0&0&0&0&0\\
0&1&-1&0&1&0&-1&0&-1&1&0&0&0&0\\
1&-1&0&0&0&1&0&-1&0&-1&1&0&0&0\\
-1&0&0&0&0&0&1&1&0&0&-1&0&0&0\\
0&0&0&0&-1&1&0&0&0&-1&0&1&0&0\\
0&0&0&0&0&-1&1&0&1&0&-1&-1&1&0\\
0&0&0&0&0&0&-1&0&0&1&1&0&-1&0\\
0&0&0&0&0&0&0&0&-1&1&0&0&-1&1\\
0&0&0&0&0&0&0&0&0&-1&0&1&1&-1\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&1
\end{pmatrix}\end{array}$$

Let $A:=(A_{-4},A_{-3},A_{-2},A_{-1},A_1,A_2,A_3,A_4,A_5,A_6,A_7,A_8,A_9,A_{10})\in\mathcal{A}_{\Sigma_{\mathbf{i}_0}}=(\mathbb{C}^{\times})^{14}$, M be the matrix in Theorem 3.12, and $p=p_M:\mathcal{A}_{\Sigma_{\mathbf{i}_0}}\to\mathcal{X}_{\Sigma_{\mathbf{i}_0}}$ be the map in Proposition 3.4. We set $(P_{-4},P_{-3},P_{-2},P_{-1},P_1,\ldots,P_{10})=p(A)$.

In this example, we shall calculate $(a_{\Sigma_{\mathbf{i}_0}}^{-1}\circ(\iota\circ\zeta^{u,e})^{-1}\circ e_1^{c}\circ(\iota\circ\zeta^{u,e})\circ a_{\Sigma_{\mathbf{i}_0}}A)_k$ via two ways: (1) A way using Theorem 6.1, (2) a way using Theorem 7.3.

First, let us calculate it using Theorem 6.1. Taking Theorem 5.1 into account, we get

$$\begin{aligned}
&x_{\Sigma_i}^{-1} \circ e_1^c \circ x_{\Sigma_i} \circ p(A)\\
&= \Bigg(P_{-4}, P_{-3}, P_{-2}\frac{P_1P_5P_8+P_5P_8+P_8+1}{cP_1P_5P_8+P_5P_8+P_8+1}, P_{-1}\frac{cP_1P_5P_8+P_5P_8+P_8+1}{P_1P_5P_8+P_5P_8+P_8+1},\\
&P_1\frac{c(P_1P_5P_8+P_5P_8)+P_8+1}{P_1P_5P_8+P_5P_8+P_8+1}, P_2\frac{cP_1P_5P_8+P_5P_8+P_8+1}{c(P_1P_5P_8+P_5P_8)+P_8+1}, P_3, P_4,\\
&P_5\frac{c(P_1P_5P_8+P_5P_8+P_8)+1}{cP_1P_5P_8+P_5P_8+P_8+1}, P_6\frac{c(P_1P_5P_8+P_5P_8)+P_8+1}{c(P_1P_5P_8+P_5P_8+P_8)+1},\\
&P_7, P_8\frac{c(P_1P_5P_8+P_5P_8+P_8+1)}{c(P_1P_5P_8+P_5P_8)+P_8+1}, P_9\frac{c(P_1P_5P_8+P_5P_8)+P_8+1}{cP_1P_5P_8+P_5P_8+P_8+1},\\
&P_{10}\frac{c(P_1P_5P_8+P_5P_8+P_8+1)}{c(P_1P_5P_8+P_5P_8+P_8)+1}\Bigg).
\end{aligned}$$

Hence, we can calculate

$$\begin{aligned}
&(a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_{10^-}\\
&= (a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_8\\
&= A_8 c^{D_{10,10}} \frac{P_{10}}{(x_{\Sigma_i}^{-1} \circ e_1^c \circ x_{\Sigma_i} \circ p(A))_{10}}\\
&= A_8\left(\frac{c(P_1P_5P_8+P_5P_8+P_8)+1}{P_1P_5P_8+P_5P_8+P_8+1}\right).
\end{aligned}\tag{7.7}$$

Similarly, we obtain

$$\begin{aligned}
&(a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_{9^-}\\
&= (a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_6\\
&= A_6 c^{D_{\{9,10\},\{8,10\}}} \frac{P_9}{(x_{\Sigma_i}^{-1} \circ e_1^c \circ x_{\Sigma_i} \circ p(A))_9}\left(\frac{P_{10}}{(x_{\Sigma_i}^{-1} \circ e_1^c \circ x_{\Sigma_i} \circ p(A))_{10}}\right)^{D_{9,8}}\\
&= A_6,
\end{aligned}\tag{7.8}$$

and

$$(a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_5 = A_5\left(\frac{c(P_1P_5P_8+P_5P_8)+P_8+1}{P_1P_5P_8+P_5P_8+P_8+1}\right),\tag{7.9}$$

$$(a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_3 = A_3,\ (a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_2 = A_2,\tag{7.10}$$

$$(a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_1 = A_1\left(\frac{cP_1P_5P_8+P_5P_8+P_8+1}{P_1P_5P_8+P_5P_8+P_8+1}\right),\tag{7.11}$$

$$(a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_{-j} = A_{-j}\ \ (1\le j\le 4),\tag{7.12}$$

$$(a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_k = c^{\delta_{k,10}}A_k\ \ (k=4,7,9,10).\tag{7.13}$$

Next, by Theorem 7.3, we get

$$\begin{aligned}
(a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_8 &= (a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_{3,1}\\
&= A_{3,1}\frac{c(P_{1,1}P_{2,1}P_{3,1}+P_{2,1}P_{3,1}+P_{3,1})+1}{P_{1,1}P_{2,1}P_{3,1}+P_{2,1}P_{3,1}+P_{3,1}+1}\\
&= A_8\frac{c(P_1P_5P_8+P_5P_8+P_8)+1}{P_1P_5P_8+P_5P_8+P_8+1},
\end{aligned}$$

$$\begin{aligned}
(a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_6 &= (a_{\Sigma_{\mathbf{i}_0}}^{-1} \circ (\iota\circ\zeta^{u,e})^{-1} \circ e_1^c \circ (\iota\circ\zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_{2,2}\\
&= A_{2,2} = A_6,
\end{aligned}$$

and these results coincide with (7.7) and (7.8). Similarly, we can verify the results of calculations for $(a^{-1}_{\Sigma_{\mathbf{i}_0}} \circ (\iota \circ \zeta^{u,e})^{-1} \circ e^c_1 \circ (\iota \circ \zeta^{u,e}) \circ a_{\Sigma_{\mathbf{i}_0}} A)_k$ by using Theorem 7.3 coincide with (7.9)–(7.13) ($k = -4, -3, -2, -1, 1, 2, 3, 4, 5, 7, 9, 10$).

Acknowledgements

We would like to appreciate David Hernandez, Michael Gekhtman, Gleb Koshevoy, Beranrd Leclerc, and Hironori Oya for fruitful discussions and valuable suggestions. We also thank the faculty and staffs at Universit'e Paris Diderot during their stay in 2017–2018. and T.N. thanks the faculty and staffs at IHES for his stay in 2017. Y.K. is supported by JSPS, Japan KAKENHI Grant Number JP17H07103, and T.N. is supported in part by JSPS, Japan Grants in Aid for Scientific Research Number 15K04794.

References

[1] A. Berenstein, S. Fomin, A. Zelevinsky, Cluster algebras III : Upper bounds and double Bruhat cells, Duke Math. J. 126 (1) (2005) 1–52.
[2] A. Berenstein, D. Kazhdan, Geometric and unipotent crystals, Geom. Funct. Anal. Special Volume (Part I) (2000) 188—236.
[3] A. Berenstein, D. Kazhdan, Geometric and unipotent crystals. II. From unipotent bicrystals to crystal bases, in: Quantum Groups, in: Contemp. Math., vol. 433, Amer. Math. Soc., Providence, RI, 2007, pp. 13–88.
[4] V.V. Fock, A.B. Goncharov, Cluster ensembles, quantization and the dilogarithm, arXiv:math/0311245.
[5] V.V. Fock, A.B. Goncharov, Cluster $\mathcal{X}$-varieties, amalgamation, and Poisson–Lie groups, in: Algebraic Geometry and Number Theory, in: Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 2006, pp. 27—68.
[6] M. Gross, P. Hacking, S. Keel, Birational geometry of cluster algebras, Algebr. Geom. 2 (2) (2015) 137—175.
[7] M. Gross, P. Hacking, S. Keel, M. Kontsevich, Canonical bases for cluster algebras, J. Amer. Math. Soc. 31 (2) (2018) 497—608.
[8] V. Kac, D. Peterson, Infinite flag varieties and conjugacy theorems, Proc. Natl. Acad. Sci. USA 80 (6i) (1983) 1778—1782.
[9] M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, Duke Math. J. 63 (2) (1991) 465—516.
[10] M. Kashiwara, The crystal base and Littelmann's refined Demazure character formula, Duke Math. J. 71 (3) (1993) 839–858.
[11] M. Kashiwara, T. Nakashima, Crystal graphs for representations of the q-analogue of classical Lie algebras, J. Algebra 165 (2) (1994) 295–345.
[12] Y. Kimura, H. Oya, Twist automorphisms on Quantum unipotent cells and Dual canonical bases, arXiv:1701.02268, pp. 1–52.
[13] S. Kumar, Kac–Moody groups, their flag varieties and representation theory, in: Progress in Mathematics, vol. 204, Birkhäuser Boston, Inc., Boston, MA, 2002.
[14] T. Nakashima, Geometric crystals on schubert varieties, J. Geom. Phys. 53 (2) (2005) 197–225.
[15] T. Nakashima, Decorations on geometric crystals and monomial realizations of crystal bases for classical groups, J. Algebra 399 (2014) 712–769.
[16] H. Williams, Cluster ensembles and Kac–Moody groups, Adv. Math. 247 (2013) 1—40.