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# Integration of twisted Poisson structures

Alberto S. Cattaneo<sup>a,\*</sup>, Ping Xu<sup>b</sup>

<sup>a</sup> *Institut für Mathematik, Universität Zürich-Irchel, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland*

<sup>b</sup> *Department of Mathematics, Penn State University, University Park, PA 16802, USA*

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## Abstract

Poisson manifolds may be regarded as the infinitesimal form of symplectic groupoids. Twisted Poisson manifolds considered by Ševera and Weinstein [Prog. Theor. Phys. Suppl. 144 (2001) 145] are a natural generalization of the former which also arises in string theory. In this note it is proved that twisted Poisson manifolds are in bijection with a (possibly singular) twisted version of symplectic groupoids.

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## 1. Introduction

Poisson manifolds may be regarded as the infinitesimal form of symplectic groupoids [6], i.e., Lie groupoids endowed with a multiplicative symplectic form. Up to singularities, Poisson manifolds may be integrated to symplectic groupoids as described in [2] (conditions under which integration with no singularities is possible are given in [4]). In this paper we generalize this result to the case when the two structures (of symplectic groupoid and of Poisson manifold) are twisted by a closed 3-form.

Let  $M$  be a smooth manifold. A pair  $(\pi, \phi)$ , where  $\pi$  is a bivector field and  $\phi$  is a closed 3-form, is called a *twisted Poisson structure* if it satisfies the equation

$$[\pi, \pi] = \frac{1}{2} \wedge^3 \pi^\# \phi, \quad (1.1)$$

\* Corresponding author.

E-mail addresses: [asc@math.unizh.ch](mailto:asc@math.unizh.ch) (A. S. Cattaneo), [ping@math.psu.edu](mailto:ping@math.psu.edu) (P. Xu).

where  $[\cdot, \cdot]$  denotes the Schouten–Nijenhuis bracket and  $\pi^\#$  is the vector bundle homomorphism  $T^*M \rightarrow TM$  induced by  $\pi$  (viz.,  $\pi^\#(x)(\sigma) := \pi(x)(\sigma, \bullet)$ , with  $x \in M$ ,  $\sigma \in T_x^*M$ ). According to [14], one also says that  $\pi$  is a  $\phi$ -Poisson tensor. In the case  $\phi = 0$  one recovers the usual notions of Poisson tensor and Poisson manifold. Twisted Poisson structures have been extensively studied in the physics literature, e.g. [5,9,11].

As explained in [14], a twisted Poisson structure induces a Lie algebroid structure on  $T^*M$  with anchor map  $\pi^\#$  and Lie bracket of sections  $\sigma$  and  $\tau$  defined by

$$[\sigma, \tau] := L_{\pi^\# \sigma} \tau - L_{\pi^\# \tau} \sigma - d\pi(\sigma, \tau) + \phi(\pi^\# \sigma, \pi^\# \tau, \bullet). \quad (1.2)$$

In particular,  $\forall f, g \in C^\infty(M)$  we have:

$$[df, dg] = d\{f, g\} + \phi(X_f, X_g, \bullet), \quad (1.3)$$

and

$$[X_f, X_g] = X_{\{f, g\}} + \pi^\#(\phi(X_f, X_g, \bullet)), \quad (1.4)$$

with  $\{f, g\} = \pi(df, dg)$  and  $X_f = \pi^\# df$ .

We will denote this Lie algebroid by  $T^*M_{(\pi, \phi)}$ . Sections of its exterior algebra are ordinary differential forms. One may define a derivation  $\delta$  deforming the de Rham differential by  $\phi$ ; viz., we define a graded derivation  $\delta : \Omega^*(M) \rightarrow \Omega^{*+1}(M)$  by setting  $\delta f = df$  if  $f \in C^\infty(M)$  and

$$\delta\sigma = d\sigma - \iota_{\pi^\# \sigma} \phi,$$

if  $\sigma \in \Omega^1(M)$ . It turns out that

$$\delta[\sigma, \tau] = [\delta\sigma, \tau] + [\sigma, \delta\tau], \quad \forall \sigma, \tau \in \Omega^1(M),$$

and that  $\delta^2 = [\phi, \bullet]$  (where we have extended the Lie bracket to the whole of  $\Omega^*(M)$  as a biderivation). So  $(T^*M_{(\pi, \phi)}, \delta)$  constitutes an example of a quasi-Lie bialgebroid [8,12], a generalization of Drinfeld's quasi-Lie bialgebras [7,10].

If  $T^*M_{(\pi, \phi)}$  may be integrated to a Lie groupoid  $(G \rightrightarrows M, \alpha, \beta)$  (i.e., if it exists a Lie groupoid  $G$  whose Lie algebroid is  $T^*M_{(\pi, \phi)}$ ), the differential  $\delta$  induces extra structure on  $G$ . Namely, denoting by  $\alpha$  and  $\beta$  the source and target maps of  $G$ , then  $G$  may be endowed with a non-degenerate, multiplicative 2-form  $\omega$  that satisfies

$$d\omega = \alpha^* \phi - \beta^* \phi.$$

In other words,  $(\omega, \phi)$  is a 3-cocycle for the double complex  $\Omega^*(G^{(*)})$ , where  $G^{(0)} = M$ ,  $G^{(1)} = G$  and elements of  $G^{(k)}$  are  $k$ -tuples of elements of  $G$  that may be multiplied (in the given order). One differential is de Rham and the other is the groupoid-complex differential. Observe that in the true Poisson case (i.e.,  $\phi = 0$ ),  $\omega$  is closed, so  $G$  is an ordinary symplectic groupoid. In the general case,  $G$  is called a *non-degenerate twisted symplectic groupoid*, and the non-degenerate 2-form  $\omega$  is said to be *relatively  $\phi$ -closed*. The main theorem of the paper (conjectured in [14]) is the following one.

**Theorem.** *There is a bijection between integrable twisted Poisson structures and source-simply connected non-degenerate twisted symplectic groupoids.*

Here “integrable twisted Poisson structure” means that the associated Lie algebroid is integrable.

In Section 2 we give an introduction to non-degenerate twisted symplectic groupoids and prove that they induce twisted Poisson structures on the base manifolds (Theorem 2.6 on page 7).

In Section 5 we prove the theorem, though in a more general setting. In fact, as shown in the generalization [3] (see also [13]) of the construction given in [2]), to any Lie algebroid  $A$  one can associate a topological source-simply connected groupoid  $G(A)$ , which is the Lie groupoid integrating  $A$  whenever  $A$  is integrable. The topological groupoid  $G(A)$  is defined as the leaf space of a smooth foliation, as we recall in Section 3; so it makes sense to define on it a notion of smooth functions and forms. In the case when  $A$  is  $T^*M_{(\pi, \phi)}$ , we prove that  $G(A)$  may always be endowed with a non-degenerate, multiplicative, relatively  $\phi$ -closed 2-form  $\omega$ . The construction is a modification, described in Section 4, of the method developed in [2], where the true Poisson case (i.e.,  $\phi = 0$ ) was dealt with.

As a final remark, we mention that general multiplicative 2-forms, their infinitesimal counterparts and their integrations are being treated in [1].

## 2. Non-degenerate twisted symplectic groupoids

**Definition 2.1.** A non-degenerate twisted symplectic groupoid is a Lie groupoid  $(G \rightrightarrows M, \alpha, \beta)$  equipped with a non-degenerate 2-form  $\omega \in \Omega^2(G)$  and a 3-form  $\phi \in \Omega^3(M)$  such that:

1.  $d\phi = 0$ ;
2.  $d\omega = \alpha^*\phi - \beta^*\phi$ ;
3.  $\omega$  is multiplicative, i.e., the 2-form  $(\omega, \omega, -\omega)$  vanishes when being restricted to the graph of the groupoid multiplication  $\Lambda \subset G \times G \times G$ .

Let  $\pi_G$  denote the bivector field on  $G$  corresponding to  $\omega$ . Then  $(\pi_G, \Omega)$ , where  $\Omega = \alpha^*\phi - \beta^*\phi$ , defines a twisted Poisson structure on  $G$  in the sense of [14].

For any  $\xi \in \Gamma(A)$ , by  $\vec{\xi}$  and  $\tilde{\xi}$  we denote its corresponding right and left invariant vector fields on the groupoid  $G$ , respectively. The following properties can be easily verified.

### Proposition 2.2.

1.  $\epsilon^*\omega = 0$ , where  $\epsilon : M \rightarrow G$  is the natural embedding;
2.  $i^*\omega = -\omega$ , where  $i : G \rightarrow G$  is the groupoid inversion;
3. for any  $\xi, \eta \in \Gamma(A)$ ,  $\omega(\vec{\xi}, \vec{\eta})$  is a right invariant function on  $G$ , and  $\omega(\tilde{\xi}, \tilde{\eta})$  is a left invariant function on  $G$ ;
4.  $\omega(\vec{\xi}, \tilde{\eta}) = 0$ ;
5.  $\omega(\vec{\xi}, \vec{\eta})(x) = -\omega(\tilde{\xi}, \tilde{\eta})(x^{-1})$ .

**Proof.** The proof is standard, and essentially follows from the multiplicativity of  $\omega$ :

1. For any  $\delta'_m, \delta''_m \in T_m M$ , since  $(\delta'_m, \delta'_m, \delta'_m), (\delta''_m, \delta''_m, \delta''_m) \in T\Lambda$ , it follows that  $\omega(\delta'_m, \delta''_m) = 0$ .

2.  $\forall x \in G$  and  $\forall \delta'_x, \delta''_x \in T_x G$ , it is clear that  $(\delta'_x, i_* \delta'_x, \alpha_* \delta'_x), (\delta''_x, i_* \delta''_x, \alpha_* \delta''_x) \in T\Lambda$ . Thus, by (1), we have

$$\omega(\delta'_x, \delta''_x) + \omega(i_* \delta'_x, i_* \delta''_x) = 0,$$

and (2) follows.

3. For any  $\xi, \eta \in \Gamma(A)$ ,  $(\vec{\xi}(x), 0_y, \vec{\xi}(xy)), (\vec{\eta}(x), 0_y, \vec{\eta}(xy)) \in T\Lambda$ . Thus

$$\omega(\vec{\xi}(x), \vec{\eta}(x)) - \omega(\vec{\xi}(xy), \vec{\eta}(xy)) = 0.$$

Hence  $\omega(\vec{\xi}, \vec{\eta})$  is a right invariant function on  $G$ . Similarly,  $\omega(\tilde{\xi}, \tilde{\eta})$  is a left invariant function on  $G$ .

4. By considering the vectors  $(\vec{\xi}(x), 0_{\beta(x)}, \vec{\xi}(x))$  and  $(0_x, \tilde{\eta}(\beta(x)), \tilde{\eta}(x)) \in T\Lambda$ , we obtain  $\omega(\vec{\xi}(x), \tilde{\eta}(x)) = 0$ .
5. Follows from (2) and the fact that  $i_* \tilde{\xi} = -\tilde{\xi}$ . □

Define a section  $\gamma \in \Gamma(\wedge^2 A^*)$  and a bundle map:  $\lambda : A \rightarrow T^*M$  by

$$\omega(\vec{\xi}, \vec{\eta}) = \alpha^* \gamma(\xi, \eta), \quad \forall \xi, \eta \in \Gamma(A), \quad (2.1)$$

and

$$\langle \lambda(\xi), v \rangle = \omega(\vec{\xi}(m), v), \quad \forall \xi \in A|_m, v \in T_m M. \quad (2.2)$$

### Lemma 2.3.

1.  $\omega(\tilde{\xi}, \tilde{\eta}) = -\beta^* \gamma(\xi, \eta), \forall \xi, \eta \in \Gamma(A)$ ;
2. for all  $\xi, \eta \in \Gamma(A)$ ,
 
$$\gamma(\xi, \eta) = \langle \rho(\xi), \lambda(\eta) \rangle; \quad (2.3)$$
3.  $\lambda : A \rightarrow T^*M$  is a vector bundle isomorphism.

### Proof.

1. Follows from Proposition 2.2 (5).
2. We have

$$\omega(\vec{\xi}, \vec{\eta}) = \omega(\vec{\xi} - \tilde{\xi}, \vec{\eta}) = \omega(\vec{\eta}, \rho(\xi)) = \langle \rho(\xi), \lambda(\eta) \rangle.$$

3. Assume that  $\lambda(\xi) = 0$ . That is,  $\omega(\vec{\xi}(m), v) = 0, \forall v \in T_m M$ , which implies that  $\vec{\xi}(m)\omega = 0$  by Proposition 2.2 (4). Hence  $\xi = 0$  since  $\omega$  is non-degenerate. This means that  $\lambda$  is injective. On the other hand, assume that  $v \in (\lambda(A|_m))^\perp$ . Then  $\omega(\vec{\xi}(m), v) = 0, \forall \xi \in A|_m$ . Thus  $v\omega = 0$  using Proposition 2.2 (1), which implies that  $v = 0$ . Therefore  $\lambda$  is surjective. □

### Lemma 2.4.

For any  $f \in C^\infty(M)$

$$\overrightarrow{\lambda^{-1}(df)} = X_{\alpha^* f}; \quad \overleftarrow{\lambda^{-1}(df)} = X_{\beta^* f}. \quad (2.4)$$

**Proof.** First, one shows that  $X_{\alpha^* f}$  is a right invariant vector field on  $G$  and  $X_{\beta^* f}$  is a left invariant vector field. This can be shown using the same argument as in the case of symplectic

groupoids [6]. Namely the multiplicativity of  $\omega$  together with dimension counting implies that the graph  $\Lambda$  is coisotropic with respect to  $(\pi_G, \pi_G, -\pi_G)$ . The later implies that  $X_{\alpha^*f}$  is a right invariant vector field on  $G$  and  $X_{\beta^*f}$  is a left invariant vector field.

Second, for any  $v \in T_m M$ , we have

$$\omega(X_{\alpha^*f}(m), v) = \langle \alpha^* df(m), v \rangle = \langle df(m), \alpha_* v \rangle = \langle df(m), v \rangle.$$

It thus follows that  $\lambda(X_{\alpha^*f}) = df$ , or  $\overrightarrow{\lambda^{-1}(df)} = X_{\alpha^*f}$ . The other equation can be proved similarly.  $\square$

By pulling back the 2-form  $\gamma \in \Gamma(\wedge^2 A^*)$  via  $\lambda^{-1}$ , one obtains a bivector field  $\pi \in \Gamma(\wedge^2 TM)$ . We introduce a bracket and Hamiltonian vector fields by the usual definitions, i.e.,  $\{f, g\} = \pi(df, dg)$  and  $X_f = \pi^\#(df)$ .

### Corollary 2.5.

$$\alpha_* \pi_G = \pi; \quad \beta_* \pi_G = -\pi; \quad (2.5)$$

or equivalently

$$\alpha_* X_{\alpha^*f} = X_f; \quad \beta_* X_{\beta^*f} = -X_f, \quad \forall f \in C^\infty(M). \quad (2.6)$$

**Proof.** For any  $f, g \in C^\infty(M)$ ,

$$\{\alpha^*f, \alpha^*g\} = \omega(X_{\alpha^*f}, X_{\alpha^*g}) = \omega(\overrightarrow{\lambda^{-1}(df)}, \overrightarrow{\lambda^{-1}(dg)}) = \alpha^*(\pi(df, dg)) = \alpha^*\{f, g\}.$$

Similarly, we have  $\{\beta^*f, \beta^*g\} = -\beta^*\{f, g\}$ .  $\square$

We are now ready to prove the main result of the section.

### Theorem 2.6.

1.  $\pi$  is a  $\phi$ -Poisson tensor in the sense of [14], i.e., it satisfies (1.1).
2. The bundle map  $\lambda : A \rightarrow T^*M$  establishes a Lie algebroid isomorphism, where the Lie algebroid on  $T^*M$  is induced by the twisted Poisson tensor  $\pi$  as given by Eq. (1.2).

**Proof.** Let  $\Omega = \alpha^*\phi - \beta^*\phi$ . Thus  $\forall f, g \in C^\infty(M)$

$$(X_{\alpha^*f} \wedge X_{\alpha^*g})\Omega = (X_{\alpha^*f} \wedge X_{\alpha^*g})\alpha^*\phi = \alpha^*[(\alpha_* X_{\alpha^*f} \wedge \alpha_* X_{\alpha^*g})\phi] = \alpha^*[X_f \wedge X_g\phi].$$

Thus by Eq. (1.4)

$$[X_{\alpha^*f}, X_{\alpha^*g}] - X_{\{\alpha^*f, \alpha^*g\}} = \pi_G^\#(\Omega(X_{\alpha^*f}, X_{\alpha^*g}, \bullet)) = \pi_G^\#(\alpha^*\phi(X_f, X_g, \bullet)).$$

Thus it follows that

$$\lambda[X_{\alpha^*f}, X_{\alpha^*g}] = d\{f, g\} + \phi(X_f, X_g, \bullet).$$

Note that  $\lambda$  intertwines the anchors:  $\pi^\# \circ \lambda = \rho$ , according to Eq. (2.3). Therefore, using Lie algebroid properties, one shows that the push forward Lie algebroid on  $T^*M$  via  $\lambda$  is

given by Eq. (1.2). This forces, by the Jacobi identity,  $\pi$  to be  $\phi$ -Poisson, and  $\lambda$  is a Lie algebroid isomorphism between  $A$  and  $(T^*M)_{\pi,\phi}$ .  $\square$

### 3. Integration of $T^*M_{(\pi,\phi)}$

We briefly describe the integration procedure for Lie algebroids of [3,13], adapted to the case of  $T^*M_{(\pi,\phi)}$ . First one defines the manifold  $P(T^*M_{(\pi,\phi)})$  of  $C^1$ -Lie algebroid morphisms  $TI \rightarrow T^*M_{(\pi,\phi)}$ , where  $I$  is the interval  $[0, 1]$  and  $TI$  is given its canonical Lie algebroid structure. An element of  $PT^*M_{(\pi,\phi)}$  consists of a  $C^2$ -path  $X : I \rightarrow M$  together with a section  $\eta$  of  $T^*I \otimes X^*T^*M$  satisfying

$$dX = \pi^\#(X)\eta.$$

On this manifold one may consider as equivalent two elements which are related by a Lie algebroid morphism  $T(I \times I) \rightarrow T^*M_{(\pi,\phi)}$  that fixes the endpoints. The quotient space  $G(T^*M_{(\pi,\phi)})$  may be given a groupoid structure. For our purposes it is however better to use a different description of  $G(T^*M_{(\pi,\phi)})$ , i.e., as the leaf space of a foliation. Namely, let  $P_0\Gamma(T^*M_{(\pi,\phi)})$  be the space of  $C^2$ -paths in the Lie algebra of sections of  $T^*M_{(\pi,\phi)}$  with endpoints at zero. We give this space the structure of a Lie algebra by the pointwise Lie bracket. One may then define an infinitesimal action of this Lie algebra on  $P(T^*M_{(\pi,\phi)})$ . To describe it, we prefer to introduce local coordinates  $\{x^i\}$  on  $M$  (alternatively, one may use a torsion-free connection). Since  $\{dx^i\}$  is a local basis of sections of  $T^*M_{(\pi,\phi)}$ , we may define structure functions  $f$  by

$$[dx^i, dx^j] = f_k^{ij} dx^k,$$

where a sum over repeated indices is understood. If we write locally  $\pi = \pi^{ij}\partial_i\partial_j$  and  $\phi = \phi_{ijk} dx^i dx^j dx^k$ , we may compute:

$$f_k^{ij} = \partial_k \pi^{ij} + \pi^{mi} \pi^{nj} \phi_{mnk}.$$

The action is then as follows. To  $B \in P_0\Gamma(T^*M_{(\pi,\phi)})$  we associate a vector field  $\xi_B$  on  $P(T^*M_{(\pi,\phi)})$ . We can always write  $\xi_B = \xi_B^h + \xi_B^v$  with  $\xi_B^h(X, \eta) \in \Gamma(I, X^*TM)$  and  $\xi_B^v(X, \eta) \in \Gamma(I, T^*I \otimes X^*T^*M)$ . We set then

$$(\xi_B^h(X, \eta))^i = -\pi^{ij}(X) (B_X)_j, \quad (3.1a)$$

$$(\xi_B^v(X, \eta))_i = -d(B_X)_i - f_i^{rs}(X) \eta_r (B_X)_s, \quad (3.1b)$$

where  $B_X$  is the section of  $X^*T^*M$  defined by  $B_X(t) = B(t)(X(t))$ .

Thus, the infinitesimal action of  $P_0\Gamma(T^*M_{(\pi,\phi)})$  defines a foliation on  $P(T^*M_{(\pi,\phi)})$  and  $G(T^*M_{(\pi,\phi)})$  is its quotient space. Let us briefly recall its groupoid structure. The target map  $\alpha$  associates to a class of morphisms  $(X, \eta)$  the value of  $X$  at 0, while the source map  $\beta$  associates it to the value of  $X$  at 1 (observe that the infinitesimal action preserves the endpoints of  $X$ ). The identity section associates to a point  $m$  in  $M$  the class  $\epsilon(m)$  of the constant path at  $m$  with  $\eta = 0$ . The product is obtained by joining the base paths and restricting the fiber maps consequently (the product is more precisely defined on smooth representatives such that  $\eta$  vanishes with its derivatives at the endpoints).

#### 4. Quasi-symplectic reduction

In this section we describe how to obtain  $G(T^*M_{(\pi,\phi)})$  by some sort of symplectic reduction, though our replacement for a symplectic form will be a non-degenerate but not necessarily closed 2-form.

Let  $T^*PM$  denote the manifold of  $C^1$ -bundle maps  $TI \rightarrow T^*M$  (over  $C^2$ -maps). This space is morally a cotangent bundle and as such it has a canonical symplectic structure  $\Omega_0$ . Explicitly, a point in  $T^*PM$  is a pair  $(X, \eta)$ , where  $X$  is a  $C^2$ -path  $I \rightarrow M$  and  $\eta$  is a  $C^1$ -section of  $T^*I \otimes X^*T^*M$ . The tangent space at  $(X, \eta)$  is the direct sum of  $T^h_{(X,\eta)}T^*PM = \Gamma(I, X^*TM)$  and  $T^v_{(X,\eta)}T^*PM = \Gamma(I, T^*I \otimes X^*T^*M)$ . Using this splitting, we write

$$\Omega_0(X, \eta)(\xi_1 \oplus e_1, \xi_2 \oplus e_2) = \int_I \langle e_1, \xi_2 \rangle - \langle e_2, \xi_1 \rangle, \quad (4.1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the canonical pairing between tangent and cotangent fibers of  $M$ .

Using the 3-form  $\phi$  on  $M$  we may also define a second 2-form on  $T^*PM$ :

$$\Omega_1(X, \eta)(\xi_1 \oplus e_1, \xi_2 \oplus e_2) = \frac{1}{2} \int_I \phi(X)(\pi^\#(X)\eta, \xi_1, \xi_2). \quad (4.2)$$

The 2-form  $\Omega = \Omega_0 + \Omega_1$  is still non-degenerate but no longer closed.

The manifold  $P(T^*M_{(\pi,\phi)})$  introduced in the previous section may be regarded as a submanifold of  $T^*PM$ . If we introduce “momentum maps”  $H : T^*PM \rightarrow P_0\Gamma(T^*M_{(\pi,\phi)})^*$  by

$$H_B(X, \eta) = \int_I \langle B_X, dX - \pi^\#(X)\eta \rangle,$$

then  $P(T^*M_{(\pi,\phi)})$  is  $H^{-1}(0)$ . One may check that  $dH_B$  lies in the image of  $\Omega$  for any  $B \in P_0\Gamma(T^*M_{(\pi,\phi)})$ ; so, since  $\Omega$  is non-degenerate, one may define a map  $B \rightarrow \hat{\xi}_B$  that associates a vector field  $\hat{\xi}_B$  on  $T^*PM$  to  $B$  by

$$\iota_{\hat{\xi}_B}\Omega = dH_B. \quad (4.3)$$

One may easily check that the restriction of  $\hat{\xi}_B$  to  $P(T^*M_{(\pi,\phi)})$  is tangent to it. More to the point, one may check that the vector field on  $P(T^*M_{(\pi,\phi)})$  so obtained is precisely the  $\xi_B$  of (3.1) which defines the infinitesimal action of  $P_0\Gamma(T^*M_{(\pi,\phi)})$  on  $P(T^*M_{(\pi,\phi)})$ .

#### 5. Proof of the theorem

In the setting of the previous section, we want to prove that the restriction  $\underline{\Omega}$  of  $\Omega$  to  $P(T^*M_{(\pi,\phi)})$  is basic w.r.t. to the projection  $p : P(T^*M_{(\pi,\phi)}) \rightarrow G(T^*M_{(\pi,\phi)})$ , viz.,  $\underline{\Omega} = p^*\omega$ ; moreover, we want to prove that  $\omega$  satisfies all the required conditions.

Observe that  $\underline{\Omega}$  is automatically horizontal by (4.3). On the other hand, unlike the usual symplectic case, it is not clear that  $\underline{\Omega}$  is also invariant; in fact, at first, we may only see that  $L_{\xi_B}\underline{\Omega} = \iota_{\xi_B}d\underline{\Omega} = \iota_{\xi_B}d\underline{\Omega}_1$ , where  $\underline{\Omega}_1$  denotes the restriction of  $\Omega_1$  to  $P(T^*M_{(\pi,\phi)})$ . To proceed, we must understand  $\underline{\Omega}_1$  better.

Let  $PM$  be the manifold of  $C^2$ -paths in  $M$ . Let  $\text{ev} : I \times PM \rightarrow M$  be the evaluation map and  $\text{pr} : I \times PM \rightarrow PM$  the projection to the second factor. Define  $\Phi = \text{pr}_* \text{ev}^* \phi \in \Omega^2(PM)$ , where  $\text{pr}_*$  denotes integration along the fiber. If we finally denote by  $q : P(T^*M_{(\pi, \phi)}) \rightarrow PM$  the map that retains only the base map of the Lie algebroid morphism, we realize immediately that

$$\underline{\Omega}_1 = q^* \Phi.$$

By the generalized Stokes' theorem and the fact that  $\phi$  is closed, we obtain  $d\Phi = \alpha^* \phi - \beta^* \phi$ , where  $\alpha$  and  $\beta$  are the maps  $PM \rightarrow M$  that assign to a path its values at 0 and 1, respectively. Thus

$$d\underline{\Omega} = q^*(\alpha^* \phi - \beta^* \phi).$$

Since the vector field  $\xi_B$  does not move the endpoints, we conclude that  $\iota_{\xi_B} d\underline{\Omega} = 0$ , viz., that  $\underline{\Omega}$  is invariant as well. We write then  $\underline{\Omega} = p^* \omega$  as at the beginning of the section. The 2-form  $\omega$  on  $G(T^*M_{(\pi, \phi)})$  is clearly multiplicative since the product is defined by joining the paths and  $\underline{\Omega}$  is defined as an integral. Moreover, recalling the definition of the source and target map  $\beta$  and  $\alpha$ , we observe that  $\alpha \circ q = \alpha \circ p$  and  $\beta \circ q = \beta \circ p$ . So we may write the equation above as

$$d\underline{\Omega} = p^*(\alpha^* \phi - \beta^* \phi).$$

Since  $d\underline{\Omega} = p^* d\omega$  and  $p$  is a surjection, this shows that  $\omega$  is relatively  $\phi$ -closed.

Finally, we need to prove that the 2-form  $\omega$  is non-degenerate. It is clear from the construction that  $\omega$  is non-degenerate along the identity  $M$ . The claim thus follows from the following lemma.

**Lemma 5.1.** *A multiplicative 2-form  $\omega \in \Omega^2(G)$  on a Lie groupoid  $G \rightrightarrows M$  is non-degenerate if and only if it is non-degenerate along the identity  $M$ .*

**Proof.** First of all, note that for any  $\delta_x \in T_x G$ , and  $\xi \in \Gamma(A)$ , we have

$$\omega(\tilde{\xi}(x), \delta_x) = \omega(\tilde{\xi}(v), \beta_* \delta_x), \quad (5.1)$$

$$\omega(\vec{\xi}(x), \delta_x) = \omega(\vec{\xi}(u), \alpha_* \delta_x), \quad (5.2)$$

where  $u = \alpha(x)$  and  $v = \beta(x)$ . Eq. (5.1), for instance, follows from the fact that both  $(\delta_x, \delta_x, \beta_* \delta_x)$ , and  $(0, \tilde{\xi}(x), \tilde{\xi}(v))$  are tangent to the graph of the groupoid multiplication  $\Lambda \subset G \times G \times G$ . Eq. (5.2) can be proved similarly. Now assume that  $\delta_x \in \ker \omega_x$ . It follows from Eq. (5.1) that  $\beta_* \delta_x \in \ker \omega_v$  since  $M$  is isotropic with respect to  $\omega$ . Therefore  $\beta_* \delta_x = 0$  by assumption. Hence  $\delta_x = \tilde{\eta}(x)$ . On the other hand, according to Eq. (5.2), one has  $\omega(\tilde{\eta}(u), T_u M) = 0$  since  $\alpha$  is a submersion. This implies that  $\tilde{\eta}(u) \in \ker \omega_u$ . Therefore  $\tilde{\eta}(u) = 0$  by assumption. This implies that  $\delta_x = \tilde{\eta}(x) = 0$ . This concludes the proof.  $\square$

We need now to prove that the correspondence between  $\phi$ -twisted Poisson structures and twisted symplectic groupoids is a bijection. The proof is divided into two steps:



Step 1. By construction (see [2,3]) the Lie algebroid of  $G(T^*M_{(\pi,\phi)})$  is  $T^*M_{(\pi,\phi)}$ . As discussed in Section 2, the relatively  $\phi$ -closed, multiplicative, non-degenerate 2-form  $\omega$  determines an automorphism  $\lambda$  of  $T^*M$  and a bivector field  $\gamma$  on  $M$  as in Eqs. (2.1) and (2.2). We have to show that  $\lambda$  is the identity and that  $\gamma = \pi$ . First of all we observe that it is enough to consider (2.1) at the unit element  $\epsilon(m) \in G(T^*M_{(\pi,\phi)})$  corresponding to  $m \in M$ :

$$\omega(\epsilon(m))(\vec{\xi}_1(\epsilon(m)), \vec{\xi}_2(\epsilon(m))) = \gamma(m)(\xi_1, \xi_2), \quad \forall \xi_1, \xi_2 \in A|_m.$$

By construction  $\epsilon(m)$  is the equivalence class of the path  $X(t) = m$ ,  $\eta(t) = 0$ ,  $\forall t \in I = [0, 1]$ . The vector field  $\vec{\xi}_i$ ,  $i = 1, 2$ , evaluated at  $\epsilon(m)$  is the projection to  $T_{\epsilon(m)}G(T^*M_{(\pi,\phi)})$  of the vector  $\hat{\xi}_i \in T_{(m,0)}P(T^*M_{(\pi,\phi)})$  defined by  $\hat{\xi}_i(t) = (\pi^\#(m)\xi_i t, \xi_i dt)$ . Observing then that for  $\eta = 0$  the 2-form  $\Omega_1$  of Eq. (4.2) vanishes, we get, also using (4.1)

$$\begin{aligned} \omega(\epsilon(m))(\vec{\xi}_1(\epsilon(m)), \vec{\xi}_2(\epsilon(m))) &= \Omega_0(m, 0)(\hat{\xi}_1, \hat{\xi}_2) = 2 \int_0^1 \pi(m)(\xi_1, \xi_2) t \, dt \\ &= \pi(m)(\xi_1, \xi_2), \end{aligned}$$

which shows  $\gamma = \pi$ . As for (2.2), observe that  $\omega(\epsilon(m))(\vec{\xi}_1(\epsilon(m)), v)$  is just  $\Omega_0(m, 0)(\hat{\xi}_1, \hat{v})$  with  $\hat{v}(t) = (v, 0)$ . As a consequence

$$\omega(\epsilon(m))(\vec{\xi}_1(\epsilon(m)), v) = \int_0^1 \langle \xi_1, v \rangle \, dt = \langle \xi_1, v \rangle,$$

which shows that  $\lambda$  is the identity.

Step 2. Assume that  $(G \rightrightarrows M, \omega + \phi)$  is an  $\alpha$ -simply connected non-degenerate twisted symplectic groupoid. Let  $\pi$  be its induced  $\phi$ -twisted Poisson structure on  $M$ . Then the above integration process integrates the Lie algebroid  $T^*M_{(\pi,\phi)}$  into a Lie groupoid, which is known to be isomorphic to  $G \rightrightarrows M$ , and a multiplicative 2-form  $\omega'$  on that groupoid. By identifying this groupoid with  $G \rightrightarrows M$ , therefore one may think  $\omega'$  as a multiplicative 2-form on  $G$ . One needs to show that  $\omega' = \omega$ . By Step 1, we conclude that  $\omega'$  and  $\omega$  must coincide along the identity space  $M$ . Let  $\tilde{\omega} = \omega - \omega'$ . Then  $\tilde{\omega}$  is a multiplicative closed 2-form on  $G$  and  $\tilde{\omega}|_M = 0$ . Given any  $\xi \in \Gamma(A)$ , it is easy to see that  $(\vec{\xi}(\alpha(x)), 0, \vec{\xi}(x))$  is tangent to the graph  $\Lambda$  of groupoid multiplication. On the other hand, for any  $\delta_x \in T_x G$ , it is also clear that  $(\alpha_* \delta_x, \delta_x, \delta_x) \in T\Lambda$ . It thus follows that

$$\tilde{\omega}(\vec{\xi}(\alpha(x)), \alpha_* \delta_x) - \tilde{\omega}(\vec{\xi}(x), \delta_x) = 0.$$

Therefore we have  $\vec{\xi} \tilde{\omega} = 0$ . Thus

$$L_{\vec{\xi}} \tilde{\omega} = (di_{\vec{\xi}} + i_{\vec{\xi}} d) \tilde{\omega} = 0,$$

which implies that  $\tilde{\omega} = 0$  since any point in  $G$  can be reached by a product of (local) bisections generated by  $\vec{\xi}$ . This concludes the proof.

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