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Loop group decompositions in almost split real forms and applications to soliton theory and geometry

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ABSTRACT

We prove a global Birkhoff decomposition for almost split real forms of loop groups, when an underlying finite dimensional Lie group is compact. Among applications, this shows that the dressing action – by the whole subgroup of loops which extend holomorphically to the exterior disc – on the U-hierarchy of the ZS-AKNS systems, on curved flats and on various other integrable systems, is global for compact cases. It also implies a global infinite dimensional Weierstrass-type representation for Lorentzian harmonic maps ($1+1$ wave maps) from surfaces into compact symmetric spaces. An "Iwasawa-type" decomposition of the same type of real form, with respect to a fixed point subgroup of an involution of the second kind, is also proved, and an application given.

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1. Introduction

Let G be a compact connected semisimple Lie group, given as the fixed point subgroup $G^{\mathbb{C}}_{\rho}$, where ρ is a complex antilinear involution and $G^{\mathbb{C}}$ is a complexification of G. We assume G is embedded as a subgroup of some matrix group. Let $\Lambda G^{\mathbb{C}}$ denote the complex Banach Lie group of loops in $G^{\mathbb{C}}$, with some H^s-topology, $s > 1/2$. In the study of loop groups by Pressley and Segal [23], among the many interesting facts proved are two loop group decompositions which have been very useful in applications to integrable systems. The *Birkhoff decomposition* states, in essence, that the set

$$\mathcal{B} := \Lambda^- G^{\mathbb{C}} \cdot \Lambda^+ G^{\mathbb{C}} \tag{1}$$

is open and dense in $\left[\Lambda G^{\mathbb{C}}\right]_e$, where $\Lambda^{\pm} G^{\mathbb{C}}$ denotes the subgroup of loops which are boundary values for holomorphic maps $\mathbb{D}^{\pm} \to G^{\mathbb{C}}$, $\mathbb{D}^+$ denotes the unit disc, $\mathbb{D}^-$ the complement of the closure of $\mathbb{D}^+$ in the Riemann sphere, and $[X]_e$ denotes the identity component of any group X. The set $\mathcal{B}$ is called the *big cell*. The *Iwasawa decomposition* is

$$\Lambda G^{\mathbb{C}} = \Omega G \cdot \Lambda^+ G^{\mathbb{C}}, \tag{2}$$

where $\Omega G \subset \Lambda G$ is the subgroup of *based* loops in G, that is loops which map 1 to the identity element.

A real form $\Lambda G^{\mathbb{C}}_{\hat{\theta}}$ of $\Lambda G^{\mathbb{C}}$ is defined to be the fixed point subgroup with respect to a complex antilinear involution $\hat{\theta}$ of $\Lambda G^{\mathbb{C}}$. As part of a natural theory of affine Kac–Moody Lie groups, finite order (complex linear or complex antilinear)

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automorphisms of loop groups have been studied and classified, based on the work of many people — see [1,16,18,20] and associated references.

After passing to an isomorphic loop group (see [16]), one can assume that an antilinear involution $\hat{\theta}$ of $\Lambda G^{\mathbb{C}}$ has one of the following forms:

$$(\hat{\theta}x)(\lambda) := \theta(x(\varepsilon(\bar{\lambda})^j)), \quad \varepsilon,\ j \in \{1, -1\},$$

where θ is an antilinear involution of $G^{\mathbb{C}}$, λ is the loop parameter, and $x \in \Lambda G^{\mathbb{C}}$. Note that, since λ is an $\mathbb{S}^1$ parameter, one could equivalently write λ^{-1} instead of $\bar{\lambda}$, but our choice of notation, which is commonly used, gives the right formula when one considers holomorphic extensions of loops away from $\mathbb{S}^1$. For example, if θ is just complex conjugation, and $x = \sum_i a_i\lambda^i$, and we take $\varepsilon = j = 1$, then, according to our choice, $(\hat{\theta}x)(\lambda) = \sum_i \overline{a_i\bar{\lambda}^i} = \sum_i \overline{a_i}\lambda^i$. Otherwise one obtains $\sum_i \overline{a_i}\bar{\lambda}^{-i}$, which is not holomorphic.

Complex linear involutions can be put into a similar standard form, where θ is $\mathbb{C}$-linear, and λ is not conjugated. We consider involutions only in these standard forms. They have some redundancy: for example, in the $\mathbb{C}$-antilinear case above, one need only consider $\varepsilon = 1$, as the fixed point subgroups determined from the two cases are isomorphic. This is not true for the $\mathbb{C}$-linear case, however.

Automorphisms come in two types, those of the *first kind*, which, in the standard forms used in this article, preserve $\Lambda^+G^{\mathbb{C}}$, and those of the *second kind*, which take $\Lambda^+G^{\mathbb{C}} \to \Lambda^-G^{\mathbb{C}}$. An involution is of the first kind if $j = 1$ and of the second kind if $j = -1$.

A real form is said to be *almost split* if the involution is of the first kind, and *almost compact* if it is of the second kind. The real form ΛG is almost compact, as it is the fixed point subgroup of $\hat{\theta}$, where $(\hat{\theta}x)(\lambda) = \rho(x(\bar{\lambda}^{-1}))$. There is thus clearly no hope of obtaining an analogue of the Birkhoff decomposition theorem for ΛG, as elements of $\Lambda^{\pm}G$ are just constant loops. On the other hand, the theorem does hold for *almost split* real forms – in fact, more generally, it is not difficult to show (see [6]) that, for any finite order automorphism of the first kind, $\hat{\theta}$ (given in a standard form analogous to that above), the set $\Lambda^-G^{\mathbb{C}}_{\hat{\theta}} \cdot \Lambda^+G^{\mathbb{C}}_{\hat{\theta}}$ is open and dense in $\left[\Lambda G^{\mathbb{C}}_{\hat{\theta}}\right]_e$.

1.1. Results

In this note we prove that this open dense set is actually the whole of $\Lambda G^{\mathbb{C}}_{\hat{\theta}}$, for almost split real forms when G is compact: suppose $\hat{\rho}$ is an extension of ρ to an involution of the first kind of $\Lambda G^{\mathbb{C}}$, given in standard form, i.e. by one of the formulae

$$(\hat{\rho}x)(\lambda) := \rho(x(\varepsilon\bar{\lambda})), \quad \varepsilon = \pm 1. \tag{3}$$

Set

$$\mathcal{H} := \Lambda G^{\mathbb{C}}_{\hat{\rho}}, \qquad \mathcal{H}^{\pm} := \mathcal{H} \cap \Lambda^{\pm}G^{\mathbb{C}}, \qquad \mathcal{H}^0 := \mathcal{H} \cap G^{\mathbb{C}} = G.$$

Let $\Lambda^-_*G^{\mathbb{C}}$ and $\Lambda^+_*G^{\mathbb{C}}$ denote the subgroups of loops in $\Lambda^{\pm}G^{\mathbb{C}}$ which respectively map the elements ∞ and 0 to the identity element, and define $\mathcal{H}^{\pm}_* := \mathcal{H} \cap \Lambda^{\pm}_*G^{\mathbb{C}}$.

Theorem 1. *There exists a decomposition*

$$\mathcal{H} = \mathcal{H}^-_* \cdot \mathcal{H}^+.$$

The map $\mathcal{H}^-_* \times \mathcal{H}^+ \to \mathcal{H}$, given by $(x, y) \mapsto xy$, is a real analytic diffeomorphism. The subgroup $G = \mathcal{H}^0$ is a deformation retract of $\mathcal{H}$. In particular, $\mathcal{H}$ is connected.

We derive Theorem 1 directly, from the Birkhoff factorization of Pressley/Segal [23], for the case $G = U(n)$, and then use the fact that any compact Lie group can be embedded in some $U(n)$. The result does not hold if G is not compact — see Remark 2. There is, moreover, a generalization, Theorem 6 to certain subgroups of $\mathcal{H}$, which is useful for applications, as described in Section 4.

Let $\hat{\tau} : \Lambda G^{\mathbb{C}} \to \Lambda G^{\mathbb{C}}$ be an arbitrary (either $\mathbb{C}$-linear or $\mathbb{C}$-antilinear) involution of the *second* kind which restricts to an involution of $\mathcal{H}$. Let $\mathcal{H}_{\hat{\tau}}$ denote the fixed point subgroup. As a corollary of Theorem 1 we obtain the following global Iwasawa-type decomposition:

Theorem 2. *Every element $x \in \mathcal{H}$ can be decomposed:*

$$x = z_\tau y_+, \quad z_\tau \in \mathcal{H}_{\hat{\tau}},\ y_+ \in \mathcal{H}^+. \tag{4}$$

The element z_τ is unique up to right multiplication by an element of $\mathcal{H}^0_{\hat{\tau}} := \mathcal{H}_{\hat{\tau}} \cap G^{\mathbb{C}}$, and the projection $\mathcal{H} \to \frac{\mathcal{H}_{\hat{\tau}}}{\mathcal{H}^0_{\hat{\tau}}}$, given by $x \mapsto [z_\tau]$, the equivalence class of z_τ modulo this right multiplication, is real analytic.

There is also a generalization of Theorem 2 to subgroups, given by Theorem 8.

1.2. Applications

1.2.1. Applications of Theorem 1

The Birkhoff decomposition in almost split real forms is used in important techniques for producing solutions to integrable systems: one such method is the *dressing technique*, originating in the work of Zakharov and Shabat [33]. The dressing technique can easily be shown to apply in general to a large class of integrable systems, discussed in [6]. In the present context, this is an (at least local) action by the group $\Lambda^- G^{\mathbb{C}}_{\hat\rho}$ on the space of solutions, where G is not necessarily compact but $\hat\rho$ is of the first kind. Special cases of dressing are the classical Bäcklund and Ribaucour transformations.

Terng and Uhlenbeck [28] studied the dressing action on a class of rapidly decaying solutions for certain integrable systems associated with the group $G = U(n)$. They showed that the subgroup of *rational* loops in $\Lambda^- G^{\mathbb{C}}_{\hat\rho}$ acts globally on this class of solutions. Dressing by rational loops is also treated more generally in almost split real forms by Donaldson, Fox and Goertsches [12].

The decomposition Theorem 1 strengthens the dressing result of Terng/Uhlenbeck by showing that the dressing action of the whole subgroup $\Lambda^- G^{\mathbb{C}}_{\hat\rho}$ is well defined globally, and that this holds for any compact G.

There are many examples to which this applies: for example, any of the integrable systems described in [27] which satisfy the "U reality condition" – which is the case $\varepsilon = 1$ in (3) – if U is compact. A large geometrical class are *curved flats* in symmetric spaces, defined by Ferus and Pedit [14]. Curved flats themselves contain, as special cases, many problems in geometry, such as isometric immersions of space forms [14], and *isothermic surfaces* [8,11], examples of which are surfaces of revolution, quadrics and constant mean curvature surfaces: a good survey can be found in Burstall [9].

Another important class of maps to which dressing has been applied successfully are *harmonic maps* from Riemann surfaces into compact symmetric spaces. For Riemannian harmonic maps, the loop group is ΛG, and dressing is done via the standard Iwasawa splitting of Pressley and Segal. This allowed various interesting results to be proved [2,10,31]. On the other hand, *Lorentzian* harmonic maps (i.e. maps which are harmonic with respect to a Lorentzian metric, also called $1+1$ *wave maps*), from surfaces into symmetric spaces, have a loop group formulation [26] into an almost split real form, and dressing can be done via a Birkhoff decomposition. Dressing of these maps by certain rational loops (simple elements) is studied in [29]. The significance of Theorem 1 is, again, that the dressing action by the whole of $\Lambda^- G^{\mathbb{C}}_{\hat\rho}$ is global, for compact targets, just as in the Riemannian case.

A second technique which uses the Birkhoff decomposition in an almost split real form is the method of Dorfmeister, Pedit and Wu (DPW) [13], when applied to Lorentzian harmonic maps from a surface into a symmetric space. This technique, while depending on a loop group factorization, differs from the dressing technique in that it gives a way to produce *all* solutions to the problem at hand from simple data. The DPW method was shown to apply in this way to Lorentzian harmonic maps by Toda [30]. Its application to pseudospherical surfaces is based on the fact that the Gauss map of a pseudospherical surface is harmonic with respect to the Lorentzian metric given by its second fundamental form. The DPW technique allows one to construct all solutions from arbitrary pairs of curves. A similar result was also obtained for the sine–Gordon equation (solutions of which correspond to pseudospherical surfaces) by Krichever [19]. Since the group here is compact, Theorem 1 shows that the construction is global. It does not (and cannot) allow one to produce *complete* pseudospherical surfaces, however, as Hilbert proved [17] that none exist.

1.2.2. Applications of Theorem 2

The Iwasawa-type decomposition Theorem 2 is relevant to both the dressing action and the generalized DPW method, given in [6], for the case of isometric immersions of space forms (as formulated by Ferus and Pedit [15]), as well as to constant curvature Lagrangian submanifolds of $\mathbb{C}P^n$ and $\mathbb{C}\mathbb{H}^n$, and, more generally, any solutions of the 3-involution loop group system studied in [4]. For isometric immersions of space forms, an interesting point here is that, even when the isometry group is *non*-compact, it turns out that in some cases the relevant loop group also satisfies a reality condition of the first kind for some *compact* Lie group, which means that all solutions can be constructed globally from curved flats, and that the dressing action is global. This example is discussed in more detail in Section 4.

2. Proof of Theorem 1

We will later be concerned with automorphisms of the first kind of order n, which, for $\mathbb{C}$-linear and $\mathbb{C}$-antilinear cases respectively, we assume are in a standard form:

$$\begin{aligned}(\hat\theta x)(\lambda) &:= \theta(x(\mathrm{e}^{\frac{2k\pi \mathrm{i}}{n}}\lambda)), \quad k \in \mathbb{Z}, \theta\ \mathbb{C}\text{-linear},\\ (\hat\theta x)(\lambda) &:= \theta(x(\varepsilon\bar\lambda)), \quad \varepsilon = \pm 1, \theta\ \mathbb{C}\text{-antilinear},\end{aligned} \tag{5}$$

where θ is a finite order automorphism of $G^{\mathbb{C}}$. The antilinear case occurs only for even values of n.

We first discuss the topology of some subgroups of $\Lambda^{\pm} G^{\mathbb{C}}$. For any subgroup $\mathcal{K}$, define $\mathcal{K}^{\pm} = \mathcal{K} \cap \Lambda^{\pm} G^{\mathbb{C}}$ and $\mathcal{K}^{\pm}_* = \mathcal{K} \cap \Lambda^{\pm}_* G^{\mathbb{C}}$.

Lemma 3. *Let $\hat\theta_j$, for $j = 1, \dots, k$, be a collection of mutually commuting finite order automorphisms of the first kind of $\Lambda G^{\mathbb{C}}$, each given as an extension of a $\mathbb{C}$-linear or $\mathbb{C}$-antilinear automorphism, θ_j of $G^{\mathbb{C}}$, in the form (5). Let*

$$\mathcal{K} := \Lambda G^{\mathbb{C}}_{\hat\theta_1 \dots \hat\theta_k},$$

be the subgroup of elements which are fixed by all k automorphisms. Then:

(1) *The subgroups $\mathcal{K}^{\pm}_*$ are contractible.*
(2) *The constant subgroup $\mathcal{K}^0 = G^{\mathbb{C}}_{\theta_1 \dots \theta_k}$ is a deformation retract of each of $\mathcal{K}^{\pm}$.*

Proof. We give the details for $\mathcal{K}^+$, the argument being the same for the other cases. Let x be an element of $\mathcal{K}^+$. By definition, x has a holomorphic extension to a map $\hat x : \mathbb{D}^+ \to G^{\mathbb{C}}$. Now for $\lambda \in \mathbb{S}^1$, $\hat x$ is fixed by each of $\hat\theta_j$, that is, if $\hat\theta_j$ is $\mathbb{C}$-linear,

$$\hat x(\lambda)\,[\theta_j(\hat x(\mathrm{e}^{\frac{2k\pi \mathrm{i}}{n}}\lambda))]^{-1} - I = 0,$$

and an analogous expression holds if it is antilinear. Either of these expressions is holomorphic in λ on $\mathbb{D}^+$, and therefore equivalent to zero on $\mathbb{D}^+$. Now consider, for t in the interval $[0, 1] \subset \mathbb{R}$, the family of loops $\gamma_t : \mathbb{S}^1 \to \Lambda G^{\mathbb{C}}$ defined by

$$\gamma_t(\lambda) := \hat x(t\lambda).$$

This family depends continuously on t because $\hat x$ is holomorphic on $\mathbb{D}^+$. Clearly $\gamma_t \in \Lambda^+ G^{\mathbb{C}}$ for any $t \in [0, 1]$. Moreover, we just showed that, for the two respective cases, we have

$$\hat x(z) = \theta_j(\hat x(\mathrm{e}^{\frac{2k\pi \mathrm{i}}{n}} z)), \qquad \hat x(z) = \theta_j(\hat x(\varepsilon \bar z)),$$

for any value $z \in \mathbb{D}^+$. In particular, for $\lambda \in \mathbb{S}^1$, and real $t \in [0, 1]$, we have, for the linear case,

$$\gamma_t(\lambda) := \hat x(t\lambda) = \theta_j(\hat x(t\mathrm{e}^{\frac{2k\pi \mathrm{i}}{n}}\lambda)) =: (\hat\theta_j \gamma_t)(\lambda),$$

and, similarly, $\gamma_t(\lambda) = (\hat\theta_j \gamma_t)(\lambda)$ for the antilinear case. In other words, γ_t is fixed by all of the involutions, so $\gamma_t \in \mathcal{K}^+$ for all $t \in [0, 1]$.

Finally, $\gamma_1 = x$, γ_0 is just the constant loop $\hat x(0) \in \mathcal{K}^0$, and if $x \in \mathcal{K}^0$ then $\gamma_t = x$ for all t. Hence $\Gamma : [0, 1] \times \mathcal{H}^+ \to \mathcal{H}^+$, given by $(t, x) \mapsto \gamma_t$, is a deformation retract of $\mathcal{H}^+$ onto $\mathcal{H}^0$.

The proof of Theorem 1 will be derived from the Birkhoff theorem of Pressley and Segal, which states:

Theorem 4 (*Birkhoff Factorization Theorem [23].*). *Every element $x \in \Lambda GL(n, \mathbb{C})$ has a decomposition:*

$$x = x_- D x_+, \qquad D = \mathrm{diag}(\lambda^{k_1}, \dots, \lambda^{k_n}), \tag{6}$$

where k_i are integers, $k_1 \geq \cdots \geq k_n$, and $x_\pm \in \Lambda^{\pm} GL(n, \mathbb{C})$. The set $\mathcal{B}$ (the big cell), on which $D = I$, is open and dense in the identity component $[\Lambda GL(n, \mathbb{C})]_e$. The multiplication map $\Lambda^-_* GL(n, \mathbb{C}) \times \Lambda^+ GL(n, \mathbb{C}) \to \mathcal{B}$ is a real analytic diffeomorphism.

The same statement holds on replacing $GL(n, \mathbb{C})$ with the complexification $G^{\mathbb{C}}$ of any connected compact semisimple Lie group G, and replacing the middle term D with a homomorphism from $\mathbb{S}^1$ into the complexification of a maximal torus of G.

Now let G and ρ be as in the introduction; suppose that $\hat\rho$ is given by (3), and $\mathcal{H} := \Lambda G^{\mathbb{C}}_{\hat\rho}$. For the subgroup $\mathcal{H}$, define the *big cell* to be the set

$$\mathcal{B}_{\hat\rho} := \mathcal{B} \cap \mathcal{H}.$$

We first recall some properties of $\mathcal{H}$, which are proved in [6], but can be verified by the reader without much difficulty:

Lemma 5 (*[6]*).

(1) *The group $\mathcal{H}$ is a closed Banach Lie subgroup of $\Lambda G^{\mathbb{C}}$.*
(2) *If $x = x_- x_+$ is the Birkhoff splitting of an element $x \in \mathcal{B}_{\hat\rho} \subset \Lambda G^{\mathbb{C}}$, with $x_- \in \Lambda^-_* G^{\mathbb{C}}$ and $x_+ \in \Lambda^+ G^{\mathbb{C}}$, then both x_- and x_+ are elements of $\mathcal{H}$.*
(3) *Let x be an element of $\mathcal{H}$, and let $D(x)$ denote the largest domain in $\mathbb{C} \cup \{\infty\}$ to which x extends analytically. Then x is G-valued for values of $\lambda \in \mathbb{R} \cap D(x)$ if $\varepsilon = 1$, and for $\lambda \in i\mathbb{R} \cap D(x)$ if $\varepsilon = -1$.*

The proof of Theorem 1 will follow from the next proposition:

Proposition 1. $\mathcal{B}_{\hat\rho} = \mathcal{H}$.

Proof. First consider the case $G = U(n)$. Set $G^{\mathbb{C}} := GL(n, \mathbb{C})$. The real form $U(n)$ is given by the involution $\rho x := (\bar{x}^t)^{-1}$. We describe the case $\varepsilon = 1$ in (3), the proof for $\varepsilon = -1$ being essentially identical. Extending ρ to an involution of the loop group by the formula

$$(\hat{\rho}x)(\lambda) := \rho(x(\bar{\lambda})) = \overline{(x(\bar{\lambda}))^t}^{-1} =: (x^*)^{-1}, \tag{7}$$

it is simple to check that loops in $\Lambda G^{\mathbb{C}}_{\hat{\rho}}$ take values in $U(n)$ when λ is real. According to the Birkhoff factorization theorem of [23], every element of $\left[\Lambda G^{\mathbb{C}}\right]_e$ which is *not* in the big cell can be expressed in the form (6), where at least one k_i is nonzero, and $x_\pm \in \Lambda^{\pm} G^{\mathbb{C}}$. We need to prove that such an element cannot be in the real form $\Lambda G^{\mathbb{C}}_{\hat{\rho}}$, in other words cannot be fixed by $\hat{\rho}$. It then follows that every element in $\mathcal{H}$ is in fact in the big cell. The condition $x = \hat{\rho}x$ is

$$x_- \operatorname{diag}(\lambda^{k_1}, \ldots, \lambda^{k_n})\, x_+ = (x_-^*)^{-1} \operatorname{diag}(\lambda^{-k_1}, \ldots, \lambda^{-k_n})\, (x_+^*)^{-1}.$$

Using the assumption that at least one k_i is non-zero, together with the holomorphic properties of $x_\pm$, we quickly obtain a contradiction as follows: Either $k_1 > 0$ or $k_n < 0$ (or both hold). If $k_n < 0$, rearrange the equation to

$$x_-^* x_- \operatorname{diag}(\lambda^{k_1}, \ldots, \lambda^{k_n}) = \operatorname{diag}(\lambda^{-k_1}, \ldots, \lambda^{-k_n})\, (x_+^*)^{-1} x_+^{-1}. \tag{8}$$

But the components of the matrix $x_-^* x_-$ are power series in λ^{-1}, while those of the matrix $(x_+^*)^{-1}x_+^{-1}$ are power series in λ. Since $k_n < 0$, it follows that the (n, n) component of (8) is identically zero. Writing the last row of x_-^* as $(a_1, \ldots, a_n)$ we thus obtain, from the (n, n) component on the left hand side, the identity

$$a_1^* a_1 + \cdots + a_n^* a_n = 0,$$

where, for a scalar function $y(\lambda)$, we use $(y^*)(\lambda) := \overline{y(\bar{\lambda})}$. Since, for real values of λ, we have $a_i^* = \overline{a_i}$, this implies that $a_1 = a_2 = \cdots = a_n = 0$, at $\lambda = 1$, which is impossible, as x_- is an invertible matrix for all λ in the unit circle.

If $k_1 > 0$, we instead write

$$\operatorname{diag}(\lambda^{k_1}, \ldots, \lambda^{k_n})\, x_+ x_+^* = x_-^{-1} (x_-^*)^{-1} \operatorname{diag}(\lambda^{-k_1}, \ldots, \lambda^{-k_n}),$$

and similarly deduce that a row (this time the first) of x_-^{-1} is zero.

The general case now follows from what we have just shown: by the Peter–Weyl theorem, the compact connected semisimple Lie group G can be embedded as a closed subgroup of the unitary group $U(n)$ for some n. While there are many possible complexifications of G, they are all isomorphic. Given the embedding in $U(n)$, and the complexification $U(n)^{\mathbb{C}} = GL(n, \mathbb{C})$, there is a natural complexification $G^{\mathbb{C}}$ as follows: if $\mathfrak{g}$ is the Lie algebra, set $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} + i\mathfrak{g}$ in $\mathfrak{gl}(n, \mathbb{C})$, and $G^{\mathbb{C}} := \exp(\mathfrak{g}^{\mathbb{C}})$ in $GL(n, \mathbb{C})$. It follows from the corresponding properties of the Lie algebras that the involution of $G^{\mathbb{C}}$ which determines G is just the restriction to $G^{\mathbb{C}}$ of that which determines $U(n)$, and we denote both by ρ. Thus $G = G^{\mathbb{C}}_{\rho} = G^{\mathbb{C}} \cap U(n)$.

Now $\Lambda G^{\mathbb{C}}$ is a subgroup of $\Lambda GL(n, \mathbb{C})$, and, by assumption, ρ is extended to $\Lambda G^{\mathbb{C}}$ by the formula (3). Hence this extension is also the restriction to $\Lambda G^{\mathbb{C}}$ of the involution $\hat{\rho}$ discussed in the $U(n)$ case. We denote both extensions by $\hat{\rho}$. Thus $\Lambda G^{\mathbb{C}}_{\hat{\rho}} = \Lambda G^{\mathbb{C}} \cap \Lambda GL(n, \mathbb{C})_{\hat{\rho}}$.

Theorem 4 says that any element $x \in \Lambda G^{\mathbb{C}}$ which is not in the big cell has a factorization of the form (6), where now D is a non-trivial homomorphism from $\mathbb{S}^1$ into the complexification of a maximal torus of G. Since any torus of G is contained in a maximal torus of $U(n)$, it follows that D is a non-trivial homomorphism into the complexification of a maximal torus of $U(n)$. Since $\Lambda G^{\mathbb{C}}_{\hat{\rho}} \subset \Lambda GL(n, \mathbb{C})_{\hat{\rho}}$ and $\Lambda^{\pm} G^{\mathbb{C}} \subset \Lambda^{\pm} GL(n, \mathbb{C})$, the $U(n)$ case implies that x cannot be in $\mathcal{H}$.

Proof of Theorem 1. It follows from the Birkhoff factorization Theorem 4, from Proposition 1, and from the second item of Lemma 5, together with the fact that $\mathcal{H}^-_*$ and $\mathcal{H}^+$ are closed Banach submanifolds, that the multiplication map $\mathcal{H}^-_* \times \mathcal{H}^+ \to \mathcal{H}$ is a real analytic diffeomorphism.

It remains to show that $G = \mathcal{H}^0$ is a deformation retract of $\mathcal{H}$. Since $\mathcal{H}$ is diffeomorphic to the product $\mathcal{H}^-_* \times \mathcal{H}^+$, this follows from Lemma 3. □

Remark 1. There is obviously an analogue of Theorem 1, switching $+$ and $-$.

Remark 2. The decomposition of Theorem 1 does not hold (even if one restricts to the identity component) if the real form G is non-compact. For example, taking $G^{\mathbb{C}} = SL(2, \mathbb{C})$ and $G = SL(2, \mathbb{R})$, the loop $x = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ is not in the big cell, but does take values in $SL(2, \mathbb{R})$ for $\lambda \in \mathbb{R} -$ more specifically, it is an element of the identity component of $\Lambda SL(2, \mathbb{C})_{\hat{\rho}}$, where $(\hat{\rho}x)(\lambda) := \overline{x(\bar{\lambda})}$ defines an involution of the first kind.

Neither is it true, if $G^{\mathbb{C}}_{\hat{\rho}}$ is non-compact, that $\mathcal{H}^0$ has the same number of connected components as $\mathcal{H}$. The connected components of $\Lambda GL(n, \mathbb{C})$ correspond to the winding numbers of the determinants of the elements. If one considers $G = GL(2, \mathbb{R})$ then the loops $x = \begin{pmatrix} \lambda^k & 0 \\ 0 & 1 \end{pmatrix}$, which have winding number k, are all elements of $\Lambda GL(2, \mathbb{C})_{\hat{\rho}}$, where again $(\hat{\rho}x)(\lambda) := \overline{x(\bar{\lambda})}$ is of the first kind. Thus $\mathcal{H} = \Lambda GL(2, \mathbb{C})_{\hat{\rho}}$ has infinitely many components, while $\mathcal{H}^0 = GL(2, \mathbb{R})$ has only two.

Remark 3. Note that Theorem 1 implies that the winding number, given by the determinant of the matrix, for any loop in $\Lambda GL(n,\mathbb{C})_{\hat\rho}$, where $\hat\rho$ is given by (7), is zero. This follows because the theorem implies that $\Lambda GL(n,\mathbb{C})_{\hat\rho}$ is contained in the identity component of $\Lambda GL(n,\mathbb{C})$.

2.1. Generalization of Theorem 1

Let G and $\hat\rho$ be as in previous sections. Let $\hat\theta_i$, $i = 1,\dots,k$, be a collection of mutually commuting finite order automorphisms of the first kind (either linear or antilinear), given in the form (5), and all of which commute with $\hat\rho$. Set

$$\widehat{\mathcal{H}} := \Lambda G^{\mathbb{C}}_{\hat\rho\hat\theta_1\dots\hat\theta_k},$$

the subgroup of $\Lambda G^{\mathbb{C}}$ consisting of elements which are fixed by all of $\hat\rho$ and $\hat\theta_i$.

Theorem 6. *There exists a decomposition*

$$\widehat{\mathcal{H}} = \widehat{\mathcal{H}}^-_* \cdot \widehat{\mathcal{H}}^+.$$

The map $\widehat{\mathcal{H}}^-_* \times \widehat{\mathcal{H}}^+ \to \mathcal{H}$, given by $(x,y)\mapsto xy$, is a real analytic diffeomorphism. The constant subgroup $\widehat{\mathcal{H}}^0 = G_{\hat\theta_1\dots\hat\theta_k}$ is a deformation retract of $\widehat{\mathcal{H}}$.

Proof. Set $\widehat{\mathcal{B}} = \widehat{\mathcal{H}} \cap \mathcal{B}$. It is not difficult to show (see [6], Proposition 1) that the first two properties of Lemma 5 also hold for $\widehat{\mathcal{H}}$. Since $\widehat{\mathcal{H}}$ is contained in $\mathcal{H}$, Proposition 1 also applies to $\widehat{\mathcal{B}}$. Hence the proof again follows from these facts, together with the Pressley/Segal Birkhoff factorization and the topology of $\widehat{\mathcal{H}}^-_*$ and $\widehat{\mathcal{H}}^+$ given by Lemma 3.

Note that in some cases, such as when $\hat\theta$ is a $\mathbb{C}$-linear involution of the first kind which restricts to an inner automorphism of $G^{\mathbb{C}}$, one has an isomorphism $\Psi : \Lambda G^{\mathbb{C}} \to \Lambda G^{\mathbb{C}}_{\hat\theta}$. However, the Birkhoff splitting obtained from this isomorphism is not the same as the one just given, because $\Psi(\Lambda^{\pm}G^{C}) \neq \Lambda^{\pm}G^{\mathbb{C}}_{\hat\theta}$. For practical applications, such as dressing and the DPW method, we are really interested in decompositions of the form $\Lambda^- G^{\mathbb{C}}_{\hat\theta} \cdot \Lambda^+ G^{\mathbb{C}}_{\hat\theta}$, as described here.

3. Proof of Theorem 2

Generalizations of the standard Iwasawa decomposition of $\Lambda G^{\mathbb{C}}$, namely (2), to subgroups $\mathcal{K}$ of $\Lambda G^{\mathbb{C}}$ and arbitrary involutions $\hat\tau$ of the second kind, were considered in [6]. That is, decompositions of the form

$$\mathcal{K} = \mathcal{K}_{\hat\tau} \cdot \mathcal{K}^+, \tag{9}$$

where $\mathcal{K}_{\hat\tau}$ is the fixed point subgroup, $\mathcal{K}^+ = \mathcal{K} \cap \Lambda^+ G^{\mathbb{C}}$, and where the $\mathcal{K}_{\hat\tau}$ factor is unique up to right multiplication by the constant subgroup $\mathcal{K}^0_{\hat\tau} := \mathcal{K}_{\hat\tau} \cap G^{\mathbb{C}}$. In general such a decomposition exists on a neighbourhood of the identity, but not globally. For example, in the standard Iwasawa decomposition (2), if the real form were non-compact, then the decomposition would not be global, even on the identity component of $\Lambda G^{\mathbb{C}}$.

Theorem 2 states that the decomposition is global for $\mathcal{K} = \mathcal{H} := \Lambda G^{\mathbb{C}}_{\hat\rho}$. Note that this is not just a special case of the standard Iwasawa decomposition (2). Taking the case $\mathcal{K} = \left[\Lambda G^{\mathbb{C}}\right]_e$, and $\hat\tau$ as an antilinear involution of the second kind for a *non*-compact real form, then, as was already mentioned, the decomposition (9) is not global. See for example, the case of $\Lambda SU(1,1)$, studied in [7]. But if one now restricts this to $\Lambda G^{\mathbb{C}}_{\hat\rho} \subset \mathcal{K}$, assuming that $\hat\rho$ commutes with $\hat\tau$, then the theorem holds.

We need a result from [6]. Note that a Birkhoff decomposable subgroup $\mathcal{K}$ of $\Lambda G^{\mathbb{C}}$ means a connected subgroup which has the property that $\left([\mathcal{K}^-]_e \cdot [\mathcal{K}^+]_e\right) \cap \left([\mathcal{K}^+]_e \cdot [\mathcal{K}^-]_e\right)$ is open and dense in $[\mathcal{K}]_e$, and this property is satisfied by our group $\mathcal{H}$. Define the *right big cell* of $\Lambda G^{\mathbb{C}}$ in analogy with the (left) big cell $\mathcal{B}$, interchanging $+$ and $-$, i.e. $\mathcal{RB} := \Lambda^+ G^{\mathbb{C}} \cdot \Lambda^- G^{\mathbb{C}}$.

Theorem 7. *(From Theorem 3 in [6].) Suppose $\mathcal{K}$ is any Birkhoff decomposable subgroup of $\Lambda G^{\mathbb{C}}$, and $\hat\tau$ is any involution of the second kind of $\Lambda G^{\mathbb{C}}$, which restricts to an involution of $\mathcal{K}$. Set*

$$\mathcal{U} := \{x \in \mathcal{K} \mid \exists\, z_\tau \in \mathcal{K}_{\hat\tau},\ y_+ \in \mathcal{K}^+,\ \text{s.t. } x = z_\tau y_+\}.$$

(1) *If the constant subgroup $\mathcal{K}^0$ is connected and compact, then every element $x \in \mathcal{K}$ which has the property that $x^{-1}\hat\tau x$ is in the identity component of $\mathcal{K} \cap \mathcal{RB}$ is also an element of $\mathcal{U}$.*
(2) *If $x = z_\tau y_+ \in \mathcal{U}$, then the factor z_τ is unique up to right multiplication by an element of $\mathcal{K}^0$.*
(3) *The map $\mathcal{U} \to \frac{\mathcal{K}_{\hat\tau}}{\mathcal{K}^0}$ given by $x = z_\tau y_+ \mapsto [z_\tau]$ is real analytic.*

Proof of Theorem 2. In Theorem 7, for the case $\mathcal{K} = \mathcal{H}$, we want to show that $\mathcal{U} = \mathcal{H}$. By Theorem 1, and Remark 1, the right big cell is the whole group, i.e. $\mathcal{H} \cap \mathcal{RB} = \mathcal{H}$. Note that $\mathcal{H}^0 = G$, which is assumed connected. Hence the conditions of Theorem 7 for x to be an element of $\mathcal{U}$ are met for any $x \in \mathcal{H}$. □

3.1. Generalization of Theorem 2

The following generalization has the identical proof of Theorem 2, using Theorem 6:

Theorem 8. *Let $\widehat{\mathcal{H}} = \Lambda G^{\mathbb{C}}_{\hat\rho\hat\theta_1\dots\hat\theta_k}$, as defined in Theorem 6. Suppose that $\hat\tau$ is a $\mathbb{C}$-linear or $\mathbb{C}$-antilinear involution of the second kind which commutes with all of $\hat\rho$ and $\hat\theta_i$. If the constant subgroup $\widehat{\mathcal{H}}^0 = G_{\theta_1\dots\theta_k}$ is connected then Theorem 2 is also valid for $\widehat{\mathcal{H}}$.*

4. An application of the theorems

As mentioned in Section 1.2, Theorems 1 and 2 are relevant to the application of the DPW method to isometric immersions of space forms (and generalizations). Isometric immersions of the hyperbolic space $\mathbb{H}^n$ into $\mathbb{E}^{2n-1}$ – equivalent to the case $k = n-1$ in the discussion below – were shown to be an integrable system in work of Terng and Tenenblat in [25,24], where Bäcklund transformations and soliton solutions are defined. The loop group formulation used here is due to Ferus and Pedit [15], where a generalized AKS theory was developed to produce "finite-type" solutions.

It was shown in [6] that the loop group maps for isometric immersions with flat normal bundle of space forms locally correspond, in a one to one manner, with curved flats via the Birkhoff and generalized Iwasawa splittings. These have been studied further in [3].

4.1. Specific case

An interesting problem is the case of isometric immersions with flat normal bundle of the hyperbolic space $\mathbb{H}^n$ into either a hyperbolic space $\mathbb{H}^{n+k}_c$, where $-1 < c$, or into a sphere $\mathbb{S}^{n+k}$. Solutions to both of these problems are obtained from just one map into a loop group, as follows (for details not proved here, see [3]): let $G^{\mathbb{C}} = SO(n+k+1,\mathbb{C})$, where $k \geq n-1$, and define the following three involutions, $\hat\sigma$, $\hat\rho$ and $\hat\tau$, on $\Lambda G^{\mathbb{C}}$ by the formulae

$$(\hat\sigma x)(\lambda) := \sigma(x(-\lambda)), \qquad (\hat\rho x)(\lambda) := \rho(x(-\bar\lambda)), \qquad (\hat\tau x)(\lambda) := \tau(x(\lambda^{-1})),$$

where ρ is complex conjugation and

$$\sigma = \mathrm{Ad}_P, \quad P = \begin{pmatrix} I_n & 0 \\ 0 & -I_{k+1} \end{pmatrix},$$

$$\tau = \mathrm{Ad}_Q, \quad Q = \begin{pmatrix} I_{n+1} & 0 \\ 0 & -I_k \end{pmatrix},$$

and I_r denotes an $r \times r$ identity matrix. Note that all three involutions commute. Set

$$\mathcal{H} = \Lambda G^{\mathbb{C}}_{\hat\rho\hat\sigma},$$

the fixed point subgroup with respect to $\hat\rho$ and $\hat\sigma$. Now $\hat\rho$ is an antilinear involution of the first kind, $G^{\mathbb{C}}_\rho = SO(n+k+1,\mathbb{R}) =: G$ is compact, and $\hat\sigma$ is also an involution of the first kind. Thus we can apply Theorem 6 to $\mathcal{H}$. Moreover, $\mathcal{H}^0 = G_\sigma = SO(n)\times SO(k+1)$, which is connected, so we can also apply Theorem 8 to $\mathcal{H}$, with respect to the involution of the second kind $\hat\tau$.

Since all three involutions commute, $\hat\tau$ restricts to an involution of $\mathcal{H}$. Let $\mathcal{H}_{\hat\tau}$ denote the fixed point subgroup.

Consider the set of $\mathcal{C}^\infty$ immersions $f : \mathbb{R}^n \to \mathcal{H}_{\hat\tau}/\mathcal{H}^0_{\hat\tau}$, such that, for any frame $F : \mathbb{R}^n \to \mathcal{H}_{\hat\tau}$ for f, the Maurer–Cartan form, $F^{-1}\mathrm{d}F$, has a Fourier expansion in λ which is polynomial of the form $\alpha_{-1}\lambda^{-1} + \alpha_0 + \alpha_1\lambda$, where α_i are $\mathfrak{g}^{\mathbb{C}}$-valued 1-forms. This condition does not depend on the choice of frame. Denote the set of such immersions by

$$\mathcal{S} := \left\{ f : \mathbb{R}^n \to \frac{\mathcal{H}_{\hat\tau}}{\mathcal{H}^0_{\hat\tau}} \,|\, f \text{ regular},\ F^{-1}\mathrm{d}F = \sum_{i=-1}^{1} \alpha_i\lambda^i,\ \text{for any frame } F \right\}.$$

If F is a frame for $f \in \mathcal{S}$, then F can be extended holomorphically in λ to $\mathbb{C}^*$. For a fixed pure imaginary value of λ, $\lambda_0 \in i\mathbb{R}$, the map $F_{\lambda_0} : \mathbb{R}^n \to G^{\mathbb{C}}$ takes values in the real form $SO(n+k+1)$ (cf. Lemma 5). One can also verify that for a fixed value of λ on the *unit circle*, $\lambda_0 \in \mathbb{S}^1$, F_{λ_0} takes values in a group isomorphic to the non-compact real form $SO(n+k,1)$.

Let $\tilde f_\lambda$ denote the $(n+1)$'st column of F_λ. Multiplication on the right by $\mathcal{H}^0_{\hat\tau} = SO(n)\times\{I\}\times SO(k)$ leaves this column fixed, so $\tilde f_\lambda$ is well defined by $f \in \mathcal{S}$. Assume that $\tilde f_\lambda$ is an immersion (a generic condition in these dimensions). Then:

(1) For $\lambda_0 \in i\mathbb{R}$, the map $\tilde f_{\lambda_0} : \mathbb{R}^n \to \mathbb{S}^{n+k}$, with the induced metric, has flat normal bundle and constant curvature $c_{\lambda_0} \in (-\infty, 0)$.
(2) For $\lambda_0 \in \mathbb{S}^1$, the map $\tilde f_{\lambda_0} : \mathbb{R}^n \to \mathbb{H}^{n+k}$ has flat normal bundle and constant curvature $c_{\lambda_0} \in (-\infty, -1)$.

Conversely, any immersion of the two types just described is associated with an element of $\mathcal{S}$ whose $(n+1)$'st column is an immersion.

4.1.1. The dressing action on $\mathcal{S}$

We describe how to use the Iwasawa-type decomposition, Theorem 8, to obtain a group action by $\mathcal{H}^-$ on $\mathcal{S}$. Given $g_- \in \mathcal{H}^-$ and a frame F for $f \in \mathcal{S}$, define a new frame $\hat{F}$ by the (pointwise at $x \in \mathbb{R}^n$) Iwasawa decomposition

$$g_- F(x) = \hat{F}(x)\, g_+(x), \quad \hat{F}(x) \in \mathcal{H}_{\hat{\tau}},\, g_+(x) \in \mathcal{H}^+.$$

The equivalence class $[\hat{F}(x)] \in \mathcal{H}_{\hat{\tau}}/\mathcal{H}^0_{\hat{\tau}}$ is well defined and depends smoothly on $g_-F(x)$, so we can choose $\hat{F}$ smoothly on the domain of F. Since $\mathcal{H}^0_{\hat{\tau}}$ consists of constant loops, it is also clear that $[\hat{F}]$ is well defined, independently of the choice of frame F for $[F]$. For a smooth choice of $\hat{F}$, g_+ is also smooth, and we can compute the Maurer–Cartan form of $\hat{F}$:

$$\hat{F}^{-1}\mathrm{d}\hat{F} = g_+^{-1}F^{-1}\mathrm{d}F g_+ + g_+^{-1}\mathrm{d}g_+.$$

We have $F^{-1}\mathrm{d}F = \sum_{i=-1}^{1} \alpha_i \lambda^i$ and we can expand $g_+ = a_0 + a_1\lambda + \cdots$. Hence

$$\hat{F}^{-1}\mathrm{d}\hat{F} = a_0^{-1}\alpha_{-1}a_0\,\lambda^{-1} + \cdots.$$

Since $\hat{F}$ is fixed by $\hat{\tau}$, which takes $\lambda \mapsto \lambda^{-1}$, it follows that

$$\hat{F}^{-1}\mathrm{d}\hat{F} = \hat{\alpha}_{-1}\lambda^{-1} + \hat{\alpha}_0 + \hat{\alpha}_1\lambda.$$

The condition that $[\hat{F}]$ is an immersion into $\mathcal{H}_{\hat{\tau}}/\mathcal{H}^0_{\hat{\tau}}$ can be read off from the 1-form $\hat{\alpha}_-$, and one can deduce that $[\hat{F}]$ is immersed if $[F]$ is immersed, which is to say $[\hat{F}] \in \mathcal{S}$. It is straightforward to check that this defines a group action by $\mathcal{H}^-$ on $\mathcal{S}$.

One cannot guarantee that the projection to the $(n+1)$'st column of the dressed solution will be everywhere regular. This is true with respect to all the loop group methods for this particular problem (including the generalized AKS theory of [15]), and is not related to the methods themselves, which globally produce immersions in the set $\mathcal{S}$. Thus from the loop group point of view, the natural object to consider here is $\mathcal{S}$, whose elements correspond to constant curvature submanifolds which may be non-immersed at some points, but which lift to immersions into $\frac{SO(n+k+1)}{SO(n)\times\{1\}\times SO(k)}$ and $\frac{SO(n+k,1)}{SO(n)\times\{1\}\times SO(k)}$ respectively for cases (1) and (2) above.

4.1.2. The DPW method for $\mathcal{S}$

The DPW method, described in [6], uses the "non-global" versions of Theorems 6 and 8 to establish, after fixing a basepoint in $\mathbb{R}^n$, a (local) bijection between elements of $\mathcal{S}$ and regular curved flats in $SO(n+k+1)/(SO(n)\times SO(k+1))$. This bijection can now be seen to be global.

These conclusions are of interest because it is an open problem whether or not *complete* solutions exist for the immersions of types (1) and (2) above. This is equivalent to the problem of the existence of complete isometric immersions with flat normal bundle of the hyperbolic space $\mathbb{H}^n$ into the Euclidean space $\mathbb{E}^{n+k}$, a natural generalization of Hilbert's non-immersibility theorem of the hyperbolic plane [17]. In the higher dimensional generalization, non-immersibility has been proven for a *non*-simply connected hyperbolic space in a series of papers [21,22,32]. See the discussion in [5] for other possible generalizations to which these comments also apply. We can conclude that any singularities arising in the immersions constructed are not related to the failure of a loop group decomposition. This is in contrast to the case of constant mean curvature surfaces in Minkowski 3-space, which are studied in [7], where *all* singularities (and there are many), other than branch points, arise as a result of the lack of the global property for the Iwasawa decomposition used.

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