Journal of Geometry and Physics 117 (2017) 84–98

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

# Symmetries of second order differential equations on Lie algebroids

Liviu Popescu

*University of Craiova, Department of Statistics and Economic Informatics, 13, Al. I. Cuza, st., Craiova 200585, Romania*

ARTICLE INFO

*Article history:*
Received 24 January 2017
Received in revised form 18 February 2017
Accepted 10 March 2017
Available online 18 March 2017

*MSC:*
17B66
34A26
53C05
70S10

*Keywords:*
Lie algebroids
Symmetries
Semispray
Nonlinear connection
Dynamical covariant derivative
Jacobi endomorphism

ABSTRACT

In this paper we investigate the relations between semispray, nonlinear connection, dynamical covariant derivative and Jacobi endomorphism on Lie algebroids. Using these geometric structures, we study the symmetries of second order differential equations in the general framework of Lie algebroids.



## 1. Introduction

The geometry of second order differential equations (SODE) on the tangent bundle $TM$ of a differentiable manifold $M$ is closely related to the geometry of nonlinear connections [1,2]. The system of SODE can by represented using the notion of semispray, which together with the nonlinear connection induce two important concepts, the dynamical covariant derivative and Jacobi endomorphism [3–9]. The notion of dynamical covariant derivative was introduced for the first time in the case of tangent bundle by J. Cariñena and E. Martínez [10] as a derivation of degree 0 along the tangent bundle projection. The notion of symmetry in fields theory using various geometric framework is intensely studied (see for instance [4,11–18]). The notion of Lie algebroid is a natural generalization of the tangent bundle and Lie algebra. In the last decades the Lie algebroids [19,20] are the objects of intensive studies with applications to mechanical systems or optimal control [21–34] and are the natural framework in which one can develop the theory of differential equations, where the notion of symmetry plays a very important role.

In this paper we study some properties of semispray and generalize the notion of symmetry for second order differential equations on Lie algebroids and characterize its properties using the dynamical covariant derivative and Jacobi endomorphism. The paper is organized as follows. In Section 2 the preliminary geometric structures on Lie algebroids are introduced and some relations between them are given. We present the Jacobi endomorphism on Lie algebroids and find the relation with the curvature tensor of Ehresmann nonlinear connection. In Section 3 we study the dynamical covariant derivative on

*E-mail addresses:* liviupopescu@central.ucv.ro, liviunew@yahoo.com.

http://dx.doi.org/10.1016/j.geomphys.2017.03.006

Lie algebroids. Using a semispray and an arbitrary nonlinear connection, we introduce the dynamical covariant derivative on Lie algebroids as a tensor derivation and prove that the compatibility condition with the tangent structure fixes the canonical nonlinear connection. In the case of the canonical nonlinear connection induced by a semispray, more properties of dynamical covariant derivative are added. In the case of homogeneous second order differential equations (spray) the relation between the dynamical covariant derivative and Berwald linear connection is given. In the last section we study the dynamical symmetries, Lie symmetries, Newtonoid sections and Cartan symmetries on Lie algebroids and find the relations between them. These structures are studied for the first time on the tangent bundle by G. Prince in [16,17]. Also, we prove that an exact Cartan symmetry induces a conservation law and conversely, which extends the work developed in [35]. Moreover, we find the invariant equations of dynamical symmetries, Lie symmetries and Newtonoid sections in terms of dynamical covariant derivative and Jacobi endomorphism, which generalize some results from [4,16,17]. We have to mention that the Noether type theorems for Lagrangian systems on Lie algebroids can be found in [26,36] and Jacobi sections for second order differential equations on Lie algebroids are studied in [37]. Finally, using an example from optimal control theory (driftless control affine systems), we prove that the framework of Lie algebroids is more useful than the tangent bundle in order to find the symmetries of the dynamics induced by a Lagrangian function. Also, using the $k$-symplectic formalism on Lie algebroids developed in [38] one can study the symmetries in this new framework, which generalize the results from [12].

## 2. Lie algebroids

Let $M$ be a real, $C^\infty$-differentiable, $n$-dimensional manifold and $(TM, \pi_M, M)$ its tangent bundle. A Lie algebroid over a manifold $M$ is a triple $(E, [\cdot,\cdot]_E, \sigma)$, where $(E, \pi, M)$ is a vector bundle of rank $m$ over $M$, which satisfies the following conditions:

(a) $C^\infty(M)$-module of sections $\Gamma(E)$ is equipped with a Lie algebra structure $[\cdot,\cdot]_E$.
(b) $\sigma : E \to TM$ is a bundle map (called the anchor) which induces a Lie algebra homomorphism (also denoted $\sigma$) from the Lie algebra of sections $(\Gamma(E), [\cdot,\cdot]_E)$ to the Lie algebra of vector fields $(\chi(M), [\cdot,\cdot])$ satisfying the Leibnitz rule

$$[s_1, fs_2]_E = f[s_1, s_2]_E + (\sigma(s_1)f)s_2, \ \forall s_1, s_2 \in \Gamma(E), \ f \in C^\infty(M). \tag{1}$$

From the above definition it results,

1° $[\cdot,\cdot]_E$ is a $\mathbb{R}$-bilinear operation,
2° $[\cdot,\cdot]_E$ is skew-symmetric, i.e. $[s_1, s_2]_E = -[s_2, s_1]_E, \quad \forall s_1, s_2 \in \Gamma(E)$,
3° $[\cdot,\cdot]_E$ verifies the Jacobi identity

$$[s_1, [s_2, s_3]_E]_E + [s_2, [s_3, s_1]_E]_E + [s_3, [s_1, s_2]_E]_E = 0,$$

and $\sigma$ being a Lie algebra homomorphism, means that $\sigma[s_1, s_2]_E = [\sigma(s_1), \sigma(s_2)]$.

The existence of a Lie bracket on the space of sections of a Lie algebroid leads to a calculus on its sections analogous to the usual Cartan calculus on differential forms. If $f$ is a function on $M$, then $df(x) \in E_x^*$ is given by $\langle df(x), a\rangle = \sigma(a)f$, for $\forall a \in E_x$. For $\omega \in \bigwedge^k(E^*)$ the *exterior derivative* $d^E\omega \in \bigwedge^{k+1}(E^*)$ is given by the formula

$$\begin{aligned} d^E\omega(s_1, \ldots, s_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}\sigma(s_i)\omega(s_1, \ldots, \hat{s}_i, \ldots, s_{k+1}) \\ &+ \sum_{1\le i<j\le k+1}(-1)^{i+j}\omega([s_i, s_j]_E, s_1, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots s_{k+1}), \end{aligned}$$

where $s_i \in \Gamma(E)$, $i = \overline{1, k+1}$, and the hat over an argument means the absence of the argument. It results that

$$(d^E)^2 = 0, \quad d^E(\omega_1 \wedge \omega_2) = d^E\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1}\omega_1 \wedge d^E\omega_2.$$

The cohomology associated with $d^E$ is called the *Lie algebroid cohomology* of $E$. Also, for $\xi \in \Gamma(E)$ one can define the *Lie derivative* with respect to $\xi$, given by $\mathcal{L}_\xi = i_\xi \circ d^E + d^E \circ i_\xi$, where $i_\xi$ is the contraction with $\xi$. We recall that if $L$ and $K$ are $(1,1)$-type tensor field, Frölicher–Nijenhuis bracket $[L, K]$ is the vector valued 2-form [39]

$$[L, K]_E(X, Y) = [LX, KY]_E + [KX, LY]_E + (LK + KL)[X, Y]_E - L[X, KY]_E - K[X, LY]_E - L[KX, Y]_E - K[LX, Y]_E,$$

and the Nijenhuis tensor of $L$ is given by

$$\mathbf{N}_L(X, Y) = \frac{1}{2}[L, L]_E = [LX, LY]_E + L^2[X, Y]_E - L[X, LY]_E - L[LX, Y]_E.$$

For a vector field in $\mathcal{X}(E)$ and a $(1,1)$-type tensor field $L$ on $E$ the Frölicher–Nijenhuis bracket $[X, L]_E = \mathcal{L}_X L$ is the $(1,1)$-type tensor field on $E$ given by

$$\mathcal{L}_X L = \mathcal{L}_X \circ L - L \circ \mathcal{L}_X,$$

where $\mathcal{L}_X$ is the usual Lie derivative. If we take the local coordinates $(x^i)$ on an open $U \subset M$, a local basis $\{s_\alpha\}$ of the sections of the bundle $\pi^{-1}(U) \to U$ generates local coordinates $(x^i, y^\alpha)$ on $E$. The local functions $\sigma^i_\alpha(x)$, $L^\gamma_{\alpha\beta}(x)$ on $M$ given by

$$\sigma(s_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta]_E = L^\gamma_{\alpha\beta} s_\gamma, \quad i = \overline{1, n}, \quad \alpha, \beta, \gamma = \overline{1, m}, \tag{2}$$

are called the *structure functions of the Lie algebroid*, and satisfy the *structure equations* on Lie algebroids

$$\sum_{(\alpha,\beta,\gamma)} \left( \sigma^i_\alpha \frac{\partial L^\delta_{\beta\gamma}}{\partial x^i} + L^\delta_{\alpha\eta} L^\eta_{\beta\gamma} \right) = 0, \quad \sigma^j_\alpha \frac{\partial \sigma^i_\beta}{\partial x^j} - \sigma^j_\beta \frac{\partial \sigma^i_\alpha}{\partial x^j} = \sigma^i_\gamma L^\gamma_{\alpha\beta}.$$

Locally, if $f \in C^\infty(M)$ then $d^E f = \frac{\partial f}{\partial x^i} \sigma^i_\alpha s^\alpha$, where $\{s^\alpha\}$ is the dual basis of $\{s_\alpha\}$ and if $\theta \in \Gamma(E^*)$, $\theta = \theta_\alpha s^\alpha$ then

$$d^E \theta = \left( \sigma^i_\alpha \frac{\partial \theta_\beta}{\partial x^i} - \frac{1}{2} \theta_\gamma L^\gamma_{\alpha\beta} \right) s^\alpha \wedge s^\beta.$$

Particularly, we get $d^E x^i = \sigma^i_\alpha s^\alpha$ and $d^E s^\alpha = -\frac{1}{2} L^\alpha_{\beta\gamma} s^\beta \wedge s^\gamma$.

### *2.1. The prolongation of a Lie algebroid over the vector bundle projection*

Let $(E, \pi, M)$ be a vector bundle. For the projection $\pi : E \to M$ we can construct the prolongation of $E$ (see [24,26,40]). The associated vector bundle is $(\mathcal{T}E, \pi_2, E)$ where

$$\mathcal{T}E = \cup_{w \in E} \mathcal{T}_w E, \quad \mathcal{T}_w E = \{(u_x, v_w) \in E_x \times T_w E \mid \sigma(u_x) = T_w \pi(v_w), \quad \pi(w) = x \in M\},$$

and the projection $\pi_2(u_x, v_w) = \pi_E(v_w) = w$, where $\pi_E : TE \to E$ is the tangent projection. We also have the canonical projection $\pi_1 : \mathcal{T}E \to E$ given by $\pi_1(u, v) = u$. The projection onto the second factor $\sigma^1 : \mathcal{T}E \to TE$, $\sigma^1(u, v) = v$ will be the anchor of a new Lie algebroid over the manifold $E$. An element of $\mathcal{T}E$ is said to be vertical if it is in the kernel of the projection $\pi_1$. We will denote $(V\mathcal{T}E, \pi_{2|_{V\mathcal{T}E}}, E)$ the vertical bundle of $(\mathcal{T}E, \pi_2, E)$ and $\sigma^1|_{V\mathcal{T}E} : V\mathcal{T}E \to VTE$ is an isomorphism. If $f \in C^\infty(M)$ we will denote by $f^c$ and $f^v$ the *complete and vertical lift* to $E$ of $f$ defined by

$$f^c(u) = \sigma(u)(f), \quad f^v(u) = f(\pi(u)), \quad u \in E.$$

For $s \in \Gamma(E)$ we can consider the *vertical lift* of $s$ given by $s^v(u) = s(\pi(u))^v_u$, for $u \in E$, where ${}^v_u : E_{\pi(u)} \to T_u(E_{\pi(u)})$ is the canonical isomorphism. There exists a unique vector field $s^c$ on $E$, the *complete lift* of $s$ satisfying the following conditions:

(i) $s^c$ is $\pi$-projectable on $\sigma(s)$,
(ii) $s^c(\hat{\alpha}) = \widehat{\mathcal{L}_s \alpha}$,

for all $\alpha \in \Gamma(E^*)$, where $\hat{\alpha}(u) = \alpha(\pi(u))(u)$, $u \in E$ (see [41,42]).

Considering the prolongation $\mathcal{T}E$ of $E$ [26], we may introduce the *vertical lift* $s^{\mathrm{v}}$ and the *complete lift* $s^{\mathrm{c}}$ of a section $s \in \Gamma(E)$ as the sections of $\mathcal{T}E \to E$ given by

$$s^{\mathrm{v}}(u) = (0, s^v(u)), \quad s^{\mathrm{c}}(u) = (s(\pi(u)), s^c(u)), \quad u \in E.$$

Other two canonical objects on $\mathcal{T}E$ are the *Euler section* $\mathbb{C}$ and the *tangent structure* (*vertical endomorphism*) $J$. The Euler section $\mathbb{C}$ is the section of $\mathcal{T}E \to E$ defined by $\mathbb{C}(u) = (0, u^v_u)$, $\forall u \in E$. The vertical endomorphism is the section of $(\mathcal{T}E) \oplus (\mathcal{T}E)^* \to E$ characterized by $J(s^{\mathrm{v}}) = 0$, $J(s^{\mathrm{c}}) = s^{\mathrm{v}}$, $s \in \Gamma(E)$ which satisfies

$$J^2 = 0, \; ImJ = \ker J = V\mathcal{T}E, \quad [\mathbb{C}, J]_{\mathcal{T}E} = -J.$$

A section $\mathcal{S}$ of $\mathcal{T}E \to E$ is called *semispray* (*second order differential equation—SODE*) on $E$ if $J(\mathcal{S}) = \mathbb{C}$. The local basis of $\Gamma(\mathcal{T}E)$ is given by $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$, where

$$\mathcal{X}_\alpha(u) = \left( s_\alpha(\pi(u)), \; \sigma^i_\alpha \frac{\partial}{\partial x^i}\bigg|_u \right), \quad \mathcal{V}_\alpha(u) = \left( 0, \; \frac{\partial}{\partial y^\alpha}\bigg|_u \right), \tag{3}$$

and $(\partial/\partial x^i, \partial/\partial y^\alpha)$ is the local basis on $TE$ (see [26]). The structure functions of $\mathcal{T}E$ are given by the following formulas

$$\sigma^1(\mathcal{X}_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad \sigma^1(\mathcal{V}_\alpha) = \frac{\partial}{\partial y^\alpha}, \tag{4}$$

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta]_{\mathcal{T}E} = L^\gamma_{\alpha\beta} \mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{V}_\beta]_{\mathcal{T}E} = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta]_{\mathcal{T}E} = 0. \tag{5}$$

The vertical lift of a section $\rho = \rho^\alpha s_\alpha$ is $\rho^{\mathrm{v}} = \rho^\alpha \mathcal{V}_\alpha$. The coordinate expression of Euler section is $\mathbb{C} = y^\alpha \mathcal{V}_\alpha$ and the local expression of $J$ is given by $J = \mathcal{X}^\alpha \otimes \mathcal{V}_\alpha$, where $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ denotes the corresponding dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$. The Nijenhuis tensor of the vertical endomorphism vanishes and it results that $J$ is integrable. The expression of the complete lift of a section $\rho = \rho^\alpha s_\alpha$ is

$$\rho^{c} = \rho^\alpha \mathcal{X}_\alpha + \left(\sigma^i_\varepsilon \frac{\partial \rho^\alpha}{\partial x^i} - L^\alpha_{\beta\varepsilon}\rho^\beta\right) y^\varepsilon \mathcal{V}_\alpha. \tag{6}$$

In particular $s^{v}_\alpha = \mathcal{V}_\alpha$, $s^{c}_\alpha = \mathcal{X}_\alpha - L^\beta_{\alpha\varepsilon} y^\varepsilon \mathcal{V}_\beta$. The local expression of the differential of a function $L$ on $\mathcal{T}E$ is $d^E L = \sigma^i_\alpha \frac{\partial L}{\partial x^i}\mathcal{X}^\alpha + \frac{\partial L}{\partial y^\alpha}\mathcal{V}^\alpha$ and we have $d^E x^i = \sigma^i_\alpha \mathcal{X}^\alpha$, $d^E y^\alpha = \mathcal{V}^\alpha$. The differential of sections of $(\mathcal{T}E)^*$ is determined by

$$d^E \mathcal{X}^\alpha = -\frac{1}{2} L^\alpha_{\beta\gamma} \mathcal{X}^\beta \wedge \mathcal{X}^\gamma, \quad d^E \mathcal{V}^\alpha = 0.$$

In local coordinates a semispray has the expression

$$\mathcal{S}(x, y) = y^\alpha \mathcal{X}_\alpha + \mathcal{S}^\alpha(x, y)\mathcal{V}_\alpha, \tag{7}$$

and the following equality holds

$$J[\mathcal{S}, JX]_{\mathcal{T}E} = -JX, \ X \in \Gamma(E). \tag{8}$$

The integral curves of $\sigma^1(\mathcal{S})$ satisfy the differential equations

$$\frac{dx^i}{dt} = \sigma^i_\alpha(x) y^\alpha, \quad \frac{dy^\alpha}{dt} = \mathcal{S}^\alpha(x, y).$$

If we have the relation $[\mathbb{C}, \mathcal{S}]_{\mathcal{T}E} = \mathcal{S}$ then $\mathcal{S}$ is called spray and the functions $\mathcal{S}^\alpha$ are homogeneous functions of degree 2 in $y^\alpha$. Let us consider a regular Lagrangian $L$ on $E$, that is the matrix

$$g_{\alpha\beta} = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta},$$

has constant rank $m$. We have the Cartan 1-section $\theta_L = \frac{\partial L}{\partial y^\alpha}\mathcal{X}^\alpha$ and the Cartan 2-section $\omega_L = d^E\theta_L$, which is a symplectic structure induced by $L$ given by [26]

$$\omega_L = g_{\alpha\beta}\mathcal{V}^\beta \wedge \mathcal{X}^\alpha + \frac{1}{2}\left(\sigma^i_\alpha \frac{\partial^2 L}{\partial x^i \partial y^\beta} - \sigma^i_\beta \frac{\partial^2 L}{\partial x^i \partial y^\alpha} - \frac{\partial L}{\partial y^\varepsilon} L^\varepsilon_{\alpha\beta}\right)\mathcal{X}^\alpha \wedge \mathcal{X}^\beta.$$

Considering the energy function $E_L = \mathbb{C}(L) - L$, with local expression

$$E_L = y^\alpha \frac{\partial L}{\partial y^\alpha} - L,$$

then the symplectic equation

$$i_S \omega_L = -d^E E_L,$$

determines the components of the canonical semispray [26]

$$S^\varepsilon = g^{\varepsilon\beta}\left(\sigma^i_\beta \frac{\partial L}{\partial x^i} - \sigma^i_\alpha \frac{\partial^2 L}{\partial x^i \partial y^\beta} y^\alpha - L^\gamma_{\beta\alpha} y^\alpha \frac{\partial L}{\partial y^\gamma}\right), \tag{9}$$

where $g_{\alpha\beta} g^{\beta\gamma} = \delta^\gamma_\alpha$, which depends only on the regular Lagrangian and the structure function of the Lie algebroid.

### *2.2. Nonlinear connections on Lie algebroids*

A nonlinear connection is an important tool in the geometry of systems of second order differential equations. The system of SODE can by represented using the notion of semispray, which together with a nonlinear connection induce two important concepts (the dynamical covariant derivative and Jacobi endomorphism) which are used in order to find the invariant equations of the symmetries of SODE.

**Definition 1.** A nonlinear connection on $\mathcal{T}E$ is an almost product structure $\mathcal{N}$ on $\pi_2 : \mathcal{T}E \to E$ (i.e. a bundle morphism $\mathcal{N} : \mathcal{T}E \to \mathcal{T}E$, such that $\mathcal{N}^2 = Id$) smooth on $\mathcal{T}E \setminus \{0\}$ such that $V\mathcal{T}E = \ker(Id + \mathcal{N})$.

If $\mathcal{N}$ is a connection on $\mathcal{T}E$ then $H\mathcal{T}E = \ker(Id - \mathcal{N})$ is the horizontal subbundle associated to $\mathcal{N}$ and $\mathcal{T}E = V\mathcal{T}E \oplus H\mathcal{T}E$. Each $\rho \in \Gamma(\mathcal{T}E)$ can be written as $\rho = \rho^{h} + \rho^{v}$, where $\rho^{h}$, $\rho^{v}$ are sections in the horizontal and respective vertical subbundles. If $\rho^{h} = 0$, then $\rho$ is called *vertical* and if $\rho^{v} = 0$, then $\rho$ is called *horizontal*. A connection $\mathcal{N}$ on $\mathcal{T}E$ induces two projectors

$\mathrm{h}, \mathrm{v} : \mathcal{T}E \to \mathcal{T}E$ such that $\mathrm{h}(\rho) = \rho^{\mathrm{h}}$ and $\mathrm{v}(\rho) = \rho^{\mathrm{v}}$ for every $\rho \in \Gamma(\mathcal{T}E)$. We have

$$\mathrm{h} = \frac{1}{2}(Id + \mathcal{N}), \quad \mathrm{v} = \frac{1}{2}(Id - \mathcal{N}), \quad \ker \mathrm{h} = Im\mathrm{v} = V\mathcal{T}E, \quad Im\mathrm{h} = \ker \mathrm{v} = H\mathcal{T}E.$$

$$\mathrm{h}^2 = \mathrm{h}, \quad \mathrm{v}^2 = \mathrm{v}, \quad \mathrm{hv} = \mathrm{vh} = 0, \quad \mathrm{h} + \mathrm{v} = Id, \quad \mathrm{h} - \mathrm{v} = \mathcal{N}.$$

$$J\mathrm{h} = J, \quad \mathrm{h}J = 0, \quad J\mathrm{v} = 0, \quad \mathrm{v}J = J.$$

Locally, a connection can be expressed as $\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - 2\mathcal{N}_\alpha^\beta \mathcal{V}_\beta$, $\mathcal{N}(\mathcal{V}_\beta) = -\mathcal{V}_\beta$, where $\mathcal{N}_\alpha^\beta = \mathcal{N}_\alpha^\beta(x, y)$ are the local coefficients of $\mathcal{N}$. The sections

$$\delta_\alpha = \mathrm{h}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - \mathcal{N}_\alpha^\beta \mathcal{V}_\beta,$$

generate a basis of $H\mathcal{T}E$. The frame $\{\delta_\alpha, \mathcal{V}_\alpha\}$ is a local basis of $\mathcal{T}E$ called Berwald basis. The dual adapted basis is $\{\mathcal{X}^\alpha, \delta\mathcal{V}^\alpha\}$ where $\delta\mathcal{V}^\alpha = \mathcal{V}^\alpha - \mathcal{N}_\beta^\alpha \mathcal{X}^\beta$. The Lie brackets of the adapted basis $\{\delta_\alpha, \mathcal{V}_\alpha\}$ are [30]

$$[\delta_\alpha, \delta_\beta]_{\mathcal{T}E} = L_{\alpha\beta}^\gamma \delta_\gamma + \mathcal{R}_{\alpha\beta}^\gamma \mathcal{V}_\gamma, \quad [\delta_\alpha, \mathcal{V}_\beta]_{\mathcal{T}E} = \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\beta} \mathcal{V}_\gamma, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta]_{\mathcal{T}E} = 0, \tag{10}$$

$$\mathcal{R}_{\alpha\beta}^\gamma = \delta_\beta(\mathcal{N}_\alpha^\gamma) - \delta_\alpha(\mathcal{N}_\beta^\gamma) + L_{\alpha\beta}^\varepsilon \mathcal{N}_\varepsilon^\gamma. \tag{11}$$

**Definition 2.** The curvature of the nonlinear connection $\mathcal{N}$ on $\mathcal{T}E$ is $\Omega = -\mathbf{N}_{\mathrm{h}}$ where h is the horizontal projector and $\mathbf{N}_{\mathrm{h}}$ is the Nijenhuis tensor of h.

In local coordinates we have

$$\Omega = -\frac{1}{2}\mathcal{R}_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma,$$

where $\mathcal{R}_{\alpha\beta}^\gamma$ are given by (11) and represent the local coordinate functions of the curvature tensor. The curvature of the nonlinear connection is an obstruction to the integrability of $H\mathcal{T}E$, understanding that a vanishing curvature entails that horizontal sections are closed under the Lie algebroid bracket of $\mathcal{T}E$. The horizontal distribution $H\mathcal{T}E$ is integrable if and only if the curvature $\Omega$ of the nonlinear connection vanishes. Also, from the Jacobi identity we obtain

$$[\mathrm{h}, \Omega]_{\mathcal{T}E} = 0.$$

Let us consider a semispray $\mathcal{S}$ and an arbitrary nonlinear connection $\mathcal{N}$ with induced (h, v) projectors. Then we set (see also [32]).

**Definition 3.** The vertically valued $(1, 1)$-type tensor field on Lie algebroid $\mathcal{T}E$ given by

$$\Phi = -\mathrm{v} \circ \mathcal{L}_{\mathcal{S}}\mathrm{v}, \tag{12}$$

will be called the Jacobi endomorphism.

The Jacobi endomorphism $\Phi$ has been used in the study of Jacobi equations for SODE on Lie algebroids in [37] and to express one of the Helmholtz conditions of the inverse problem of the calculus of variations on Lie algebroids [32] (see also [43]). We obtain

$$\Phi = -\mathrm{v} \circ \mathcal{L}_{\mathcal{S}}\mathrm{v} = \mathrm{v} \circ \mathcal{L}_{\mathcal{S}}\mathrm{h} = \mathrm{v} \circ (\mathcal{L}_{\mathcal{S}} \circ \mathrm{h} - \mathrm{h} \circ \mathcal{L}_{\mathcal{S}}) = \mathrm{v} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{h},$$

and in local coordinates the action of Lie derivative on the Berwald basis is given by

$$\mathcal{L}_{\mathcal{S}}\delta_\beta = \left(\mathcal{N}_\beta^\alpha - L_{\beta\varepsilon}^\alpha y^\varepsilon\right)\delta_\alpha + \mathcal{R}_\beta^\gamma \mathcal{V}_\gamma, \quad \mathcal{L}_{\mathcal{S}}\mathcal{V}_\beta = -\delta_\beta - \left(\mathcal{N}_\beta^\alpha + \frac{\partial \mathcal{S}^\alpha}{\partial y^\beta}\right)\mathcal{V}_\alpha. \tag{13}$$

The Jacobi endomorphism has the local form

$$\Phi = \mathcal{R}_\beta^\alpha \mathcal{V}_\alpha \otimes \mathcal{X}^\beta, \quad \mathcal{R}_\beta^\gamma = -\sigma_\beta^i \frac{\partial \mathcal{S}^\gamma}{\partial x^i} - \mathcal{S}(\mathcal{N}_\beta^\gamma) + \mathcal{N}_\beta^\alpha \mathcal{N}_\alpha^\gamma + \mathcal{N}_\beta^\alpha \frac{\partial \mathcal{S}^\gamma}{\partial y^\alpha} + \mathcal{N}_\varepsilon^\gamma L_{\alpha\beta}^\varepsilon y^\alpha. \tag{14}$$

**Proposition 1.** *The following formula holds*

$$\Phi = i_{\mathcal{S}}\Omega + \mathrm{v} \circ \mathcal{L}_{\mathrm{v}\mathcal{S}}\mathrm{h}. \tag{15}$$

**Proof.** Indeed, $\Phi(\rho) = \mathrm{v} \circ \mathcal{L}_{\mathcal{S}}\mathrm{h}\rho = \mathrm{v} \circ \mathcal{L}_{\mathrm{h}\mathcal{S}}\mathrm{h}\rho + \mathrm{v} \circ \mathcal{L}_{\mathrm{v}\mathcal{S}}\mathrm{h}\rho$ and $\Omega(\mathcal{S}, \rho) = \mathrm{v}[\mathrm{h}\mathcal{S}, \mathrm{h}\rho]_{\mathcal{T}E} = \mathrm{v} \circ \mathcal{L}_{\mathrm{h}\mathcal{S}}\mathrm{h}\rho$, which yields $\Phi(\rho) = \Omega(\mathcal{S}, \rho) + \mathrm{v} \circ \mathcal{L}_{\mathrm{v}\mathcal{S}}\mathrm{h}\rho$. □

For a given semispray $\mathcal{S}$ on $\mathcal{T}E$ the Lie derivative $\mathcal{L}_{\mathcal{S}}$ defines a tensor derivation on $\mathcal{T}E$, but does not preserve some of the geometric structures as tangent structure and nonlinear connection. Next, using a nonlinear connection, we introduce a tensor derivation on $\mathcal{T}E$, called the dynamical covariant derivative, that preserves some other geometric structures.

## 3. Dynamical covariant derivative on Lie algebroids

In the following we will introduce the notion of dynamical covariant derivative on Lie algebroids as a tensor derivation and study its properties. We will use the Jacobi endomorphism and the dynamical covariant derivative in the study of symmetries for SODE on Lie algebroids.

**Definition 4** ([32]). A map $\nabla : \mathfrak{T}(\mathcal{T}E \setminus \{0\}) \to \mathfrak{T}(\mathcal{T}E \setminus \{0\})$ is said to be a tensor derivation on $\mathcal{T}E \setminus \{0\}$ if the following conditions are satisfied:

(i) $\nabla$ is $\mathbb{R}$-linear
(ii) $\nabla$ is type preserving, i.e. $\nabla(\mathfrak{T}_s^r(\mathcal{T}E \setminus \{0\})) \subset \mathfrak{T}_s^r(\mathcal{T}E \setminus \{0\})$, for each $(r, s) \in \mathbb{N} \times \mathbb{N}$.
(iii) $\nabla$ obeys the Leibniz rule $\nabla(P \otimes S) = \nabla P \otimes S + P \otimes \nabla S$, for any tensors $P, S$ on $\mathcal{T}E \setminus \{0\}$.
(iv) $\nabla$ commutes with any contractions, where $\mathfrak{T}_\bullet^\bullet(\mathcal{T}E \setminus \{0\})$ is the space of tensors on $\mathcal{T}E \setminus \{0\}$.

For a semispray $\mathcal{S}$ and an arbitrary nonlinear connection $\mathcal{N}$ we consider the $\mathbb{R}$-linear map $\nabla : \Gamma(\mathcal{T}E \setminus \{0\}) \to \Gamma(\mathcal{T}E \setminus \{0\})$ given by

$$\nabla = \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{h} + \mathrm{v} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v}, \tag{16}$$

which will be called the dynamical covariant derivative induced by the semispray $\mathcal{S}$ and the nonlinear connection $\mathcal{N}$. By setting $\nabla f = \mathcal{S}(f)$, for $f \in C^\infty(E \setminus \{0\})$ using the Leibniz rule and the requirement that $\nabla$ commutes with any contraction, we can extend the action of $\nabla$ to arbitrary section on $\mathcal{T}E \setminus \{0\}$. For a section on $\mathcal{T}E \setminus \{0\}$ the dynamical covariant derivative is given by $(\nabla\varphi)(\rho) = S(\varphi)(\rho) - \varphi(\nabla\rho)$. For a $(1, 1)$-type tensor field $T$ on $\mathcal{T}E \setminus \{0\}$ the dynamical covariant derivative has the form

$$\nabla T = \nabla \circ T - T \circ \nabla, \tag{17}$$

and by direct computation using (17) we obtain

$$\nabla \mathrm{h} = \nabla \mathrm{v} = 0,$$

which means that $\nabla$ preserves the horizontal and vertical sections. Also, we get

$$\nabla \mathcal{V}_\beta = \mathrm{v}[\mathcal{S}, \mathcal{V}_\beta]_{\mathcal{T}E} = -\left(\mathcal{N}_\beta^\alpha + \frac{\partial \mathcal{S}^\alpha}{\partial y^\beta}\right)\mathcal{V}_\alpha, \quad \nabla \delta\mathcal{V}^\beta = \left(\mathcal{N}_\alpha^\beta + \frac{\partial \mathcal{S}^\beta}{\partial y^\alpha}\right)\delta\mathcal{V}^\alpha,$$

$$\nabla \delta_\beta = \mathrm{h}[\mathcal{S}, \delta_\beta]_{\mathcal{T}E} = \left(\mathcal{N}_\beta^\alpha - L_{\beta\varepsilon}^\alpha y^\varepsilon\right)\delta_\alpha, \quad \nabla \mathcal{X}^\beta = -\left(\mathcal{N}_\alpha^\beta - L_{\alpha\varepsilon}^\beta y^\varepsilon\right)\mathcal{X}^\alpha.$$

The action of the dynamical covariant derivative on the horizontal section $X = hX$ is given by following relations

$$\nabla X = \nabla\left(X^\alpha \delta_\alpha\right) = \nabla X^\alpha \delta_\alpha, \quad \nabla X^\alpha = \mathcal{S}(X^\alpha) + \left(\mathcal{N}_\beta^\alpha + y^\varepsilon L_{\varepsilon\beta}^\alpha\right)X^\beta. \tag{18}$$

**Proposition 2.** *The following results hold*

$$\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ J = -\mathrm{h}, \quad J \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} = -\mathrm{v}, \tag{19}$$

$$\nabla J = \mathcal{L}_{\mathcal{S}} J + \mathcal{N}, \quad \nabla J = -\left(\frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} - y^\varepsilon L_{\alpha\varepsilon}^\beta + 2\mathcal{N}_\alpha^\beta\right)\mathcal{V}_\beta \otimes \mathcal{X}^\alpha. \tag{20}$$

**Proof.** From (8) we get

$$J[\mathcal{S}, JX]_{\mathcal{T}E} = -JX \Rightarrow J\left([\mathcal{S}, JX]_{\mathcal{T}E} + X\right) = 0 \Rightarrow [\mathcal{S}, JX]_{\mathcal{T}E} + X \in V\mathcal{T}E,$$

$$\mathrm{h}\left([\mathcal{S}, JX]_{\mathcal{T}E} + X\right) = 0 \Rightarrow \mathrm{h}[\mathcal{S}, JX]_{\mathcal{T}E} = -\mathrm{h}X \Leftrightarrow \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ J = -\mathrm{h}.$$

Also, in $J[\mathcal{S}, JX]_{\mathcal{T}E} + JX = 0$ considering $JX = \mathrm{v}Z$ it results $J[\mathcal{S}, \mathrm{v}Z]_{\mathcal{T}E} = -\mathrm{v}Z \Leftrightarrow J \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} = -\mathrm{v}$. Next

$$\begin{aligned}\nabla \circ J &= \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{h} \circ J + \mathrm{v} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} \circ J = \mathrm{v} \circ \mathcal{L}_{\mathcal{S}} \circ J \\ &= (Id - \mathrm{h}) \circ \mathcal{L}_{\mathcal{S}} \circ J = \mathcal{L}_S \circ J - \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ J = \mathcal{L}_S \circ J + \mathrm{h}.\end{aligned}$$

But, on the other hand

$$J \circ \nabla = J \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{h} = J \circ \mathcal{L}_{\mathcal{S}} \circ (Id - \mathrm{v}) = J \circ \mathcal{L}_{\mathcal{S}} - J \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} = J \circ \mathcal{L}_{\mathcal{S}} + v.$$

and we obtain

$$\nabla \circ J - J \circ \nabla = \mathcal{L}_S \circ J + \mathrm{h} - J \circ \mathcal{L}_{\mathcal{S}} - \mathrm{v} \Rightarrow \nabla J = \mathcal{L}_{\mathcal{S}} J + \mathrm{h} - \mathrm{v} = \mathcal{L}_{\mathcal{S}} J + \mathcal{N}.$$

For the last relation, we have

$$\begin{aligned}\nabla J &= \nabla\left(\mathcal{X}^\beta\otimes\mathcal{V}_\beta\right)=\nabla\mathcal{X}^\beta\otimes\mathcal{V}_\beta+\mathcal{X}^\beta\otimes\nabla\mathcal{V}_\beta\\ &= -\left(\mathcal{N}_\alpha^\beta-L_{\alpha\varepsilon}^\beta y^\varepsilon\right)\mathcal{X}^\alpha\otimes\mathcal{V}_\beta+\mathcal{X}^\beta\otimes\left(-\mathcal{N}_\beta^\alpha-\frac{\partial\mathcal{S}^\alpha}{\partial y^\beta}\right)\mathcal{V}_\alpha\\ &= -\mathcal{N}_\alpha^\beta\mathcal{X}^\alpha\otimes\mathcal{V}_\beta+L_{\alpha\varepsilon}^\beta y^\varepsilon\mathcal{X}^\alpha\otimes\mathcal{V}_\beta-\mathcal{N}_\beta^\alpha\mathcal{X}^\beta\otimes\mathcal{V}_\alpha-\frac{\partial\mathcal{S}^\alpha}{\partial y^\beta}\mathcal{X}^\beta\otimes\mathcal{V}_\alpha\\ &= \left(L_{\alpha\varepsilon}^\beta y^\varepsilon-\frac{\partial\mathcal{S}^\beta}{\partial y^\alpha}-2\mathcal{N}_\alpha^\beta\right)\mathcal{X}^\alpha\otimes\mathcal{V}_\beta. \quad\Box\end{aligned}$$

The above proposition leads to the following result,

**Theorem 1.** *For a semispray $\mathcal{S}$, an arbitrary nonlinear connection $\mathcal{N}$ and $\nabla$ the dynamical covariant derivative induced by $\mathcal{S}$ and $\mathcal{N}$, the following conditions are equivalent,*

(i) $\nabla J=0$,
(ii) $\mathcal{L}_{\mathcal{S}}J+\mathcal{N}=0$,
(iii) $\mathcal{N}_\alpha^\beta=\frac{1}{2}\left(-\frac{\partial\mathcal{S}^\beta}{\partial y^\alpha}+y^\varepsilon L_{\alpha\varepsilon}^\beta\right)$.

**Proof.** The proof follows from the relations (20). $\Box$

This theorem shows that the compatibility condition $\nabla J=0$ of the dynamical covariant derivative with the tangent structure determines the nonlinear connection $\mathcal{N}=-\mathcal{L}_{\mathcal{S}}J$. For the particular case of tangent bundle we obtain the results from [4]. In the following we deal with this nonlinear connection induced by semispray.

*3.1. The canonical nonlinear connection induced by a semispray*

A semispray $\mathcal{S}$, together with the condition $\nabla J=0$, determines the canonical nonlinear connection $\mathcal{N}=-\mathcal{L}_{\mathcal{S}}J$ with local coefficients

$$\mathcal{N}_\alpha^\beta=\frac{1}{2}\left(-\frac{\partial\mathcal{S}^\beta}{\partial y^\alpha}+y^\varepsilon L_{\alpha\varepsilon}^\beta\right).$$

In this case the following equations hold

$$[\mathcal{S},\mathcal{V}_\beta]_{\mathcal{T}E}=-\delta_\beta+\left(\mathcal{N}_\beta^\alpha-L_{\beta\varepsilon}^\alpha y^\varepsilon\right)\mathcal{V}_\alpha,$$

$$[\mathcal{S},\delta_\beta]_{\mathcal{T}E}=\left(\mathcal{N}_\beta^\alpha-L_{\beta\varepsilon}^\alpha y^\varepsilon\right)\delta_\alpha+\mathcal{R}_\beta^\alpha\mathcal{V}_\alpha,$$

where

$$\mathcal{R}_\beta^\alpha=-\sigma_\beta^i\frac{\partial\mathcal{S}^\alpha}{\partial x^i}-\mathcal{S}(\mathcal{N}_\beta^\alpha)-\mathcal{N}_\gamma^\alpha\mathcal{N}_\beta^\gamma+(L_{\varepsilon\beta}^\gamma\mathcal{N}_\gamma^\alpha+L_{\gamma\varepsilon}^\alpha\mathcal{N}_\beta^\gamma)y^\varepsilon \tag{21}$$

are the local coefficients of the Jacobi endomorphism.

**Proposition 3.** *If $\mathcal{S}$ is a spray, then the Jacobi endomorphism is the contraction with $\mathcal{S}$ of curvature of the nonlinear connection*

$$\Phi=i_{\mathcal{S}}\Omega.$$

**Proof.** If $\mathcal{S}$ is a spray, then the coefficients $\mathcal{S}^\alpha$ are 2-homogeneous with respect to the variables $y^\beta$ and it results

$$2\mathcal{S}^\alpha=\frac{\partial\mathcal{S}^\alpha}{\partial y^\beta}y^\beta=-2\mathcal{N}_\beta^\alpha y^\beta+L_{\beta\gamma}^\alpha y^\beta y^\gamma=-2\mathcal{N}_\beta^\alpha y^\beta.$$

$$\mathcal{S}=\mathrm{h}\mathcal{S}=y^\alpha\delta_\alpha,\quad \mathrm{v}\mathcal{S}=0,\quad \mathcal{N}_\beta^\alpha=\frac{\partial\mathcal{N}_\varepsilon^\alpha}{\partial y^\beta}y^\varepsilon+L_{\beta\varepsilon}^\alpha y^\varepsilon,$$

which together with (15) yields $\Phi=i_{\mathcal{S}}\Omega$. Locally,we get $\mathcal{R}_\beta^\alpha=\mathcal{R}_{\varepsilon\beta}^\alpha y^\varepsilon$ and represents the local relation between the Jacobi endomorphism and the curvature of the nonlinear connection. Also, we have $\Phi(\mathcal{S})=0$. $\Box$

Next, we introduce the almost complex structure in order to find the decomposition formula for the dynamical covariant derivative.

**Definition 5.** The almost complex structure is given by the formula

$$\mathbb{F}=\mathrm{h}\circ\mathcal{L}_{\mathcal{S}}\mathrm{h}-J.$$

We have to show that $\mathbb{F}^2 = -Id$. Indeed, from the relation $\mathcal{L}_{\mathcal{S}}\mathrm{h} = \mathcal{L}_{\mathcal{S}} \circ \mathrm{h} - \mathrm{h} \circ \mathcal{L}_{\mathcal{S}}$ we obtain $\mathbb{F} = \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{h} - \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} - J = \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ (\mathrm{h} - Id) - J = -\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} - J$ and $\mathbb{F}^2 = (-\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} - J) \circ (-\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} - J) = \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} \circ \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} + \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} \circ J + J \circ \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} + J^2 = \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ J + J \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} = -\mathrm{h} - \mathrm{v} = -Id$.

**Proposition 4.** *The following results hold*

$$\mathbb{F} \circ J = \mathrm{h}, \quad J \circ \mathbb{F} = \mathrm{v}, \quad \mathrm{v} \circ \mathbb{F} = \mathbb{F} \circ \mathrm{h} = -J,$$
$$\mathrm{h} \circ \mathbb{F} = \mathbb{F} \circ \mathrm{v} = \mathbb{F} + J, \quad \mathcal{N} \circ \mathbb{F} = \mathbb{F} + 2J, \quad \Phi = \mathcal{L}_{\mathcal{S}}\mathrm{h} - \mathbb{F} - J.$$

**Proof.** Using the relations (19) we obtain $\mathbb{F} \circ J = (-\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} - J) \circ J = -\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} \circ J - J^2 = -\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ J = \mathrm{h}$, $J \circ \mathbb{F} = -J \circ (\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} + J) = -J \circ \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} - J^2 = -J \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} = \mathrm{v}$, $\mathrm{v} \circ \mathbb{F} = \mathrm{v} \circ (\mathrm{h} \circ \mathcal{L}_{\mathcal{S}}\mathrm{h} - J) = -\mathrm{v} \circ J = -J$, $\mathbb{F} \circ \mathrm{h} = (-\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} - J) \circ \mathrm{h} = -J \circ \mathrm{h} = -J$, $\mathrm{h} \circ \mathbb{F} = \mathrm{h} \circ (\mathrm{h} \circ \mathcal{L}_{\mathcal{S}}\mathrm{h} - J) = \mathrm{h} \circ \mathcal{L}_{\mathcal{S}}\mathrm{h} = \mathbb{F} + J$, $\mathbb{F} \circ \mathrm{v} = (-\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} - J) \circ \mathrm{v} = -\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} = \mathbb{F} + J$. In the same way, the other relations can be proved. □

In local coordinates we have

$$\mathbb{F} = -\mathcal{V}_\alpha \otimes \mathcal{X}^\alpha + \delta_\alpha \otimes \delta\mathcal{V}^\alpha.$$

For a semispray $\mathcal{S}$ and the associated nonlinear connection we consider the $\mathbb{R}$-linear map $\nabla_0 : \Gamma(\mathcal{T}E \setminus \{0\}) \to \Gamma(\mathcal{T}E \setminus \{0\})$ given by

$$\nabla_0\rho = \mathrm{h}[\mathcal{S}, \mathrm{h}\rho]_{\mathcal{T}E} + \mathrm{v}[\mathcal{S}, \mathrm{v}\rho]_{\mathcal{T}E}, \quad \forall \rho \in \Gamma(\mathcal{T}E \setminus \{0\}).$$

It results that

$$\nabla_0(f\rho) = \mathcal{S}(f)\rho + f\nabla_0\rho, \quad \forall f \in C^\infty(E),\ \rho \in \Gamma(\mathcal{T}E \setminus \{0\}).$$

Any tensor derivation on $\mathcal{T}E \setminus \{0\}$ is completely determined by its actions on smooth functions and sections on $\mathcal{T}E \setminus \{0\}$ (see [44] generalized Willmore's theorem). Therefore, there exists a unique tensor derivation $\nabla$ on $\mathcal{T}E \setminus \{0\}$ such that

$$\nabla|_{C^\infty(E)} = \mathcal{S}, \quad \nabla|_{\Gamma(\mathcal{T}E\setminus\{0\})} = \nabla_0.$$

We will call the tensor derivation $\nabla$, the *dynamical covariant derivative* induced by the semispray $\mathcal{S}$ (see [3] for the tangent bundle case).

**Proposition 5.** *The dynamical covariant derivative has the following decomposition*

$$\nabla = \mathcal{L}_{\mathcal{S}} + \mathbb{F} + J - \Phi. \tag{22}$$

**Proof.** Using the formula (16) and the expressions of $\mathbb{F}$ and $\Phi$ we obtain

$$\begin{aligned}\nabla &= \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{h} + \mathrm{v} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v}\\ &= \mathrm{h} \circ (\mathcal{L}_{\mathcal{S}}\mathrm{h} + \mathrm{h} \circ \mathcal{L}_{\mathcal{S}}) + \mathrm{v} \circ (\mathcal{L}_{\mathcal{S}}\mathrm{v} + \mathrm{v} \circ \mathcal{L}_{\mathcal{S}})\\ &= \mathrm{h} \circ \mathcal{L}_{\mathcal{S}}\mathrm{h} + \mathrm{v} \circ \mathcal{L}_{\mathcal{S}}\mathrm{v} + (\mathrm{h} + \mathrm{v}) \circ \mathcal{L}_{\mathcal{S}} = \mathcal{L}_{\mathcal{S}} + \mathrm{h} \circ \mathcal{L}_{\mathcal{S}}\mathrm{h} + \mathrm{v} \circ \mathcal{L}_{\mathcal{S}}\mathrm{v}\\ &= \mathcal{L}_{\mathcal{S}} + \mathbb{F} + J - \Phi. \quad \square\end{aligned}$$

In this case the dynamical covariant derivative is characterized by the following formulas

$$\nabla\mathcal{V}_\beta = \mathrm{v}[\mathcal{S}, \mathcal{V}_\beta]_{\mathcal{T}E} = \left(\mathcal{N}^\alpha_\beta - L^\alpha_{\beta\varepsilon}y^\varepsilon\right)\mathcal{V}_\alpha = -\frac{1}{2}\left(\frac{\partial\mathcal{S}^\alpha}{\partial y^\beta} + L^\alpha_{\beta\varepsilon}y^\varepsilon\right)\mathcal{V}_\alpha,$$

$$\nabla\delta_\beta = \mathrm{h}[\mathcal{S}, \delta_\beta]_{\mathcal{T}E} = \left(\mathcal{N}^\alpha_\beta - L^\alpha_{\beta\varepsilon}y^\varepsilon\right)\delta_\alpha = -\frac{1}{2}\left(\frac{\partial\mathcal{S}^\alpha}{\partial y^\beta} + L^\alpha_{\beta\varepsilon}y^\varepsilon\right)\delta_\alpha.$$

The next result shows that $\nabla$ acts identically on both vertical and horizontal distributions, that is enough to find the action of $\nabla$ on either one of the two distributions.

**Proposition 6.** *The dynamical covariant derivative induced by the semispray $\mathcal{S}$ is compatible with $J$ and $\mathbb{F}$, that is*

$$\nabla J = 0,\ \nabla\mathbb{F} = 0.$$

**Proof.** $\nabla J = 0$ follows from (20). Using the formula $\mathbb{F} = -\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} - J$ and $\nabla\mathbb{F} = \nabla \circ \mathbb{F} - \mathbb{F} \circ \nabla$ we obtain

$$\begin{aligned}\nabla\mathbb{F} &= (\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{h} + \mathrm{v} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v}) \circ (-\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v}) - (-\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v}) \circ (\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{h} + \mathrm{v} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v})\\ &= -\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} + \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v}\\ &= \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ (\mathrm{v} - \mathrm{h}) \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} = \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathcal{L}_{\mathcal{S}}J \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v}\\ &= \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ (\mathcal{L}_{\mathcal{S}} \circ J - J \circ \mathcal{L}_{\mathcal{S}}) \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v}\\ &= \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathcal{L}_{\mathcal{S}} \circ (J \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v}) - (\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ J) \circ \mathcal{L}_{\mathcal{S}} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v}\\ &= -\mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} + \mathrm{h} \circ \mathcal{L}_{\mathcal{S}} \circ \mathcal{L}_{\mathcal{S}} \circ \mathrm{v} = 0. \quad \square\end{aligned}$$

The next proposition proves that in the case of spray $\nabla$ has more properties.

**Proposition 7.** *If the dynamical covariant derivative is induced by a spray $\mathcal{S}$ then*

$$\nabla\mathcal{S}=0,\ \nabla\mathbb{C}=0.$$

**Proof.** Indeed, if $\mathcal{S}$ is a spray then we have $\mathcal{S}=\mathrm{h}\mathcal{S}$ and $\mathrm{v}\mathcal{S}=0$ and it results $\nabla\mathcal{S}=\mathrm{h}\circ\mathcal{L}_{\mathcal{S}}\circ\mathrm{h}\mathcal{S}+\mathrm{v}\circ\mathcal{L}_{\mathcal{S}}\circ\mathrm{v}\mathcal{S}=\mathrm{h}\circ\mathcal{L}_{\mathcal{S}}\circ\mathcal{S}=0$. Also $\nabla\mathbb{C}=0$ follows from $\mathrm{h}\mathbb{C}=0$, $\mathrm{v}\mathbb{C}=\mathbb{C}$ and $[\mathbb{C},\mathcal{S}]_{\mathcal{T}E}=\mathcal{S}$. □

Next, we introduce the Berwald linear connection induced by a nonlinear connection and prove that in the case of homogeneous second order differential equations it coincides with the dynamical covariant derivative. The Berwald linear connection is given by

$$\mathcal{D}:\Gamma(\mathcal{T}E\setminus\{0\})\times\Gamma(\mathcal{T}E\setminus\{0\})\to\Gamma(\mathcal{T}E\setminus\{0\}),$$

$$\mathcal{D}_XY=\mathrm{v}[\mathrm{h}X,\mathrm{v}Y]_{\mathcal{T}E}+\mathrm{h}[\mathrm{v}X,\mathrm{h}Y]_{\mathcal{T}E}+J[\mathrm{v}X,(\mathbb{F}+J)Y]_{\mathcal{T}E}+(\mathbb{F}+J)[\mathrm{h}X,JY]_{\mathcal{T}E}.$$

**Proposition 8.** *The Berwald linear connection has the following properties*

$$\mathcal{D}\mathrm{h}=0,\quad \mathcal{D}\mathrm{v}=0,\quad \mathcal{D}J=0,\quad \mathcal{D}\mathbb{F}=0.$$

**Proof.** Using the properties of the vertical and horizontal projectors we obtain $\mathcal{D}_X\mathrm{v}Y=\mathrm{v}[\mathrm{h}X,\mathrm{v}Y]_{\mathcal{T}E}+J[\mathrm{v}X,(\mathbb{F}+J)Y]_{\mathcal{T}E}$ and $\mathrm{v}(\mathcal{D}_XY)=\mathrm{v}[\mathrm{h}X,\mathrm{v}Y]_{\mathcal{T}E}+J[\mathrm{v}X,(\mathbb{F}+J)Y]_{\mathcal{T}E}$ which yields $\mathcal{D}\mathrm{v}=0$. Also, $\mathcal{D}_X\mathrm{h}Y=\mathrm{h}[\mathrm{v}X,\mathrm{h}Y]_{\mathcal{T}E}+(\mathbb{F}+J)[\mathrm{h}X,JY]_{\mathcal{T}E}=\mathrm{h}(\mathcal{D}_XY)$ and it results $\mathcal{D}\mathrm{h}=0$. Moreover, $\mathcal{D}_XJY=\mathrm{v}[\mathrm{h}X,JY]_{\mathcal{T}E}+J[\mathrm{v}X,\mathrm{h}Y]_{\mathcal{T}E}$ and $J(\mathcal{D}_XY)=J[\mathrm{v}X,\mathrm{h}Y]_{\mathcal{T}E}+\mathrm{v}[\mathrm{h}X,JY]_{\mathcal{T}E}$ and we obtain $\mathcal{D}J=0$. From $\mathcal{D}_X\mathbb{F}Y=\mathrm{v}[\mathrm{h}X,-JY]_{\mathcal{T}E}+\mathrm{h}[\mathrm{v}X,(\mathbb{F}+J)Y]_{\mathcal{T}E}+J[\mathrm{v}X,-\mathrm{h}Y]_{\mathcal{T}E}+(\mathbb{F}+J)[\mathrm{h}X,\mathrm{v}Y]_{\mathcal{T}E}$ and $\mathbb{F}(\mathcal{D}_XY)=(\mathbb{F}+J)[\mathrm{h}X,\mathrm{v}Y]_{\mathcal{T}E}-J[\mathrm{v}X,\mathrm{h}Y]_{\mathcal{T}E}+\mathrm{h}[\mathrm{v}X,(\mathbb{F}+J)Y]_{\mathcal{T}E}-v[\mathrm{h}X,JY]_{\mathcal{T}E}=\mathcal{D}_X\mathbb{F}Y$ we get $\mathcal{D}\mathbb{F}=0$. □

It results that the Berwald connection preserves both horizontal and vertical sections. Moreover, $\mathcal{D}$ has the same action on horizontal and vertical distributions and locally we have the following formulas

$$\mathcal{D}_{\delta_\alpha}\delta_\beta=\frac{\partial\mathcal{N}_\alpha^\gamma}{\partial y^\beta}\delta_\gamma,\quad \mathcal{D}_{\delta_\alpha}\mathcal{V}_\beta=\frac{\partial\mathcal{N}_\alpha^\gamma}{\partial y^\beta}\mathcal{V}_\gamma,\quad \mathcal{D}_{\mathcal{V}_\alpha}\delta_\beta=0,\quad \mathcal{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta=0.$$

We can see that the dynamical covariant derivative has the same properties and this leads to the next result.

**Proposition 9.** *If $\mathcal{S}$ is a spray then the following equality holds*

$$\nabla=\mathcal{D}_{\mathcal{S}}.$$

**Proof.** If $\mathcal{S}$ is a spray then $\mathcal{S}=\mathrm{h}\mathcal{S}$ and $\mathrm{v}\mathcal{S}=0$ which implies

$$\mathcal{D}_{\mathcal{S}}Y=\mathrm{v}[\mathcal{S},\mathrm{v}Y]_{\mathcal{T}E}+(\mathbb{F}+J)[\mathcal{S},JY]_{\mathcal{T}E}.$$

But $\nabla Y=\mathrm{h}[\mathcal{S},\mathrm{h}Y]_{\mathcal{T}E}+\mathrm{v}[\mathcal{S},\mathrm{v}Y]_{\mathcal{T}E}$ and we will prove that $\mathrm{h}[\mathcal{S},\mathrm{h}Y]_{\mathcal{T}E}=(\mathbb{F}+J)[\mathcal{S},JY]_{\mathcal{T}E}$ using the computation in local coordinates. Let us consider $Y=X^\alpha(x,y)\mathcal{X}_\alpha+Y^\beta(x,y)\mathcal{V}_\beta$ and using (10) we get

$$[\mathcal{S},\mathrm{h}Y]_{\mathcal{T}E}=[y^\alpha\delta_\alpha,X^\beta\delta_\beta]_{\mathcal{T}E}=y^\alpha X^\beta\mathcal{R}_{\alpha\beta}^\varepsilon\mathcal{V}_\varepsilon+y^\alpha X^\beta L_{\alpha\beta}^\varepsilon\delta_\varepsilon+y^\alpha\delta_\alpha(X^\beta)\delta_\beta+X^\beta\mathcal{N}_\beta^\alpha\delta_\alpha,$$

$$\mathrm{h}[\mathcal{S},\mathrm{h}Y]_{\mathcal{T}E}=\left(y^\alpha\delta_\alpha(X^\beta)+X^\alpha\mathcal{N}_\alpha^\beta+y^\alpha X^\varepsilon L_{\alpha\varepsilon}^\beta\right)\delta_\beta.$$

Next

$$[\mathcal{S},JY]_{\mathcal{T}E}=[y^\alpha\delta_\alpha,X^\beta\mathcal{V}_\beta]_{\mathcal{T}E}=y^\alpha X^\beta\frac{\partial\mathcal{N}_\alpha^\varepsilon}{\partial y^\beta}\mathcal{V}_\varepsilon+y^\alpha\delta_\alpha(X^\beta)\mathcal{V}_\beta-X^\beta\delta_\beta.$$

Also, we have

$$y^\alpha X^\beta\frac{\partial\mathcal{N}_\alpha^\varepsilon}{\partial y^\beta}=\mathcal{N}_\beta^\varepsilon X^\beta-L_{\beta\alpha}^\varepsilon y^\alpha X^\beta,$$

and using the relations $(\mathbb{F}+J)(\mathcal{V}_\alpha)=\delta_\alpha$, $(\mathbb{F}+J)(\delta_\alpha)=0$ we obtain the result which ends the proof. □

Moreover, $\nabla\mathcal{S}=\mathcal{D}_{\mathcal{S}}\mathcal{S}=0$ and it results that the integral curves of the spray are geodesics of the Berwald linear connection.

## 4. Symmetries for semispray

In this section we study the symmetries of SODE on Lie algebroids and prove that the canonical nonlinear connection can be determined by these symmetries. We find the relations between dynamical symmetries, Lie symmetries, Newtonoid sections, Cartan symmetries and conservation laws, and show when one of them will imply the others. Also, we obtain the invariant equations of these symmetries, using the dynamical covariant derivative and Jacobi endomorphism. In the particular case of the tangent bundle some results from [4,16,17,35] are obtained.

**Definition 6.** A section $X \in \Gamma(\mathcal{T}E \setminus \{0\})$ is a dynamical symmetry of semispray $\mathcal{S}$ if $[\mathcal{S}, X]_{\mathcal{T}E} = 0$.

In local coordinates for $X = X^\alpha(x, y)\mathcal{X}_\alpha + Y^\alpha(x, y)\mathcal{V}_\alpha$ we obtain

$$[\mathcal{S}, X]_{\mathcal{T}E} = \left(y^\alpha L^\beta_{\alpha\gamma} X^\gamma - Y^\beta + \mathcal{S}(X^\beta)\right)\mathcal{X}_\beta + \left(\mathcal{S}(Y^\beta) - X(\mathcal{S}^\beta)\right)\mathcal{V}_\beta,$$

and it results that the dynamical symmetry is characterized by the equations

$$Y^\alpha = \mathcal{S}(X^\alpha) + y^\varepsilon L^\alpha_{\varepsilon\beta} X^\beta, \tag{23}$$

$$\mathcal{S}(Y^\alpha) - X(\mathcal{S}^\alpha) = 0. \tag{24}$$

Introducing (23) into (24) we obtain

$$\mathcal{S}^2(X^\alpha) - X(\mathcal{S}^\alpha) = \left(\sigma^i_\gamma \frac{\partial L^\alpha_{\varepsilon\beta}}{\partial x^i} X^\beta + L^\alpha_{\varepsilon\beta}\sigma^i_\gamma \frac{\partial X^\beta}{\partial x^i}\right) y^\gamma y^\varepsilon + \mathcal{S}^\gamma \left(L^\alpha_{\gamma\beta} X^\beta + y^\varepsilon L^\alpha_{\varepsilon\beta} \frac{\partial X^\beta}{\partial y^\gamma}\right).$$

**Definition 7.** A section $\widetilde{X} = \widetilde{X}^\alpha(x, y)s_\alpha$ on $E \setminus \{0\}$ is a Lie symmetry of a semispray if its complete lift $\widetilde{X}^c$ is a dynamical symmetry, that is $[\mathcal{S}, \widetilde{X}^c]_{\mathcal{T}E} = 0$.

**Proposition 10.** *The local expression of a Lie symmetry is given by*

$$\mathcal{S}^\alpha \frac{\partial \widetilde{X}^\beta}{\partial y^\alpha} = 0,$$
$$\mathcal{S}^\alpha \widetilde{X}^\beta_{|_\alpha} + y^\alpha y^\varepsilon \sigma^i_\alpha \frac{\partial \widetilde{X}^\beta_{|_\varepsilon}}{\partial x^i} - \widetilde{X}^\alpha \sigma^i_\alpha \frac{\partial \mathcal{S}^\beta}{\partial x^i} - y^\varepsilon \widetilde{X}^\alpha_{|_\varepsilon} \frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} + \mathcal{S}^\alpha y^\varepsilon \left(\sigma^i_\varepsilon \frac{\partial^2 \widetilde{X}^\beta}{\partial y^\alpha \partial x^i} - L^\beta_{\gamma\varepsilon} \frac{\partial \widetilde{X}^\gamma}{\partial y^\alpha}\right) = 0,$$

*where*

$$\widetilde{X}^\alpha_{|_\varepsilon} := \sigma^i_\varepsilon \frac{\partial \widetilde{X}^\alpha}{\partial x^i} - L^\alpha_{\beta\varepsilon} \widetilde{X}^\beta.$$

**Proof.** Considering $\widetilde{X}^c = \widetilde{X}^\alpha \mathcal{X}_\alpha + y^\varepsilon \widetilde{X}^\alpha_{|_\varepsilon} \mathcal{V}_\alpha$ and using (1) we obtain

$$[\mathcal{S}, \widetilde{X}^c]_{\mathcal{T}E} = \left(\widetilde{X}^\alpha y^\varepsilon L^\beta_{\varepsilon\alpha} + y^\alpha \sigma^i_\alpha \frac{\partial \widetilde{X}^\beta}{\partial x^i} - y^\varepsilon \widetilde{X}^\beta_{|_\varepsilon} + \mathcal{S}^\alpha \frac{\partial \widetilde{X}^\beta}{\partial y^\alpha}\right)\mathcal{X}_\beta$$
$$+ \left(y^\alpha y^\varepsilon \sigma^i_\alpha \frac{\partial \widetilde{X}^\beta_{|_\varepsilon}}{\partial x^i} - \widetilde{X}^\alpha \sigma^i_\alpha \frac{\partial \mathcal{S}^\beta}{\partial x^i} + \mathcal{S}^\alpha \widetilde{X}^\beta_{|_\alpha} - y^\varepsilon \widetilde{X}^\alpha_{|_\varepsilon} \frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} + \mathcal{S}^\alpha y^\varepsilon \left(\sigma^i_\varepsilon \frac{\partial^2 \widetilde{X}^\beta}{\partial y^\alpha \partial x^i} - L^\beta_{\gamma\varepsilon} \frac{\partial \widetilde{X}^\gamma}{\partial y^\alpha}\right)\right)\mathcal{V}_\beta.$$

We deduce that $\widetilde{X}^\alpha y^\varepsilon L^\beta_{\varepsilon\alpha} + y^\alpha \sigma^i_\alpha \frac{\partial \widetilde{X}^\beta}{\partial x^i} - y^\varepsilon \widetilde{X}^\beta_{|_\varepsilon} = 0$ and it results the local expression of a Lie symmetry. $\square$

We have to remark that a section $\widetilde{X} = \widetilde{X}^\alpha(x)s_\alpha$ on $E \setminus \{0\}$ is a Lie symmetry if and only if (see also [45])

$$y^\alpha y^\varepsilon \sigma^i_\alpha \frac{\partial \widetilde{X}^\beta_{|_\varepsilon}}{\partial x^i} - \widetilde{X}^\alpha \sigma^i_\alpha \frac{\partial \mathcal{S}^\beta}{\partial x^i} + \mathcal{S}^\alpha \widetilde{X}^\beta_{|_\alpha} - y^\varepsilon \widetilde{X}^\alpha_{|_\varepsilon} \frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} = 0,$$

and it results, by direct computation, that the components $\widetilde{X}^\alpha(x)$ satisfy Eqs. (23), (24).

**Definition 8.** A section $X \in \Gamma(\mathcal{T}E \setminus \{0\})$ is called Newtonoid if $J[\mathcal{S}, X]_{\mathcal{T}E} = 0$.

In local coordinates we obtain

$$J[\mathcal{S}, X]_{\mathcal{T}E} = \left(\mathcal{S}(X^\alpha) - Y^\alpha + y^\varepsilon L^\alpha_{\varepsilon\beta} X^\beta\right)\mathcal{V}_\alpha,$$

which yields

$$Y^\alpha = \mathcal{S}(X^\alpha) + y^\varepsilon L^\alpha_{\varepsilon\beta} X^\beta, \quad X = X^\alpha \mathcal{X}_\alpha + \left(\mathcal{S}(X^\alpha) + y^\varepsilon L^\alpha_{\varepsilon\beta} X^\beta\right)\mathcal{V}_\alpha. \tag{25}$$

We remark that a section $X \in \Gamma(\mathcal{T}E \setminus \{0\})$ is a dynamical symmetry if and only if it is a Newtonoid and satisfies Eq. (24). The set of Newtonoid sections denoted $\mathfrak{X}_{\mathcal{S}}$ is given by

$$\mathfrak{X}_{\mathcal{S}} = Ker(J \circ \mathcal{L}_{\mathcal{S}}) = Im(Id + J \circ \mathcal{L}_{\mathcal{S}}).$$

In the following we will use the dynamical covariant derivative in order to find the invariant equations of Newtonoid sections and dynamical symmetries on Lie algebroids. Let $\mathcal{S}$ be a semispray, $\mathcal{N}$ an arbitrary nonlinear connection and $\nabla$ the induced dynamical covariant derivative. We set,

**Proposition 11.** *A section $X \in \Gamma(\mathcal{T}E \setminus \{0\})$ is a Newtonoid if and only if*

$$\mathrm{v}(X) = J(\nabla X), \tag{26}$$

*which locally yields*

$$X = X^\alpha \delta_\alpha + \nabla X^\alpha \mathcal{V}_\alpha,$$

*with $\nabla X^\alpha$ given by formula (18).*

**Proof.** We know that $J \circ \nabla = J \circ \mathcal{L}_{\mathcal{S}} + \mathrm{v}$ and it results $J[\mathcal{S}, X]_{\mathcal{T}E} = 0$ if and only if $\mathrm{v}(X) = J(\nabla X)$. In local coordinates we obtain

$$\begin{aligned} X &= X^\alpha \left(\delta_\alpha + \mathcal{N}_\alpha^\beta \mathcal{V}_\beta\right) + \left(\mathcal{S}(X^\alpha) + y^\varepsilon L_{\varepsilon\beta}^\alpha X^\beta\right) \mathcal{V}_\alpha \\ &= X^\alpha \delta_\alpha + \left(\mathcal{S}(X^\alpha) + X^\beta(\mathcal{N}_\beta^\alpha + y^\varepsilon L_{\varepsilon\beta}^\alpha)\right) \mathcal{V}_\alpha \\ &= X^\alpha \delta_\alpha + \nabla X^\alpha \mathcal{V}_\alpha. \quad \square \end{aligned}$$

**Proposition 12.** *A section $X \in \Gamma(\mathcal{T}E \setminus \{0\})$ is a dynamical symmetry if and only if $X$ is a Newtonoid and*

$$\nabla(J\nabla X) + \Phi(X) = 0. \tag{27}$$

**Proof.** If $X$ is a dynamical symmetry then $\mathrm{h}[\mathcal{S}, X]_{\mathcal{T}E} = \mathrm{v}[\mathcal{S}, X]_{\mathcal{T}E} = 0$ and composing by $J$ we get $J[\mathcal{S}, X]_{\mathcal{T}E} = 0$ that means $X$ is a Newtonoid. Therefore, $\mathrm{v}[\mathcal{S}, X]_{\mathcal{T}E} = \mathrm{v}[\mathcal{S}, \mathrm{v}X]_{\mathcal{T}E} + \mathrm{v}[\mathcal{S}, \mathrm{h}X]_{\mathcal{T}E} = \nabla(\mathrm{v}X) + \Phi(X)$ and using (26) we get $\nabla(J\nabla X) + \Phi(X) = 0$. $\square$

For $f \in C^\infty(E)$ and $X \in \Gamma(\mathcal{T}E \setminus \{0\})$ we define the product

$$f * X = (Id + J \circ \mathcal{L}_{\mathcal{S}})(fX) = fX + fJ[\mathcal{S}, X]_{\mathcal{T}E} + \mathcal{S}(f)JX,$$

and remark that a section $X \in \Gamma(\mathcal{T}E \setminus \{0\})$ is a Newtonoid if and only if

$$X = X^\alpha(x, y) * \mathcal{X}_\alpha.$$

If $X \in \mathfrak{X}_{\mathcal{S}}$ then

$$f * X = fX + \mathcal{S}(f)JX.$$

The next result proves that the canonical nonlinear connection can by determined by symmetry.

**Proposition 13.** *Let us consider a semispray $\mathcal{S}$, an arbitrary nonlinear connection $\mathcal{N}$ and $\nabla$ the dynamical covariant derivative. The following conditions are equivalent,*

(i) *$\nabla$ restricts to $\nabla : \mathfrak{X}_{\mathcal{S}} \to \mathfrak{X}_{\mathcal{S}}$ satisfies the Leibnitz rule with respect to the $*$ product.*
(ii) $\nabla J = 0$,
(iii) $\mathcal{L}_{\mathcal{S}} J + \mathcal{N} = 0$,
(iv) $\mathcal{N}_\alpha^\beta = \frac{1}{2}\left(-\frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} + y^\varepsilon L_{\alpha\varepsilon}^\beta\right)$.

**Proof.** For (ii) $\Rightarrow$ (i) we consider $X \in \mathfrak{X}_{\mathcal{S}}$ and using (26) we have $\mathrm{v}X = J(\nabla X)$ which leads to $\nabla(\mathrm{v}X) = \nabla(J\nabla X)$. It results $(\nabla \mathrm{v})X + \mathrm{v}(\nabla X) = (\nabla J)(\nabla X) + J\nabla(\nabla X)$ and using the relations $\nabla \mathrm{v} = 0$ and $\nabla J = 0$ we obtain $\mathrm{v}(\nabla X) = J\nabla(\nabla X)$ which implies $\nabla X \in \mathfrak{X}_{\mathcal{S}}$ . For $X \in \mathfrak{X}_{\mathcal{S}}$ we have

$$\nabla\,(f * X) = \nabla(fX + \mathcal{S}(f)JX) = \mathcal{S}(f)X + f\nabla X + \mathcal{S}^2(f)JX + \mathcal{S}(f)\nabla(JX),$$

$$\nabla f * X + f * \nabla X = \mathcal{S}(f)X + \mathcal{S}^2(f)JX + f\nabla X + \mathcal{S}(f)J(\nabla X).$$

But $\nabla(JX) = (\nabla J)X + J(\nabla X)$ and from $\nabla J = 0$ we obtain $\nabla(JX) = J(\nabla X)$ which leads to $\nabla\,(f * X) = \nabla f * X + f * \nabla X$.

For (i) $\Rightarrow$ (ii) we consider the set $\mathfrak{X}_{\mathcal{S}} \cup \Gamma^{\mathrm{v}}(\mathcal{T}E \setminus \{0\})$ which is a set of generators for $\Gamma(\mathcal{T}E \setminus \{0\})$. We have $\nabla J(X) = 0$ for $X \in \Gamma^{\mathrm{v}}(\mathcal{T}E \setminus \{0\})$ and for $X \in \mathfrak{X}_{\mathcal{S}}$ using $\nabla\,(f * X) = \nabla f * X + f * \nabla X$ it results $\mathcal{S}(f)\nabla(JX) = \mathcal{S}(f)J(\nabla X)$, which implies $\mathcal{S}(f)(\nabla J)X = 0$, for an arbitrary function $f \in C^\infty(E \setminus \{0\})$. Therefore, $\nabla J = 0$ on $\mathfrak{X}_{\mathcal{S}}$ which ends the proof. The equivalence of the conditions (ii) , (iii), (iv) has been proved in the Theorem 1. $\square$

Next, we consider the dynamical covariant derivative $\nabla$ induced by the semispray $\mathcal{S}$, the canonical nonlinear connection $\mathcal{N} = -\mathcal{L}_{\mathcal{S}}J$ and find the invariant equations of dynamical and Lie symmetries.

**Proposition 14.** *A section $X \in \Gamma(\mathcal{T}E \setminus \{0\})$ is a dynamical symmetry if and only if $X$ is a Newtonoid and*

$$\nabla^2 JX + \Phi(X) = 0, \tag{28}$$

*which locally yields*

$$\nabla^2 X^\alpha + \mathcal{R}^\alpha_\beta X^\beta = 0.$$

**Proof.** From (20) it results $\nabla J = 0$ and using (27) and (17) we get (28). Next, using (25) and (14) the local components of the vertical section $\nabla^2 JX + \Phi(X)$ is $\nabla^2 X^\alpha + \mathcal{R}^\alpha_\beta X^\beta$. □

**Proposition 15.** *A section $\widetilde{X} \in \Gamma(E \setminus \{0\})$ is a Lie symmetry of $\mathcal{S}$ if and only if*

$$\nabla^2 \widetilde{X}^v + \Phi(\widetilde{X}^c) = 0. \tag{29}$$

**Proof.** Using (28) and the relation $J(\widetilde{X}^c) = \widetilde{X}^v$ we obtain (29). □

Let us consider in the following a regular Lagrangian $L$ on $E$, the Cartan 1-section $\theta_L$, the symplectic structure $\omega_L = d^E\theta_L$, the energy function $E_L$ and the induced canonical semispray $\mathcal{S}$ with the components given by the relation (9).

**Proposition 16.** *If $\widetilde{X}$ is a section on $E$ such that $\mathcal{L}_{\widetilde{X}^c}\theta_L$ is closed and $d^E(\widetilde{X}^c E_L) = 0$, then $\widetilde{X}$ is a Lie symmetry of the canonical semispray $\mathcal{S}$ induced by $L$.*

**Proof.** We have

$$\begin{aligned} i_{[\widetilde{X}^c,\mathcal{S}]}\omega_L &= \mathcal{L}_{\widetilde{X}^c}(i_{\mathcal{S}}\omega_L) - i_{\mathcal{S}}(\mathcal{L}_{\widetilde{X}^c}\omega_L) = -\mathcal{L}_{\widetilde{X}^c}d^E E_L - i_{\mathcal{S}}(\mathcal{L}_{\widetilde{X}^c}d^E\theta_L) \\ &= -d^E\mathcal{L}_{\widetilde{X}^c}E_L - i_{\mathcal{S}}d^E(\mathcal{L}_{\widetilde{X}^c}\theta_L) = -d^E(\widetilde{X}^c E_L) - i_{\mathcal{S}}d^E(\mathcal{L}_{\widetilde{X}^c}\theta_L) = 0. \end{aligned}$$

But $\omega_L$ is a symplectic structure ($L$ is regular) and we get $[\widetilde{X}^c, \mathcal{S}] = 0$ which ends the proof. □

**Definition 9.**

(a) A section $X \in \Gamma(\mathcal{T}E \setminus \{0\})$ is called a Cartan symmetry of the Lagrangian $L$, if $\mathcal{L}_X\omega_L = 0$ and $\mathcal{L}_X E_L = 0$.
(b) A function $f \in C^\infty(E)$ is a constant of motion (or a conservation law) for the Lagrangian $L$ if $\mathcal{S}(f) = 0$.

**Proposition 17.** *The canonical semispray induced by the regular Lagrangian $L$ is a Cartan symmetry.*

**Proof.** Using the relation $i_{\mathcal{S}}\omega_L = -d^E E_L$ and the skew symmetry of the symplectic 2-section $\omega_L$ we obtain

$$0 = i_{\mathcal{S}}\omega_L(\mathcal{S}) = -d^E E_L(\mathcal{S}) = -\mathcal{S}(E_L) = -\mathcal{L}_{\mathcal{S}}E_L.$$

Also, from $d^E\omega_L = 0$ we get

$$\mathcal{L}_{\mathcal{S}}\omega_L = d^E i_{\mathcal{S}}\omega_L + i_{\mathcal{S}}d^E\omega_L = -d^E(d^E E_L) = 0,$$

and it results that the semispray $\mathcal{S}$ is a Cartan symmetry. □

**Proposition 18.** *A Cartan symmetry $X$ of the Lagrangian $L$ is a dynamical symmetry for the canonical semispray $\mathcal{S}$.*

**Proof.** From the symplectic equation $i_{\mathcal{S}}\omega_L = -d^E E_L$, applying the Lie derivative in both sides, we obtain

$$\mathcal{L}_X(i_{\mathcal{S}}\omega_L) = -\mathcal{L}_X d^E E_L = -d^E\mathcal{L}_X E_L = 0.$$

Also, using the formula $i_{[X,Y]_{\mathcal{T}E}} = \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X$ it results

$$\mathcal{L}_X(i_{\mathcal{S}}\omega_L) = -i_{[\mathcal{S},X]_{\mathcal{T}E}}\omega_L + i_{\mathcal{S}}\mathcal{L}_X\omega_L = -i_{[\mathcal{S},X]_{\mathcal{T}E}}\omega_L$$

which yields

$$i_{[\mathcal{S},X]_{\mathcal{T}E}}\omega_L = 0. \tag{30}$$

But $\omega_L$ is a symplectic 2-section and we conclude that $[\mathcal{S}, X]_{\mathcal{T}E} = 0$, so $X$ is a dynamical symmetry. □

Since Lie and exterior derivatives commute, we obtain

$$d^E\mathcal{L}_X\theta_L = \mathcal{L}_X d^E\theta_L = \mathcal{L}_X\omega_L = 0.$$

It results that, for a Cartan symmetry, the 1-section $\mathcal{L}_X\theta_L$ is a closed 1-section.

**Definition 10.** A Cartan symmetry $X$ is said to be an exact Cartan symmetry if the 1-section $\mathcal{L}_X\theta_L$ is exact.

The next result proves that there is a one to one correspondence between exact Cartan symmetries and conservation laws. Also, if $X$ is an exact Cartan symmetry, then there is a function $f \in C^\infty(E)$ such that $\mathcal{L}_X\theta_L = d^E f$.

**Proposition 19.** *If $X$ is an exact Cartan symmetry, then $f - \theta_L(X)$ is a conservation law for the Lagrangian $L$. Conversely, if $f \in C^\infty(E)$ is a conservation law for $L$, then $X \in \Gamma(\mathcal{T}E \setminus \{0\})$ the unique solution of the equation $i_X\omega_L = -d^E f$ is an exact Cartan symmetry.*

**Proof.** We have $\mathcal{S}(f-\theta_L(X)) = d^E(f-\theta_L(X))(\mathcal{S}) = \left(\mathcal{L}_X\theta_L - d^E i_X(\theta_L)\right)(\mathcal{S}) = i_X d^E\theta_L(\mathcal{S}) = i_X\omega_L(\mathcal{S}) = -i_{\mathcal{S}}\omega_L(X) = d^E E_L(X) = 0$, and it results that $f-\theta_L(X)$ is a conservation law for the dynamics associated to the regular Lagrangian $L$. Conversely, if $X$ is the solution of the equation $i_X\omega_L = -d^E f$ then $\mathcal{L}_X\theta_L = i_X\omega_L$ is an exact 1-section. Consequently, $0 = d^E\mathcal{L}_X\theta_L = \mathcal{L}_X d^E\theta_L = \mathcal{L}_X\omega_L$. Also, $f$ is a conservation law, and we have $0 = \mathcal{S}(f) = d^E f(\mathcal{S}) = -i_X\omega_L(\mathcal{S}) = i_{\mathcal{S}}\omega_L(X) = -d^E E_L(X) = -X(E_L)$. Therefore , we obtain $\mathcal{L}_X E_L = 0$ and $X$ is an exact Cartan symmetry. □

We have to mention that the Noether type theorems for Lagrangian systems on Lie algebroids are studied in [26,36] and Jacobi sections for second order differential equations on Lie algebroids are investigated in [37].

### 4.1. Example

Next, we consider an example from optimal control theory and prove that the framework of Lie algebroids is more useful that the tangent bundle in order to calculate some symmetries of the dynamics induced by a Lagrangian function. Let us consider the following distributional system in $\mathbb{R}^3$ (driftless control affine system) [31],

$$\begin{cases} \dot{x}^1 = u^1 + u^2x^1 \\ \dot{x}^2 = u^2x^2 \\ \dot{x}^3 = u^2. \end{cases}$$

Let $x_0$ and $x_1$ be two points in $\mathbb{R}^3$. An optimal control problem consists of finding the trajectories of our control system which connect $x_0$ and $x_1$ and minimizing the Lagrangian

$$\min \int_0^T \mathcal{L}(u(t))dt,\ \mathcal{L}(u) = \frac{1}{2}\left((u^1)^2 + (u^2)^2\right), \quad x(0) = x_0,\ x(T) = x_1,$$

where $\dot{x}^i = \frac{dx^i}{dt}$ and $u^1, u^2$ are control variables. From the system of differential equations we obtain $u^2 = \dot{x}^3$, $u^1 = \dot{x}^1 - \dot{x}^3x^1$. The Lagrangian function on the tangent bundle $T\mathbb{R}^3$ has the form

$$\mathcal{L} = \frac{1}{2}\left((\dot{x}^1 - \dot{x}^3x^1)^2 + (\dot{x}^3)^2\right),$$

with the constraint

$$\dot{x}^2 = \dot{x}^3x^2.$$

Then, using the Lagrange multiplier $\lambda = \lambda(t)$, we obtain the total Lagrangian (including the constraints) given by

$$L(x,\dot{x}) = \mathcal{L}(x,\dot{x}) + \lambda\left(\dot{x}^2 - \dot{x}^3x^2\right) = \frac{1}{2}\left((\dot{x}^1 - \dot{x}^3x^1)^2 + (\dot{x}^3)^2\right) + \lambda\left(\dot{x}^2 - \dot{x}^3x^2\right).$$

We observe that the Hessian matrix of $L$ is singular, and $L$ is a degenerate Lagrangian (not regular). The corresponding Euler–Lagrange equations lead to a complicated system of second order differential equations. Moreover, because the Lagrangian is not regular, we cannot obtain the explicit coefficients of the semispray $\mathcal{S}$ from the equation $i_{\mathcal{S}}\omega_L = -dE_L$ and it is difficult to study the symmetries of SODE in this case.

For this reason, we will use a different approach, considering the framework of Lie algebroids. The system can be written in the next form

$$\dot{x} = u^1X_1 + u^2X_2, \quad x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in \mathbb{R}^3,\ X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ X_2 = \begin{pmatrix} x^1 \\ x^2 \\ 1 \end{pmatrix}.$$

The associated distribution $\Delta = span\{X_1, X_2\}$ has the constant rank 2 and is holonomic, because

$$X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \quad [X_1, X_2] = X_1.$$

From the Frobenius theorem, the distribution $\Delta$ is integrable, it determines a foliation on $T\mathbb{R}^3$ and two points can be joined by a optimal trajectory if and only if they are situated on the same leaf (see [31]). In order to apply the theory of Lie algebroids, we consider the Lie algebroid being just the distribution, $E = \Delta$ and the anchor $\sigma : E \to T\mathbb{R}^3$ is the inclusion, with the components

$$\sigma_\alpha^i = \begin{pmatrix} 1 & x^1 \\ 0 & x^2 \\ 0 & 1 \end{pmatrix}.$$

From the relation

$$[X_\alpha, X_\beta] = L_{\alpha\beta}^\gamma X_\gamma, \quad \alpha, \beta, \gamma = 1, 2,$$

we obtain the non-zero structure functions

$$L_{12}^1 = 1,\ L_{21}^1 = -1.$$

The components of the semispray from (9) are given by

$$\mathcal{S}^1 = -u^1 u^2,\ \mathcal{S}^2 = (u^1)^2.$$

The functions $\mathcal{S}^\alpha$ are homogeneous of degree 2 in $u$ and it results that $\mathcal{S}$ is a spray. By straightforward computation we obtain the expression of the canonical spray induced by $\mathcal{L}$

$$\mathcal{S}(x, u) = (u^1 + u^2 x^1)\frac{\partial}{\partial x^1} + u^2 x^2 \frac{\partial}{\partial x^2} + u^2 \frac{\partial}{\partial x^3} - u^1 u^2 \frac{\partial}{\partial u^1} + (u^1)^2 \frac{\partial}{\partial u^2}.$$

From Proposition 17 it results that $\mathcal{S}(x, u)$ is a Cartan symmetry of the dynamics associated to the regular Lagrangian $\mathcal{L}$ on Lie algebroids.

The coefficients of the canonical nonlinear connection $\mathcal{N} = -\mathcal{L}_\mathcal{S} J$ are given by

$$\mathcal{N}_1^1 = u^2, \mathcal{N}_2^1 = 0,\ \mathcal{N}_1^2 = u^1,\ \mathcal{N}_2^2 = 0,$$

and the components of the Jacobi endomorphism from (21) have the form

$$\mathcal{R}_1^1 = -(u^2)^2, \mathcal{R}_1^2 = -u^1 u^2,\ \mathcal{R}_2^1 = u^1 u^2, \mathcal{R}_2^2 = (u^1)^2.$$

Also, the non-zero coefficients of the curvature from (11) of $\mathcal{N}$ are

$$\mathcal{R}_{12}^1 = u^2,\ \mathcal{R}_{12}^2 = u^1,\ \mathcal{R}_{21}^1 = -u^2,\ \mathcal{R}_{21}^2 = -u^1,$$

and we obtain that the Jacobi endomorphism is the contraction with $\mathcal{S}$ of the curvature of $\mathcal{N}$, or locally $\mathcal{R}_\beta^\alpha = \mathcal{R}_{\varepsilon\beta}^\alpha u^\varepsilon$.

The Euler–Lagrange equations on Lie algebroids given by (see [34])

$$\frac{dx^i}{dt} = \sigma_\alpha^i u^\alpha, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u^\alpha}\right) = \sigma_\alpha^i \frac{\partial \mathcal{L}}{\partial x^i} - L_{\alpha\beta}^\varepsilon u^\beta \frac{\partial \mathcal{L}}{\partial u^\varepsilon},$$

lead to the following differential equations

$$\dot{u}^1 = -u^1 u^2, \quad \dot{u}^2 = (u^1)^2,$$

which can be written in the form

$$\frac{dx^i}{dt} = \sigma_\alpha^i u^\alpha, \quad \frac{du^\alpha}{dt} = \mathcal{S}^\alpha(x, u),$$

and give the integral curves of $\mathcal{S}$. The Cartan 1-section $\theta_\mathcal{L}$ has the form

$$\theta_\mathcal{L} = u^1 dx^1 + u^2(x^1 dx^1 + x^2 dx^2 + dx^3),$$

and the symplectic structure is $\omega_\mathcal{L} = d^E \theta_\mathcal{L}$. The energy of Lagrangian $\mathcal{L}$ is

$$E_\mathcal{L} = \frac{1}{2}\left((u^1)^2 + (u^2)^2\right).$$

For the optimal solution of the control system (using the framework of Lie algebroids) see [31].

## Conclusions

The main purpose of this paper is to study the symmetries of SODE on Lie algebroids and relations between them, using the dynamical covariant derivative and Jacobi endomorphism. The existence of a semispray $\mathcal{S}$ together with an arbitrary nonlinear connection $\mathcal{N}$ defines a dynamical covariant derivative and the Jacobi endomorphism. Let us remark that at this point we do not have any relation between $\mathcal{S}$ and the nonlinear connection $\mathcal{N}$. This will be given considering the compatibility condition between the dynamical covariant derivative and the tangent structure, $\nabla J = 0$, which fix the canonical nonlinear connection $\mathcal{N} = -\mathcal{L}_{\mathcal{S}}J$. This canonical nonlinear connection depends only on semispray. In this case we have the decomposition $\nabla = \mathcal{L}_{\mathcal{S}} + \mathbb{F} + \mathcal{J} - \Phi$ which can be compared with the tangent case from [4,7]. Also, in the case of homogeneous SODE (spray), the dynamical covariant derivative coincides with Berwald linear connection and the Jacobi endomorphism is the contraction with $\mathcal{S}$ of the curvature of the nonlinear connection. We study the dynamical symmetry, Lie symmetry, Newtonoid section and Cartan symmetry on Lie algebroids and find their invariant equations with the help of dynamical covariant derivative and Jacobi endomorphism. Finally, we give an example from optimal control theory which proves that the framework of Lie algebroids is more useful than the tangent bundle in order to find the symmetries of the dynamics induced by a Lagrangian function. For further developments one can study the symmetries using the $k$-symplectic formalism on Lie algebroids given in [38].

**Acknowledgments**

The author wishes to express his thanks to the referees for useful comments and suggestions concerning this paper.

**References**

[1] M. Crampin, Tangent bundle geometry for Lagrangian dynamics, J. Phys. A: Math. Gen. 16 (1983) 3755–3772.
[2] J. Grifone, Structure presque tangente et connections I, Ann. Inst. Fourier 22 (1) (1972) 287–334.
[3] I. Bucătaru, M.F. Dahl, Semi-bazic 1-form and Helmholtz conditions for the inverse problem of the calculus of variations, J. Geom. Mech. 1 (2) (2009) 159–180.
[4] I. Bucătaru, O. Constantinescu, M.F. Dahl, A geometric setting for systems of ordinary differential equations, Int. J. Geom. Methods Mod. Phys. 08 (2011) 12019.
[5] M. Crampin, E. Martínez, W. Sarlet, Linear connections for system of second-order ordinary differential equations, Ann. Inst. Henry Poincare 65 (2) (1996) 223–249.
[6] J. Grifone, Z. Muzsnay, Variational Principle for Second-order Differential Equations, World Scientific, 2000.
[7] E. Cariñema, J.F. Martínez, W. Sarlet, Derivations of differential forms along the tangent bundle projection II, Differential Geom. Appl. 3 (1) (1993) 1–29.
[8] W. Sarlet, Linear connections along the tangent bundle projection, in: Variations, Geometry and Physics, Nova Science Publishers, 2008.
[9] M. Jerie, G. Prince, Jacobi fields and linear connections for arbitrary second-order ODEs, J. Geom. Phys. 43 (2002) 351–370.
[10] J.F. Cariñena, E. Martínez, Generalized Jacobi equation and inverse problem in classical mechanics, in: V.V. Dodonov, V.I. Manko (Eds.), Group Theoretical Methods in Physics, in: Proceedings of the 18th International Colloquim 1990, Moskow, vol. 2, Nova Science Publishers, New York, 1991.
[11] R. Abraham, J. Marsden, Foundation of Mechanics, Benjamin, New-York, 1978.
[12] L. Bua, I. Bucataru, M. Salgado, Symmetries, Newtonoid vector fields and conservation laws on the Lagrangian k-symplectic formalism, Rev. Math. Phys. 24 (2012) 1250030.
[13] X. Gràcia, J.M. Pons, Symmetries and infinitesimal symmetries of singular differential equations, J. Phys. A: Math. Gen. 35 (2002) 5059–5077.
[14] M. de León, D. Martín de Diego, Symmetries and constants of the motion for singular Lagrangian systems, Internat. J. Theoret. Phys. 35 (5) (1996) 975–1011.
[15] M. De León, A. Martín de Diego, D. Santamaria-Merino, Symmetries in classical field theories, Int. J. Geom. Methods Mod. Phys. 5 (2004) 651–710.
[16] G. Prince, Toward a classification of dynamical symmetries in classical mechanics, Bull. Aust.. Math. Soc. 27 (1983) 53–71.
[17] G. Prince, A complete classification of dynamical symmetries in classical mechanics, Bull. Aust. Math. Soc. 32 (1985) 299–308.
[18] N. Román-Roy, M. Salgado, S. Vilariño, Symmetries and conservation laws in Günter k-symplectic formalism of field theory, Rev. Math. Phys. 19 (10) (2007) 1117–1147.
[19] K. Mackenzie, Lie groupoids and Lie algebroids in differential geometry, in: London Mathematical Society Lecture Note Series, vol. 124, 1987.
[20] K. Mackenzie, General theory of Lie groupoids and Lie algebroids, in: London Mathematical Society Lecture Note Series, vol. 123, 2005.
[21] C.M. Arcuş, Mechanical systems in the generalized Lie algebroids framework, Int. J. Geom. Methods Mod. Phys. 11 (2014) 1450023.
[22] J. Cortes, E. Martínez, Mechanical control systems on Lie algebroids, IMA J. Math. Control Inform. 21 (2004) 457–492.
[23] R.L. Fernandes, Lie algebroids, holonomy and characteristic classes, Adv. Math. 170 (2002) 119–179.
[24] M. de León, J.C. Marrero, E. Martínez, Lagrangian submanifolds and dynamics on Lie algebroids, J. Phys. A: Math. Gen. 38 (2005) 241–308.
[25] P. Libermann, Lie algebroids and mechanics, Arch. Math. (Brno) 32 (1996) 147–162.
[26] E. Martínez, Lagrangian mechanics on Lie algebroids, Acta Appl. Math. 67 (2001) 295–320.
[27] E. Mestdag, T. Martínez, W. Sarlet, Lie algebroid structures and Lagrangian systems on affine bundles, J. Geom. Phys. 44 (1) (2002) 70–95.
[28] T. Mestdag, B. Langerock, A Lie algebroid framework for non-holonomic systems, J. Phys. A: Math. Gen. 38 (5) (2005) 1097–1111.
[29] L. Popescu, The geometry of Lie algebroids and applications to optimal control, Annals. Univ. Al. I. Cuza, Iasi, series I., Math., 51 (2005) 155–170.
[30] L. Popescu, Geometrical structures on Lie algebroids, Publ. Math. Debrecen 72 (1-2) (2008) 95–109.
[31] L. Popescu, Lie algebroids framework for distributional systems, Annals Univ. Al. I. Cuza, Iasi, series I., Math. 55 (2) (2009) 373–390.
[32] L. Popescu, Metric nonlinear connections on Lie algebroids, Balkan J. Geom. Appl. 16 (1) (2011) 111–121.
[33] L. Popescu, Dual structures on the prolongations of a Lie algebroid, Annals Univ. Al. I. Cuza, Iasi, series I., Math. 59 (2) (2013) 357–374.
[34] A. Weinstein, Lagrangian mechanics and groupoids, Fields Inst. Commun. 7 (1996) 206–231.
[35] G. Marmo, N. Mukunda, Symmetries and constant of the motion in the Lagrangian formalism on $TQ$: beyond point transformations, Nuovo Cim. B. 92 (1986) 1–12.
[36] J.F. Cariñena, M. Rodríguez-Olmos, Gauge equivalence and conserved quantities for Lagrangian systems on Lie algebroids, J. Phys. A 42 (2009) 265209.
[37] J.F. Cariñena, I. Gheorghiu, E. Martínez, Jacobi fields for second-order differential equations on Lie algebroids, in: Dynamical Systems, Differential Equations and Applications AIMS Proceedings, 2015, pp. 213–222.
[38] M. de León, D. Martín de Diego, M. Salgado, S. Vilariño, k-symplectic formalism on Lie algebroids., J. Phys. A 42 (2009) 385209.
[39] A. Frölicher, A. Nijenhuis, Theory of vector-valued differential forms, Nederl. Akad. Wetensch. Proc. Ser. A. 59 (1956) 338–359.
[40] P.J. Higgins, K. Mackenzie, Algebraic constructions in the category of Lie algebroids, J. Algebra 129 (1990) 194–230.
[41] J. Grabowski, P. Urbanski, Tangent and cotangent lift and graded Lie algebra associated with Lie algebroids, Ann. Global Anal. Geom. 15 (1997) 447–486.
[42] J. Grabowski, P. Urbanski, Lie algebroids and Poisson-Nijenhuis structures, Rep. Math. Phys. 40 (1997) 195–208.
[43] M. Barbero-Liñán, M. Farré Puiggalí, D. Martín de Diego, Inverse problem for Lagrangian systems on Lie algebroids and applications to reduction by symmetries, Monatsh. Math. 180 (4) (2016) 665–691.
[44] J. Szilasi, A setting for spray and Finsler geometry, in: Handbook of Finsler Geometry, vol. 2, Kluwer Acad. Publ, 2003, pp. 1183–1426.
[45] E. Peyghan, Berwald-type and Yano-type connections on Lie algebroids, Int. J. Geom. Methods Mod. Phys. 12 (10) (2015) 1550125.