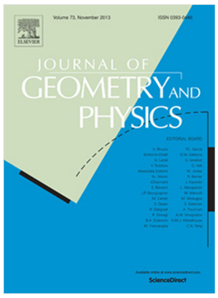
# Accepted Manuscript




Prescribed curvature tensor in locally conformally flat manifolds

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PII: S0393-0440(17)30233-4
DOI: https://doi.org/10.1016/j.geomphys.2017.09.014
Reference: GEOPHY 3078

To appear in: *Journal of Geometry and Physics*

Received date : 29 August 2016
Revised date : 26 September 2017
Accepted date : 28 September 2017

Please cite this article as: R. Pina, M. Pieterzack, Prescribed curvature tensor in locally conformally flat manifolds, *Journal of Geometry and Physics* (2017), https://doi.org/10.1016/j.geomphys.2017.09.014

# PRESCRIBED CURVATURE TENSOR IN LOCALLY CONFORMALLY FLAT MANIFOLDS

ROMILDO PINA AND MAURICIO PIETERZACK

Abstract. A global existence theorem for the prescribed curvature tensor problem in locally conformally flat manifolds is proved for a special class of tensors $R$. Necessary and sufficient conditions for the existence of a metric $\bar{g}$, conformal to Euclidean $g$, are determined such that $\bar{R} = R$, where $\bar{R}$ is the Riemannian curvature tensor of the metric $\bar{g}$. The solution to this problem is given explicitly for special cases of the tensor $R$, including the case where the metric $\bar{g}$ is complete on $\mathbb{R}^n$. Similar problems are considered for locally conformally flat manifolds.

## 1. Introduction

Over the last decades several authors have considered the following problem:

(P) Given a smooth function $\bar{K} : M \to \mathbb{R}$ on a manifold $(M, g)$ is there a metric $\bar{g}$ conformal to $g$ whose scalar curvature is $\bar{K}$?

This problem has been studied by various authors. Particularly, when $\bar{K}$ is a constant it is known as the Yamabe Problem. If $M = \mathbb{R}^n$ with $n \geq 3$ and $g$ is the Euclidean metric, various results can be found in [1], [2], [3] and in their references.

An interesting problem related to problem (P), that is currently under extensive investigation is the prescribed Ricci curvature equation. It can be formulated as follows:

(P1) Given a symmetric $(0, 2)$-tensor $T$, defined on a manifold $M^n$, $n \geq 3$, does there exist a Riemannian metric $g$ such that $Ric\, g = T$?

When $T$ is nonsingular, that is, its determinant does not vanish, a local solution of the Ricci equation always exists, as shown by DeTurck in [4]. When $T$ is singular, the Ricci equation still admits local solutions, provided that $T$ has constant rank and satisfies certain conditions [5]. Rotationally symmetric nonsingular tensors were considered in [6]. Related results can be found in [5], [7], [8], [9], [10], [14], [11], [12], and the references therein. Recent developments on problem (P1) can be found in [15], [16], [17], and [18].

Another problem related to problem (P1) is the *Prescribed Curvature Tensor problem*, which can be formulated as follows:

(P2) Given a $(0, 4)$-tensor $R$, defined on a manifold $M^n$, $n \geq 3$, does there exist a Riemannian metric $g$ such that $R_g = R$, where $R_g$ is the Riemannian curvature tensor of the metric $g$?

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*Date*: August, 2016 and, in revised form, .

2010 *Mathematics Subject Classification.* Primary 53C21; Secondary 35N10.

*Key words and phrases.* Conformal metric, Riemannian curvature tensor, scalar curvature, Ricci curvature.

The authors were supported in part by CAPES/PROCAD and FAPEG/GO.

We observe that solving problem (P2) is equivalent to solving a nonlinear system of partial differential equations of second order.

There are very few studies in the literature concerning this problem. Considering $n = 3$ and $R$ a $(0, 4)$-tensor nondegenerate as a quadratic form on $\Lambda^2 T^* M$, Robert Bryant proved in the early 1980s that when $R$ is real-analytic, there always exist local solutions to the equation $R_g = R$. Then, he proved that the combined overdetermined system of six second-order equations and four first-order equations for $g$ is involutive; therefore, an application of the Cartan–Kähler Theorem proves local solvability. He never published his proof; however, DeTurck and Yang later studied the overdetermined system and published a proof of its local solvability in the smooth category [19].

To the best of our knowledge, the existence of global solutions to problem (P2) has not yet been considered. Thus, this study is intended as a contribution in this research direction.

In an attempt to solve (P2), we consider the *Prescribed Curvature Tensor problem* with conformal change of metric. That is, the following problem:

(P3) Let $(M^n, g)$, $n \geq 3$, be a Riemannian manifold. Given a $(0, 4)$-tensor $R$, defined on $M$, does there exist a Riemannian metric $\bar{g}$ conformal to $g$ such that $R_{\bar{g}} = R$?

The curvature tensor of the metric $g$ on $M$ can be decomposed as follows:

$$R_g = W_g + A_g \odot g,$$

where $R_g$ is the Riemannian curvature tensor, $\odot$ is the Kulkarni–Nomizu product, and $A_g$ and $W_g$ are the Schouten and Weyl tensors of $g$, respectively [13].

If $g$ is locally conformally flat, then $W_g = 0$, as the Weyl tensor is conformally invariant. Therefore, the Riemann curvature tensor is determined by the Schouten tensor, and the decomposition is

$$R_g = A_g \odot g. \tag{1.1}$$

In this paper, we consider a $(0, 4)$-tensor $R = T \odot g$ on the Euclidean space $(\mathbb{R}^n, g)$, $n \geq 3$, where $T$ is a diagonal $(0, 2)$-tensor given by $T = \sum_i f_i(x) dx_i^2$, and $f_i(x)$ are smooth functions. Our interest is to study the *Prescribed Curvature Tensor problem* in this particular context.

In Theorem 3.2 we provide necessary and sufficient conditions on the tensor $R$ for the existence of a metric $\bar{g}$ conformal to $g$ such that $R_{\bar{g}} = R$. We extend this result to locally conformally flat manifolds in Theorem 3.8. We consider particular cases for the tensor $R$ when the solutions for the *Prescribed Curvature Tensor problem* are given explicitly. In Theorem 3.3 we consider the case $R = f(g \odot g)$, where $f$ is a smooth function and $g$ is the Euclidean metric. Unfortunately, in this case the metric $\bar{g}$ is not complete. In Theorem 3.5, we obtain a non-existence result. In Theorem 3.6, we consider tensors $T$ depending only on one fixed variable. From this, we exhibit examples of complete metrics on $\mathbb{R}^n$ with prescribed Riemannian curvature tensor. In Theorem 3.8, these results are generalized to locally conformally flat manifolds.

## 2. Preliminaires

Our purpose is to study the *Prescribed Curvature Tensor problem* in locally conformally flat manifolds. In this section, we set forth the notation and review the necessary background to state the results about this problem in the next section.

Let $(M^n, g)$ be a flat manifold and $R = T \odot g$ be a $(0,4)$-tensor defined on $M$, where $T$ is a $(0,2)$-tensor. We wish to find $\bar{g} = \frac{1}{\varphi^2} g$ such that $\bar{R} = R$, where $\bar{R}$ is the Riemannian curvature tensor of the metric $\bar{g}$. That is, we study the following problem:

$$\left\{ \begin{array}{l} \bar{g} = \frac{1}{\varphi^2} g \\ \bar{R} = R \end{array} \right. . \tag{2.1}$$

Using the decomposition of the Riemannian curvature tensor $\bar{R}$ in (1.1), we obtain that $\bar{R} = R$ is equivalent to $A_{\bar{g}} \odot \bar{g} = T \odot g$. As $\bar{g} = \frac{1}{\varphi^2} g$, this equation is equivalent to

$$A_{\bar{g}} \odot \left( \frac{g}{\varphi^2} \right) = T \odot g.$$

Thus, $\bar{R} = R$ is equivalent to

$$A_{\bar{g}} \odot g = \left( \varphi^2 T \right) \odot g.$$

As the Kulkarni–Nomizu product is injective (see Lemma 1.113 in [13]), problem (2.1) is equivalent to

$$\left\{ \begin{array}{l} \bar{g} = \frac{1}{\varphi^2} g \\ A_{\bar{g}} = \varphi^2 T \end{array} \right. . \tag{2.2}$$

Henceforth, we consider the Euclidean space $(\mathbb{R}^n, g)$, $n \geq 3$, with coordinates $x = (x_1, .., x_n)$ and $g_{ij} = \delta_{ij}$. Given a $(0,4)$-tensor $R = T \odot g$, where $T$ is a diagonal $(0,2)$-tensor defined by $T = \sum_i f_i(x) dx_i^2$ and $f_i(x)$ are smooth functions, we seek necessary and sufficient conditions on the tensor $R = T \odot g$ for the existence of a metric $\bar{g} = \frac{1}{\varphi^2} g$ such that $\bar{R} = R$.

The Schouten tensor of $\bar{g}$ is defined by

$$A_{\bar{g}} = \frac{1}{n-2} \left( Ric_{\bar{g}} - \frac{\bar{K}}{2(n-1)} \bar{g} \right),$$

where $Ric_{\bar{g}}$ and $\bar{K}$ are the Ricci tensor and the scalar curvature of $\bar{g}$, respectively.

As $\bar{g}$ is conformal to the Euclidean metric $g$, the Ricci tensor of $\bar{g}$ is given by

$$\mathrm{Ric}_{\bar{g}} = \frac{1}{\varphi^2} \left\{ (n-2) \varphi Hess_g \varphi + \left( \varphi \Delta_g \varphi - (n-1) |\nabla_g \varphi|^2 \right) g \right\}, \tag{2.3}$$

and the scalar curvature of $\bar{g}$ is given by

$$\bar{K} = (n-1) \left( 2 \varphi \Delta_g \varphi - n |\nabla_g \varphi|^2 \right), \tag{2.4}$$

where $\Delta_g$ and $\nabla_g$ denote the Laplacian and the gradient in the Euclidean metric $g$, respectively [13].

Using (2.3) and (2.4), the Schouten tensor of $\bar{g}$ can be expressed by

$$A_{\bar{g}} = \frac{Hess_g\varphi}{\varphi} - \frac{|\nabla_g\varphi|^2}{2\varphi^2}g. \tag{2.5}$$

We will denote by $\varphi_{x_k}$ and $f_{i,x_k}$ the derivatives of $\varphi$ and $f_i$ with respect to $x_k$, respectively. Likewise, $\varphi_{x_ix_j}$ and $f_{i,x_ix_j}$ are the second order derivatives of $\varphi$ and $f_i$ with respect to $x_ix_j$, respectively. As $g$ is the Euclidean metric in $\mathbb{R}^n$, $n \geq 3$, studying problem (2.2) when $T = \sum_i f_i(x)dx_i^2$ and $f_i(x)$ are smooth functions, is equivalent to studying the following system of equations:

$$\begin{cases} \dfrac{\varphi_{x_ix_i}}{\varphi} - \dfrac{|\nabla_g\varphi|^2}{2\varphi^2} = \varphi^2 f_i, \quad \forall \quad i:1,...,n. \\ \varphi_{x_ix_j} = 0, \quad \forall \quad i \neq j. \end{cases} \tag{2.6}$$

From the second equation of (2.6), it follows that $\varphi$ can be expressed as a sum of functions, each of which depends only on one of the variables $x_i$; thus we will write

$$\varphi(x) = \sum_{i=1}^{n} \varphi_i(x_i).$$

We will study system (2.6) with the additional condition that $3f_i(x)+f_j(x) \neq 0$, for all $x \in \mathbb{R}^n$ and all $i \neq j$.

## 3. Main results

We now state our main results. We start with a lemma that will be used in the proofs to follow.

**Lemma 3.1.** *Let $\varphi(x_1,...,x_n)$ be a solution of (2.6). Then*

$$\frac{\varphi_{x_j}}{\varphi} = -\frac{f_{i,x_j}}{3f_i+f_j}, \qquad \forall i \neq j \tag{3.1}$$

*and*

$$\frac{f_{k,x_j}}{3f_k+f_j} = \frac{f_{i,x_j}}{3f_i+f_j}, \tag{3.2}$$

*for distinct $i,j,k$.*

*Proof.* From the first equation of (2.6) we obtain

$$\varphi_{x_ix_i} = \varphi^3 f_i + \frac{\sum_k (\varphi_{x_k})^2}{2\varphi}.$$

Taking the derivative with respect to $x_j$, $j \neq i$, and using again the first equation of (2.6) we obtain

$$\begin{aligned} 0 &= 3\varphi^2\varphi_{x_j}f_i + \varphi^3 f_{i,x_j} + \frac{2\varphi_{x_j}\varphi_{x_jx_j}2\varphi - 2\varphi_{x_j}|\nabla_g\varphi|^2}{4\varphi^2} \\ &= 3\varphi^2\varphi_{x_j}f_i + \varphi^3 f_{i,x_j} + \varphi_{x_j}\frac{\varphi_{x_jx_j}}{\varphi} - \frac{\varphi_{x_j}|\nabla_g\varphi|^2}{2\varphi^2} \\ &= 3\varphi^2\varphi_{x_j}f_i + \varphi^3 f_{i,x_j} + \varphi_{x_j}\varphi^2 f_j + \frac{\varphi_{x_j}|\nabla_g\varphi|^2}{2\varphi^2} - \frac{\varphi_{x_j}|\nabla_g\varphi|^2}{2\varphi^2} \\ &= \varphi^2(\varphi_{x_j}(3f_i + f_j) + \varphi f_{i,x_j}). \end{aligned}$$

As $\varphi \neq 0$, we obtain

$$\varphi_{x_j}(3f_i + f_j) + \varphi f_{i,x_j} = 0. \tag{3.3}$$

Consequently, we have (3.1). Equation (3.2) follows immediately from (3.1). □

The relationship between the conformal factor $\varphi$ and the functions $f_i$ that compose the tensor $T$ in this Lemma is fundamental for establishing the results of this study.

The next theorem is our main result and provides necessary and sufficient conditions for problem (2.1) to have a solution.

**Theorem 3.2.** *Let* $(\mathbb{R}^n, g)$, $n \geq 3$, *be the Euclidean space, with coordinates* $x_1, ..., x_n$, *and metric* $g_{ij} = \delta_{ij}$. *We consider a* $(0,4)$*-tensor* $R = T \odot g$, *where* $T = \sum_{i=1}^{n} f_i(x)dx_i^2$ *and* $f_i(x)$ *are smooth functions such that* $3f_i(x) + f_j(x) \neq 0$ *for all* $x \in \mathbb{R}^n$ *and all* $i \neq j$. *Then, there exists a positive function* $\varphi$ *such that the metric* $\bar{g} = \frac{1}{\varphi^2}g$ *satisfies* $\bar{R} = R$ *if and only if the functions* $f_i$ *satisfy the following system of differential equations:*

$$\left\{ \begin{array}{l} \dfrac{f_{i,x_j}}{3f_i + f_j} = \dfrac{f_{k,x_j}}{3f_k + f_j}, \quad i \neq j, \quad k \neq j, \\ \left(\dfrac{f_{j,x_i}}{3f_j + f_i}\right)_{x_k} = \left(\dfrac{f_{j,x_k}}{3f_j + f_k}\right)_{x_i}, \quad i \neq j, \quad k \neq j, \\ \left(\dfrac{f_{i,x_j}}{3f_i + f_j}\right)_{x_i} = \left(\dfrac{f_{j,x_i}}{3f_j + f_i}\right)_{x_j}, \quad i \neq j, \\ \dfrac{1}{2}\left(\dfrac{f_{j,x_i}}{3f_j + f_i}\right)^2 - \left(\dfrac{f_{j,x_i}}{3f_j + f_i}\right)_{x_i} - \dfrac{1}{2}\displaystyle\sum_{k \neq i}\left(\dfrac{f_{j,x_k}}{3f_j + f_k}\right)^2 = h_i, \quad i \neq j, \\ \dfrac{f_{j,x_i}}{3f_j + f_i}\,\dfrac{f_{i,x_j}}{3f_i + f_j} = \left(\dfrac{f_{i,x_j}}{3f_i + f_j}\right)_{x_i}, \quad i \neq j, \end{array} \right. \tag{3.4}$$

*where* $h_i(x) = f_i \quad e^{-2\int \frac{f_{i,x_j}}{3f_i + f_j}dx_j + \psi_j(x_1,...,x_{j-1},x_{j+1},...,x_n)}$ *does not depend on* $j$, *the function* $\psi_j(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)$ *satisfies the following system of* $(n-1)$ *differential equations*

$$\psi_{j,x_i} = \int \left(\frac{f_{i,x_j}}{3f_i + f_j}\right)_{x_i} dx_j - \frac{f_{j,x_i}}{3f_j + f_i}, \quad \textit{for} \quad i \neq j. \tag{3.5}$$

*The expression* $\int \frac{f_{i,x_j}}{3f_i+f_j}dx_j + \psi_j(x_1,...,x_{j-1},x_{j+1},...,x_n)$ *is independent of* $i$ *and* $j$ *as long as* $i \neq j$, *and up to a multiplicative constant, the function* $\varphi$ *is given by*

$$\varphi(x) = \exp\left(-\int \frac{f_{i,x_j}}{3f_i+f_j}dx_j + \psi_j(x_1,...,x_{j-1},x_{j+1},...,x_n)\right).$$

*Proof.* We assume that $\bar{g} = g/\varphi^2$ is a solution of (2.2). Then, by Lemma 3.1, the first equation of (3.4) is satisfied, and we obtain, for a fixed $j = 1, \ldots, n$, that

$$\varphi(x) = \exp\left(-\int \frac{f_{i,x_j}}{3f_i+f_j}dx_j + \psi_j(x_1,...,x_{j-1},x_{j+1},...,x_n)\right), \quad i \neq j,$$

where the function $\psi_j(x_1,...,x_{j-1},x_{j+1},...,x_n)$ does not depend on $x_j$.

We will show that the expression of $\varphi$, and consequently of $h_i$, is independent of the variable of integration and the function $\psi_j$ is well-defined.

Taking the derivative of $\varphi$ with respect to $x_i$, $i \neq j$, we obtain

$$\psi_{j,x_i} = \int \left(\frac{f_{i,x_j}}{3f_i+f_j}\right)_{x_i} dx_j + \frac{\varphi_{x_i}}{\varphi} = \int \left(\frac{f_{i,x_j}}{3f_i+f_j}\right)_{x_i} dx_j - \frac{f_{j,x_i}}{3f_j+f_i}.$$

Taking now the derivative of this expression with respect to $x_k$, $k \neq j$, we obtain

$$\psi_{j,x_i x_k} = \int \left(\frac{f_{i,x_j}}{3f_i+f_j}\right)_{x_i x_k} dx_j - \left(\frac{f_{j,x_i}}{3f_j+f_i}\right)_{x_k}.$$

Similarly, we obtain that $\psi_{j,x_k x_i} = \int \left(\frac{f_{i,x_j}}{3f_i+f_j}\right)_{x_k x_i} dx_j - \left(\frac{f_{j,x_k}}{3f_j+f_k}\right)_{x_i}$.

Thus, $\psi_{j,x_i x_k} = \psi_{j,x_k x_i}$ if and only if

$$\left(\frac{f_{j,x_i}}{3f_j+f_i}\right)_{x_k} = \left(\frac{f_{j,x_k}}{3f_j+f_k}\right)_{x_i}.$$

Therefore, the second equation in (3.4) is satisfied.

Taking the derivative of $\psi_{j,x_i}$ with respect to $x_j$, $i \neq j$, we obtain

$$\psi_{j,x_i x_j} = \left(\frac{f_{i,x_j}}{3f_i+f_j}\right)_{x_i} - \left(\frac{f_{j,x_i}}{3f_j+f_i}\right)_{x_j}.$$

As $\psi_{j,x_i x_j} = \psi_{j,x_j x_i} = 0$, the third equation in (3.4) is satisfied.

Using Equation (3.1), we have that, for a fixed $j = 1, \ldots, n$,

$$\ln \varphi(x) = -\int \frac{f_{i,x_j}}{3f_i+f_j}dx_j + \psi_j(x_1,...,x_{j-1},x_{j+1},...,x_n), \quad i \neq j,$$

where the function $\psi_j(x_1,...,x_{j-1},x_{j+1},...,x_n)$ does not depend on $x_j$.

Integrating Equation (3.1) with respect to another variable $x_s$, for a fixed $s = 1, \ldots, n$, $s \neq j$, we obtain

$$\ln \tilde{\varphi}(x) = -\int \frac{f_{i,x_s}}{3f_i+f_s}dx_s + \psi_s(x_1,...,x_{s-1},x_{s+1},...,x_n), \quad i \neq s,$$

where the function $\psi_s(x_1,...,x_{s-1},x_{s+1},...,x_n)$ does not depend on $x_s$.

Setting $A = \ln \varphi(x) - \ln \tilde{\varphi}(x)$, we will show that $A$ is constant.

Taking the derivative with respect to $x_s$, we obtain

$$A_{x_s} = -\int \left(\frac{f_{i,x_j}}{3f_i + f_j}\right)_{x_s} dx_j + \psi_{j,x_s} + \frac{f_{i,x_s}}{3f_i + f_s}.$$

Using Equation (3.2) and that $\psi_{j,x_s} = \int \left(\frac{f_{i,x_j}}{3f_i + f_j}\right)_{x_s} dx_j - \frac{f_{i,x_s}}{3f_i + f_s}$, we conclude that $A_{x_s} = 0$. Similarly, taking the derivative of $A$ with respect to $x_j$, and using Equation (3.2) and the expression of $\psi_{s,x_j}$, we conclude that $A_{x_j} = 0$.

For $k \neq s$ and $k \neq j$, using the expressions of the derivatives of $\psi_j$ and $\psi_s$, and Equation (3.2) in Lemma 3.1, we obtain

$$\begin{aligned}
A_{x_k} &= -\int \left(\frac{f_{i,x_j}}{3f_i + f_j}\right)_{x_k} dx_j + \psi_{j,x_k} + \int \left(\frac{f_{i,x_s}}{3f_i + f_s}\right)_{x_k} dx_s - \psi_{s,x_k} \\
&= -\int \left(\frac{f_{i,x_j}}{3f_i + f_j}\right)_{x_k} dx_j + \int \left(\frac{f_{i,x_j}}{3f_i + f_j}\right)_{x_k} dx_j + \frac{\varphi_{x_k}}{\varphi} \\
&\quad + \int \left(\frac{f_{i,x_s}}{3f_i + f_s}\right)_{x_k} dx_s - \int \left(\frac{f_{i,x_s}}{3f_i + f_s}\right)_{x_k} dx_s - \frac{\tilde{\varphi}_{x_k}}{\tilde{\varphi}} \\
&= -\frac{f_{i,x_k}}{3f_i + f_k} + \frac{f_{i,x_k}}{3f_i + f_k} = 0.
\end{aligned}$$

Thus, $\varphi$ is well-defined. As $\varphi$ is a solution of (2.2), it satisfies (2.6). Lemma 3.1 implies that for $i \neq j$, we have $\varphi_{x_i} = -\varphi \frac{f_{j,x_i}}{3f_j + f_i}$. Taking the derivative with respect to $x_i$, we obtain

$$\frac{\varphi_{x_i x_i}}{\varphi} = -\frac{\varphi_{x_i}}{\varphi}\frac{f_{j,x_i}}{3f_j + f_i} - \left(\frac{f_{j,x_i}}{3f_j + f_i}\right)_{x_i} = \left(\frac{f_{j,x_i}}{3f_j + f_i}\right)^2 - \left(\frac{f_{j,x_i}}{3f_j + f_i}\right)_{x_i}.$$

Thus, using that $|\nabla_g \varphi|^2 = \sum_{k=1}^{n} (\varphi_{x_k})^2 = \sum_{k \neq i} \varphi^2 \left(\frac{f_{j,x_k}}{3f_j + f_k}\right)^2 + (\varphi_{x_i})^2$ and the expression above, the first equation in (2.6) is equivalent to

$$\left(\frac{f_{j,x_i}}{3f_j + f_i}\right)^2 - \left(\frac{f_{j,x_i}}{3f_j + f_i}\right)_{x_i} - \frac{1}{2}\sum_{k \neq i}\left(\frac{f_{j,x_k}}{3f_j + f_k}\right)^2 - \frac{1}{2}\left(\frac{f_{j,x_i}}{3f_j + f_i}\right)^2 = \varphi^2 f_i.$$

Simplifying this expression, we obtain

$$\frac{1}{2}\left(\frac{f_{j,x_i}}{3f_j + f_i}\right)^2 - \left(\frac{f_{j,x_i}}{3f_j + f_i}\right)_{x_i} - \sum_{k \neq i}\left(\frac{f_{j,x_k}}{3f_j + f_k}\right)^2 =$$

$$f_i \, e^{-2\int \frac{f_{i,x_j}}{3f_i + f_j} dx_j + \psi_j(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)}.$$

This proves the fourth equality of (3.4).

Lemma 3.1 implies that for $i \neq j$, we have

$$\begin{aligned}
\varphi_{x_j x_i} &= -\varphi_{x_i}\frac{f_{i,x_j}}{3f_i + f_j} - \varphi\left(\frac{f_{i,x_j}}{3f_i + f_j}\right)_{x_i} \\
&= \varphi\left\{\frac{f_{j,x_i}}{3f_j + f_i}\frac{f_{i,x_j}}{3f_i + f_j} - \left(\frac{f_{i,x_j}}{3f_i + f_j}\right)_{x_i}\right\} = 0.
\end{aligned}$$

As $\varphi \neq 0$, the other expression equals zero, which proves the fifth and last equality of (3.4). The converse is a straightforward computation. $\square$

To provide explicit examples of metrics satisfying $\bar{R} = R$, we shall consider particular cases for $T$.

**Theorem 3.3.** *Let $(\mathbb{R}^n, g)$, $n \geq 3$, be the Euclidean space, with coordinates $x_1, ..., x_n$, and metric $g_{ij} = \delta_{ij}$. Then, there exists a metric $\bar{g} = \dfrac{1}{\varphi^2} g$ such that $\bar{R} = R = f(g \odot g)$, where $f$ is a nonvanishing smooth function, if and only if*

$$f(x) = \frac{-\lambda}{2(\sum_{i=1}^{n}(ax_i^2 + b_i x_i) + c)^4}, \tag{3.6}$$

*where $a, b_i, c$ are constants, $\lambda = \sum_{i=1}^{n} b_i^2 4ac$, and*

$$\varphi(x) = \sum_{i=1}^{n}(ax_i^2 + b_i x_i) + c. \tag{3.7}$$

*Any such metric $\bar{g}$ is unique up to homothety. Moreover, we have:*

(1) *If $\lambda < 0$, then $\bar{g}$ is globally defined on $\mathbb{R}^n$.*
(2) *If $\lambda \geq 0$, then the metric $\bar{g}$ is defined:*
  (a) *on all $\mathbb{R}^n$ if $\lambda = 0$ and $a = 0$;*
  (b) *in $\mathbb{R}^n$ minus a point if $\lambda = 0$ and $a \neq 0$;*
  (c) *in $\mathbb{R}^n \setminus L$ if $\lambda > 0$ and $a = 0$ , where $L$ is a hyperplane;*
  (d) *in $\mathbb{R}^n \setminus \mathbb{S}$ if $\lambda > 0$ and $a \neq 0$, where $\mathbb{S}$ is an $(n-1)$-dimensional sphere.*

*Proof.* As $f_i = f_j$, for all $i, j$, Lemma 3.1 implies that

$$\frac{\varphi_{x_j}}{\varphi} = -\frac{f_{x_j}}{4\,f} \quad \text{for all} \quad j. \tag{3.8}$$

Therefore, there exists a constant $\lambda$ such that $\varphi^4 f = -\dfrac{\lambda}{2}$ and $f = -\dfrac{\lambda}{2\varphi^4}$.

(2.6) implies that

$$\varphi_{x_i x_i} = f\varphi^3 + \frac{|\nabla_g \varphi|^2}{2\varphi}, \quad \text{for all} \quad i = 1, ..., n.$$

Then

$$\varphi_{x_i x_i} = \varphi_{x_j x_j}$$

for all $i, j$. Thus, for every $i = 1, ..., n$, $\varphi_i(x_i) = ax_i^2 + b_i x_i + c_i$ and

$$\varphi(x) = \sum_{i=1}^{n} \varphi_i(x_i) = \sum_{i=1}^{n}(ax_i^2 + b_i x_i) + c.$$

Using the above relation between $\varphi$ and $f$, we obtain that

$$f(x) = \frac{-\lambda}{2(\sum_{i=1}^{n}(ax_i^2 + b_i x_i) + c)^4}.$$

Calculating the expressions in the first equality in (2.6) we obtain that $\lambda = \sum_i b_i^2 - 4ac$. Analyzing the expression of $\varphi$, we arrive at the conclusions concerning the domain of $\varphi$. Particularly, if $\lambda < 0$, the function $\varphi$ does not vanish, and the metric $\bar{g}$ is globally defined on $\mathbb{R}^n$. □

*Remark* 3.4. In this theorem, we directly used system (2.6). However, Equations (3.4) of Theorem 3.2 are satisfied with $f_i = f\ \forall i = 1, ..., n$ and $\varphi$ given by (3.7) with $\psi(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)$ constant. This shows that there exist examples of tensors in $\mathbb{R}^n$ that are solutions of the equations of Theorem 3.2.

We now present a non-existence result for conformal metrics for a special tensor $R$.

**Theorem 3.5.** *Let $(\mathbb{R}^n, g)$, $n \geq 3$, be the Euclidean space, with coordinates $x_1, ..., x_n$, and metric $g_{ij} = \delta_{ij}$. We consider the $(0,4)$-tensor $R = T \odot g$, where $T = \sum_{i=1}^{n} f_i(x_i) dx_i^2$ and $f_i(x_i)$ are smooth functions that depend on only the variable $x_i$ such that $3f_i(x_i) + f_j(x_j) \neq 0$ for all $(x_i, x_j) \in \mathbb{R} \times \mathbb{R}$, $i \neq j$. Then, there is no metric $\bar{g} = \frac{1}{\varphi^2} g$ such that $\bar{R} = R$.*

*Proof.* As $f_i$ do not depend on the variables $x_j$, $j \neq i$, (3.1) implies that

$$\frac{\varphi_{x_j}}{\varphi} = -\frac{f_{i,x_j}}{3f_i + f_j} = 0 \quad \text{for all} \quad j \neq i, \tag{3.9}$$

and

$$\frac{\varphi_{x_i}}{\varphi} = -\frac{f_{j,x_i}}{3f_j + f_i} = 0 \quad \text{for all} \quad i \neq j. \tag{3.10}$$

Thus, $\varphi_{x_k} = 0$ for all $k$; therefore $\varphi$ is constant. Using (2.6) we conclude that $f_i = 0$, for all $i = 1, \dots, n$, which contradicts $3f_i + f_j \neq 0$, for $(x_i, x_j) \in \mathbb{R} \times \mathbb{R}$, $i \neq j$. Hence, there does not exist a metric $\bar{g}$ such that $\bar{R} = R$. □

In the particular case in which the components of the tensor $T$ depend only on one variable, we have the following result.

**Theorem 3.6.** *Let $(\mathbb{R}^n, g)$, $n \geq 3$, be the Euclidean space, with coordinates $x_1, ..., x_n$, and metric $g_{ij} = \delta_{ij}$. We consider a $(0,4)$-tensor $R = T \odot g$, where $T$ is a diagonal $(0,2)$-tensor given by $T = \sum_{i=1}^{n} f_i(x_k) dx_i^2$ and $f_i(x_k)$ are smooth functions that depend only on $x_k$, for some fixed $k$, $1 \leq k \leq n$, such that $3f_i(x_k) + f_j(x_k) \neq 0$ for all $x_k \in \mathbb{R}$ and $i \neq j$. There exists a metric $\bar{g} = \frac{1}{\varphi^2} g$ such that $\bar{R} = R$ if and only if all functions $f_i$ for $i \neq k$ are equal to a function $f$, and $f$ and $f_k$ satisfy the system*

$$\begin{cases} \dfrac{1}{2}\left(\dfrac{f_{x_k}}{3f + f_k}\right)^2 - \left(\dfrac{f_{x_k}}{3f + f_k}\right)_{x_k} = C^2 f_k\, v \\ -\left(\dfrac{f_{x_k}}{3f + f_k}\right)^2 = 2C^2 f\, v \end{cases} \tag{3.11}$$

*where $v = v(x_k) = e^{-2\int \frac{f_{x_k}}{3f + f_k} dx_k}$. Moreover, if $f_k$ and $f$ satisfy these conditions, then $\varphi$ depends only on $x_k$ and is given by*

$$\varphi(x_k) = C \exp\left(-\int \frac{f_{x_k}}{3f + f_k} dx_k\right), \tag{3.12}$$

*where $C$ is a positive constant.*

*Proof.* As $f_i = f_i(x_k)$ for some fixed $k$, Lemma 3.1 implies that $\varphi_{x_j} = 0$ for every $j \neq k$; therefore $\varphi = \varphi(x_k)$. Furthermore, from (2.6) we have that $f_i = -\dfrac{|\nabla_g \varphi|^2}{2\varphi^4}$ for all $i \neq k$; hence, $f_i = f_j$, if $i \neq k$ and $j \neq k$. Hence, the expression of $\varphi$ in (3.12) and the system in (3.11) are consequences of Theorem 3.2. □

**Corollary 3.7.** Let $(\mathbb{R}^n, g)$, $n \geq 3$, be the Euclidean space, with coordinates $x_1, ..., x_n$, and metric $g_{ij} = \delta_{ij}$. We consider the $(0,2)$-tensor

$$T = f_k(x_k)dx_k^2 + f(x_k)\sum_{i \neq k} dx_i^2,$$

where $f_k(x_k) = \dfrac{h^2 - 2h_{x_k}}{2C^2} e^{2\int h(x_k)dx_k}$, $f(x_k) = -\dfrac{h^2}{2c^2} e^{2\int h(x_k)dx_k}$, and $h = h(x_k)$ is a smooth function that depends only on $x_k$, for some fixed $k$, $1 \leq k \leq n$. Then, there exists a conformal metric $\bar{g} = g/\varphi^2$ such that $\bar{R} = R$ and

$$\varphi(x_k) = C \exp(-\int h(x_k)\ dx_k), \tag{3.13}$$

where $C$ is a positive constant.

If, in addition, $0 \leq \left| \int h(x_k)\ dx_k \right| \leq L$, for a finite constant $L$, then the metric $\bar{g}$ is complete on $\mathbb{R}^n$.

*Proof.* It follows immediately from Theorem 3.6, considering $h(x_k) = \dfrac{f_{x_k}}{3f + f_k}$. The equalities in (3.11) are trivially satisfied and the expressions of $f$ and $f_k$ are exactly the components of the tensor $T$. □

We can extend Theorem 3.2 to locally conformally flat manifolds. Let $(M^n, g)$ be a locally conformally flat Riemannian manifold. We may consider problem (2.1) for a neighborhood $V \subset M$ with local coordinates $(x_1, x_2, ..., x_n)$ such that $g_{ij} = \delta_{ij}/F^2$, where $F$ is a non-vanishing smooth function on $V$.

**Theorem 3.8.** *Let $(M^n, g)$, $n \geq 3$, be a locally conformally flat Riemannian manifold. Let $V$ be an open subset of $M$ with coordinates $x = (x_1, x_2, ..., x_n)$ and $g_{ij} = \delta_{ij}/F^2$. We consider a $(0,4)$-tensor $R = T \odot g$, where $T$ is a diagonal $(0,2)$-tensor given by $T = \sum_{i=1}^{n} f_i(x)dx_i^2$ and $f_i$ are smooth functions such that $3f_i(x) + f_j(x) \neq 0$ for all $x \in V$ and all $i \neq j$. Then, there exists a metric $\bar{g} = \dfrac{1}{\phi^2} g$ such that $\bar{R} = R$ if and only if the functions $f_i$, $\varphi$, and $\psi$ are given as in Theorem 3.2 and $\phi = \dfrac{\varphi}{F}$.*

*Proof.* We consider $\varphi = \phi F$ and apply Theorem 3.2. □

*Remark* 3.9. In a similar fashion, we can extend Theorem 3.6 for locally conformally flat manifolds.

As an application of Theorem 3.8, we show that given a $(0,4)$-tensor $R$ in $\mathbb{R}^n_+$, there exists a metric $\bar{g}$, conformal to the metric of the hyperbolic space, whose Riemannian curvature tensor is $R$.

**Example 3.10.** Let $\mathbb{H}^n = (\mathbb{R}^n_+, g)$ be the hyperbolic space, where $g = \dfrac{1}{x_n^2} g_0$, $(g_0)_{ij} = \delta_{ij}$ is the Euclidean metric, and $\mathbb{R}^n_+ = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n / x_n > 0\}$.

Given the $(0,4)$-tensor $R = T \odot g$, where $T$ is the diagonal (0,2)-tensor

$$T = -\frac{(2x_n^2 - 1)^2}{2x_n^4} e^{2x_n^2} \sum_{i \neq n} dx_i^2 + \frac{4x_n^4 - 8x_n^2 - 1}{2x_n^4} e^{2x_n^2} dx_n^2$$

defined in $\mathbb{R}^n_+$, Theorem 3.8 implies the existence of a metric $\bar{g} = \dfrac{1}{\phi^2} g$, where $\phi(x) = \dfrac{\varphi(x)}{F(x)} = e^{-x_n^2}$ such that $\bar{R} = R$.

Moreover, as $\phi(x)$ is bounded, $(\mathbb{R}^n_+, \bar{g})$ is a complete Riemannian manifold, conformal to the hyperbolic space.

The scalar curvature of $(\mathbb{R}^n_+, \bar{g})$ is not constant and given by

$$\bar{K} = (n-1) e^{-2x_n^2} (4(2-n)x_n^4 + 4(n-3)x_n^2 - n),$$

and the Ricci tensor of the metric $\bar{g}$ is

$$Ric_{\bar{g}} = \frac{4(2-n)x_n^4 + 2(2n-5)x_n^2 + 1 - n}{x_n^2} \sum_{i \neq n} dx_i^2 + (n-1) \frac{4x_n^4 - 4x_n^2 - 1}{x_n^2} dx_n^2.$$

Likewise, the sectional curvature of $(\mathbb{R}^n_+, \bar{g})$ is non-constant and given by

$$K\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = -(1 - 2x_n^2)^2 e^{-2x_n^2} \leq 0,$$

if $i, j \neq n$, and

$$K\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_n}\right) = 2x_n^2 (2x_n^2 - 3) e^{-2x_n^2}.$$

**Example 3.11.** Corollary 3.7 also provides examples of complete, conformally flat manifolds with prescribed Riemannian curvature tensor and non-constant curvatures.

(1) In the Euclidean space $(\mathbb{R}^n, g)$, $n \geq 3$, we consider the $(0,4)$-tensor $R = T \odot g$, where $T$ is a diagonal (0,2)-tensor given by

$$T = \left(\frac{\sinh^2 x_k - 2\cosh x_k}{2C^2} e^{2\cosh x_k}\right) dx_k^2 - \frac{\sinh^2 x_k}{2C^2} e^{2\cosh x_k} \sum_{i \neq k} dx_i^2,$$

where $C$ is a positive constant.

Corollary 3.7 implies the existence of a metric $\bar{g} = \dfrac{1}{\varphi^2} g$, conformal to the Euclidean metric, such that $\bar{R} = R = T \odot g$ is the Riemannian curvature tensor of the metric $\bar{g}$. In particular, we have that

$$\varphi(x_k) = Ce^{-\cosh x_k},$$

where $C$ is a positive constant. The manifold $(\mathbb{R}^n, \bar{g})$ is complete and has negative scalar curvature given by

$$\bar{K} = -(n-1) Ce^{-\cosh x_k} (2\cosh^2 x_k + (n-2)\sinh^2 x_k)$$

and negative Ricci curvature whose Ricci tensor is negative definite and given by

$$Ric_{\bar{g}} = -(n-1)\cosh x_k dx_k^2 - (\cosh x_k + (n-2)\sinh^2 x_k)\sum_{i\neq k} dx_i^2.$$

Moreover, $(\mathbb{R}^n, \bar{g})$ has non-positive sectional curvature given by

$$K\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = -C^2 \sinh^2 x_k e^{-2\cosh x_k},$$

if $i, j \neq k$, and

$$K\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}\right) = -C^2 \cosh x_k e^{-2\cosh x_k}.$$

(2) In the Euclidean space $(\mathbb{R}^n, g)$, $n \geq 3$, we consider the $(0,4)$-tensor $R = T \odot g$, where $T$ is a diagonal $(0,2)$-tensor given by

$$T = \frac{(4x_k^2 - 2)}{C^2} dx_k^2 - \frac{2x_k^2}{C^2}\sum_{i\neq k} dx_i^2,$$

where $C$ is a positive constant. Corollary 3.7 implies the existence of a metric $\bar{g} = \dfrac{1}{\varphi^2} g$, conformal to the Euclidean metric, such that $\bar{R} = R = T \odot g$ is the Riemannian curvature tensor of the metric $\bar{g}$. In particular, we have

$$\varphi(x_k) = \frac{C}{1 + x_k^2},$$

where $C$ is a positive constant. The manifold $(\mathbb{R}^n, \bar{g})$ is complete, it has negative scalar curvature given by

$$\bar{K} = -\frac{4(n-1)C^2}{(1+x_k^2)^2}(1 + (n-3)x_k^2),$$

and the Ricci tensor of $\bar{g}$ is given by

$$Ric_{\bar{g}} = \frac{2(n-1)(x_k^2 - 1)}{(1+x_k^2)^2} dx_k^2 + \frac{(10-4n)x_k^2 - 2}{(1+x_k^2)^2}\sum_{i\neq k} dx_i^2.$$

Moreover, $(\mathbb{R}^n, \bar{g})$ has sectional curvature given by

$$K\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = -\frac{4C^2 x_k^2}{(1+x_k^2)^4},$$

if $i, j \neq k$, and

$$K\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}\right) = -\frac{2C^2(1 - x_k^2)}{(1+x_k^2)^4}.$$

(3) In the Euclidean space $(\mathbb{R}^n, g)$, we consider the $(0,4)$-tensor $R = T \odot g$, where $T$ is a diagonal $(0,2)$-tensor given by

$$T = \frac{2(x_k^2 - 1)}{C^2} e^{2x_k^2} dx_k^2 - \frac{2x_k^2}{C^2} e^{2x_k^2}\sum_{i\neq k} dx_i^2,$$

where $C$ is a positive constant. Corollary 3.7 implies the existence of a metric $\bar{g} = \dfrac{1}{\varphi^2} g$, conformal to the Euclidean metric, such that $\bar{R} = R =$

$T \odot g$ is the Riemannian curvature tensor of the metric $\bar{g}$. In particular, we have that

$$\varphi(x_k) = Ce^{-x_k^2},$$

where $C$ is a positive constant. The manifold $(\mathbb{R}^n, \bar{g})$ is complete and has negative scalar curvature given by

$$\bar{K} = -4(n-1)C^2 e^{-2x_k^2}(1 + (n-2)x_k^2)$$

and negative Ricci curvature, whose Ricci tensor is given by

$$Ric_{\bar{g}} = -2(n-1)dx_k^2 - 2(1 - 2(n-2)x_k^2)\sum_{i \neq k} dx_i^2.$$

The sectional curvature of $(\mathbb{R}^n, \bar{g})$ is non-positive and given by the expressions

$$K\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = -4x_k^2 C^2 e^{-2x_k^2},$$

if $i, j \neq k$, and

$$K\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}\right) = -2C^2 e^{-2x_k^2}.$$

We observe that although there are points where the tensor $R$ is zero, there still exists a complete metric such that the curvature tensor of this metric is the prescribed tensor $R$.

Acknowledgment: We would like to thank the reviewer for their detailed comments and suggestions for the manuscript.

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