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Invariants of contact Lie algebras

G. Salgado-González *

Fac. de Ciencias, UASLP, Av. Salvador Nava s/n, Zona Universitaria,, CP 78290, San Luis Potosí, S.L.P., Mexico
Centro de Investigación en Matemáticas, Guanajuato, Mexico

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ABSTRACT

The aim of this work is to prove that any contact Lie algebra has a semi-invariant. We explicitly compute it and give conditions for it to be an invariant of the contact Lie algebra. As a consequence we can prove that any perfect contact Lie algebra has trivial center. Moreover, we prove that a perfect Lie algebra is a contact Lie algebra if, and only if, it has only one invariant.

1. Introduction

There is a deep interplay between mathematics and physics. The theory of Lie groups and Lie algebras, as well as their representation theory, are good examples of this. Another good example is given by Frobenius Lie groups or Frobenius Lie algebras as they provide "integrable models" via their relations with the classical Yang–Baxter equation, more exactly with the so-called rational solutions of the CYBE. First relation between Frobenius algebras and the CYBE were discovered by Belavin and Drinfeld in their famous paper [5]. Final relations between rational solutions and Frobenius Lie algebras were established in [21].

In particular, the study of invariant polynomials is an useful tool since they represent important physical quantities in quantum mechanics such as angular momentum, a relativistic elementary particle's mass and spin or the Hamiltonian of a particle undergoing geodesic motion.

Invariant polynomials of a semisimple Lie algebra were found first by G. Racah, who gave an explicit construction of them. Currently, invariant polynomials are known as Casimir invariants or Casimir operators. Formally, given a Lie algebra $\mathfrak{g}$, a Casimir invariant is a polynomial in the universal enveloping algebra of $\mathfrak{g}$ that commutes with every element of $\mathfrak{g}$, i.e. the Casimir invariant is a central element of the universal enveloping algebra of $\mathfrak{g}$. If the Lie algebra $\mathfrak{g}$ is semisimple, it is well-known that the number of invariants of $\mathfrak{g}$ is equal to its rank, i.e., each one of its invariants is Casimir. As we can expect, the non-semisimple case is not easy to work with, and one can find plenty of works on this topic, for instance: [1,6–12,15,16,18,23], among others.

For example, in [9] and [12] R. Campoamor describes the invariants for perfect Lie algebras, proving that each one of them is a Casimir invariant, and giving a bound for the number of them. On the other hand, in [15] and [16] explicit

* Correspondence to: Fac. de Ciencias, UASLP, Av. Salvador Nava s/n, Zona Universitaria,, CP 78290, San Luis Potosí, S.L.P., Mexico.
E-mail addresses: gsalgado@fciencias.uaslp.mx, gil.salgado@cimat.mx.

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calculations are made over particular families of Lie algebras, for example, Lie algebras with abelian nilradical, or Lie algebras with nilpotent radical. In these cases, the invariants are not necessarily of polynomial type: they can be rational functions, or even transcendental ones.

Our motivation to determine the invariants of contact Lie algebras comes from [17]. In this work, A.I. Ooms presents a deep study on Frobenius Lie algebras; in particular, he proves that a Frobenius Lie algebra only admits one semi-invariant, and he explicitly shows how to compute it in terms of the Pfaffian associated to the Frobenius structure of such a Lie algebra. On the other hand, there exists a close relationship between Frobenius and contact Lie algebras. There exists a parallelism between the two theories that has been developed in [2,4,13,14] and [20]. Thus, following the ideas of [17], we can associate a Pfaffian $\mathrm{Pf}_{\mathfrak{g}}$ to each contact structure on a Lie algebra $\mathfrak{g}$, and we can prove that $\mathrm{Pf}_{\mathfrak{g}}$ is a semi-invariant. If in addition $\mathfrak{g}$ is perfect, then $\mathrm{Pf}_{\mathfrak{g}}$ is an invariant of $\mathfrak{g}$; moreover, $\mathrm{Pf}_{\mathfrak{g}}$ is a Casimir invariant and it is the only invariant that such a $\mathfrak{g}$ admits. We also prove that perfect contact Lie algebras with non-trivial center do not exist. In order to do this, for the sake of completeness we describe the structure of a contact Lie algebra with non-trivial center, and as a consequence we obtain that any perfect Lie algebra with non-trivial center has at least two invariants, and hence we can conclude that every perfect contact Lie algebra has trivial center. Therefore, as a corollary we can prove that a perfect Lie algebra is a contact Lie algebra if, and only if, it has only one invariant.

This work is organized as follows. In Section 2, we provide basic definitions on contact and Frobenius Lie algebras; we also prove that a Pfaffian can be associated to any contact Lie algebra. For the sake of completeness, in Section 3 we present some previous results on contact and Frobenius solvable Lie algebras with abelian nilradical (see [2] for more details), and we state a characterization of contact Lie algebras in terms of its extended structure matrix. In Section 4 we show how to associate to every contact Lie algebra a semi-invariant, and give conditions for it to be an invariant. As a consequence, in Section 5 we prove that contact perfect Lie algebras with non-trivial center do not exist. Moreover, a perfect Lie algebra is a contact Lie algebra if, and only if, it has only one invariant. Finally, this family of contact Lie algebras are interesting because they provide examples of such algebras that can not be expressed as contact double extensions of contact Lie algebras of codimension 2 (see [3] and [19]).

2. Preliminaries

Throughout this paper $\mathfrak{g}$ will be a finite-dimensional Lie algebra over a field $\mathbb{F}$. Here and subsequently, $\mathbb{F}$ will denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

2.1. Contact and Frobenius Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Given an element $\varphi \in \mathfrak{g}^*$ we can construct the following skew-symmetric bilinear form on $\mathfrak{g}$:

$$B_\varphi(x, y) := \varphi([x, y]),$$

Let, as usual, $\mathrm{Rad}(B_\varphi) = \{x \in \mathfrak{g} \mid B_\varphi(x, y) = 0,\ \forall y \in \mathfrak{g}\}$. It follows that $Z(\mathfrak{g}) \subseteq \mathrm{Rad}(B_\varphi)$. From the definition we get that $B_\varphi([x, y], z) + B_\varphi([y, z], x) + B_\varphi([z, x], y) = 0$ for all $x, y, z \in \mathfrak{g}$, i.e., B_φ is a 2-cocycle for the scalar cohomology of $\mathfrak{g}$.

Definition 2.1. Let $\mathfrak{g}$ be a Lie algebra. We say that $\mathfrak{g}$ is a **Frobenius** Lie algebra, or a **symplectic exact** Lie algebra, if there exists $\varphi \in \mathfrak{g}^*$ such that B_φ is a non-degenerate skew-symmetric bilinear form on $\mathfrak{g}$. If $\dim \mathfrak{g} = 2n + 1$, $\mathfrak{g}$ is a **contact** Lie algebra if there exists $\varphi \in \mathfrak{g}^*$ such that $\varphi \wedge (d\varphi)^n \neq 0$.

Remarks 2.2.

(1) Let $\mathfrak{g}$ be a Lie algebra such that $\dim \mathfrak{g} = 2n$, and let $\varphi \in \mathfrak{g}^*$. Then B_φ is non-degenerate if, and only if, $(d\varphi)^n \neq 0$.
(2) Let $\mathfrak{g}$ be a Lie algebra such that $\dim \mathfrak{g} = 2n + 1$, and let $\varphi \in \mathfrak{g}^*$. Then $\varphi \wedge (d\varphi)^n \neq 0$ implies $\dim(\mathrm{Rad}(B_\varphi)) = 1$.

2.2. Invariants on Lie algebras

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, $D(\mathfrak{g})$ the division ring of quotients of $U(\mathfrak{g})$ and let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$.

Definition 2.3. Let $\mathfrak{g}$ be a Lie algebra, $\lambda \in \mathfrak{g}^*$. The set of all $f \in D(\mathfrak{g})$ such that $[x, f] = \lambda(x)f$ for all $x \in \mathfrak{g}$ is denoted by $D(\mathfrak{g})_\lambda$. Its elements are called **semi-invariants** of $D(\mathfrak{g})$ relative to λ. We put $U(\mathfrak{g})_\lambda = U(\mathfrak{g}) \cap D(\mathfrak{g})_\lambda$. More over, when $\lambda = 0$, f will be called an **invariant**.

Given a Lie algebra $\mathfrak{g}$, the following task is to construct or determine the basis of the functionally independent invariants (semi-invariants). To do that, we consider the realization of $\mathfrak{g}$ in the space $C^\infty(\mathfrak{g}^*)$ determinated by the differential operators

$$\widehat{X}_i := \sum_{k=1}^{n} \sum_{j=1}^{n} C_{i,j}^k x_k \frac{\partial}{\partial x_j},$$

where $C_{i,j}^k$ are the structure constants of $\mathfrak{g}$, i.e., if $\{e_1, \ldots, e_n\}$ is a basis for $\mathfrak{g}$, then

$$[e_i, e_j]_{\mathfrak{g}} = \sum_{k=1}^{n} C_{i,j}^k e_k,$$

for all $1 \le i, j \le n$, and $\{x_1, \ldots, x_n\}$ are the coordinates in $\mathfrak{g}^*$ associated with the basis dual to the basis $\{e_1, \ldots, e_n\}$. Now, $f(x_1, \ldots, x_n) \in C^\infty(\mathfrak{g}^*)$ is an invariant whenever

$$\widehat{X}_i(f) = 0, \quad \forall\, i, \quad 1 \le i \le n. \tag{1}$$

This shows, in some way, that polynomial invariants of $\mathfrak{g}$ are completely determined by its structure constants. The solutions of (1) are called **generalized Casimir invariants**. A maximal set of functionally independent solutions of (1) will be called a **fundamental set of invariants**. Let $N(\mathfrak{g})$ be maximal possible number of functionally independent invariants. Now, it has been proved (cf. [6]) that:

$$N(\mathfrak{g}) = \dim \mathfrak{g} - \sup\{\operatorname{Rank} C(x) |\; C(x)_{i,j} = \sum_{k=1}^{n} C_{i,j}^k x_k\}, \tag{2}$$

where $x = (x_1, \ldots, x_n)$.

Given a polynomial invariant $f(x)$ one finds the corresponding invariant of the Lie algebra $\mathfrak{g}$ as symmetrization, $\operatorname{Sym}(f)$, of f. More precisely, the symmetrization operator Sym acts only on the monomials of the forms $x_{i_1} \cdots x_{i_r}$, where there are non-commuting elements among $x_{i_1}, \ldots, x_{i_r}$, and is defined by the formula

$$\operatorname{Sym}(x_{i_1} \cdots x_{i_r}) = \frac{1}{r!} \sum_{\sigma \in S_r} x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(r)}}.$$

For polynomial invariant $f(x)$, $\operatorname{Sym}(f(x))$ is a Casimir operator replacing the variables x_i by the corresponding generator e_i.

2.3. Pfaffian associated to Lie algebras

We recall the following very well known facts: Let $n = 2k$ and A be the skew-symmetric $n \times n$ matrix such that $(A)_{ij} := x_{ij}$ for $i < j$. Then there exists a unique polynomial Pf_A with integer coefficients in the $\frac{1}{2}n(n-1)$ independent indeterminates x_{ij} and such that:

$$\det(A) = (\mathrm{Pf}_A)^2, \quad \text{and} \quad \mathrm{Pf}_B = 1,$$

for $B = \operatorname{Diag}(J, J, \ldots, J)$, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Observe that Pf_A is homogeneous of degree $\frac{n}{2}$.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and let $C(\mathfrak{g})$ be its structure matrix defined by:

$$C(\mathfrak{g})_{ij} := [e_i, e_j]$$

where $\beta = \{e_1, \ldots, e_n\}$ is a basis for $\mathfrak{g}$. Clearly $C(\mathfrak{g}) \in \operatorname{Mat}(n \times n, S(\mathfrak{g}))$, where $S(\mathfrak{g})$ denotes the symmetric algebra of $\mathfrak{g}$, and $C(\mathfrak{g})^T = -C(\mathfrak{g})$.

We can associate to a Lie algebra $\mathfrak{g}$ its Pfaffian (with respect to a fixed basis β) in the following way:

$$\mathrm{Pf}_{\mathfrak{g}} := \mathrm{Pf}_{C(\mathfrak{g})}, \quad \text{where} \quad \det(C(\mathfrak{g})) = \mathrm{Pf}^2_{C(\mathfrak{g})}.$$

Obviously, if $\mathfrak{g}$ is an odd-dimensional Lie algebra then $\mathrm{Pf}_{\mathfrak{g}} = 0$.

3. Solvable contact and Frobenius Lie algebras

In order to get some insight this section is devoted to collecting some previous results that appear in [4].

3.1. Contact and Frobenius solvable Lie algebras with abelian nilradical

From now on, we shall use the convention that the space $\mathbb{F}^n$ will be considered as an n-dimensional *abelian* Lie algebra, and we will denote it by $\mathfrak{a}_n$. Let $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{a}_n)$ be an n-dimensional Lie subalgebra of $\mathfrak{gl}(\mathfrak{a}_n)$.

Theorem 3.1 (*See Thm. 3.3 in [4] or Prop. 1 in [22]*). *Let $\mathfrak{a}_n$ be a finite dimensional vector space, and $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{a}_n)$ be a Lie subalgebra such that $\dim \mathfrak{h} = \dim \mathfrak{a}_n = n$. The semidirect sum $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{a}_n$ of $\mathfrak{h}$ and $\mathfrak{a}_n$ is a Frobenius Lie algebra if, and only if, there exists $\varphi \in \mathfrak{a}_n^*$ such that $\det(\varphi(N)) \neq 0$.*

Where $C(\mathfrak{g}) = \begin{pmatrix} C(\mathfrak{h}) & N \\ -N^T & 0 \end{pmatrix}$.

Remark 3.2. In fact, Proposition 2.1 in [22]

Corollary 3.3. *Under the same hypotheses* $\mathrm{Pf}_{\mathfrak{g}} = (-1)^{\frac{n(n+1)}{2}} \det(N)$. *More over,* $\mathrm{Pf}_{\mathfrak{g}} \in S(\mathfrak{a}_n)$.

For contact Lie algebras we have:

Theorem 3.4 (*See Thm. 3.5 in [4]*)**.** *Let* $\mathfrak{a}_{n+1}$ *be a finite dimensional vector space, and* $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{a}_{n+1})$ *be a Lie subalgebra such that* $\dim \mathfrak{h} + 1 = \dim \mathfrak{a}_{n+1}$. *The semidirect sum* $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{a}_{n+1}$ *is a contact Lie algebra if, and only if, there exists* $\varphi \in \mathfrak{a}_{n+1}^*$ *such that*

$$\det \begin{pmatrix} [\varphi]^T \\ \varphi(N) \end{pmatrix} \neq 0,$$

where $[\varphi]^T$ *is the associated matrix of* φ *in a given basis of* $\mathfrak{a}_{n+1}^*$.

Moreover, there is a deep relationship between these two types of Lie algebras. Let us define a linear transformation $e_0 : \mathfrak{h} \ltimes \mathfrak{a}_{n+1} \to \mathfrak{h} \ltimes \mathfrak{a}_{n+1}$ in the following way: $e_0|_{\mathfrak{h}} \equiv 0$ and $e_0|_{\mathfrak{a}_{n+1}} = \mathrm{Id}_{\mathfrak{a}_{n+1}}$. It is easy to verify that $e_0 \in \mathrm{Der}(\mathfrak{h} \ltimes \mathfrak{a}_{n+1})$. We can consider the Lie algebra defined by the semidirect sum of $\langle e_0 \rangle$ and $\mathfrak{h} \ltimes \mathfrak{a}_{n+1}$,

$$\begin{aligned} \langle e_0 \rangle \ltimes (\mathfrak{h} \times \mathfrak{a}_{n+1}) &\simeq (\langle e_0 \rangle \oplus \mathfrak{h}) \ltimes \mathfrak{a}_{n+1} \\ &\simeq \mathfrak{a}_{n+1} \ltimes \mathfrak{a}_{n+1} \end{aligned}$$

Theorem 3.5 (*See Thm. 3.6 in [4]*)**.** $(\mathfrak{h} \ltimes \mathfrak{a}_{n+1}, \varphi)$ *is a contact Lie algebra if, and only if,* $\big((\langle e_0 \rangle \oplus \mathfrak{h}) \ltimes \mathfrak{a}_{n+1}, \varphi\big)$ *is a Frobenius Lie algebra, where* $\varphi \in \mathfrak{a}_{n+1}^*$.

3.2. Characterizing contact Lie algebras

Let $\mathfrak{g}$ be a contact Lie algebra with contact structure given by $\varphi \in \mathfrak{g}^*$, i.e., $\varphi \wedge (d\varphi)^n \neq 0$. Let $C(\mathfrak{g})$ be its structure matrix, and B_φ its associated skew-symmetric bilinear form, then

$$[B_\varphi] = \varphi(C(\mathfrak{g})).$$

We will denote by $[\varphi] = \begin{pmatrix} x_1 & \cdots & x_{2n+1} \end{pmatrix}^t$ the coordinate vector in $\mathbb{F}^{2n+1}$ such that $\varphi = \sum_{i=1}^{2n+1} x_i e^i$ where $\{e^1, \dots, e^{2n+1}\}$ is the dual basis associated to the fixed basis $\{e_1, \dots, e_{2n+1}\}$ of $\mathfrak{g}$. We define

$$[\widehat{B}_\varphi] = \begin{pmatrix} 0 & [\varphi]^T \\ -[\varphi] & \varphi(C(\mathfrak{g})) \end{pmatrix} \in \mathrm{Mat}((2n+2) \times (2n+2), \mathbb{F}).$$

Moreover, $[\widehat{B}_\varphi]^T = -[\widehat{B}_\varphi]$. A straightforward computation proves:

Lemma 3.6. *Let* $\mathfrak{g}$ *be a Lie algebra with* $\dim \mathfrak{g} = 2n+1$ *and* $\varphi \in \mathfrak{g}^*$. *With the same notation as before:*

$$\varphi \wedge (d\varphi)^n = \det([\widehat{B}_\varphi])\, e^1 \wedge \cdots \wedge e^{2n+1}.$$

And, we get:

Theorem 3.7. *Let* $\mathfrak{g}$ *be a Lie algebra with* $\dim \mathfrak{g} = 2n+1$. *Then* $\varphi \in \mathfrak{g}^*$ *is a contact structure on* $\mathfrak{g}$ *if, and only if,* $\det([\widehat{B}_\varphi]) \neq 0$.

Remark 3.8.

(1) It follows that we can think of $\det([\widehat{B}_\varphi])$ as a homogeneous polynomial in the variables $\{x_i\}$, therefore $\det([\widehat{B}_\varphi]) \in \mathbb{F}[x_1, \dots, x_{2m+1}]$.
(2) Moreover, this suggests that we can define a Pfaffian associated to any contact structure on a Lie algebra.

4. Invariants of contact Lie algebras

The aim of this section is to prove that every contact Lie algebra admits one semi-invariant, since we can define a Pfaffian associated to its contact structure.

Let $\mathfrak{g}$ be a Lie algebra, $\varphi \in \mathfrak{g}^*$, and $C(\mathfrak{g})$ its structure matrix, i.e., $C(\mathfrak{g})_{i,j} := [e_i, e_j]$, where $\mathfrak{g} = \langle e_1, \dots, e_m \rangle_{\mathbb{F}}$ and $\mathfrak{g}^* = \langle e^1, \dots, e^m \rangle_{\mathbb{F}}$, with $e^i(e_j) = \delta^i_j$, $\varphi = \sum_{l=1}^m x_l e^l$ and $[\varphi]^t = \begin{pmatrix} x_1 & \cdots & x_m \end{pmatrix}$.

We have two cases, if $\dim \mathfrak{g} = 2k$, we define $\widehat{C}(\mathfrak{g}) := C(\mathfrak{g})$. If $\dim \mathfrak{g} = 2k+1$, then

$$\widehat{C}(\mathfrak{g}) := \begin{pmatrix} 0 & e_1 & \cdots & e_m \\ -e_1 & & & \\ \vdots & & C(\mathfrak{g}) & \\ -e_m & & & \end{pmatrix} \in \mathrm{Mat}((2k+2) \times (2k+2), S(\mathfrak{g})),$$

where, as usual, $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$. Given $\varphi \in \mathfrak{g}^*$, we can define

$$\widehat{C}(\mathfrak{g}) \longmapsto \varphi(\widehat{C}(\mathfrak{g})) =: \widehat{B}_\varphi.$$

Now, we define $\Phi : \mathfrak{g}^* \to \mathbb{F}[x_1, \ldots, x_n]$ by:

$$\varphi \longmapsto \det(\widehat{B}_\varphi) \longmapsto \mathrm{Pf}_\varphi .$$

Let $\widehat{\mathfrak{g}}$ the vector space defined by $\widehat{\mathfrak{g}} := \langle \mathrm{Id}_\mathfrak{g} \rangle \oplus \mathfrak{g}$. Then $\widehat{B}_\varphi$ defines a skew-symmetric bilinear form on $\mathfrak{g}$, $\widehat{B}_\varphi : \widehat{\mathfrak{g}} \times \widehat{\mathfrak{g}} \to \mathbb{F}$ by:

$$(\mathrm{Id}_\mathfrak{g}, x) \longmapsto \varphi(\mathrm{Id}_\mathfrak{g}(x)), \quad \text{and} \quad (x, y) \longmapsto \varphi([x, y]_\mathfrak{g})$$

for all $x, y \in \mathfrak{g}$.

Let $T : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ be the linear isomorphism defined by $\mathrm{Id}_\mathfrak{g} \mapsto \mathrm{Id}_\mathfrak{g}$, and $x \mapsto A(x)$, where $x \in \mathfrak{g}$ and $A \in \mathrm{Aut}(\mathfrak{g})$. Using T, we can define $\widehat{B}_{\varphi,T}(u, v) := \widehat{B}_\varphi(Tu, Tv)$ for all $u, v \in \widehat{\mathfrak{g}}$, therefore:

$$[\widehat{B}_{\varphi,T}] = [T]^t [\widehat{B}_\varphi][T]$$

and we get:

$$\det([\widehat{B}_{\varphi,T}]) = \det([T])^2 \det([\widehat{B}_\varphi])$$

This last equation implies the following:

Proposition 4.1. *Let $\mathfrak{g}$ be a Lie algebra, $\dim \mathfrak{g} = 2k + 1$, $\varphi \in \mathfrak{g}^*$ and $\widehat{B}_\varphi$ its associated skew-symmetric bilinear form. Let $g \in \mathrm{Aut}(\mathfrak{g})$ be an automorphism of $\mathfrak{g}$. Then:*

$$\mathrm{Pf}_{\widehat{B}_{\varphi,g}} = \det(g)\, \mathrm{Pf}_{\widehat{B}_\varphi}$$

Proof. Given $g \in \mathrm{Aut}(\mathfrak{g})$, we can define a linear isomorphism $T_g : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ letting $T_g(\mathrm{Id}_\mathfrak{g}) = \mathrm{Id}_\mathfrak{g}$ and $T_g(x) = g(x)$ if $x \in \mathfrak{g}$. □

Corollary 4.2. *Let $\mathfrak{g}$ be a Lie algebra, $\dim \mathfrak{g} = 2k + 1$, $\varphi \in \mathfrak{g}^*$ and $g \in \mathrm{Aut}(\mathfrak{g})$. Then*

$$g\,.\, \mathrm{Pf}_\varphi := \mathrm{Pf}_{\widehat{B}_{\varphi,g}} = \det(g)\, \mathrm{Pf}_\varphi .$$

As direct consequences:

Theorem 4.3. *Let $\mathfrak{g}$ be a Lie algebra, $\dim \mathfrak{g} = 2k + 1$, $\varphi \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$. Then:*

$$[x, \mathrm{Pf}_\varphi]_{U(\mathfrak{g})} = x(\mathrm{Pf}_\varphi) = \mathrm{Tr}(x)\, \mathrm{Pf}_\varphi .$$

Theorem 4.4. *Let $\mathfrak{g}$ be a Lie algebra, $\varphi \in \mathfrak{g}^*$ and Pf_φ its associated Pfaffian. Then $(\mathfrak{g}, \varphi)$ is a Frobenius or contact Lie algebra if, and only if, $\mathrm{Pf}_\varphi \neq 0$.*

Corollary 4.5. *Let $\mathfrak{g}$ be a Lie algebra, $\varphi \in \mathfrak{g}^*$. Then:*

(1) *If $(\mathfrak{g}, \varphi)$ is a Frobenius Lie algebra, then $N(\mathfrak{g}) = 0$.*
(2) *If $(\mathfrak{g}, \varphi)$ is a contact Lie algebra, then $N(\mathfrak{g}) \geq 1$.*

The main results of this section are:

Theorem 4.6. *Let $\mathfrak{g}$ be a contact Lie algebra with contact structure given by $\varphi \in \mathfrak{g}^*$. Then Pf_φ is a* **semi-invariant** *of $\mathfrak{g}$.*

Moreover:

Theorem 4.7. *Let $\mathfrak{g}$ be a perfect contact Lie algebra, i.e., $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, with contact structure $\varphi \in \mathfrak{g}^*$. Then Pf_φ is an* **invariant** *of $\mathfrak{g}$.*

Proof. It follows from Theorem 4.3. □

Theorem 4.8. *Let $\mathfrak{g}$ be a contact Lie algebra with contact structure given by $\varphi \in \mathfrak{g}^*$. Suppose that $\mathfrak{a}_\mathfrak{g}$ is a maximal abelian ideal in $\mathfrak{g}$ such that $\dim(\mathfrak{a}_\mathfrak{g}) = n + 1$. Then Pf_φ is a polynomial in the variables of $\mathfrak{a}_\mathfrak{g}$.*

Example 4.9. Let $\mathfrak{g} = \mathfrak{sl}_2 \ltimes \mathbb{F}^4$, where $\mathfrak{sl}_2 = \langle H, E, F \rangle$ and $\mathbb{F}^4 = \langle e_1, e_2, e_3, e_4 \rangle$. As usual:

$$[H] = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \quad [E] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad [F] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and $\beta(\mathfrak{g}) = 4$ and $\mathfrak{a}_{\mathfrak{g}} = \mathbb{F}^4$. Let $\varphi = x_1e^1 + x_2e^2 + x_3e^3 + x_4e^4 \in (\mathbb{F}^4)^*$. Then

$$\widehat{N} = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ 3e_1 & e_2 & -e_3 & -3e_4 \\ 0 & e_1 & 2e_2 & 3e_3 \\ 3e_2 & 2e_3 & e_4 & 0 \end{pmatrix}.$$

We get:

$$\mathrm{Pf}_{\varphi} = \det(\varphi(\widehat{N})) = x_1^2x_4^2 - 6x_1x_2x_3x_4 + 4x_2^3x_4 + 4x_1x_3^3 - 3x_2^2x_3^2.$$

An easy calculation proves that $\widehat{X}(\mathrm{Pf}_{\varphi}) = 0$, where $X = H, E, F$, and, for example, $\widehat{H} = 3x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} - 3x_4\frac{\partial}{\partial x_4}$.

5. Contact perfect Lie algebras

5.1. *Perfect Lie algebras* $\mathfrak{g}$ *with* $\dim Z(\mathfrak{g}) = 1$

Let $\mathfrak{g}$ be a perfect Lie algebra, i.e., $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional linear representation of $\mathfrak{g}$ to V, and suppose that in V there is a skew-symmetric bilinear form $B : V \times V \to \mathbb{F}$ such that:

$$B(\rho(x)u, v) + B(u, \rho(x)v) = 0,$$

for all $x \in \mathfrak{g}$, $u, v \in V$.

We can define a Lie algebra structure on the vector space $\widehat{V} := V \oplus \langle h\rangle$, where h is a new element, $h \notin V$, $h \notin \mathfrak{g}$, by letting

$$[u, v]_{\widehat{V}} := B(u, v)h, \qquad [u, h]_{\widehat{V}} := 0$$

for all, $u, v \in V$.

Lemma 5.1. *Let $\widehat{V}$ be as above. Then, there exists an abelian Lie algebra $\mathfrak{a}$, such that $\widehat{V} = \mathfrak{a} \oplus \mathfrak{h}_n$, where $\mathfrak{h}_n$ is the Heisenberg Lie algebra of dimension $2n + 1$. B is non-degenerate if, and only if, $\widehat{V} = \mathfrak{h}_n$.*

Proof. Let $\mathfrak{a}$ be the radical of B, i.e, $\mathfrak{a} := \mathrm{Rad}(B) = \{u \in V \mid B(u, v) = 0,\ \forall v \in V\}$, and let W be a complementary subspace of $\mathfrak{a}$ such that $B|_{W\times W}$ is a non-degenerate skew-symmetric bilinear form, then $\mathfrak{h}_n = \widehat{W}$ and $\widehat{V} = \mathfrak{a} \oplus \mathfrak{h}_n$. □

With this structure $\widehat{V}$ is a 2-step nilpotent Lie algebra, which implies that $Z_{\widehat{V}}(\widehat{V}) \neq \{0\}$. In fact, $Z_{\widehat{V}}(\widehat{V}) = \mathfrak{a} \oplus \langle h\rangle$. And we can extend ρ to $\widehat{\rho} : \mathfrak{g} \to \mathfrak{gl}(\widehat{V})$ defining $\widehat{\rho}(x)h = 0$, for all $x \in \mathfrak{g}$. But this definition is forced upon us because $\mathfrak{g}$ is a perfect Lie algebra and $h \in Z_{\widehat{V}}(\widehat{V})$.

Let $\mathfrak{g} \ltimes V$ and $\mathfrak{g} \ltimes \widehat{V}$ be Lie algebras as before.

Lemma 5.2. *$\mathfrak{g} \ltimes V$ is a perfect Lie algebra if, and only if $\mathfrak{g} \ltimes \widehat{V}$ is a perfect Lie algebra.*

Proof. We observe that

$$\begin{aligned} \mathfrak{g} \ltimes \widehat{V} &= \mathfrak{g} \oplus V \oplus \langle h\rangle \\ &= [\mathfrak{g} \oplus V \oplus \langle h\rangle, \mathfrak{g} \oplus V \oplus \langle h\rangle] \\ &= [\mathfrak{g}, \mathfrak{g}] \oplus [\mathfrak{g}, V] \oplus [V, V] \\ &= \mathfrak{g} \oplus [\mathfrak{g}, V] \oplus \langle h\rangle \end{aligned}$$

if, and only if, $[\mathfrak{g}, V] = V$, so $[\mathfrak{g} \ltimes V, \mathfrak{g} \ltimes V] = \mathfrak{g} \ltimes V$. □

Corollary 5.3. *Let $\mathfrak{g} \ltimes \widehat{V}$ and $\mathfrak{g} \ltimes V$ be as before. Then $\mathfrak{g} \ltimes \widehat{V}$ is a central extension of $\mathfrak{g} \ltimes V$.*

On the other hand, if $\mathfrak{a} = \mathrm{Rad}(B) \neq \{0\}$, we get $\mathfrak{a} \cap [\mathfrak{g} \ltimes \widehat{V}, \mathfrak{g} \ltimes \widehat{V}] = \{0\}$, i.e., $\mathfrak{g} \ltimes \widehat{V}$ is a non-perfect Lie algebra.

Corollary 5.4. *Let $\mathfrak{g} \ltimes \widehat{V}$ as before. Then $\mathfrak{g} \ltimes \widehat{V}$ is a perfect Lie algebra if, and only if, B is non-degenerate. Furthermore, $\widehat{V} = \mathfrak{h}_n$.*

From now on, $\mathfrak{g}_L$ will denote a semisimple Lie algebra and we suppose that $\mathfrak{g} = \mathfrak{g}_L \ltimes \mathrm{NilRad}(\mathfrak{g})$ is a perfect Lie algebra such that $\dim Z(\mathrm{NilRad}(\mathfrak{g})) = 1$. Let $Z(\mathrm{NilRad}(\mathfrak{g})) = \langle h\rangle$. It is straightforward to see that:

$$[x, h]_{\mathfrak{g}} = 0, \qquad \forall\, x \in \mathfrak{g}_L.$$

Therefore, $Z(\mathfrak{g}) = Z(\mathrm{NilRad}(\mathfrak{g})) = \langle h\rangle$; in other words, $Z(\mathfrak{g})$ is the only copy of the trivial $\mathfrak{g}_L$-module in $\mathrm{NilRad}(\mathfrak{g})$. Let

$$\begin{aligned} \mathrm{NilRad}(\mathfrak{g}) &= V_1 \oplus \cdots \oplus V_r \oplus Z(\mathfrak{g}), \\ &= V_1 \oplus \cdots \oplus V_r \oplus \langle h\rangle, \end{aligned}$$

the $\mathfrak{g}_L$-module decomposition of NilRad($\mathfrak{g}$). Using that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ we have two possibilities:

(1) there exists an index i, $1 \le i \le r$, such that $\Lambda^2(V_i)$ has a copy of $Z(\mathfrak{g})$ in its $\mathfrak{g}_L$-module decomposition; or
(2) there are two indices a and b, $1 \le a < b \le r$, such that we can define a non-degenerate skew-symmetric bilinear form $B_{a,b} : (V_a \oplus V_b) \times (V_a \oplus V_b) \to \mathbb{F}$, and for the representation $\rho_{a,b} = \rho_a \oplus \rho_b : \mathfrak{g}_L \to \mathfrak{gl}(V_a \oplus V_b)$ we get

$$B_{a,b}(\rho_{a,b}(x)u, v) + B_{a,b}(u, \rho_{a,b}(v)) = 0$$

for all $x \in \mathfrak{g}_L$, $u, v \in V_a \oplus V_b$; moreover, $B_{a,b}(V_k, V_k) = 0$, for $k = a, b$.

In the second case, the existence of this bilinear form implies that $V_b \simeq V_a^*$ as $\mathfrak{g}_L$-module, therefore, we can rewrite NilRad($\mathfrak{g}$) as:

$$\mathfrak{n} := \mathrm{NilRad}(\mathfrak{g}) = \sum_{p=1}^{l} V_{i_p} \oplus \sum_{q=1}^{m} (V_{a_q} \oplus V_{b_q}) \oplus Z(\mathfrak{g}) \oplus (V_{c_1} \oplus \cdots \oplus V_{c_t}),$$

At this point, we must note that:

$$\sum_{p=1}^{l} V_{i_p} \oplus \sum_{q=1}^{m} (V_{a_q} \oplus V_{b_q}) \oplus Z(\mathfrak{g}) \simeq \mathfrak{h}_n$$

for some $n \in \mathbb{N}$. Combining $\mathfrak{n}^k = \{0\}$, $\mathfrak{n}^{k-1} = Z(\mathfrak{g}) = Z(\mathfrak{n})$ and $\mathfrak{n}^2 \subsetneq \mathfrak{n}$, we get

$$\mathfrak{n}^2 := \sum_{p=1}^{l_1} V_{i_p} \oplus \sum_{q=1}^{m_1} (V_{a_q} \oplus V_{b_q}) \oplus Z(\mathfrak{g}) \oplus \sum_{s=1}^{t_1} V_{c_s},$$

with:

$$l_1 < l, \qquad m_1 < m, \qquad t_1 < t, \quad \text{and} \quad 0 < l_1 + m_1,$$

and we are also reordering indices in such a way that the last one does not appear in the first derived ideal. We conclude that, at most,

$$\mathfrak{n}^t = \sum_{p=1}^{l_{t-1}} V_{i_p} \oplus \sum_{q=1}^{m_{t-1}} (V_{a_q} \oplus V_{b_q}) \oplus Z(\mathfrak{g}),$$

and, obviously (at most) $\mathfrak{n}^{t+2} = \{0\}$. In particular, $t < l + m$ and $k \le t + 2$.

5.2. Perfect Lie algebras with trivial center

Let $\mathfrak{g}$ be a perfect Lie algebra with abelian nilradical and Levi–Malcev decomposition

$$\mathfrak{g} = \mathfrak{g}_L \ltimes \mathrm{NilRad}(\mathfrak{g}),$$

let $\mathfrak{a}_{\mathfrak{g}} := \mathrm{NilRad}(\mathfrak{g})$ be the abelian nilradical. The $\mathfrak{g}_L$-module decomposition of $\mathfrak{a}_{\mathfrak{g}}$ is given by:

$$\mathfrak{a}_{\mathfrak{g}} = V_1 \oplus \cdots \oplus V_r,$$

where $\dim V_i \ge 1$, $1 \le i \le r$.

Lemma 5.5. *Let $\mathfrak{g}$ be a perfect Lie algebra. Then $Z(\mathfrak{g}) = \{0\}$ if, and only if, $\dim V_i > 1$ for all $1 \le i \le r$ and $\mathfrak{a}_{\mathfrak{g}} = V_1 \oplus \cdots \oplus V_r$ is an abelian Lie algebra.*

Proof. If NilRad($\mathfrak{g}$) is not an abelian Lie algebra, each $x \in Z(\mathrm{NilRad}(\mathfrak{g}))$ defines a trivial 1-dimensional $\mathfrak{g}_L$-module, and therefore $x \in Z(\mathfrak{g})$. □

From now on, we will suppose that:

$$1 < \dim V_r \le \cdots \le \dim V_2 \le \dim V_1,$$

5.3. Contact perfect Lie algebras

The aim of this subsection is to prove that contact perfect Lie algebras are completely characterized by the equation $N(\mathfrak{g}) = 1$.

Theorem 5.6. *Let $\mathfrak{g}$ be a perfect Lie algebra with non-trivial center. Then $N(\mathfrak{g}) \ge 2$.*

Proof. Note that $N(\mathfrak{g}) \geq \dim Z(\mathfrak{g})$. If $\dim Z(\mathfrak{g}) \geq 2$ we are done. Suppose that $\dim Z(\mathfrak{g}) = 1$.

Case $\dim \mathfrak{g} = 2k$.

Let $C(\mathfrak{g})$ be the skew-symmetric matrix defined by $C(\mathfrak{g})_{i,j} := [e_i, e_j]$, where $\mathfrak{g} = \langle e_1, \ldots, e_{2k}\rangle$, and $Z(\mathfrak{g}) = \langle e_{2k}\rangle$. This implies that the last row and column are trivial in $C(\mathfrak{g})$, but then $\mathrm{Rank}(C(\mathfrak{g})) \leq n - 2$. From the fact that

$$N(\mathfrak{g}) = \dim \mathfrak{g} - \sup\{\mathrm{Rank}(C(\mathfrak{g})) \mid \{e_1, \ldots, e_{2k}\} \text{ is a basis of } \mathfrak{g}\}, \tag{3}$$

we conclude $N(\mathfrak{g}) \geq 2$.

Case $\dim \mathfrak{g} = 2k + 1$.

Let $C(\mathfrak{g})$ be the skew-symmetric matrix defined by $C(\mathfrak{g})_{i,j} := [e_i, e_j]$, where $\mathfrak{g} = \langle e_1, \ldots, e_{2k+1}\rangle$, and $Z(\mathfrak{g}) = \langle e_{2k+1}\rangle$. Let $\widehat{C}(\mathfrak{g})$ be the submatrix of $C(\mathfrak{g})$ obtained by removing the last row and the last column. Obviously $\widehat{C}(\mathfrak{g})$ is a skew-symmetric matrix of $(2k) \times (2k)$ size. If $\det \widehat{C}(\mathfrak{g}) \neq 0$ then its associated Pfaffian $\mathrm{Pf}_{\widehat{C}}$ is another non-trivial invariant. In any other case apply (3) to get $N(\mathfrak{g}) \geq 2$. □

Corollary 5.7. *Let $\mathfrak{g}$ be a perfect Lie algebra with $\dim \mathfrak{g} = 2n$. Then $N(\mathfrak{g}) = 2r$.*

As an immediate consequence:

Theorem 5.8. *Let $\mathfrak{g}$ be a perfect Lie algebra with non-trivial center. Then $\mathfrak{g}$ is not a contact Lie algebra.*

Our final result is:

Theorem 5.9. *Let $\mathfrak{g}$ be perfect Lie algebra. Then $\mathfrak{g}$ is a contact Lie algebra if, and only if $N(\mathfrak{g}) = 1$.*

Proof. Clearly if $\mathfrak{g}$ is a contact Lie algebra, then $N(\mathfrak{g}) = 1$. Let $\mathfrak{g}$ be a perfect Lie algebra with $N(\mathfrak{g}) = 1$. From (3):

$$\begin{aligned}\dim \mathfrak{g} &= N(\mathfrak{g}) + \sup\{\mathrm{Rank}(C(\mathfrak{g})) \mid \{e_1, \ldots, e_{2k}\} \text{ is a basis of } \mathfrak{g}\}\\ &= 1 + \sup\{\mathrm{Rank}(C(\mathfrak{g})) \mid \{e_1, \ldots, e_{2k}\} \text{ is a basis of } \mathfrak{g}\}\\ &= 1 + 2n,\end{aligned}$$

where $C(\mathfrak{g}) \in \mathrm{Mat}((2n+1) \times (2n+1), S(\mathfrak{g}))$, $C(\mathfrak{g})^T = -C(\mathfrak{g})$. Let $\varphi \in \mathfrak{g}^*$ as before, such that $\varphi = \sum_{l=1}^{2n+1} x_l e^l$, where $\mathfrak{g} = \langle e_1, \ldots, e_{2n+1}\rangle$, $\mathfrak{g}^* = \langle e^1, \ldots, e^{2n+1}\rangle$ and $e^i(e_j) = \delta_i^j$. Let $[\varphi]^t = \begin{pmatrix} x_1 & \cdots & x_{2n+1}\end{pmatrix}$ and

$$[\widehat{B}_\varphi] := \begin{pmatrix} 0 & [\varphi]^T \\ -[\varphi] & \varphi(C(\mathfrak{g}))\end{pmatrix} \in \mathrm{Mat}((2n+2) \times (2n+2), \mathbb{F}).$$

From $\mathrm{Rank}(C(\mathfrak{g})) = 2n$ it is straightforward to prove that there exists $\varphi \in \mathfrak{g}^*$ such that $\det([\widehat{B}_\varphi]) \neq 0$. Thus, from Theorem 3.7, $(\mathfrak{g}, \varphi)$ is a contact perfect Lie algebra. □

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