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Integrable motions of curves in $S^1 \times \mathbb{R}$

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Abstract

In this paper, motions of non-stretching curves in some Klein geometries, determined by the transformation groups acting on $S^1 \times \mathbb{R}$, are studied. It is shown that several $1+1$-dimensional integrable equations including the KdV, mKdV, defocusing mKdV, Sawada–Kotera, Burgers equations and their hierarchies arise naturally from motions of non-stretching curves in such geometries.

Keywords: Integrable equation; Invariant geometric flow; Klein geometry; Differential invariant

1. Introduction

Invariant geometric flows play an important role in image processing and computer vision (see [1] and references therein). It has been known for a long time that integrable equations that can be solved by the inverse scattering method are closely related to invariant geometric flows in certain geometries. Many interesting results have been obtained on this subject. Hasimoto transformation [2] sets up a one-to-one correspondence between the integrable Schrödinger equation and the binormal motion of a space curve driven by its curvature and torsion (see [3] for the history of this equation, also known as the Betchov–Da Rios equation). More developments in this direction were obtained later in [4–8]. Motions of curves in S^2 and S^3 were considered by Doliwa and Santini [9]. Lakshmanan et al. [10,11] interpreted the dynamics of a nonlinear string of fixed length in $\mathbb{R}^3$ through the consideration of the motion of an arbitrary rigid body along it, deriving the AKNS spectral problem without spectral parameter. Nakayama [12,13] showed that the defocusing nonlinear Schrödinger equation, the Regge–Lund equation, a coupled system of KdV equations and their hyperbolic type arise from motions of curves in hyperboloids in the Minkowski space. Motion of plane curves in Minkowski space was also discussed in [14]. In an intriguing paper [15], Goldstein and Petrich related the mKdV equation and its hierarchies to a motion of a non-stretching closed curves on the plane. Nakayama, Segur and Wadati [16] obtained the sine–Gordon equation by considering a nonlocal motion of plane curves. Recently, Chou, Qu, Zhang and Zhang [17–22] studied systematically the motion of non-stretching curves in Klein geometries in $\mathbb{R}^2$ or $\mathbb{R}^3$, and they showed that a number of $1+1$-dimensional integrable equations including

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Table 1
Lie algebras of vector fields of some geometries in $S^1 \times \mathbb{R}$

Name	Generators	Dimension
C_1	$\partial_\theta, \sin\theta\partial_\theta + \frac{1}{2}\cos\theta u\partial_u, \cos\theta\partial_\theta - \frac{1}{2}u\sin\theta\partial_u$	3
C_2	$\partial_\theta, \sin\theta\partial_\theta + \frac{1}{2}\cos\theta u\partial_u, \cos\theta\partial_\theta - \frac{1}{2}u\sin\theta\partial_u, u\partial_u$	4
C_3	$\partial_\theta, \sin\theta\partial_\theta - (2\sin\theta + u\cos\theta)\partial_u, \cos\theta\partial_\theta - (2\cos\theta - u\sin\theta)\partial_u$	3
E_1	$\partial_\theta, \sin\theta\partial_u, \cos\theta\partial_u$	3
S_1	$\partial_\theta, \sin\theta\partial_u, \cos\theta\partial_u, u\partial_u$	4
S_2	$\partial_\theta, \sin\theta\partial_u, \cos\theta\partial_u, \partial_u, u\partial_u$	5
A_1	$\partial_\theta, \sin\theta\partial_u, \cos\theta\partial_u, \cos 2\theta\partial_\theta - \sin 2\theta u\partial_u, \sin 2\theta\partial_\theta + \cos 2\theta u\partial_u$	5
A_2	$\partial_\theta, \sin\theta\partial_u, \cos\theta\partial_u, \cos 2\theta\partial_\theta - \sin 2\theta u\partial_u, \sin 2\theta\partial_\theta + \cos 2\theta u\partial_u, u\partial_u$	6
C_4	$\partial_\theta, \sin 2\theta\partial_u, \cos 2\theta\partial_u, \cos 2\theta\partial_\theta - 2\sin 2\theta u\partial_u, \sin 2\theta\partial_\theta + 2\cos 2\theta u\partial_u, \partial_u$	6
E_2	$\partial_\theta, u\sin\theta\partial_\theta - (au^3\sin\theta - u^2\cos\theta)\partial_u, u\cos\theta\partial_\theta - (au^3\cos\theta + u^2\sin\theta)\partial_u$	3
H_1	$\partial_\theta, u\sin\theta\partial_\theta - [(u^2-1)^{3/2}\sin\theta - (u^2-1)\cos\theta]\partial_u, u\cos\theta\partial_\theta - [(u^2-1)^{3/2}\cos\theta + (u^2-1)\sin\theta]\partial_u$	3

the KdV, mKdV, Camassa–Holm, Kaup–Kupershmidt, Harry–Dym, Burgers, defocusing mKdV equations and their hierarchies arise naturally from the motion of inextensible curves in Klein geometries. In [23], Ivey provided an approach for classifying integrable geometric evolution equations for plane curves. Beffa, Wang and Sanders [24] discussed motion of inextensible curves in Riemannian manifold and established the correspondence of the invariant curve flows and certain integrable equations.

It is well known that the existence of bi-Hamiltonian structure of a nonlinear evolution equation can identify the integrability of the equation. Beffa, González-López and Hernández [25–29] in a series of papers investigated the relationship between invariant geometric flows and Hamiltonian structure. On the basis of the moving coframe method, they established a general setting for generating compatible Hamiltonian structure from the geometry of curves in flat homogeneous space of the form $\mathcal{G}/\mathcal{H}$, where $\mathcal{G}$ is a semisimple Lie group, and $\mathcal{H}$ is a subgroup of $\mathcal{G}$. Recently, Anco [30,31] obtained the bi-Hamiltonian operators and associated hierarchies of multi-component soliton equations from invariant geometric flows of non-stretching curves in constant curvature manifolds and Lie groups.

The purpose of this paper is to study motion of plane curves in Klein geometries in $S^1 \times \mathbb{R} = \{(\theta, u), \theta \in S^1, u \in \mathbb{R}\}$, where S^1 is the circle. These geometries are characterized by their associated Lie algebras of vector fields in $S^1 \times \mathbb{R}$. Each Lie algebra generates the isometry group of its corresponding Klein geometry. To the best of our knowledge, there is no complete classification of Lie groups or the Lie algebra of vector fields acting on $S^1 \times \mathbb{R}$. Throughout the paper, we assume that the geometries are invariant under translation with respect to $\theta \in S^1$. This means that ∂_θ is an element of the corresponding Lie algebra of the isometry group of the geometry considered. The outline of this paper is as follows. In Section 2, we provide some background material and notions. In Section 3, we discuss motion of inextensible curves in the geometries in Table 1, which admit three-dimensional isometry groups. In Sections 4–6, we consider respectively the four-, five- and six-dimensional cases. Section 7 contains concluding remarks on this work.

2. Preliminaries and definitions

In this section, we provide the background notions and results from differential geometry and invariant theory used throughout the paper. Our basic references are the papers [33–35] and the books [32,36,37]. The reader is referred to these works for definitions and proofs of results stated below.

2.1. Klein geometry and differential invariant

Let us briefly recall some basic facts about Klein geometry, which was introduced by Klein in 1872 and was further developed by Killing [37]. For any Lie group $\mathcal{G}$ acting transitively and effectively on a manifold, there is an associated Klein geometry. A Klein geometry is the theory on geometric invariants of the transformation group $\mathcal{G}$. For example, the curvatures and arc-lengths of curves are invariants under the isometry group of the Klein geometry.

Assume $\mathcal{G}$ be a Lie transformation group acting transitively and effectively on $S^1 \times \mathbb{R}$. Its Lie algebra $\mathfrak{g}$ can be identified with a subalgebra of the Lie algebra of all smooth vector fields in $S^1 \times \mathbb{R}$ with the usual Poisson bracket.

According to the Erlanger Programme, every $\mathcal{G}$ or $\mathfrak{g}$ determines a Klein geometry for curves via its invariants. To describe the invariants, let us assume that a curve γ and its image γ' under a typical element g in $\mathcal{G}$ are represented as graphs $(\theta, u(\theta))$ and $(\vartheta, \tilde{u}(\vartheta))$ over S^1. For the space $M = S^1 \times \mathbb{R}$, the n-th jet space denoted by $J^{(n)} = S^1 \times \mathbb{R}^{(n)}$ is a $n+2$-dimensional Euclidean space [32].

Definition 2.1. A differential invariant of $\mathcal{G}$ is a C^n function Φ defined on the n-th jet space $J^{(n)} = S^1 \times \mathbb{R}^{(n)}$, for some $n \geq 1$, which is invariant with respect to each element of $\mathcal{G}$.

Definition 2.2. An invariant one-form, or more precisely a horizontal contact-invariant form, is a one-form defined in the n-th jet space $S^1 \times \mathbb{R}^{(n)}$, which is invariant with respect to each element of $\mathcal{G}$.

Therefore, if $\Phi(\theta, u(\theta), \ldots, u^{(n)}(\theta))$ is a differential invariant then it satisfies $\Phi(\theta, u(\theta), \ldots, u^{(n)}(\theta)) = \Phi(\vartheta, \tilde{u}(\vartheta), \ldots, \tilde{u}^{(n)}(\vartheta))$ for all $g \in \mathcal{G}$. An invariant one-form, given in the form $\mathrm{d}\sigma = P(\theta, u(\theta), \ldots, u^{(n)}(\theta))\mathrm{d}\theta$, satisfies

$$\int_I P(\theta, u(\theta), \ldots, u^{(n)}(\theta))\mathrm{d}\theta = \int_{\tilde{I}} P(\vartheta, \tilde{u}(\vartheta), \ldots, \tilde{u}^{(n)}(\vartheta))\mathrm{d}\vartheta, \tag{1}$$

for any $I \subseteq [0, 2\pi]$ and g in $\mathcal{G}$, and $\tilde{I}$ is the image of I under $g \in \mathcal{G}$. Let

$$\mathbf{v} = \tilde{\xi}(\theta, u)\frac{\partial}{\partial\theta} + \tilde{\eta}(\theta, u)\frac{\partial}{\partial u} \tag{2}$$

be an arbitrary vector field in $\mathfrak{g}$. We denote its n-th prolongation vector field on $S^1 \times \mathbb{R}^{(n)}$ by $\mathbf{pr}^{(n)}\mathbf{v}$. The following propositions provide a method for computing the differential invariant and invariant one-form of the group $\mathcal{G}$.

Proposition 2.1. *A function Φ: $J^{(n)} \to \mathbb{R}$ is a differential invariant for a connected group $\mathcal{G}$ if and only if*

$$\mathbf{pr}^{(n)}\mathbf{v}(\Phi) = 0$$

holds for every $\mathbf{v} \in \mathfrak{g}$ of the form (2).

Proposition 2.2. *A differential one-form $\mathrm{d}\sigma = P(\theta, u(\theta), \ldots, u^{(n)}(\theta))\mathrm{d}\theta$ is an invariant one-form for a connected group $\mathcal{G}$ if and only if P satisfies*

$$\mathbf{pr}^{(n)}\mathbf{v}(P) + P\mathrm{div}\,\tilde{\xi} = 0 \tag{3}$$

for every $\mathbf{v} \in \mathfrak{g}$ of the form (2).

Given a group, the existence of differential invariants is guaranteed by the following theorem.

Theorem 2.3 (*[32,36]*). *For any Lie transformation group $\mathcal{G}$ acting transitively and effectively on $S^1 \times \mathbb{R}$, there exist an invariant one-form $\mathrm{d}\sigma = P(\theta, u(\theta), \ldots, u^{(n)}(\theta))\mathrm{d}\theta$ and a differential invariant Φ, both of lowest order, such that every differential invariant can be written as a function of Φ and its derivatives $D\Phi$, $D^2\Phi, \ldots,$ where*

$$D = \frac{1}{P}\frac{\mathrm{d}}{\mathrm{d}\theta}.$$

Moreover, every invariant one-form is of the form $Q\mathrm{d}\sigma$, where Q is an arbitrary differential invariant of $\mathcal{G}$.

Invariant one-forms of lowest order are unique up to multiplication by a non-zero constant, and differential invariants of lowest order are unique up to compositions of nonconstant functions. For each concrete geometry, we shall specify one such differential form and call it the *arc-length* and one such invariant and call it the *curvature*. Usually the latter is chosen so that it is linear in the highest order of u.

Fels and Olver [34,35] developed a practical and simple method for computing moving frames, differential invariants, invariant one-forms and invariant differential operators. Their approach is generally called the method of "moving coframe". Indeed, a complete set of differential invariants of a given group can be found among the coefficients of a moving coframe. In particular, it can be used to compute differential invariants and invariant one-forms of the isometry groups of the geometries in $S^1 \times \mathbb{R}$.

Table 2
Arc-length and curvature for the geometries in Table 1

Name	Arc-length	Curvature
C_1	$u^{-2}\mathrm{d}\theta$	$k = u^3(u_{\theta\theta} + \frac{1}{4}u)$
C_2	$k^{\frac{1}{2}}u^{-2}\mathrm{d}\theta$	$u^2 k^{-\frac{3}{2}} k_\theta$
C_3	$(u_\theta + \frac{1}{4}u^2 + 1)^{\frac{1}{2}}\mathrm{d}\theta$	$\frac{u_{\theta\theta}+\frac{3}{2}uu_\theta+\frac{1}{4}u^3+u}{(u_\theta+\frac{1}{4}u^2+1)^{\frac{3}{2}}}$
E_1	$u\mathrm{d}\theta$	$v \equiv u_{\theta\theta} + u$
S_1	$\mathrm{d}\theta$	$\frac{v_\theta}{v}$
S_2	$\mathrm{d}\theta$	$\frac{v_{\theta\theta}}{v_\theta}$
A_1	$v^{\frac{2}{3}}\mathrm{d}\theta$	$\mu \equiv v^{-1}[(v^{-\frac{1}{3}})_{\theta\theta} + v^{-\frac{1}{3}}]$
A_2	$v^{\frac{2}{3}}\mu^{\frac{1}{2}}\mathrm{d}\theta$	$v^{-\frac{2}{3}}\mu^{-\frac{3}{2}}\mu_\theta$
C_4	$v_\theta^{\frac{1}{2}}\mathrm{d}\theta$	$4v_\theta^{-2}v_{\theta\theta\theta} - 5v_\theta^{-3}v_{\theta\theta}^2 - 4v_\theta^{-1}$
E_2	$\frac{1}{u}[(\frac{u_\theta}{u} + au)^2 + 1]^{\frac{1}{2}}\mathrm{d}\theta$	$\frac{u_{\theta\theta}+3auu_\theta+a^2u^3+u}{[(\frac{u_\theta}{u}+au)^2+1]^{\frac{3}{2}}}$
H_1	$\iota(u^2-1)^{-1/2}\mathrm{d}\theta$, $\iota \equiv u_\theta/\sqrt{u^2-1} + u$	$\frac{u_{\theta\theta}+\frac{3u^2u_\theta}{\sqrt{u^2-1}}+u^3+u-\frac{3}{2}\iota}{\iota^3}$

2.2. *Geometries in* $S^1 \times \mathbb{R}$

In this paper, we are interested in some Klein geometries acting on $S^1 \times \mathbb{R}$, $(\theta, u) \in S^1 \times \mathbb{R}$. Those geometries are determined by the corresponding Lie algebras of the isometry groups listed in Table 1.

In Table 2, we obtain the curvature and arc-length for a curve in the geometries listed in Table 1. In the following we shall consider motion of curves in the geometries listed in Table 1.

2.3. *Invariant curve flows*

The nice thing about Klein geometry for curves is that we have a very precise notion of arc-length, curvature, tangent and normal vectors [36,37]. For any parametrized curve γ in a Klein geometry, we can define its *tangent* $\mathbf{T} = \gamma_s$ and *normal* $\mathbf{N}$, where $\mathbf{N}$ is chosen to be independent of $\mathbf{T}$, and s is the arc-length of the curve.

The invariant curve motion flow in a Klein geometry is governed by

$$\gamma_t = f\mathbf{N} + g\mathbf{T}, \tag{4}$$

where f and g are functions depending on the curvature and its derivatives with respect to the arc-length. We shall follow the procedure in [18,19] to derive invariant curve motions whose corresponding equations for the curvature are integrable. The procedure is as follows: First, by differentiating (4) with respect to time t, we obtain the evolution of the frame vectors

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_t = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}. \tag{5}$$

Next we compute the first variation of the perimeter and obtain the conditions for f and g by requiring that the derivative with respect to the arc-length commutes with the derivative with respect to time and that the perimeter of a closed curve is invariant when the curve evolves according to (4). Finally, we obtain the evolution of the curvature, i.e., the equation for the curvature. Very often the recursion operator for a certain integrable hierarchy comes out naturally in the equation of the curvature.

3. Motion of curves in geometries C_1, C_3, E_2 and H_1

The geometries in Table 1 with three-dimensional isometry groups in $S^1 \times \mathbb{R}$ contain conformal geometries C_1 and C_3, the extended Euclidean geometry E_2 and the hyperbolic geometry H_1. First, we consider the motion of plane curves in conformal geometry C_1, where the curve flow is driven by (4). One can show using the expressions in row 1 of Table 1 in Section 2.2 that the arc-length and curvature are given respectively by

$$\mathrm{d}s = u^{-2}\mathrm{d}\theta$$

and

$$k = u^3\left(u_{\theta\theta} + \frac{1}{4}u\right).$$

The tangent vector is $\mathbf{T} = u^2(1, u_\theta)$ and the normal vector is chosen to be $\mathbf{N} = u(0, 1)$. Then the curve flow (4) becomes

$$\gamma_t = fu(0, 1) + gu^2(1, u_\theta). \tag{6}$$

It gives the time evolution for θ and u

$$\begin{cases}\theta_t = u^2 g,\\ u_t = uf + u^2 u_\theta g.\end{cases} \tag{7}$$

In this paper, the curve that we will consider is non-stretching during the motion, that is, the distance between any two points of the curve does not vary in time. So arc-length s and time t commute. Let p be a parameter which does not depend on time t. We can represent $\mathrm{d}\theta$ by $\tilde{g}\mathrm{d}p$ in terms of p. For any functions υ, we have

$$\begin{aligned}\upsilon_{\theta t} &= (\upsilon_p\tilde{g}^{-1})_t = \upsilon_{t\theta} - \frac{\tilde{g}_t}{\tilde{g}}\upsilon_\theta,\\ \upsilon_{st} &= (u^2\tilde{g}^{-1}\upsilon_p)_t\\ &= \upsilon_{ts} + (2u\tilde{g}^{-1}u_t - u^2\tilde{g}^{-2}\tilde{g}_t)\tilde{g}\upsilon_s\\ &= \upsilon_{ts} + \left(2\frac{u_t}{u} - \frac{\tilde{g}_t}{\tilde{g}}\right)\upsilon_s.\end{aligned} \tag{8}$$

Also

$$\tilde{g}_t = \frac{\partial\theta_p}{\partial t} = \frac{\partial\theta_t}{\partial p} = (u^2 g)_\theta\tilde{g}.$$

Using (7) and above expressions, we deduce

$$\begin{aligned}2\frac{u_t}{u} - \frac{\tilde{g}_t}{\tilde{g}} &= u^{-1}(2uf + 2u^2u_\theta g) - (u^2 g_\theta + 2uu_\theta g)\\ &= -(g_s - 2f).\end{aligned} \tag{9}$$

It follows from (8) and (9) that commutativity of the derivatives with respect to s and t implies

$$g_s = 2f,$$

which gives

$$g = 2\partial_s^{-1} f. \tag{10}$$

The perimeter of a closed curve is

$$L = \oint \mathrm{d}s = \oint u^{-2}\mathrm{d}\theta. \tag{11}$$

Using (8)–(11), one obtains the first variation of the perimeter

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t} &= \oint (u^{-2}\theta_p)_t\mathrm{d}p\\ &= \oint (g_s - 2f)\mathrm{d}s.\end{aligned} \tag{12}$$

Assuming that the perimeter of a closed curve is invariant under the curve motion flow, we then obtain

$$\oint f\,\mathrm{d}s = 0. \tag{13}$$

The evolution of the curvature is

$$\begin{aligned} k_t &= \left[u^3\left(u_{\theta\theta} + \frac{1}{4}u\right)\right]_t \\ &= 3ku^{-1}u_t + u^3\left(u_{\theta\theta t} + \frac{1}{4}u_t\right). \end{aligned} \tag{14}$$

A straightforward computation gives

$$\begin{aligned} u_{\theta t} &= u_{t\theta} - (u^2 g)_\theta u_\theta \\ &= u f_\theta + u_\theta f + u^2 u_{\theta\theta} g, \\ u_{\theta\theta t} &= u_{\theta t\theta} - (u^2 g)_\theta u_{\theta\theta} \\ &= u f_{\theta\theta} + 2u_\theta f_\theta + u_{\theta\theta} f + u^2 u_{\theta\theta\theta} g. \end{aligned} \tag{15}$$

Utilizing (14) and (15), we obtain

$$\begin{aligned} k_t &= 3k(f + uu_\theta g) + \frac{1}{4}u^4(f + uu_\theta g) + u^3(uf_{\theta\theta} + 2u_\theta f_\theta + u_{\theta\theta} f + u^2 u_{\theta\theta\theta} g) \\ &= u^4 f_{\theta\theta} + 2u^3 u_\theta f_\theta + 4kf + u^2 k_\theta g \\ &= f_{ss} + 4kf + k_s g. \end{aligned}$$

It turns out that k satisfies

$$k_t = (\partial_s^2 + 4k + 2k_s\partial_s^{-1})f \equiv \Omega_1 f, \tag{16}$$

after using (10), where Ω_1 is the recursion operator of the KdV equation. Letting $f = -k_s$ in (16), we obtain the KdV equation

$$k_t + k_{sss} + 6kk_s = 0.$$

Letting $f = -\Omega_1^{n-2}k_s$ $(n \geq 2)$ in (16), we obtain the KdV hierarchy with the recursion operator Ω_1.

Next we consider motion of a curve in the conformal geometry C_3. The arc-length and curvature of the curve are given respectively by

$$\mathrm{d}\tilde{s} = \left(u_\theta + \frac{1}{4}u^2 + 1\right)^{\frac{1}{2}} \mathrm{d}\theta$$

and

$$\phi = \frac{u_{\theta\theta} + \frac{3}{2}uu_\theta + \frac{1}{4}u^3 + u}{(u_\theta + \frac{1}{4}u^2 + 1)^{\frac{3}{2}}}.$$

The tangent and normal vectors are

$$\mathbf{T} = l^{-\frac{1}{2}}(1, u_\theta) \quad \text{and} \quad \mathbf{N} = u(0, 1),$$

where $l = u_\theta + \frac{1}{4}u^2 + 1$. So the curve motion flow reads as

$$\gamma_t = fu(0, 1) + gl^{-\frac{1}{2}}(1, u_\theta).$$

This implies

$$\begin{cases}\theta_t = l^{-\frac{1}{2}} g,\\ u_t = l^{\frac{1}{2}} f + l^{-\frac{1}{2}} u_\theta g.\end{cases} \tag{17}$$

The perimeter for a closed curve is

$$L = \oint \mathrm{d}\tilde{s} = \oint l^{\frac{1}{2}} \mathrm{d}\theta. \tag{18}$$

Differentiating (17), we obtain

$$\begin{aligned}
u_{\theta t} &= u_{t\theta} - (l^{-\frac{1}{2}} g)_\theta u_\theta \\
&= l^{\frac{1}{2}} f_\theta + (l^{\frac{1}{2}})_\theta f + l^{-\frac{1}{2}} u_{\theta\theta} g,\\
u_{\theta\theta t} &= u_{\theta t\theta} - \left(l^{-\frac{1}{2}} g\right)_\theta u_{\theta\theta}\\
&= l^{\frac{1}{2}} f_{\theta\theta} + l^{-\frac{1}{2}} l_\theta f_\theta + \left(\frac{1}{2} l^{-\frac{1}{2}} l_\theta\right)_\theta f + l^{-\frac{1}{2}} u_{\theta\theta\theta} g,\\
l_t &= l^{\frac{1}{2}} f_\theta + \left[(l^{\frac{1}{2}})_\theta + \frac{1}{2} u l^{\frac{1}{2}}\right] f + l^{-\frac{1}{2}} (u_{\theta\theta} + u_\theta) g,\\
u_{\theta\theta} &= l^{\frac{3}{2}} \phi - \frac{3}{2} u u_\theta - \frac{1}{4} u^3 - u,\\
u_{\theta\theta\theta} &= l^2 \phi_{\tilde{s}} + \frac{3}{2} l^{\frac{1}{2}} l_\theta \phi - \frac{3}{2}(u u_{\theta\theta} + u_\theta^2) - \frac{3}{4} u^2 u_\theta - u_\theta.
\end{aligned}$$

Using the above expressions and (17), we get the first variation of the perimeter

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t} &= \oint \left[\frac{1}{2} l^{-\frac{1}{2}} l_t + l^{\frac{1}{2}} (l^{-\frac{1}{2}} g)_\theta\right] \mathrm{d}\theta\\
&= \oint \left[g_\theta + \frac{1}{2} f_\theta + \left(\frac{1}{2} l^{-\frac{1}{2}} (l^{\frac{1}{2}})_\theta + \frac{1}{4} u\right) f\right] \mathrm{d}\theta\\
&= \oint \left(g_{\tilde{s}} + \frac{1}{2} f_{\tilde{s}} + \frac{1}{4} \phi f\right) \mathrm{d}\tilde{s}.
\end{aligned}$$

This leads to the inextensibility condition

$$g_{\tilde{s}} + \frac{1}{2} f_{\tilde{s}} + \frac{1}{4} \phi f = 0$$

and

$$\oint \phi f \,\mathrm{d}\tilde{s} = 0.$$

We compute the curvature evolution to get

$$\begin{aligned}
\phi_t &= \left[l^{-\frac{3}{2}} \left(u_{\theta\theta} + \frac{3}{2} u u_\theta + \frac{1}{4} u^3 + u\right)\right]_t\\
&= -\frac{3}{2} l^{-1} \phi \left[l f_{\tilde{s}} + \frac{1}{2} l \phi f + l^{-\frac{1}{2}} \left(u_{\theta\theta} + \frac{1}{2} u u_\theta\right) g\right] + l^{-\frac{3}{2}} \left[l^{\frac{3}{2}} f_{\tilde{s}\tilde{s}} + \frac{3}{2} l_\theta f_{\tilde{s}}\right.\\
&\quad \left. + \frac{1}{2} (l^{-\frac{1}{2}} l_\theta)_\theta f + l^{-\frac{1}{2}} u_{\theta\theta\theta} g + \left(\frac{3}{2} u_\theta + \frac{3}{4} u^2 + 1\right) (l^{\frac{1}{2}} f + l^{-\frac{1}{2}} u_\theta g) + \frac{3}{2} u (l f_{\tilde{s}} + (l^{\frac{1}{2}})_\theta f + l^{-\frac{1}{2}} u_{\theta\theta} g)\right]\\
&= f_{\tilde{s}\tilde{s}} + c_1(\phi) f + c_2(\phi) g.
\end{aligned} \tag{19}$$

The coefficients $c_i(\phi)$ in ϕ_t can be determined as

$$\begin{aligned}
c_1(\phi) &= -\frac{3}{4}\phi^2 + l^{-\frac{3}{2}}\left[\left(\frac{1}{2}l^{-\frac{1}{2}}l_\theta\right)_\theta + \frac{3}{2}l^{\frac{1}{2}}u_\theta + \frac{3}{2}u(l^{\frac{1}{2}})_\theta + \frac{3}{4}u^2 l^{\frac{1}{2}} + l^{\frac{1}{2}}\right] \\
&= -\frac{3}{4}\phi^2 + l^{-\frac{3}{2}}\left[\frac{1}{2}\left(l^{-\frac{1}{2}}l_{\theta\theta} - \frac{1}{2}l^{-\frac{3}{2}}l_{\theta\theta}^2\right) + \frac{3}{4}l^{-\frac{1}{2}}ul_\theta + \frac{1}{2}l^{\frac{1}{2}}(u_\theta + u^2) + l^{\frac{1}{2}}\left(u_\theta + \frac{1}{4}u^2 + 1\right)\right] \\
&= -\frac{3}{4}\phi^2 + \frac{1}{2}l^{-2}\left[l^2\phi_{\tilde{s}} + \frac{3}{2}l^{\frac{1}{2}}\left(u_{\theta\theta} + \frac{1}{2}uu_\theta\right)\phi - \frac{3}{2}uu_{\theta\theta} - \frac{3}{2}u_\theta^2\right. \\
&\quad \left. - \frac{3}{4}u^2u_\theta - u_\theta\right] + \frac{1}{2}l^{-2}\left(\frac{1}{2}uu_{\theta\theta} + \frac{1}{2}u_\theta^2\right) - \frac{1}{4}l^{-3}\left(u_{\theta\theta}^2 + uu_\theta u_{\theta\theta} + \frac{1}{4}u^2u_\theta^2\right) \\
&\quad + \frac{3}{4}l^{-2}u\left(u_{\theta\theta} + \frac{1}{2}uu_\theta\right) + \frac{1}{2}l^{-1}(u_\theta + u^2) + 1 \\
&= \frac{1}{2}\phi_{\tilde{s}} - \frac{3}{4}l^{-\frac{3}{2}}\phi\left(uu_\theta + \frac{1}{4}u^3 + u\right) - \frac{1}{2}l^{-2}\left(\frac{3}{2}u_\theta^2 + u_\theta\right) \\
&\quad + \frac{1}{4}l^{-2}u\left(l^{\frac{3}{2}}\phi - \frac{3}{2}uu_\theta - \frac{1}{4}u^3 - u\right) + \frac{1}{4}l^{-2}u_\theta^2 - \frac{1}{4}\phi^2 \\
&\quad + \frac{1}{4}l^{-3}\left[2l^{\frac{3}{2}}\phi\left(\frac{3}{2}uu_\theta + \frac{1}{4}u^3 + u\right) - \left(\frac{3}{2}uu_\theta + \frac{1}{4}u^3 + u\right)^2\right] \\
&\quad - \frac{1}{4}l^{-\frac{3}{2}}uu_\theta\left(l^{\frac{3}{2}}\phi - uu_\theta - \frac{1}{4}u^3 - u\right) + \frac{1}{2}l^{-1}(u_\theta + u^2) + 1 \\
&= \frac{1}{2}\phi_{\tilde{s}} - \frac{1}{4}\phi^2 + 1, \\
c_2(\phi) &= -\frac{3}{2}l^{-1}\left(u_{\theta\theta} + \frac{1}{2}uu_\theta\right)\phi + l^{-2}\left[u_{\theta\theta\theta} + \frac{3}{2}(uu_{\theta\theta} + u_\theta^2) + \frac{3}{4}u^2u_\theta + u_\theta\right] \\
&= \phi_{\tilde{s}}.
\end{aligned} \tag{20}$$

Plugging (20) into (19), we arrive at

$$\begin{aligned}
\phi_t &= f_{\tilde{s}\tilde{s}} + \phi_{\tilde{s}}g - \left(\frac{1}{4}\phi^2 - \frac{1}{2}\phi_{\tilde{s}} - 1\right)f \\
&= \left(\partial_{\tilde{s}}^2 - \frac{1}{4}\phi^2 - \frac{1}{4}\phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}\phi + 1\right)f.
\end{aligned} \tag{21}$$

Letting $f = -\phi_{\tilde{s}}$ in (21), we have the mKdV equation

$$\phi_t + \phi_{\tilde{s}\tilde{s}\tilde{s}} - \frac{3}{8}\phi^2\phi_{\tilde{s}} + \phi_{\tilde{s}} = 0.$$

Letting $f = \Omega_2^{n-2}\phi_{\tilde{s}}$ $(n \geq 2)$ in (21), we get an mKdV hierarchy with the recursion operator $\Omega_2 = \partial_{\tilde{s}}^2 - \frac{1}{4}\phi^2 - \frac{1}{4}\phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}\phi$.

It is known that the motion of non-stretching plane curves in Euclidean geometry is closely related to the mKdV hierarchy [15]. We now consider motion of curves in the extended Euclidean geometry E_2. For this geometry, the arc-length and curvature are given respectively by

$$\mathrm{d}\omega = \lambda^{\frac{1}{2}}u^{-1}\mathrm{d}\theta \quad \text{and} \quad \varrho = \lambda^{-\frac{3}{2}}(u_{\theta\theta} + 3auu_\theta + a^2u^3 + u),$$

where $\lambda = (u^{-1}u_\theta + au)^2 + 1$ and a is a constant. The tangent and normal vectors are

$$\begin{aligned}
&\mathbf{T} = \lambda^{-\frac{1}{2}}u(1, u_\theta), \\
&\mathbf{N} = (\lambda^{-\frac{1}{2}}(u_\theta + au^2), \lambda^{-\frac{1}{2}}(u_\theta^2 + au^2u_\theta) - \lambda^{\frac{1}{2}}u^2).
\end{aligned}$$

The curve motion flow reads as

$$\gamma_t = f(\lambda^{-\frac{1}{2}}(u_\theta + au^2), \lambda^{-\frac{1}{2}}(u_\theta^2 + au^2 u_\theta) - \lambda^{\frac{1}{2}} u^2) + g\lambda^{-\frac{1}{2}} u(1, u_\theta).$$

Using it, a straightforward computation gives

$$\theta_t = \lambda^{-\frac{1}{2}}(u_\theta + au^2) f + \lambda^{-\frac{1}{2}} ug,$$

$$u_t = [\lambda^{-\frac{1}{2}}(u_\theta^2 + au^2 u_\theta) - \lambda^{\frac{1}{2}} u^2] f + \lambda^{-\frac{1}{2}} uu_\theta g,$$

$$\begin{aligned} u_{\theta t} &= u_{t\theta} - [\lambda^{-\frac{1}{2}}(u_\theta + au^2) f + \lambda^{-\frac{1}{2}} g]_\theta uu_{\theta\theta} \\ &= -\lambda^{\frac{1}{2}} u^2 f_\theta + \left[\lambda^{-\frac{1}{2}} u^2 \left(\frac{u_\theta}{u} + au\right)\left(\frac{u_\theta^2}{u^2} - au_\theta\right) - 2\lambda^{\frac{1}{2}} uu_\theta\right] f + u\lambda^{-\frac{1}{2}} u_{\theta\theta} g, \end{aligned}$$

$$\lambda_t = 2\left(\frac{u_\theta}{u} + au\right)\left(\frac{u_{\theta t}}{u} - \frac{u_\theta u_t}{u^2} + au_t\right).$$

The variation for the perimeter is

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}t} &= \oint \left[-\frac{u_t}{u} + \frac{\lambda_t}{2\lambda} + ((u_\theta + au^2)\lambda^{-\frac{1}{2}} f + \lambda^{-\frac{1}{2}} ug)_\theta\right] \mathrm{d}\omega \\ &= \oint \Big[-u^{-1}((u_\theta + au^2)\lambda^{-\frac{1}{2}} f + \lambda^{-\frac{1}{2}} ug) + \lambda^{-1}\left(\frac{u_\theta}{u} + au\right) \\ &\quad \cdot \left(\frac{u_{\theta t}}{u} - \frac{u_\theta u_t}{u^2} + au_t\right) + (\lambda^{-\frac{1}{2}}(u_\theta + au^2) f + \lambda^{-\frac{1}{2}} ug)_\theta\Big] \mathrm{d}\omega \\ &= \oint (g_\omega + \varrho f) \mathrm{d}\omega. \end{aligned}$$

As usual, we require

$$\oint \varrho f \mathrm{d}\omega = 0$$

and

$$g_\omega + \varrho f = 0.$$

The evolution of the curvature is

$$\varrho_t = \lambda^{-\frac{3}{2}}[u_{\theta\theta t} + 3auu_{\theta t} + (3au_\theta + 3a^2u^2 + 1)u_t] - \frac{3}{2}\lambda^{-1}\lambda_t. \tag{22}$$

A straightforward computation gives

$$\begin{aligned} u_{\theta\theta t} &= (\beta - \alpha u_\theta) f_{\theta\theta} + 2(\beta_\theta - \alpha_\theta u_\theta - \alpha u_{\theta\theta}) f_\theta + (\beta_{\theta\theta} - \alpha_{\theta\theta} u_\theta - 2\alpha_\theta u_{\theta\theta}) f + \lambda^{-\frac{1}{2}} uu_{\theta\theta\theta} g \\ &= \lambda(\beta - \alpha u_\theta) u^{-2} f_{\omega\omega} + [2\lambda^{\frac{1}{2}} u^{-1}(\beta_\theta - \alpha_\theta u_\theta - \alpha u_{\theta\theta}) + (u^{-1}\lambda^{\frac{1}{2}})_\theta \\ &\quad \cdot (\beta - \alpha u_\theta)] f_\omega + (\beta_{\theta\theta} - \alpha_{\theta\theta} u_\theta - 2\alpha_\theta u_{\theta\theta}) f + \lambda^{-\frac{1}{2}} uu_{\theta\theta\theta} g. \end{aligned}$$

Also we have

$$\beta - \alpha u_\theta = -\lambda^{\frac{1}{2}} u^2,$$

$$\beta_\theta - \alpha_\theta u_\theta = \lambda^{-\frac{1}{2}}\left(\frac{u_\theta}{u} + au\right)(u_\theta^2 - au^2 u_\theta) - 2\lambda^{\frac{1}{2}} uu_\theta,$$

$$\begin{aligned} \beta_{\theta\theta} - \alpha_{\theta\theta} u_\theta - 2\alpha_\theta u_{\theta\theta} &= \frac{1}{4}\lambda^{-\frac{3}{2}}\lambda_\theta^2 u^2 - \frac{1}{2}\lambda^{-\frac{1}{2}}\lambda_{\theta\theta} u^2 - 2\lambda^{-\frac{1}{2}}\lambda_\theta uu_\theta \\ &\quad - 2\lambda^{\frac{1}{2}} u_\theta^2 - 2\lambda^{\frac{1}{2}} uu_{\theta\theta} + \lambda^{-\frac{1}{2}}(u_\theta + au^2) u_{\theta\theta\theta}, \end{aligned}$$

$$\lambda_\theta = 2\left(\frac{u_\theta}{u} + au\right)\left(\frac{u_{\theta\theta}}{u} - \frac{u_\theta^2}{u^2} + au_\theta\right),$$

$$\lambda_{\theta\theta} = 2\left(\frac{u_{\theta\theta}}{u} - \frac{u_\theta^2}{u^2} + au_\theta\right)^2 + 2\left(\frac{u_\theta}{u} + au\right)\left(\frac{u_{\theta\theta\theta}}{u} - 3\frac{u_{\theta\theta}u_\theta}{u^2} + 2\frac{u_\theta^3}{u^3} + au_{\theta\theta}\right),$$

where $\alpha = \lambda^{-1/2}(u_\theta + au^2)$, $\beta = \lambda^{-1/2}(u_\theta^2 + au^2u_\theta) - \lambda^{1/2}u^2$. Substituting the expressions for $u_{\theta t}$, $u_{\theta\theta t}$ and λ_t into (22), we arrive at

$$\begin{aligned}\varrho_t &= f_{\omega\omega} + \lambda^{-3}[u_{\theta\theta}^2 + 2(3auu_\theta + a^2u^3 + u)u_{\theta\theta} + 6a^3u^4u_\theta^2 \\ &\quad + 9a^2u^2u_\theta^2 + 6au^2u_\theta + a^4u^6 + 2a^2u^4 + u^2]f + \varrho_\omega g \\ &= f_{\omega\omega} + \varrho^2 f + \varrho_\omega g = (\partial_\omega^2 + \varrho^2 - \varrho_\omega\partial_\omega^{-1}\varrho)f,\end{aligned} \tag{23}$$

after a cumbersome calculation. Letting $f = -\varrho_\omega$ in (23), we obtain the mKdV equation

$$\varrho_t + \varrho_{\omega\omega\omega} + \frac{1}{2}\varrho^2\varrho_\omega = 0.$$

Letting $f = -\Omega_3^{n-2}\varrho_\omega$ in (23), where $\Omega_3 = \partial_\omega^2 + \varrho^2 - \varrho_\omega\partial_\omega^{-1}\varrho$, $n \geq 2$, we obtain the mKdV hierarchy with the recursion operator Ω_3.

Finally, we turn to the hyperbolic geometry H_1. The arc-length and curvature are given respectively by

$$\begin{aligned}&\mathrm{d}\tilde{\sigma} = \iota(u^2-1)^{-\frac{1}{2}}\mathrm{d}\theta, \\ &\nu = \iota^{-3}\left[u_{\theta\theta} + 3(u^2-1)^{-\frac{1}{2}}u^2u_\theta + u^3 + u - \frac{3}{2}\iota\right],\end{aligned}$$

where $\iota = (u^2-1)^{-1/2}u_\theta + u$. The tangent and normal vectors are

$$\mathbf{T} = \iota^{-1}\tilde{\alpha}(1, u_\theta) \quad \text{and} \quad \mathbf{N} = (\tilde{\alpha}, \tilde{\beta}),$$

where $\tilde{\alpha} = (u^2-1)^{1/2}$, $\tilde{\beta} = -u(u^2-1)$. So the curve motion flow (4) becomes

$$\gamma_t = f((u^2-1)^{\frac{1}{2}}, -u(u^2-1)) + g\iota^{-1}(u^2-1)^{\frac{1}{2}}(1, u_\theta).$$

It yields the evolution of θ, u and u_θ:

$$\begin{aligned}&\theta_t = \tilde{\alpha}(f + \iota^{-1}g), \\ &u_t = \tilde{\beta}f + \iota^{-1}\tilde{\alpha}u_\theta g, \\ &u_{\theta t} = (\tilde{\beta} - \tilde{\alpha}u_\theta)f_\theta + (\tilde{\beta}_\theta - \tilde{\alpha}_\theta u_\theta)f + \iota^{-1}\tilde{\alpha}u_{\theta\theta}g, \\ &\iota_t = (u^2-1)^{-\frac{1}{2}}u_{\theta t} - (u^2-1)^{-\frac{3}{2}}u_\theta u_t + u_t.\end{aligned} \tag{24}$$

The perimeter for a closed curve in this geometry is $L = \oint \iota(u^2-1)^{-1/2}\mathrm{d}\theta$. Its time evolution is

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t} &= \oint\{-(u^2-1)^{-1}uu_t + \iota^{-1}\iota_t + [\tilde{\alpha}(f + \iota^{-1}g)]_\theta\}\mathrm{d}\tilde{\sigma} \\ &= \oint\{[\iota^{-1}\tilde{\alpha}^{-1}(\tilde{\beta} - \tilde{\alpha}u_\theta) + \tilde{\alpha}]f_\theta + [-\tilde{\alpha}^{-2}\tilde{\beta}u + \iota^{-1}((\tilde{\beta}_\theta - \tilde{\alpha}_\theta u_\theta) - \tilde{\beta}u(u^2-1)^{-\frac{3}{2}}u_\theta + \tilde{\beta}) + \tilde{\alpha}_\theta]f \\ &\quad + g_{\tilde{\sigma}} + [-\iota^{-1}\tilde{\alpha}^{-1}uu_\theta + \iota^{-2}(u_{\theta\theta} + (1 - (u^2-1)^{-\frac{3}{2}}u_\theta)\tilde{\alpha}u_\theta) + (\iota^{-1}\tilde{\alpha})_\theta]g\}\mathrm{d}\tilde{\sigma} \\ &= \oint(g_{\tilde{\sigma}} + f)\mathrm{d}\tilde{\sigma}.\end{aligned}$$

Hence the motion is inextensible if

$$\oint f\,\mathrm{d}\tilde{\sigma} = 0$$

and

$$g = -\partial_{\tilde{\sigma}}^{-1} f.$$

Under this inextensibility condition, we compute the evolution of the curvature to get

$$\nu_t = \iota^{-3}\left[u_{\theta\theta t} + 3u^2(u^2-1)^{-\frac{1}{2}}u_{\theta t} - (3u^3(u^2-1)^{-\frac{3}{2}}u_\theta - 6u(u^2-1)^{-\frac{1}{2}} - 3u^2 - 1)u_t - \frac{3}{2}\iota_t\right] - 3\nu\iota^{-1}\iota_t. \tag{25}$$

Also we have

$$\begin{aligned} \iota^{-1}\iota_t &= f + g_{\tilde{\sigma}} + (u^2-1)^{-1}uu_t - [\tilde{\alpha}(f + g\iota^{-1})]_\theta, \\ u_{\theta\theta t} &= \iota^2(u^2-1)^{-1}(\tilde{\beta} - \tilde{\alpha}u_\theta)f_{\tilde{\sigma}\tilde{\sigma}} + [2(u^2-1)^{-\frac{1}{2}}\iota(\tilde{\beta}_\theta - \tilde{\alpha}_\theta u_\theta - \tilde{\alpha}u_{\theta\theta}) \\ &\quad + ((u^2-1)^{-\frac{1}{2}}\iota)_\theta(\tilde{\beta} - \tilde{\alpha}u_\theta)]f_{\tilde{\sigma}} + (\tilde{\beta}_{\theta\theta} - \tilde{\alpha}_{\theta\theta}u_\theta - \tilde{\alpha}_\theta u_{\theta\theta})f, \end{aligned} \tag{26}$$

where

$$\begin{aligned} &\tilde{\beta} - \tilde{\alpha}u_\theta = -\iota(u^2-1), \\ &\tilde{\beta}_\theta - \tilde{\alpha}_\theta u_\theta - \tilde{\alpha}u_{\theta\theta} = -(u^2-1)^{\frac{1}{2}}u_{\theta\theta} + (1-3u^2)u_\theta - (u^2-1)^{-\frac{1}{2}}uu_\theta^2, \\ &\tilde{\beta}_{\theta\theta} - \tilde{\alpha}_{\theta\theta}u_\theta - \tilde{\alpha}_\theta u_{\theta\theta} = (u^2-1)^{-\frac{1}{2}}[3uu_\theta u_{\theta\theta} + (u^2(u^2-1)^{-1} - 1)u_\theta^3] + (1-3u^2)u_{\theta\theta} - 6uu_\theta^2. \end{aligned}$$

It follows that ν satisfies

$$\nu_t = -f_{\tilde{\sigma}\tilde{\sigma}} + c_1(\nu)f + c_2(\nu)g,$$

where the coefficients are determined as

$$\begin{aligned} c_1(\nu) &= \iota^{-3}[-2u_{\theta\theta} - 3u^2(u^2-1)^{-\frac{1}{2}}u_\theta + 3(u^2-1)^{-\frac{1}{2}}u_\theta + (u^2-1)^{-\frac{3}{2}}u_\theta^3 + 3u(u^2-1)^{-\frac{1}{2}}u_\theta^2 - u^3 + u] \\ &= 1 - 2\nu, \\ c_2(\nu) &= -3\nu[\iota^{-1}u(u^2-1)^{-\frac{1}{2}}u_\theta - (\iota^{-1}(u^2-1)^{-\frac{1}{2}})_t] \\ &\quad + \iota^{-4}\left[(u^2-1)^{\frac{1}{2}}(u_{\theta\theta\theta} + 3u^2u_\theta + u_\theta) + \frac{3}{2}(u^2-1)^{-1}uu_\theta u_{\theta\theta}\right. \\ &\quad \left. + \left(3u^2 - \frac{3}{2}\right)u_{\theta\theta} - 3(u^2-1)^{-1}u^3u_\theta^2 + 6uu_\theta\right] \\ &= \nu_{\tilde{\sigma}}. \end{aligned}$$

Hence we get the equation for the curvature

$$\begin{aligned} \nu_t &= -f_{\tilde{\sigma}\tilde{\sigma}} + (-2\nu + 1)f + \nu_{\tilde{\sigma}}g \\ &= -[\partial_{\tilde{\sigma}}^2 + (2\nu - 1) + \nu_{\tilde{\sigma}}\partial_{\tilde{\sigma}}^{-1}]f. \end{aligned} \tag{27}$$

Letting $f = \nu_{\tilde{\sigma}}$ in (27), we obtain the KdV equation with lower order term

$$\nu_t + \nu_{\tilde{\sigma}\tilde{\sigma}\tilde{\sigma}} + 3\nu\nu_{\tilde{\sigma}} - \nu_{\tilde{\sigma}} = 0.$$

Letting $f = \Omega_4^{n-1}\nu_{\tilde{\sigma}}$ $(n \geq 2)$ in (27), where $\Omega_4 = \partial_{\tilde{\sigma}}^2 + 2\nu - 1 + \nu_{\tilde{\sigma}}\partial_{\tilde{\sigma}}^{-1}$, we obtain the KdV hierarchy with the recursion operator Ω_4.

4. Motion of curves in geometries C_2 and S_1

The geometries with four-dimensional isometry groups contain conformal geometry C_2 and similarity geometry S_1. We first consider motion of curves in the conformal geometry C_2. The arc-length and curvature in C_2 are given respectively by

$$d\tau = u^{-2}k^{\frac{1}{2}}d\theta$$

and

$$\tilde{k} = u^2 k^{-\frac{3}{2}} k_\theta,$$

where $k = u^3(u_{\theta\theta} + \frac{1}{4}u)$. The tangent and normal vectors are

$$\mathbf{T} = u^2 k^{-\frac{1}{2}}(1, u_\theta) \quad \text{and} \quad \mathbf{N} = u(0, 1).$$

The motion of curve (4) now reads as

$$\gamma_t = f u(0, 1) + g u^2 k^{-\frac{1}{2}}(1, u_\theta).$$

Thus we have

$$\begin{cases} \theta_t = u^2 k^{-\frac{1}{2}} g, \\ u_t = u f + u^2 k^{-\frac{1}{2}} u_\theta g. \end{cases} \tag{28}$$

The perimeter of a closed curve is

$$L = \oint \mathrm{d}s = \oint u^{-2} k^{\frac{1}{2}} \mathrm{d}\theta. \tag{29}$$

The first variation of the perimeter is

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}t} &= \oint (u^{-2} k^{\frac{1}{2}} \theta_p)_t \mathrm{d}p \\ &= \oint \left[\frac{k_t}{2k} - 2\frac{u_t}{u} + (u^2 k^{-\frac{1}{2}} g)_\theta \right] \mathrm{d}\tau \equiv \oint J(\tilde{k}, \tilde{k}_\tau, \tilde{k}_{\tau\tau}, \ldots) \mathrm{d}\tau. \end{aligned} \tag{30}$$

Differentiating (28) with respect to t, we get

$$\begin{aligned} u_{\theta t} &= u_{t\theta} - (u^2 k^{-\frac{1}{2}} g)_\theta u_\theta \\ &= u f_\theta + u_\theta f + u^2 k^{-\frac{1}{2}} u_{\theta\theta} g, \\ u_{\theta\theta t} &= u_{\theta t\theta} - (u^2 k^{-\frac{1}{2}} g)_\theta u_{\theta\theta} \\ &= u f_{\theta\theta} + 2u_\theta f_\theta + u_{\theta\theta} f + k^{-\frac{1}{2}} u^2 u_{\theta\theta\theta} g. \end{aligned}$$

Also we have

$$\begin{aligned} \frac{k_t}{2k} &= \frac{1}{2k} \left[u^3 \left(u_{\theta\theta} + \frac{1}{4} u \right) \right]_t \\ &= \frac{1}{2k} \left(k f_{\tau\tau} + \frac{1}{2} u^2 k^{-\frac{1}{2}} k_\theta f_\tau + 4kf + k_\tau g \right) \\ &= \frac{1}{2} f_{\tau\tau} + \frac{1}{4} \tilde{k} f_\tau + 2f + \frac{1}{2} \tilde{k} g, \\ (g u^2 k^{-\frac{1}{2}})_\theta - 2\frac{u_t}{u} &= (u^2 k^{-\frac{1}{2}} g)_\theta - 2(f + u k^{-\frac{1}{2}} u_\theta g) \\ &= g_\tau - 2f - \frac{1}{2} \tilde{k} g. \end{aligned} \tag{31}$$

Thus we obtain

$$J = g_\tau + \frac{1}{2} f_{\tau\tau} + \frac{1}{4} \tilde{k} f_\tau.$$

So the motion is inextensible if

$$g_\tau + \frac{1}{2} f_{\tau\tau} + \frac{1}{4} \tilde{k} f_\tau = 0$$

and

$$\oint \tilde{k} f_\tau \mathrm{d}\tau = 0.$$

The evolution for the curvature is

$$\begin{aligned}\tilde{k}_t &= (k^{-\frac{3}{2}} u^2 k_\theta)_t = (k^{-1} k_\tau)_t = (k^{-1} k_t)_\tau \\ &= \left(\frac{1}{2} f_{\tau\tau} + \frac{1}{4}\tilde{k} f_\tau + 2f + \frac{1}{2}\tilde{k} g\right)_\tau \\ &= \frac{1}{4}[2\partial_\tau^3 + \tilde{k}\partial_\tau^2 + (\tilde{k}_\tau + 8)\partial_\tau] f + \frac{1}{2}\tilde{k} g_\tau + \frac{1}{2}\tilde{k}_\tau g,\end{aligned}$$

where $g = -\frac{1}{2} f_\tau - \frac{1}{4}\partial_\tau^{-1}\tilde{k} f_\tau$. Consequently, we have

$$\tilde{k}_t = \frac{1}{2}\left(\partial_\tau^2 - \frac{1}{4}\tilde{k}^2 - \frac{1}{4}\tilde{k}_\tau \partial_\tau^{-1}\tilde{k} + 4\right) f_\tau. \tag{32}$$

If we let $f = -2\tilde{k}$ in (32), we obtain the defocusing mKdV equation with lower order term

$$\tilde{k}_t + \tilde{k}_{\tau\tau\tau} - \frac{3}{8}\tilde{k}^2 \tilde{k}_\tau + 4\tilde{k}_\tau = 0.$$

In general, letting $f = -2\partial_\tau^{-1}\Omega_5 \tilde{k}_\tau$, we obtain the mKdV hierarchy with the recursion operator $\Omega_5 = \partial_\tau^2 - (1/4)\tilde{k}^2 - (1/4)\tilde{k}_\tau \partial_\tau^{-1}\tilde{k} + 4$.

Next, we consider the similarity geometry S_1. The arc-length and curvature are given by

$$\mathrm{d}\xi = \mathrm{d}\theta$$

and

$$\varphi = v^{-1} v_\theta,$$

where $v = u_{\theta\theta} + u$. The tangent and normal vectors are

$$\mathbf{T} = (1, u_\theta) \quad \text{and} \quad \mathbf{N} = v(0, 1).$$

The motion of curves is governed by

$$\gamma_t = f v(0, 1) + g(1, u_\theta).$$

This gives

$$\begin{cases}\theta_t = g, \\ u_t = vf + u_\theta g.\end{cases} \tag{33}$$

The first variation of the perimeter is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \left(\oint \mathrm{d}\theta\right)_t = \oint (\theta_p)_t \mathrm{d}p = \oint g_p \mathrm{d}p = 0. \tag{34}$$

This leads to the inextensibility condition

$$g_\theta = 0 \quad \text{i.e. } g = c,$$

where c is a constant. Since

$$\begin{aligned}v_t &= (u_{\theta\theta} + u)_t = u_{\theta\theta t} + u_t = u_{t\theta\theta} + u_t = (vf + u_\theta g)_{\theta\theta} + (vf + u_\theta g), \\ v_{\theta t} &= (u_{\theta\theta\theta} + u_\theta)_t = u_{\theta\theta\theta t} + u_{\theta t} = u_{t\theta\theta\theta} + u_{t\theta} = (vf + u_\theta g)_{\theta\theta\theta} + (vf + u_\theta g)_\theta,\end{aligned}$$

the evolution for the curvature is

$$\begin{aligned}\varphi_t &= (v^{-1}v_\theta)_t = v^{-1}v_{\theta t} - v^{-2}v_\theta v_t\\ &= v^{-1}[(vf)_{\theta\theta\theta} + (vf)_\theta + c(u_{\theta\theta\theta\theta} + u_{\theta\theta})] - v^{-2}v_\theta[(vf)_{\theta\theta} + vf - cv_\theta(u_{\theta\theta\theta} + u_\theta)]\\ &= f_{\theta\theta\theta} + 2\varphi f_{\theta\theta} + (3\varphi_\theta + \varphi^2 + 1)f_\theta + (\varphi_{\theta\theta} + 2\varphi\varphi_\theta)f + c\varphi_\xi\\ &= (\partial_\xi^3 + 2\varphi\partial_\xi^2 + (3\varphi_\xi + \varphi^2 + 1)\partial_\xi + (\varphi_{\xi\xi} + 2\varphi\varphi_\xi))f + c\varphi_\xi\\ &= [(\partial_\xi + \varphi + \varphi_\xi\partial_\xi^{-1})^2 + 1]f_\xi + c\varphi_\xi.\end{aligned} \tag{35}$$

Letting $f = 1$ in (35), we obtain the Burgers equation

$$\varphi_t = \varphi_{\xi\xi} + 2\varphi\varphi_\xi.$$

Letting $f = \varphi$ in (35), we obtain the third-order Burgers equation

$$\varphi_t = \varphi_{\xi\xi\xi} + 3\varphi\varphi_{\xi\xi} + 3\varphi_\xi^2 + 3\varphi^2\varphi_\xi + (c+1)\varphi_\xi.$$

In general, letting $f_\xi = \Omega_6^{n-2}\varphi_\xi$, we obtain the Burgers hierarchy with the recursion operator $\Omega_6 = \partial_\xi + \varphi + \varphi_\xi\partial_\xi^{-1}$.

5. Motion of curves in geometries S_2 and A_1

From Table 1, the geometries with the five-dimensional isometry group in $S^1 \times \mathbb{R}$ contain the extended similarity geometry S_2 and the special affine geometry A_1. We first look at the extended similarity geometry S_2. The arc-length and curvature are given respectively by

$$\mathrm{d}\eta = \mathrm{d}\theta$$

and

$$\psi = h^{-1}h_\theta,$$

where $h = u_{\theta\theta\theta} + u_\theta$. The tangent and normal vectors are given by

$$\mathbf{T} = (1, u_\theta) \quad \text{and} \quad \mathbf{N} = h(0,1).$$

The motion of curves is governed by

$$\gamma_t = fh(0,1) + g(1, u_\theta).$$

So we have

$$\begin{cases}\theta_t = g,\\ u_t = hf + u_\theta g.\end{cases} \tag{36}$$

The first variation of the perimeter is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \left(\oint \mathrm{d}\theta\right)_t = \oint(\theta_p)_t\mathrm{d}p = \oint g_p\mathrm{d}p = 0,$$

where p is a parameter commuting with time t. This leads to the inextensibility condition

$$g_\theta = 0.$$

One can put $g = 0$ without loss of generality. A straightforward computation gives

$$\begin{aligned}h_t &= (u_{\theta\theta\theta} + u_\theta)_t = u_{t\theta\theta\theta} + u_{t\theta} = (hf)_{\theta\theta\theta} + (hf)_\theta,\\ h_{\theta t} &= (u_{\theta\theta\theta\theta} + u_{\theta\theta})_t = u_{t\theta\theta\theta\theta} + u_{t\theta\theta} = (hf)_{\theta\theta\theta\theta} + (hf)_{\theta\theta}.\end{aligned} \tag{37}$$

Using (36) and (37), we derive the equation for the curvature

$$\psi_t = (h^{-1}h_\theta)_t = h^{-1}h_{\theta t} - h^{-2}h_\theta h_t$$

$$
\begin{aligned}
&= h^{-1}[(hf)_{\theta\theta\theta\theta} + (hf)_{\theta\theta}] - h^{-2}h_\theta[(hf)_{\theta\theta\theta} + (hf)_\theta] \\
&= f_{\theta\theta\theta\theta} + 3h^{-1}h_\theta f_{\theta\theta\theta} + (6h^{-1}h_{\theta\theta} - 3h^{-2}h_\theta^2 + 1)f_{\theta\theta} + h^{-1}[(4-3h^{-1}h_\theta)h_{\theta\theta} + h_\theta]f_\theta \\
&\quad + h^{-2}(hh_{\theta\theta\theta} - h_\theta h_{\theta\theta\theta} + hh_{\theta\theta} - h_\theta^2)f \\
&= [\partial_\eta^4 + 3\psi\partial_\eta^3 + (6\psi_\eta + 3\psi^2 + 1)\partial_\eta^2 + (4\psi_{\eta\eta} + 9\psi\psi_\eta + \psi^3 + 2\psi)\partial_\eta \\
&\quad + \psi_{\eta\eta\eta} + 3\psi_\eta^2 + 3\psi\psi_{\eta\eta} + 3\psi^2\psi_\eta + \psi_\eta]f \\
&= [(\partial_\eta + \psi + \psi_\eta\partial_\eta{}^{-1})^3 + \partial_\eta + \psi + \psi_\eta\partial_\eta^{-1}]f_\eta.
\end{aligned}
\tag{38}
$$

Letting $f = 1$ in (38), we obtain the third-order Burgers equation

$$\psi_t = \psi_{\eta\eta\eta} + 3\psi\psi_{\eta\eta} + 3\psi_\eta^2 + 3\psi^2\psi_\eta.$$

In general, we obtain the Burgers hierarchy with the recursion operator $\Omega_7 = (\partial_\eta + \psi + \psi_\eta\partial_\eta{}^{-1})^3 + \partial_\eta + \psi + \psi_\eta\partial_\eta^{-1}$.

We now consider the special affine geometry A_1. The arc-length and curvature are given respectively by

$$\mathrm{d}\rho = (u_{\theta\theta} + u)^{\frac{2}{3}}\mathrm{d}\theta = v^{\frac{2}{3}}\mathrm{d}\theta$$

and

$$\mu = v^{-1}(v^{-\frac{1}{3}})_{\theta\theta} + v^{-\frac{4}{3}},$$

where $v = u_{\theta\theta} + u$. The tangent and normal vectors are

$$\mathbf{T} = v^{-\frac{2}{3}}(1, u_\theta) \quad \text{and} \quad \mathbf{N} = v^{-\frac{1}{3}}(0, 1).$$

The curve motion flow in this geometry reads as

$$\gamma_t = f\mathbf{T} + g\mathbf{N} = fv^{-\frac{1}{3}}(0,1) + gv^{-\frac{2}{3}}(1, u_\theta).$$

This implies

$$
\begin{cases}
\theta_t = v^{-\frac{2}{3}}g, \\
u_t = v^{-\frac{1}{3}}f + v^{-\frac{2}{3}}u_\theta g.
\end{cases}
\tag{39}
$$

A straightforward computation shows

$$
\begin{aligned}
&u_{\theta t} = u_{t\theta} - (v^{-\frac{2}{3}}g)_\theta u_\theta = (v^{-\frac{1}{3}}f)_\theta + v^{-\frac{2}{3}}u_{\theta\theta}g, \\
&u_{\theta\theta t} = u_{\theta t\theta} - (v^{-\frac{2}{3}}g)_\theta u_{\theta\theta} = (v^{-\frac{1}{3}}f)_{\theta\theta} + v^{-\frac{2}{3}}u_{\theta\theta\theta}g, \\
&(v^{-\frac{1}{3}}f)_\theta = v^{\frac{1}{3}}f_\rho - \frac{1}{3}v^{\frac{2}{3}}v_\rho f, \\
&(v^{-\frac{1}{3}}f)_{\theta\theta} = vf_{\rho\rho} - \frac{1}{3}v^{\frac{2}{3}}(v^{-\frac{2}{3}}v_\rho)_\rho f, \\
&(v^{-\frac{1}{3}}f)_{\theta\theta\theta} = v^{\frac{5}{3}}f_{\rho\rho\rho} + v_\rho v^{\frac{2}{3}}f_{\rho\rho} - \frac{1}{3}v^{\frac{4}{3}}(v^{-\frac{2}{3}}v_\rho)_\rho f_\rho - \frac{1}{3}v^{\frac{2}{3}}[v^{\frac{2}{3}}(v^{-\frac{2}{3}}v_\rho)_\rho]_\rho f, \\
&(v^{-\frac{1}{3}}f)_{\theta\theta\theta\theta} = v^{\frac{7}{3}}f_{\rho\rho\rho\rho} + \frac{8}{3}v^{\frac{4}{3}}v_\rho f_{\rho\rho\rho} + [v^{\frac{2}{3}}(v^{\frac{2}{3}}v_\rho)_\rho - \frac{1}{3}v^2(v^{-\frac{2}{3}}v_\rho)_\rho]f_{\rho\rho} \\
&\qquad - \frac{1}{3}[v^{\frac{4}{3}}(v^{\frac{2}{3}}(v^{-\frac{2}{3}}v_\rho)_\rho)_\rho + v^{\frac{2}{3}}(v^{\frac{4}{3}}(v^{-\frac{2}{3}}v_\rho)_\rho)_\rho]f_\rho - \frac{1}{3}v^{\frac{2}{3}}[(v^{\frac{2}{3}}(v^{-\frac{2}{3}}v_\rho)_\rho)_\rho v^{\frac{2}{3}}]_\rho f, \\
&v_t = u_{\theta\theta t} + u_t = (v^{-\frac{1}{3}}f)_{\theta\theta} + v^{-\frac{1}{3}}f + v^{-\frac{2}{3}}v_\theta g, \\
&v_{\theta t} = v_{t\theta} - (v^{-\frac{2}{3}}g)_\theta v_\theta = (v^{-\frac{1}{3}}f)_{\theta\theta\theta} + (v^{-\frac{1}{3}})_\theta f + v^{-\frac{2}{3}}v_{\theta\theta}g, \\
&v_{\theta\theta t} = v_{\theta t\theta} - (v^{-\frac{2}{3}}g)_\theta v_{\theta\theta} = (v^{-\frac{1}{3}}f)_{\theta\theta\theta\theta} + (v^{-\frac{1}{3}}f)_{\theta\theta} + v^{-\frac{2}{3}}v_{\theta\theta\theta}g.
\end{aligned}
\tag{40}
$$

The perimeter is

$$L = \oint \mathrm{d}\rho = \oint v^{\frac{2}{3}} \mathrm{d}\theta. \tag{41}$$

Using (39) and (40), we get the first variation of the perimeter

$$\begin{aligned} L_t &= \oint \left[\frac{2}{3} v^{-1} v_t + (v^{-\frac{2}{3}} g)_\theta\right] \mathrm{d}\rho \\ &= \oint \left[g_\rho + \frac{2}{3}(f_{\rho\rho} + \mu f)\right] \mathrm{d}\rho, \end{aligned}$$

after a cumbersome calculation. As usual we obtain the inextensibility condition

$$g_\rho + \frac{2}{3}(f_{\rho\rho} + \mu f) = 0 \tag{42}$$

and

$$\oint \mu f \mathrm{d}\rho = 0.$$

We now compute the evolution for the curvature:

$$\begin{aligned} \mu_t &= [v^{-1}(v^{-\frac{1}{3}})_{\theta\theta} + v^{-\frac{4}{3}}]_t \\ &= -\frac{1}{3} v^{-\frac{7}{3}} v_{\theta\theta t} + \left(\frac{7}{9} v^{-\frac{10}{3}} v_{\theta\theta} - \frac{40}{27} v^{-\frac{13}{3}} v_\theta^2 - \frac{4}{3} v^{-\frac{7}{3}}\right) v_t + \frac{8}{9} v^{-\frac{10}{3}} v_\theta v_{\theta t} \\ &= -\frac{1}{3} v^{-\frac{7}{3}} [(v^{-\frac{1}{3}} f)_{\theta\theta\theta\theta} + (v^{-\frac{1}{3}} f)_{\theta\theta} + v^{-\frac{2}{3}} v_{\theta\theta\theta} g] \\ &\quad + \left(\frac{7}{9} v^{-\frac{10}{3}} v_{\theta\theta} - \frac{40}{27} v^{-\frac{13}{3}} v_\theta^2 - \frac{4}{3} v^{-\frac{7}{3}}\right) [(v^{-\frac{1}{3}} f)_{\theta\theta} + v^{-\frac{1}{3}} f + v^{-\frac{2}{3}} v_\theta g] \\ &\quad + \frac{8}{9} v^{-\frac{10}{3}} v_\theta [(v^{-\frac{1}{3}} f)_{\theta\theta\theta} + (v^{-\frac{1}{3}} f)_\theta + v^{-\frac{2}{3}} v_{\theta\theta} g] \\ &= -\frac{1}{3} f_{\rho\rho\rho\rho} + c_1(\mu) f_{\rho\rho} + c_2(\mu) f_\rho + c_3(\mu) f + c_4(\mu) g. \end{aligned} \tag{43}$$

The coefficients $c_i(\mu)$ in μ_t are given by

$$\begin{aligned} c_1(\mu) &= -\frac{1}{9} v^{-\frac{7}{3}} [3 v^{\frac{2}{3}} (v^{\frac{2}{3}} v_\rho)_\rho - v^2 (v^{-\frac{2}{3}} v_\rho)_\rho + 3v] + \frac{7}{9} v^{-\frac{7}{3}} v_{\theta\theta} - \frac{40}{27} v^{-\frac{10}{3}} v_\theta^2 - \frac{4}{3} v^{-\frac{4}{3}} + \frac{8}{9} v^{-\frac{8}{3}} v_\theta v_\rho \\ &= \frac{5}{3}\left(\frac{1}{3} v^{-\frac{7}{3}} v_{\theta\theta} - \frac{4}{9} v^{-\frac{10}{3}} v_\theta - v^{-\frac{4}{3}}\right) = -\frac{5}{3}\mu, \\ c_2(\mu) &= \frac{1}{9} v^{-\frac{5}{3}} [v^{\frac{3}{2}} (v^{\frac{2}{3}} (v^{-\frac{2}{3}} v_\rho)_\rho)_\rho + (v^{\frac{4}{3}} (v^{-\frac{2}{3}} v_\rho)_\rho)_\rho] - \frac{8}{9} v^{-3} v_\theta \left[\frac{1}{3} v (v^{-\frac{2}{3}} v_\rho)_\rho - 1\right] \\ &= \frac{2}{3}\left(\frac{1}{3} v^{-1} v_{\theta\theta\theta} - \frac{5}{3} v^{-4} v_\theta v_{\theta\theta} + \frac{40}{27} v^{-5} v_\theta^3 + \frac{4}{3} v^{-\frac{7}{3}} v_\theta\right) \\ &= -\frac{2}{3}\mu_\rho, \\ c_3(\mu) &= \frac{1}{3} v^{-\frac{5}{3}} \left[\frac{1}{3}((v^{\frac{2}{3}} (v^{-\frac{2}{3}} v_\rho)_\rho)_\rho v^{\frac{2}{3}})_\rho + \frac{1}{3}(v^{-\frac{2}{3}} v_\rho)_\rho\right] \\ &\quad + \left(\frac{7}{9} v^{-\frac{10}{3}} v_{\theta\theta} - \frac{40}{27} v^{-\frac{13}{3}} v_\theta^2 - \frac{4}{3} v^{-\frac{7}{3}}\right)\left[1 - \frac{1}{3} v (v^{-\frac{2}{3}} v_\rho)_\rho\right] - \frac{8}{27} v^{\frac{10}{3}} v_\theta [(v^{\frac{2}{3}} (v^{-\frac{2}{3}} v_\rho)_\rho)_\rho + v^{\frac{2}{3}} v_\rho] \\ &= -\frac{1}{3}\mu_{\rho\rho} - \frac{4}{3}\mu^2, \end{aligned}$$

$$\begin{aligned} c_4(\mu) &= -\frac{1}{3}v^{-1}v_{\theta\theta\theta} - \frac{40}{27}v^{-5}v_\theta^3 v_{\theta\theta} - \frac{4}{3}v^{-\frac{7}{3}}v_\theta + \frac{5}{3}v^{-4}v_\theta \\ &= \mu_\rho. \end{aligned}$$

Thus we get the equation for the curvature:

$$\begin{aligned} \mu_t &= -\frac{1}{3}[f_{\rho\rho\rho\rho} + 5\mu f_{\rho\rho} + 2\mu_\rho f_\rho + (\mu_{\rho\rho} + 4\mu^2)f - 3\mu_\rho g] \\ &= -\frac{1}{3}(\partial_\rho^4 + 5\mu\partial_\rho^2 + 4\mu_\rho\partial_\rho + \mu_{\rho\rho} + 4\mu^2 + 2\mu_\rho\partial_\rho^{-1}\mu)f. \end{aligned} \tag{44}$$

Letting $f = 3\mu_\rho$ in (44), we obtain the Sawada–Kotera equation [38]

$$\mu_t + \mu_{\rho\rho\rho\rho\rho} + 5\mu\mu_{\rho\rho\rho} + 5\mu_\rho\mu_{\rho\rho} + 5\mu^2\mu_\rho = 0.$$

Letting $f = 3(\partial_\mu^2 + \mu + \mu\partial_\rho^{-1})w$ in (44) with $w = \Omega_8^{n-1}\mu_\rho$, and

$$\Omega_8 = (\partial_\rho^3 + 2\mu\partial_\rho + 2\partial_\rho\mu)\left[\partial_\rho^3 + \partial_\rho^2\mu\partial_\rho^{-1} + \partial_\rho^{-1} + \frac{1}{2}(\mu^2\partial_\rho^{-1} + \partial_\rho^{-1}\mu^2)\right],$$

we obtain the Sawada–Kotera hierarchy [39].

6. Motion of curves in geometries A_2 and C_4

From Table 1, it is noted that the geometries with six-dimensional isometry groups in $S^1 \times \mathbb{R}$ contain the fully affine geometry A_2 and the conformal geometry C_4.

We first consider the fully affine geometry. The arc-length and curvature in A_2 are given by

$$\mathrm{d}\sigma = v^{\frac{2}{3}}\mu^{\frac{1}{2}}\mathrm{d}\theta,$$

and

$$\kappa = \mu^{-1}\mu_\sigma = v^{-\frac{2}{3}}\mu^{-\frac{3}{2}}\mu_\theta,$$

respectively, where $v = u_{\theta\theta} + u$, $\mu = v^{-1}[(v^{-1/3})_{\theta\theta} + v^{-1/3}]$. The fully affine tangent vector is

$$\mathbf{T} = v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}(1, u_\theta),$$

and the normal vector is chosen to be

$$\mathbf{N} = v^{-\frac{1}{3}}\mu^{-1}(0, 1).$$

The curve motion flow reads as

$$\gamma_t = f v^{-\frac{1}{3}}\mu^{-1}(0, 1) + g v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}(1, u_\theta).$$

So we have

$$\begin{cases} \theta_t = v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}g, \\ u_t = v^{-\frac{1}{3}}\mu^{-1}f + v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}u_\theta g. \end{cases} \tag{45}$$

Hence, the first variation of the perimeter is

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}t} &= \left(\oint v^{\frac{2}{3}}\mu^{\frac{1}{2}}\mathrm{d}\theta\right)_t = \oint\left[\frac{\mu_t}{2\mu} + \frac{2v_t}{3v} + (v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}g)_\theta\right]\mathrm{d}\sigma \\ &\equiv \oint J(\kappa, \kappa_\sigma, \kappa_{\sigma\sigma}, \ldots)\mathrm{d}\sigma. \end{aligned} \tag{46}$$

One has the following formulas:

$$u_{\theta t} = u_{t\theta} - (v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}g)_\theta u_\theta = (v^{-\frac{1}{3}}\mu^{-\frac{1}{2}}f)_\theta + v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}u_{\theta\theta}g,$$
$$u_{\theta\theta t} = u_{\theta t\theta} - (v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}g)_\theta u_{\theta\theta} = (v^{-\frac{1}{3}}\mu^{-1}f)_{\theta\theta} + v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}u_{\theta\theta\theta}g,$$

and

$$v_t = u_{\theta\theta t} + u_t = (v^{-\frac{1}{3}}\mu^{-1}f)_{\theta\theta} + v^{-\frac{1}{3}}\mu^{-1}f + v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}v_\theta g,$$
$$(v^{-\frac{1}{3}}\mu^{-1}f)_\theta = v^{\frac{1}{3}}\mu^{-\frac{1}{2}}f_\sigma + (v^{-\frac{1}{3}}\mu^{-1})_\theta f,$$
$$(v^{-\frac{1}{3}}\mu^{-1}f)_{\theta\theta} = vf_{\sigma\sigma} - \frac{3}{2}v^{\frac{1}{3}}\mu^{-\frac{3}{2}}\mu_\theta f_\sigma + (v^{-\frac{1}{3}}\mu^{-1})_{\theta\theta}f,$$
$$(v^{-\frac{1}{3}}\mu^{-1}f)_{\theta\theta\theta} = v^{\frac{5}{3}}\mu^{\frac{1}{2}}f_{\sigma\sigma\sigma} + \left(v_\theta - \frac{3}{2}v\mu^{-1}\mu_\theta\right)f_{\sigma\sigma} + \left[v^{\frac{2}{3}}\mu^{\frac{1}{2}}(v^{-\frac{1}{3}}\mu^{-1})_{\theta\theta} - \frac{3}{2}(v^{\frac{1}{3}}\mu^{-\frac{3}{2}}\mu_\theta)_\theta\right]f_\sigma,$$
$$\begin{aligned}(v^{-\frac{1}{3}}\mu^{-1}f)_{\theta\theta\theta\theta} \;=\;& v^{\frac{7}{3}}\mu f_{\sigma\sigma\sigma\sigma} + \left(\frac{8}{3}v^{\frac{2}{3}}\mu^{\frac{1}{2}}v_\theta - v^{\frac{5}{3}}\mu^{-\frac{1}{2}}\mu_\theta\right)f_{\sigma\sigma\sigma} + \left[\left(v_\theta - \frac{3}{2}v\mu^{-1}\mu_\theta\right)_\theta\right.\\ &\left.+\; v^{\frac{2}{3}}\mu^{\frac{1}{2}}(v^{\frac{2}{3}}\mu^{\frac{1}{2}}(v^{-\frac{1}{3}}\mu^{-1})_{\theta\theta} - \frac{3}{2}(v^{\frac{1}{3}}\mu^{-\frac{3}{2}}\mu_\theta)_\theta)\right]f_{\sigma\sigma} + \left[(v^{\frac{2}{3}}\mu^{\frac{1}{2}}(v^{-\frac{1}{3}}\mu^{-1})_{\theta\theta}\right.\\ &\left.-\; \frac{3}{2}(v^{\frac{1}{3}}\mu^{-\frac{3}{2}}\mu_\theta)_\theta)_\theta + v^{\frac{2}{3}}\mu^{\frac{1}{2}}(v^{-\frac{1}{3}}\mu^{-1})_{\theta\theta\theta}\right]f_\sigma + (v^{-\frac{1}{3}}\mu^{-1})_{\theta\theta\theta}f.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{2v_t}{3v} + (v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}g)_\theta \;=\;& \frac{2}{3}f_{\sigma\sigma} - v^{-\frac{2}{3}}\mu^{-\frac{3}{2}}\mu_\theta f_\sigma + \frac{2}{3}v^{-1}[(v^{-\frac{1}{3}}\mu^{-1})_{\theta\theta} + v^{-\frac{1}{3}}\mu^{-1}]f\\ &+ v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}g_\theta - \frac{1}{2}v^{-\frac{2}{3}}\mu^{-\frac{3}{2}}\mu_\theta g\\ =\;& \frac{2}{3}f_{\sigma\sigma} - \kappa f_\sigma - \frac{2}{3}\left(\kappa_\sigma - \frac{1}{2}\kappa^2 - 1\right)f + g_\sigma - \frac{1}{2}\kappa g,\\ \frac{\mu_t}{2\mu} \;=\;& -\frac{1}{6\mu}\left[v^{-\frac{7}{3}}v_{\theta\theta t} - \left(\frac{7}{3}v^{-\frac{10}{3}}v_{\theta\theta} - \frac{40}{9}v^{-\frac{13}{3}}v_\theta^2 - 4v^{-\frac{7}{3}}\right)v_t - \frac{8}{3}v^{-\frac{10}{3}}v_\theta v_{\theta t}\right]\\ =\;& \frac{1}{6\mu}v^{-\frac{7}{3}}\left(\frac{7}{3}v^{-1}v_{\theta\theta} - \frac{40}{9}v^{-2}v_\theta^2 - 4\right)[(v^{-\frac{1}{3}}\mu^{-1}f)_{\theta\theta} + \mu^{-1}(v^{-\frac{1}{3}}f + v^{-\frac{2}{3}}v_\theta g)]\\ &- \frac{1}{3}v^{-\frac{7}{3}}[(v^{-\frac{1}{3}}\mu^{-1}f)_{\theta\theta\theta\theta} + (v^{-\frac{1}{3}}\mu^{-1}f)_{\theta\theta} + \mu^{-1}v^{-\frac{2}{3}}v_{\theta\theta\theta}g]\\ &+ \frac{8}{9}v^{-\frac{10}{3}}v_\theta[(v^{-\frac{1}{3}}\mu^{-1}f)_{\theta\theta\theta} + (v^{-\frac{1}{3}}\mu^{-1}f)_\theta + \mu^{-1}v^{-\frac{2}{3}}v_{\theta\theta}g]\\ =\;& -\frac{1}{6}f_{\sigma\sigma\sigma\sigma} + c_1(\kappa)f_{\sigma\sigma\sigma} + c_2(\kappa)f_{\sigma\sigma} + c_3(\kappa)f_\sigma + c_4(\kappa)f + c_5(\kappa)g,\end{aligned} \tag{47}$$

where the coefficients $c_i(\kappa)$ are given by

$$\begin{aligned}c_1(\kappa) \;=\;& \frac{1}{2}\mu^{-1}\left[\frac{8}{9}v^{-\frac{5}{3}}\mu^{\frac{1}{2}}v_\theta - \frac{1}{3}v^{-\frac{7}{3}}\left(\frac{8}{3}v^{\frac{2}{3}}\mu^{\frac{1}{2}}v_\theta + v^{\frac{5}{3}}\mu^{-\frac{1}{2}}\mu_\theta\right)\right]\\ =\;& -\frac{1}{6}v^{-\frac{2}{3}}\mu^{-\frac{3}{2}}\mu_\theta = -\frac{1}{6}\kappa,\\ c_2(\kappa) \;=\;& \frac{5}{6}\mu^{-1}\left(\frac{1}{3}v^{-\frac{7}{3}}v_{\theta\theta} - \frac{4}{9}v^{-\frac{10}{3}}v_\theta - v^{\frac{4}{3}}\right) + \frac{2}{3}v^{-\frac{4}{3}}\mu^{-2}\mu_{\theta\theta} - \frac{23}{24}v^{-\frac{4}{3}}\mu^{-3}\mu_\theta^2 - \frac{4}{9}v^{-\frac{7}{3}}\mu^{-2}v_\theta\mu_\theta\\ =\;& \frac{2}{3}\kappa_\sigma + \frac{1}{24}\kappa^2 - \frac{5}{6},\end{aligned}$$

$$\begin{aligned}
c_3(\kappa) \;=\; & \frac{1}{2}\mu^{-1}\left[\frac{7}{6}\mu^{-\frac{3}{2}}v^{-2}\left(\mu_{\theta\theta\theta} - 2v^{-1}v_\theta\mu_{\theta\theta} - 5\mu^{-1}\mu_\theta\mu_{\theta\theta}\right.\right.\\
& + \left.\frac{14}{9}v^{-2}\mu_\theta v_\theta^2 + \frac{10}{3}v^{-1}\mu^{-1}v_\theta\mu_\theta^2 - \frac{2}{3}v^{-1}\mu_\theta v_{\theta\theta} + \frac{9}{2}\mu^{-2}\mu_\theta\right)\\
& - \frac{5}{2}\mu^{-\frac{3}{2}}\left(\frac{1}{3}v^{-3}v_{\theta\theta} - \frac{4}{9}v^{-4}v_\theta - v^{\frac{2}{3}}\right)\mu_\theta\\
& + \frac{1}{4}v^{-\frac{7}{3}}\mu^{-3}\mu_\theta\left(\frac{2}{3}\mu_\theta v_\theta + \frac{3}{2}v\mu_\theta^2 - \mu v\right) - \frac{1}{12}\mu^{-\frac{7}{2}}v^{-2}\mu_\theta^3\\
& + \left.\frac{2}{3}v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}\left(\frac{1}{3}v^{-\frac{7}{3}}v_{\theta\theta} - \frac{4}{9}v^{-\frac{10}{3}}v_\theta - v^{\frac{4}{3}}\right)_\theta\right]\\
=\; & \frac{7}{12}\kappa_{\sigma\sigma} - \frac{1}{8}\kappa\kappa_\sigma - \frac{1}{24}\kappa^3 + \frac{11}{12}\kappa,\\
c_4(\kappa) =\; & \frac{1}{6}\kappa_{\sigma\sigma\sigma} - \frac{1}{12}\kappa\kappa_{\sigma\sigma} - \frac{1}{6}\left(1 + \frac{1}{2}\kappa^2\right)\kappa_\sigma^2 + \frac{2}{3}\kappa_\sigma - \frac{1}{3}\kappa^2 - \frac{2}{3},\\
c_5(\kappa) =\; & \frac{1}{2}\kappa.
\end{aligned}$$

To simplify the expressions, setting $A = (2/3)f_{\sigma\sigma} - \kappa f_\sigma - (2/3)(\kappa_\sigma - (1/2)\kappa^2 - 1)f$ in (47), we have

$$\begin{aligned}
&\frac{2v_t}{3v} + (v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}g)_\theta = A + g_\sigma - \frac{1}{2}\kappa g,\\
&\frac{\mu_t}{2\mu} = -\frac{1}{4}A_{\sigma\sigma} - \frac{1}{8}\kappa A_\sigma - A + \frac{1}{2}\kappa g.
\end{aligned} \tag{48}$$

So the motion is inextensible if f and g satisfy

$$\frac{\mu_t}{2\mu} + \frac{2v_t}{3v} + (v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}g)_\theta = g_\sigma - \frac{1}{4}A_{\sigma\sigma} - \frac{1}{8}\kappa A_\sigma = 0 \tag{49}$$

and

$$\oint \kappa A_\sigma \mathrm{d}\sigma = 0.$$

Using (48) and (49), we obtain the evolution of the curvature:

$$\begin{aligned}
\kappa_t &= (v^{-\frac{2}{3}}\mu^{-\frac{3}{2}}\mu_\theta)_t\\
&= \left(\frac{\mu_t}{\mu}\right)_\sigma - \kappa\left[\frac{\mu_t}{2\mu} + \frac{2v_t}{3v} + (v^{-\frac{2}{3}}\mu^{-\frac{1}{2}}g)_\theta\right]\\
&= -\left(\frac{1}{2}A_{\sigma\sigma} + \frac{1}{4}\kappa A_\sigma + A - \kappa g\right)_\sigma.
\end{aligned} \tag{50}$$

Plugging (49) into (50), we arrive at

$$\begin{aligned}
\kappa_t &= -\frac{1}{2}A_{\sigma\sigma\sigma} + \frac{1}{8}\kappa^2 A_{\sigma\sigma} + \frac{1}{8}\partial_\sigma^{-1}(\kappa A_\sigma) - 2A_\sigma\\
&= -\frac{1}{2}\left(\partial_\sigma^2 - \frac{1}{4}\kappa^2 - \frac{1}{4}\partial_\sigma^{-1}\kappa + 4\right)A_\sigma.
\end{aligned} \tag{51}$$

Letting $A = 2\partial_\sigma^{-1}\Omega_9^{n-1}\kappa_\sigma$ in (51), we obtain the defocusing mKdV hierarchy

$$\kappa_t = -\Omega_9^{n-1}\kappa_\sigma.$$

where $\Omega_9 = D_\sigma^2 - (1/4)\kappa^2 - (1/4)\kappa_\sigma\partial_\sigma^{-1}\kappa + 4$ is a recursion operator of the defocusing mKdV equation.

Finally, we consider the motion of curves in the conformal geometry C_4. The arc-length and curvature in C_4 are given by

$$\mathrm{d}\delta = q^{\frac{1}{2}}\mathrm{d}\theta$$

and

$$\chi = 4q^{-2}q_{\theta\theta} - 5q^{-3}q_\theta^2 - 4q^{-1},$$

where $q = u_{\theta\theta\theta} + u_\theta$. The tangent and normal vectors are

$$\mathbf{T} = q^{-\frac{1}{2}}(1, u_\theta) \quad \text{and} \quad \mathbf{N} = q^{-\frac{1}{2}}(0, 1).$$

The curve motion flow is specified by

$$\gamma_t = fq^{-\frac{1}{2}}(0, 1) + gq^{-\frac{1}{2}}(1, u_\theta).$$

So we have

$$\begin{cases} \theta_t = q^{-\frac{1}{2}}g, \\ u_t = q^{-\frac{1}{2}}f + q^{-\frac{1}{2}}u_\theta g. \end{cases} \tag{52}$$

Using the expressions

$$\begin{aligned}
&u_{\theta t} = u_{t\theta} - (q^{-\frac{1}{2}}g)_\theta u_\theta = q^{-\frac{1}{2}}u_{\theta\theta}g + (q^{-\frac{1}{2}}f)_\theta, \\
&u_{\theta\theta t} = u_{\theta t\theta} - (q^{-\frac{1}{2}}g)_\theta u_{\theta\theta} = q^{-\frac{1}{2}}u_{\theta\theta\theta}g + (q^{-\frac{1}{2}}f)_{\theta\theta}, \\
&u_{\theta\theta\theta t} = u_{\theta\theta t\theta} - (q^{-\frac{1}{2}}g)_\theta u_{\theta\theta\theta} = q^{-\frac{1}{2}}u_{\theta\theta\theta\theta}g + (q^{-\frac{1}{2}}f)_{\theta\theta\theta}, \\
&q_t = u_{\theta\theta\theta t} + u_{\theta t} = q^{-\frac{1}{2}}q_\theta g + (q^{-\frac{1}{2}}f)_{\theta\theta\theta} + (q^{-\frac{1}{2}}f)_\theta, \\
&(q^{-\frac{1}{2}}f)_\theta = f_\delta - \frac{1}{2}q^{-\frac{3}{2}}q_\theta f, \\
&(q^{-\frac{1}{2}}f)_{\theta\theta} = q^{\frac{1}{2}}f_{\delta\delta} - \frac{1}{2}q^{-1}q_\theta f_\delta + (q^{-\frac{1}{2}})_{\theta\theta}f, \\
&(fq^{-\frac{1}{2}})_{\theta\theta\theta} = qf_{\delta\delta\delta} - \left(q^{-1}q_{\theta\theta} - \frac{5}{4}q^{-2}q_\theta^2\right)f_\delta + (q^{-\frac{1}{2}})_{\theta\theta\theta}f,
\end{aligned}$$

we obtain the first variation of the perimeter:

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t} &= \left(\oint q^{\frac{1}{2}}\mathrm{d}\theta\right)_t = \oint \left[\frac{q_t}{2q} + (q^{-\frac{1}{2}}g)_\theta\right]\mathrm{d}\delta \\
&= \oint \left(g_\delta + \frac{1}{2}f_{\delta\delta\delta} - \frac{1}{8}\chi f_\delta - \frac{1}{16}\chi_\delta f\right)\mathrm{d}\delta.
\end{aligned} \tag{53}$$

The motion is inextensible if f and g satisfy

$$\oint \chi_\delta f \,\mathrm{d}\delta = 0$$

and

$$g_\delta + \frac{1}{2}f_{\delta\delta\delta} - \frac{1}{8}\chi f_\delta - \frac{1}{16}\chi_\delta f = 0.$$

A straightforward computation gives

$$\begin{aligned}
&q_{\theta t} = q_{t\theta} - (q^{-\frac{1}{2}}g)_\theta q_\theta = q^{-\frac{1}{2}}q_{\theta\theta}g + (q^{-\frac{1}{2}}f)_{\theta\theta} + (q^{-\frac{1}{2}}f)_{\theta\theta\theta\theta}, \\
&q_{\theta\theta t} = q_{\theta t\theta} - (q^{-\frac{1}{2}}g)_\theta q_{\theta\theta} = q^{-\frac{1}{2}}q_{\theta\theta\theta}g + (q^{-\frac{1}{2}}f)_{\theta\theta\theta} + (q^{-\frac{1}{2}}f)_{\theta\theta\theta\theta\theta},
\end{aligned}$$

$$
\begin{aligned}
(q^{-\frac{1}{2}}f)_{\theta\theta\theta\theta} \;=\;& q^{\frac{3}{2}}f_{\delta\delta\delta\delta}+q_\theta f_{\delta\delta\delta}-\left(q^{-\frac{1}{2}}q_{\theta\theta}-\frac{5}{4}q^{-\frac{3}{2}}q_\theta^2\right)f_{\delta\delta}\\
&-\left(\frac{3}{2}q^{-1}q_{\theta\theta\theta}-\frac{23}{4}q^{-2}q_\theta q_{\theta\theta}+\frac{35}{8}q^{-3}q_\theta^3\right)f_\delta+(q^{-\frac{1}{2}})_{\theta\theta\theta\theta}f,\\
(fq^{-\frac{1}{2}})_{\theta\theta\theta\theta\theta} \;=\;& qf_{\delta\delta\delta\delta\delta}+\frac{5}{2}q^{\frac{1}{2}}q_\theta f_{\delta\delta\delta\delta}+\frac{5}{4}q^{-1}q_\theta f_{\delta\delta\delta}-\left(\frac{5}{2}q^{-\frac{1}{2}}q_{\theta\theta\theta}-\frac{35}{4}q^{-\frac{3}{2}}q_\theta q_{\theta\theta}+\frac{25}{4}q^{-\frac{5}{2}}q_\theta^3\right)f_{\delta\delta}\\
&-\left(2q^{-\frac{3}{2}}q_{\theta\theta\theta\theta}-\frac{41}{4}q^{-2}q_\theta q_{\theta\theta\theta}-8q^{-2}q_{\theta\theta}^2-\frac{287}{8}q^{-3}q_\theta^2q_{\theta\theta}-\frac{315}{16}q^{-4}q_\theta^4\right)f_\delta\\
&+(q^{-\frac{1}{2}})_{\theta\theta\theta\theta\theta}f.
\end{aligned}
$$

This implies that the curvature satisfies

$$
\begin{aligned}
\chi_t &= -8q^{-3}q_{\theta\theta}q_t+4q^{-2}q_{\theta\theta t}-10q^{-3}q_\theta q_{\theta t}+15q^{-4}q_\theta^2q_t+4q^{-2}q_t\\
&= -(8q^{-3}q_{\theta\theta}-15q^{-4}q_\theta^2-4q^{-2})[q^{-\frac{1}{2}}q_\theta g+(q^{-\frac{1}{2}}f)_\theta+(q^{-\frac{1}{2}}f)_{\theta\theta\theta}]\\
&\quad -10q^{-3}q_\theta[q^{-\frac{1}{2}}q_{\theta\theta}g+(q^{-\frac{1}{2}})_{\theta\theta}f+(q^{-\frac{1}{2}}f)_{\theta\theta\theta\theta}]\\
&\quad +4q^{-2}[q^{-\frac{1}{2}}q_{\theta\theta\theta}g+(q^{-\frac{1}{2}}f)_{\theta\theta\theta}+(q^{-\frac{1}{2}}f)_{\theta\theta\theta\theta\theta}]\\
&= 4f_{\delta\delta\delta\delta\delta}+c_1(\chi)f_{\delta\delta\delta}+c_2(\chi)f_{\delta\delta}+c_3(\chi)f_\delta+c_4(\chi)f+c_5(\chi)g.
\end{aligned}\tag{54}
$$

The coefficients in (54) can be determined as

$$
\begin{aligned}
c_1(\chi) &= -2(4q^{-2}q_{\theta\theta}-5q^{-3}q_\theta^2-4q^{-1})=-2\chi,\\
c_2(\chi) &= -4q^{-2}\left(\frac{5}{2}q^{-\frac{1}{2}}q_{\theta\theta\theta}-\frac{35}{4}q^{-\frac{3}{2}}q_\theta q_{\theta\theta}+\frac{25}{4}q^{-\frac{5}{2}}q_\theta^3\right)+10q^{-3}q_\theta\left(q^{-\frac{1}{2}}q_{\theta\theta}-\frac{5}{4}q^{-\frac{3}{2}}q_\theta^2-q^{\frac{1}{2}}\right)\\
&= -\frac{5}{2}\chi_\delta,\\
c_3(\chi) &= -8q^{-3}q_{\theta\theta\theta\theta}+56q^{-4}q_\theta q_{\theta\theta\theta}-226q^{-5}q_\theta^2q_{\theta\theta}+40q^{-4}q_{\theta\theta}^2\\
&\quad -16q^{-3}q_{\theta\theta}+\frac{565}{4}q^{-6}q_\theta^4+30q^{-4}q_\theta^2+4q^{-2}\\
&= -2\left(4q^{-3}q_{\theta\theta\theta\theta}-28q^{-4}q_\theta q_{\theta\theta\theta}-18q^{-4}q_{\theta\theta}^2+108q^{-5}q_\theta^2q_{\theta\theta}+4q^{-3}q_{\theta\theta}+\frac{135}{2}q^{-6}q_\theta^4\right)\\
&\quad +4q^{-4}q_{\theta\theta}^2-10q^{-5}q_\theta^2q_{\theta\theta}-8q^{-3}q_{\theta\theta}+\frac{25}{4}q^{-6}q_\theta^4+10q^{-4}q_\theta^2+4q^{-2}\\
&= -2\chi_{\delta\delta}+\frac{1}{4}\chi^2,\\
c_4(\chi) &= -(8q^{-3}q_{\theta\theta}-15q^{-4}q_\theta^2-4q^{-2})[(q^{-\frac{1}{2}})_{\theta\theta\theta}+(q^{-\frac{1}{2}})_\theta]\\
&\quad +4q^{-2}[(q^{-\frac{1}{2}})_{\theta\theta\theta\theta\theta}+(q^{-\frac{1}{2}})_{\theta\theta\theta}]-10q^{-3}q_\theta[(q^{-\frac{1}{2}})_{\theta\theta\theta\theta}+(q^{-\frac{1}{2}})_{\theta\theta}]\\
&= -\frac{1}{2}\chi_{\delta\delta\delta}+\frac{1}{8}\chi\chi_\delta,\\
c_5(\chi) &= q^{-\frac{1}{2}}q_\theta(-8q^{-3}q_{\theta\theta}+15q^{-4}q_\theta^2+4q^{-2})+4q^{-2}q^{-\frac{1}{2}}q_{\theta\theta\theta}-10q^{-3}q_\theta q^{-\frac{1}{2}}q_{\theta\theta}\\
&= \chi_\delta.
\end{aligned}
$$

Thus the curvature satisfies the equation

$$
\chi_t=4f_{\delta\delta\delta\delta\delta}-2\chi f_{\delta\delta\delta}-\frac{5}{2}\chi_\delta f_{\delta\delta}-\left(2\chi_{\delta\delta}-\frac{1}{4}\chi^2\right)f_\delta-\left(\frac{1}{2}\chi_{\delta\delta\delta}-\frac{1}{8}\chi\chi_\delta\right)f+\chi_\delta g.\tag{55}
$$

Inserting $g=-f_{\delta\delta}+\frac{1}{16}\chi f+\frac{1}{16}\partial_\delta^{-1}\chi f_\delta$ into (55), one obtains

$$
\chi_t=4f_{\delta\delta\delta\delta\delta}-2\chi f_{\delta\delta\delta}-\tfrac{7}{2}\chi_\delta f_{\delta\delta}-\left(2\chi_{\delta\delta}-\tfrac{1}{4}\chi^2-\tfrac{1}{16}\chi_\delta\partial_\delta^{-1}\chi\right)f_\delta-\left(\tfrac{1}{2}\chi_{\delta\delta\delta}-\tfrac{3}{16}\chi\chi_\delta\right)f.\tag{56}
$$

Letting $f = 2$ in (56), we obtain the KdV equation

$$\chi_t + \chi_{\delta\delta\delta} - \frac{1}{4}\chi\chi_\delta = 0.$$

7. Concluding remarks

In this paper, we have carried out a study of inextensible motions of curves in geometries in $S^1 \times \mathbb{R}$. A number of well-known $1+1$-dimensional integrable equations and their hierarchies can be obtained in a natural way, in the sense that the recursion operators of these integrable equations come out in the equations of the curvature. Similarly, we can discuss motion of inextensible space curves in $S^1 \times \mathbb{R}^2$ and we believe that many $1+1$-dimensional integrable systems are associated with such motions. It is important and interesting to perform a complete classification of the Lie algebra of vector fields acting on $S^1 \times \mathbb{R}^2$.

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