Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 58 (2008) 683–700
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Immersions of Lorentzian surfaces in $\mathbb{R}^{2,1}$

Marie-Amélie Lawn

University of Luxembourg, Institute of Mathematics, Campus Limpertsberg, 162A, Avenue de la Faiencerie, L-1511 Luxembourg City, Luxembourg

Received 20 July 2007; received in revised form 10 January 2008; accepted 21 January 2008
Available online 8 February 2008

Abstract

We study whether a given Lorentzian surface (M, g) can be immersed as the hypersurface of codimension one into the pseudo-Euclidean space $\mathbb{R}^{2,1}$. Using the methods of para-complex geometry and real spinor representations we succeed in proving the equivalence between the data of a spacelike conformal immersion of (M, g) into $\mathbb{R}^{2,1}$ and two spinors satisfying a Dirac-type equation on the surface. We generalize in this way with new technics a result of Friedrich [Th. Friedrich, On the spinor representation of surfaces in euclidean 3-Space, J. Geom. Phys. 28 (1–2) (1998) 143–157] to the pseudo-Riemannian context. Moreover we give a geometrically invariant representation of such surfaces using two Dirac spinors.

JGP SC: Differential geometry; Global analysis

MSC: 53C27; 53C40; 53C80; 58C40

Keywords: Dirac operator; Lorentzian surfaces; Isometric and conformal immersions; Gauss and Codazzi equations

1. Introduction

The relationship between immersions of Riemannian surfaces in Euclidean three- and four-dimensional spaces and spinors has been studied by many authors (see [1,3,14,9,17–19]). In fact the spinor representations of surfaces are not only of mathematical interest, but it is also of great importance in many areas of theoretical physics, especially soliton theory [18] and string theory [11,12].

The restriction φ of a parallel spinor field on $\mathbb{R}^n$ to a Riemannian hypersurface M^{n-1} is a solution of a generalized Killing equation

$$\nabla_X^{\Sigma M}\varphi = \frac{1}{2}A(X)\cdot\varphi, \tag{1.1}$$

where $\nabla^{\Sigma M}$ is the spin connection on M^{n-1}, A is the Weingarten tensor of the immersion and $\cdot$ is the Clifford multiplication on M^{n-1}. Conversely, Friedrich proves in [9] that, in the two-dimensional case, if there exists a generalized Killing spinor field satisfying Eq. (1.1), where A is an arbitrary field of symmetric endomorphisms of

E-mail address: marie-amelie.lawn@uni.lu.

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2008.01.007

TM, then A satisfies the Codazzi–Mainardi and Gauss equations of hypersurface theory and is consequently the Weingarten tensor of an isometric immersion of M^2 into $\mathbb{R}^3$. Moreover in this case, a solution φ of the generalized Killing equation is equivalently a solution of the Dirac equation, where $|\varphi|$ is constant.

Recently the case of pseudo-Riemannian manifolds of general dimension was examined in [6]: it was proven that if φ is solution of a generalized Killing equation with Codazzi tensor A on a pseudo-Riemannian manifold M, then the manifold can be embedded as a hypersurface into a Ricci flat manifold equipped with a parallel spinor which restricts to φ. The motivation of our work was the question if, at least in low dimensions, we can omit the condition on A to be Codazzi and generalize the result of Friedrich to the pseudo-Riemannian case.

With the methods of para-complex geometry and using real spinor representations we succeed in proving the equivalence between the data of a conformal immersion of a Lorentzian surface in $\mathbb{R}^{2,1}$ and spinors satisfying a Dirac-type equation on the surface. In fact, Lorentz surfaces can be viewed as real two-dimensional para-complex manifolds, and admit therefore an atlas $\{U, \phi\}$ such that the coordinate changes are para-holomorphic. Using first the real splitting of the tangent bundle we give a real Weierstraß representation in terms of $(0+, 1-)$- and $(1+, 0-)$-forms for arbitrary conformal immersions of Lorentz surfaces in $\mathbb{R}^{2,1}$. As in the case of $(1, 0)$-forms on complex manifolds, a para-complex $(1, 0)$-form ω on M can be written as $\omega = \phi\, dz$, where, having e as the para-complex unit, $z = x + ey$ is a para-holomorphic coordinate and ϕ is a para-complex function. We then deduce a para-complex version of this representation using a triple of para-complex $(1, 0)$-forms verifying certain conditions analogous to the complex model. This generalizes a result of Konderak (see [13]) for Lorentzian minimal surfaces.

We consider spin bundles on an oriented and time-oriented Lorentz surface M as para-complex line bundles L such that there exists an isomorphism

$$\kappa : L^2 \cong T^*M.$$

Consequently any section of L may be viewed as a square root of a para-complex $(1, 0)$-form on M. This allows us, with the help of the real Weierstrass representation described above, to give a real spinor representation for conformal immersions of M into the pseudo-Euclidean space $\mathbb{R}^{2,1}$. Then, we derive a Dirac-type equation for the two spinors related to the representation. A similar result was proven in [14] for Riemannian surfaces immersed in $\mathbb{R}^3$.

Finally we give a geometrically invariant representation of Lorentzian surfaces in $\mathbb{R}^{2,1}$ using two non-vanishing spinors φ_1 and φ_2 satisfying a coupled Dirac equation

$$D\varphi_1 = H\varphi_1, \qquad D\varphi_2 = -H\varphi_2, \qquad \langle\varphi_1, \varphi_2\rangle = 1,$$

where D is the Dirac operator on the surface, and H a real valued function.

We show that φ_1 and φ_2 are equivalently solutions of two generalized Killing equations

$$\nabla_X\varphi_1 = A(X)\cdot\varphi_1, \qquad \nabla_X\varphi_2 = -A(X)\cdot\varphi_2.$$

The Codazzi condition on A is then no more necessary to prove that these two properties are again equivalent to an isometric immersion $M \hookrightarrow \mathbb{R}^{2,1}$, with Weingarten tensor A.

2. Preliminaries

2.1. Para-complex differential geometry

We refer to [7] for a survey on para-complex geometry.

The algebra C of para-complex numbers is the real algebra generated by 1 and by the para-complex unit e with $e^2 = 1$. For all $z = x + ey \in C$, $x, y \in \mathbb{R}$ we define the para-complex conjugation $\bar{\cdot} : C \to C, x + ey \mapsto x - ey$ and the real and imaginary parts of z

$$\Re(z) := \frac{z + \bar{z}}{2} = x, \qquad \Im(z) := \frac{e(z - \bar{z})}{2} = y.$$

We notice that C is a real Clifford algebra. More precisely, we have

$$C \cong \mathbb{R} \oplus \mathbb{R} \cong Cl_{0,1}.$$

Definition 1. A para-complex structure on a real finite-dimensional vector space V is an endomorphism $J \in \mathrm{End}(V)$ such that $J^2 = Id$, $J \neq \pm Id$ and the two eigenspaces $V^{\pm} := \ker(Id \mp J)$ to the eigenvalues ± 1 of J have the same dimension. We call the pair (V, J) a para-complex vector space.

The free C-module C^n is a para-complex vector space where its para-complex structure is just the multiplication with e and the para-complex conjugation of C extends to $\bar{\cdot} : C^n \to C^n$, $v \mapsto \bar{v}$. A real scalar product of signature (n, n) may be defined on C^n by

$$\langle z, z' \rangle := \Re(z\bar{z'}) = \Re(z_1 \bar{z'_1} + \cdots + z_n \bar{z'_n}).$$

In the following we will denote by

$$C^{n*} = \{z \in C^n \,|\, \langle z, z \rangle \neq 0\}$$

the set of non-isotropic elements in C^n and by K^n the set of zero divisors. In particular note that in the one-dimensional case

$$C \supset C^* = \{\pm r \exp(e\theta) | r \in \mathbb{R}^+, \theta \in \mathbb{R}\} \cup \{\pm re \exp(e\theta) | r \in \mathbb{R}^+, \theta \in \mathbb{R}\}.$$

Analogous to the complex case, this can be seen as a para-complex polar decomposition, where $C^* \simeq \mathbb{R}^+ \times H^1$ and where H^1 are the four hyperbolas $\{z = x + ey \in C | x^2 - y^2 = \pm 1\}$.

In addition we want to define square roots of a para-complex number w as solutions z of the equation $z^2 = w$, with $z, w \in C$. We remark that these are only defined for para-complex numbers w if $\Re(w) \geq 0$. In this case there exist at most four square roots of w: More precisely w has exactly four square roots if it is non-isotropic and two square roots if it is isotropic.

Definition 2. An almost para-complex structure on a smooth manifold M is an endomorphism field $J \in \Gamma(\mathrm{End}(TM))$ such that, for all $p \in M$, J_p is a para-complex structure on T_pM. It is called integrable if the distributions $T^{\pm}M = \ker(Id \mp J)$ are integrable. An integrable almost para-complex structure on M is called a para-complex structure on M and a manifold M endowed with a para-complex structure is called a para-complex manifold. The para-complex dimension of a para-complex manifold M is the integer $n = \dim_C M := \frac{\dim M}{2}$.

As in the complex case we can define the Nijenhuis tensor N_J of an almost para-complex structure J by

$$N_J(X, Y) := [X, Y] + [JX, JY] - J[X, JY] - J[JX, Y],$$

for all vector fields X and Y on M. As shown in [8] we have the

Proposition 1. *An almost para-complex structure J is integrable if and only if $N_J = 0$.*

The splitting of the tangent bundle of a para-complex, or of an almost para-complex, manifold M into the eigenspaces $T^{\pm}M$ extends to a bi-grading on the exterior algebra:

$$\Lambda^k T^*M = \bigoplus_{k=p+q} \Lambda^{p+,q-} T^*M \tag{2.1}$$

and induces an obvious bi-grading on exterior forms with values in a vector bundle E.

In particular the corresponding decomposition of differential forms on M is given by

$$\Omega^k(M) = \bigoplus_{k=p+q} \Omega^{p+,q-}(M). \tag{2.2}$$

We consider the de Rham differential $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$. In the case where the almost para-complex structure is integrable we have the splitting $\mathrm{d} = \partial_+ + \partial_-$ with

$$\partial_+ : \Omega^{p+,q-}(M) \to \Omega^{(p+1)+,\, q-}(M), \qquad \partial_- : \Omega^{p+,q-}(M) \to \Omega^{p+,\,(q+1)-}(M).$$

Applying the Frobenius theorem to the distribution $T^{\pm}M$ we obtain, on an open neighborhood $U(p)$ of M, real functions $z^i_{\pm}$, $i = 1, \ldots, n$, which are constant on the leaves of $T^{\mp}M$ and for which the differential $\mathrm{d}z^i_{\pm}$ are linearly independent. $(z^1_+, \ldots, z^n_+, z^1_-, \ldots, z^n_-)$ is a system of local coordinates on M.

Moreover

$$x_i = \frac{z_+^i + z_-^i}{2}, \qquad y_i = \frac{z_+^i - z_-^i}{2}$$

defines a system of local real coordinates on $U(p)$.

Similarly to the complex model, we now define local para-holomorphic coordinates, for which the real coordinates x_i (resp. y_i) can be seen as the real (resp. imaginary) part:

Definition 3. Let (M, J_M), (N, J_N) be para-complex manifolds. A smooth map $\varphi : (M, J_M) \to (N, J_N)$ is called para-holomorphic if $d\varphi \circ J_M = J_N \circ d\varphi$. A para-holomorphic map $f : (M, J) \to C$ is called a para-holomorphic function.

A system of local para-holomorphic coordinates is a system of para-holomorphic functions z^i, $i = 1, \ldots, n$ defined on an open subset $U \subset M$ of a para-complex manifold where $(x^1 = \Re(z^1), \ldots, x^n = \Re(z^n), y^1 = \Im^1(z^1), \ldots y^n = \Im^n(z^n))$ is a system of real local coordinates.

The existence of a system of local para-holomorphic coordinates in an open neighborhood U of any point $p \in M$ was ensured by [8].

Hence, in contrast to the complex case there exist, due to the real splitting of the tangent bundle, three different sorts of appropriate local coordinates on M. The adapted coordinates are very important in this work.

Definition 4. Let (M, J) be a para-complex manifold. A para-complex vector bundle of rank r is a smooth real vector bundle $\pi : E \to M$ of rank $2r$ where the total space E is endowed with a fiber-wise para-complex structure $J^E \in \Gamma(\mathrm{End}(E))$. We will denote it by (E, J^E).

Given a para-complex vector bundle (E, J^E) over the para-complex manifold (M, J) the space of one-forms $\Omega^1(M, E)$ with values in E has the following decomposition

$$\Omega^1(M, E) = \Omega^{1,0}(M, E) \oplus \Omega^{0,1}(M, E) \tag{2.3}$$

where

$$\Omega^{1,0}(M, E) := \{\omega \in \Omega^1(M, E) \mid J^*\omega = J^E\omega\},$$
$$\Omega^{0,1}(M, E) := \{\omega \in \Omega^1(M, E) \mid J^*\omega = -J^E\omega\}.$$

The case $E = M \times C$ leads to a graduation of C-valued differential forms

$$\Omega^k_C(M) := \Omega^k(M, M \times C) = \Omega^k(M, C) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

Now we consider a para-complex vector space (V, J) endowed with a para-hermitian scalar product g on it, i.e. g is a pseudo-Euclidean scalar product and J is an anti-isometry for g:

$$J^*g := g(J\cdot, J\cdot) = -g.$$

A para-hermitian vector space is a para-complex vector space endowed with a para-hermitian scalar product.

The para-unitary group of a para-complex vector space (V, J) is then defined by

$$U^\pi(V) = \{A \in GL(V) | [A, J] = 0 \text{ and } A^*g = g\}.$$

Note that if V has para-complex dimension 1, i.e $V \simeq C \simeq \mathbb{R}^2$, then $U^\pi(V) = \{\pm \exp(e\theta) | \theta \in \mathbb{R}\}$, where e is the para-complex unit.

Definition 5. A para-hermitian vector bundle (E, J^E, g) on a para-complex vector bundle (E, J^E) is a para-complex vector bundle (E, J^E) together with a smooth fiber-wise para-hermitian scalar product g.

Note that if L is a para-hermitian line bundle, i.e. a para-hermitian vector bundle of dimension one, then L has obviously the structure group $Gl(1, C) \cap O(1, 1) = U^\pi(C)$.

Definition 6. A para-holomorphic vector bundle is a para-complex vector bundle $\pi : E \to M$ whose total space E is a para-complex manifold, such that the projection π is a para-holomorphic map.

3. A spinor representation for Lorentzian surfaces in $\mathbb{R}^{2,1}$

3.1. Lorentzian surfaces

We refer to [10] and [15] for spin geometry in general and to [5] for pseudo-Riemannian spin geometry.

In the following we call Lorentzian surfaces two-dimensional smooth manifolds endowed with an indefinite metric. We recall that the tangent bundle of such a manifold splits into the orthogonal direct sum $TM = \eta \oplus \xi$ of a one-dimensional spacelike bundle η and a one-dimensional timelike bundle ξ. The manifold is called time-oriented, if the bundle ξ is oriented.

Now let M be a strongly oriented, i.e. a time-oriented and oriented, Lorentzian surface. We can consider the $\mathrm{SO}_+(1,1)$-principal bundle (P_{SO_+}) of positively strongly oriented orthonormal frames over M. We recall that in this case the existence of spin structures is ensured (see [4]). Denote by P_{Spin_+} a spin structure on M.

We have

$$\mathrm{Spin}_+(1,1) \subset \mathcal{C}l^0_{1,1} \cong \mathcal{C}l_{0,1} \cong \mathbb{R} \oplus \mathbb{R} \cong C.$$

Therefore, the spin representation $\Delta_{1,1}$ splits under the action of the volume form $\omega_{1,1}$ into the direct sum of two inequivalent representations and it holds for the spinor module $\Sigma_{1,1} = \Sigma^+_{1,1} \oplus \Sigma^-_{1,1} \cong \mathbb{R} \oplus \mathbb{R} \cong C$. We remark that $\omega_{1,1}$ defines a para-complex structure on $\Sigma_{1,1}$ and we identify it in the following with the para-complex unit. Therefore the spinor bundle $\Sigma M = P_{\mathrm{Spin}} \times_{\Delta_{1,1}} \Sigma_{1,1} = P_{\mathrm{Spin}} \times_{\Delta_{1,1}} C$ of M can be identified with a para-complex line bundle.

Moreover, we have

$$\mathrm{SO}_+(1,1) = \{\exp(e\theta) | \theta \in \mathbb{R}\} \subset H^1,$$
$$\mathrm{Spin}_+(1,1) \cong \mathrm{U}^{\pi}(C) = \{\pm \exp(e\theta) | \theta \in \mathbb{R}\} \subset H^1.$$

The unique two-to-one Spin-covering of $\mathrm{SO}_+(1,1)$ is given by

$$\lambda : C^* \supset \mathrm{Spin}_+(1,1) \to \mathrm{SO}_+(1,1) \subset C^*,$$
$$z \mapsto z^2.$$

Let L be a para-hermitian line bundle over M. As seen in Section 2.1, the transition functions of L for a certain open covering $\{U_\alpha\}$ of M are of the form $\widetilde{\varphi}_{\alpha\beta}(x) = \pm \exp(\omega_{1,1}\theta_{\alpha\beta}(x))$, where $\theta_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}$, $x \in M$. This means that L is a $\mathrm{Spin}_+(1,1)$-bundle.

Consider now the product bundle $L^2 := L \otimes_C L$. This bundle has transition functions given by $\widetilde{\varphi}^2_{\alpha\beta}(x) \in \mathrm{SO}_+(1,1)$ for the same open covering $\{U_\alpha\}$. Similarly to the approach of [2] (see also [14,16]) for Riemannian surfaces the above considerations show, that the classical definition of spinor bundle reduces to the following

Definition 7. A spinor bundle on a strongly oriented Lorentzian surface M is a para-hermitian line bundle L endowed with an isomorphism $\kappa : L \otimes_C L \cong T^*M$. In the following we will denote it by ΣM.

A real formulation of Definition 7 is given by

Proposition 2. *A spinor bundle on a strongly oriented Lorentzian surface M is equivalent to the data of two real line bundles $L_\pm$ (called half spinor bundles and denoted in the following by $\Sigma^\pm M$), with a pairing $L_+ \otimes_{\mathbb{R}} L_- \to \mathbb{R}$, and isomorphisms $T^\pm M \cong L_\pm \otimes_{\mathbb{R}} L_\pm$.*

Proof. We put $L_+ \oplus L_- =: L$. Let $k^\pm_{\alpha\beta}$ be the transition functions of the bundles $L_\pm$ with respect to an open covering $\{U_\alpha\}$. Then by definition the transition functions of $L_+ \oplus L_- = L$ are given by $K_{\alpha\beta} = \begin{pmatrix} k^+_{\alpha\beta} & 0 \\ 0 & k^-_{\alpha\beta} \end{pmatrix}$.

Obviously the transition functions of the bundles $L_+ \otimes_{\mathbb{R}} L_+ \oplus L_- \otimes_{\mathbb{R}} L_-$ and $L \otimes_C L$ are the same, i.e. $\tilde{K}_{\alpha\beta} = \begin{pmatrix} (k^+_{\alpha\beta})^2 & 0 \\ 0 & (k^-_{\alpha\beta})^2 \end{pmatrix} = K^2_{\alpha\beta}$. $\square$

To illustrate this point of view, it is illuminating to consider the Minkowski space $M = \mathbb{R}^{1,1} = C = \mathbb{R}(1+e) \oplus \mathbb{R}(1-e)$.

We have $T^\pm_p M = \mathbb{R}(1 \pm e) \cong \mathbb{R}\sqrt{(1 \pm e)} \otimes_{\mathbb{R}} \mathbb{R}\sqrt{(1 \pm e)}$, $p \in M$.

The pairing

$$\mathbb{R}\sqrt{1+e} \otimes_{\mathbb{R}} \mathbb{R}\sqrt{1-e} \to \mathbb{R}$$
$$a\sqrt{1+e} \otimes b\sqrt{1-e} \mapsto 2ab,$$

with $a, b \in \mathbb{R}$, induces a Clifford multiplication on $\Sigma_p^{\pm} M = \mathbb{R}\sqrt{(1 \pm e)}$ by:

$$\rho^{\pm} : T^{\pm}M \otimes \Sigma^{\mp}M = \Sigma^{\pm}M \otimes \Sigma^{\pm}M \otimes \Sigma^{\mp}M \to \Sigma^{\pm}M$$
$$a(1 \pm e) \otimes b\sqrt{1 \mp e} \mapsto 2ab\sqrt{1 \pm e}$$

and hence a Clifford multiplication

$$\rho : TM \otimes \Sigma M \to \Sigma M \tag{3.1}$$

on $\Sigma M = \Sigma^+ M \oplus \Sigma^- M \cong M \times \mathbb{R}^2$.

Obviously $(1+e)$, resp $(1-e)$ corresponds to the multiplication by $-2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, resp $2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Let $\nabla : \Gamma(\Sigma^{\pm}) \to \Gamma(T^*M \otimes \Sigma^{\pm})$ be the covariant derivative on the spinor bundle.

As $\{1, e\}$ is an orthonormal basis we have

$$D\psi = \rho(1)\nabla_1 \psi - \rho(e)\nabla_e \psi = \frac{1}{2}\rho(1+e)\nabla_{1-e}\psi + \frac{1}{2}\rho(1-e)\nabla_{1+e}\psi,$$

where $D : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$ is the Dirac operator on $\mathbb{R}^{1,1}$ and $\psi \in \Gamma(\Sigma M)$. Hence as $\nabla_{1+e} = 2\frac{\partial}{\partial z_+}$ and $\nabla_{1-e} = 2\frac{\partial}{\partial z_-}$, the Dirac operator in the Minkowski space has the form

$$D = 2\begin{pmatrix} 0 & -\dfrac{\partial}{\partial z_+} \\ \dfrac{\partial}{\partial z_-} & 0 \end{pmatrix}. \tag{3.2}$$

Remark that for a given $w \in \mathrm{SO}_+(1,1) \subset C^*$ there exist exactly two square roots $z \in \mathrm{Spin}_+(1,1)$. We will denote the one with $\Im(z) > 0$ by $z = \sqrt{w}$. Locally we can consider the $(1,0)$-form $\mathrm{d}z$, where z is a para-holomorphic coordinate, as a section of T^*M. There exist four sections s of L (see Section 2.1) such that $\kappa(s \otimes s) = \mathrm{d}z$, as z has to be compatible with the orientation and the time orientation. Without loss of generality we can choose one of these spinors and denote it by $\varphi = \sqrt{\mathrm{d}z}$. Later we choose a trivialization of $T^{\pm}M$, which induces a trivialization of the spinor bundle. Therefore, we can express any spinor s in the form $s = f\varphi$, for which it holds $s^2 = f^2 \mathrm{d}z$.

We will use this point of view to derive a spinor representation of Lorentzian surfaces in the Minkowski space $\mathbb{R}^{2,1}$.

3.2. Weierstraß representations

Using the real splitting (2.1) of exterior forms on a para-complex manifold we give a real Weierstraß representation for Lorentzian surfaces. This generalizes a result of Konderak (see [13]) for minimal surfaces. We recall that a $(1+, 0-)$- (resp. a $(0+, 1-)$-form $\omega_{\pm}$ on M can be written as $\omega_{\pm} = \phi_{\pm}\mathrm{d}z_{\pm}$, where $z_{\pm}$ are the adapted coordinates introduced in Section 2.1 and $\phi_{\pm}$ are real functions.

Let (M, g) be a Lorentzian surface with pseudo-Riemannian metric g. In this chapter, we say that M is conformally immersed in $\mathbb{R}^{2,1}$ if and only if there exists a smooth map $F : M \to \mathbb{R}^{2,1}$, such that

$$\langle dF(X), dF(Y)\rangle_{\mathbb{R}^{2,1}} = \mu g(X, Y),$$

for all $X, Y \in TM$, and where μ is a **positive** function. Let $\{U, \phi\}$ be a local chart on M and (x, y) real local coordinates for this chart. Then in this coordinates g is conformally equivalent to $dx^2 - dy^2$, i.e.

$$g|_U = \lambda(dx^2 - dy^2), \quad \lambda > 0$$

and the above definition is equivalent to

$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle = 0, \qquad \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle = -\left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right\rangle = \lambda > 0. \tag{3.3}$$

In local coordinates (x_i, x_j) we can write $g = g_{ij}dx^i dx^j$, with $i, j, k = 1, 2$. The Laplace operator on M is defined for an arbitrary real-valued function f by taking

$$\triangle_g f = g^{ij}\left(\frac{\partial^2}{\partial x_i \partial x_j} f - \Gamma_{ij}^k \frac{\partial f}{\partial x^k}\right),$$

where we follow the Einstein summation convention and g^{ij} is the inverse of the matrix g_{ij}. Now let $F : M \to \mathbb{R}^{2,1}$ be a conformal immersion, then for the local coordinates (z_+, z_-) we can write $g = \lambda \mathrm{d}z_+ \mathrm{d}z_-$, $\lambda > 0$ or in matrix form $g = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. A simple calculation shows that the Laplacian of a real-valued function f on M is given by

$$\triangle f = \frac{2\partial^2 f}{\lambda \partial z_+ \partial z_-}, \tag{3.4}$$

where λ is the conformal factor of the metric.

Moreover it holds true for the mean curvature $H = \frac{1}{2}\mathrm{tr}\, B$ of the surface, where B is the second fundamental form of F, that

$$\frac{1}{2}H\nu = \triangle F, \tag{3.5}$$

where ν is the (spacelike) unit normal vector field of the immersion.

Theorem 1. *Let M be a Lorentzian surface. Then the two following conditions are equivalent:*

(1) *The map $F : M \to \mathbb{R}^{2,1}$ is a conformal immersion.*

(2) *There exist a triple $\omega_+ = (\omega_{1+}, \omega_{2+}, \omega_{3+})$ of $(1+, 0-)$-forms and a triple $\omega_- = (\omega_{1-}, \omega_{2-}, \omega_{3-})$ of $(0+, 1-)$-forms on M such that*

(i)

$$\begin{cases} \omega_{1+}^2 + \omega_{2+}^2 - \omega_{3+}^2 = 0, \\ \omega_{1-}^2 + \omega_{2-}^2 - \omega_{3-}^2 = 0, \end{cases} \tag{3.6}$$

(ii)

$$\omega_{1+}\omega_{1-} + \omega_{2+}\omega_{2-} - \omega_{3+}\omega_{3-} > 0, \tag{3.7}$$

(iii) *The forms ω_{i+} resp. ω_{i-} are ∂_+-exact resp. ∂_--exact.*

which satisfy the equation

$$F(q) = \int_p^q (\omega_{1+} + \omega_{1-}, \omega_{2+} + \omega_{2-}, \omega_{3+} + \omega_{3-}) + \mathit{Constant}.$$

Proof. (1) $\Rightarrow$ (2): Consider a conformal immersion $F = (F_1, F_2, F_3) : M \to \mathbb{R}^{1,2}$ and let $\phi_\pm = (\phi_{\pm 1}, \phi_{\pm 2}, \phi_{\pm 3})$, $\phi_{\pm i} = \frac{\partial F_i}{\partial z_\pm}$, $i \in \{1, 2, 3\}$. Then $\omega_{\pm i} := \phi_{\pm i}\mathrm{d}z_\pm$ are $(1+, 0-)$-forms resp. $(0+, 1-)$-forms on M, which obviously verify condition 2(iii).

Moreover we have:

$$\begin{aligned} \phi_1^{\pm 2} + \phi_2^{\pm 2} - \phi_3^{\pm 2} &= \left(\frac{\partial F_1}{\partial z_\pm}\right)^2 + \left(\frac{\partial F_2}{\partial z_\pm}\right)^2 - \left(\frac{\partial F_3}{\partial z_\pm}\right)^2 \\ &= \left(\frac{\partial F_1}{\partial x} \pm \frac{\partial F_1}{\partial y}\right)^2 + \left(\frac{\partial F_2}{\partial x} \pm \frac{\partial F_2}{\partial y}\right)^2 - \left(\frac{\partial F_3}{\partial x} \pm \frac{\partial F_3}{\partial y}\right)^2 \\ &= \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x}\right\rangle + \left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y}\right\rangle \pm 2\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right\rangle = \lambda - \lambda + 0 = 0 \end{aligned}$$

which proves 2(i). Further

$$\begin{aligned} \langle \phi^+, \phi^- \rangle &= \left\langle \frac{\partial F}{\partial z^+}, \frac{\partial F}{\partial z^-}\right\rangle = \left\langle \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}, \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right\rangle \\ &= \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x}\right\rangle - \left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y}\right\rangle = 2\lambda > 0, \end{aligned}$$

which is equivalent to condition 2(ii).

(2) $\Rightarrow$ (1). Condition (iii) yields that F is well-defined. Moreover with conditions (i) and (ii) we have

$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle + \left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right\rangle \pm 2 \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle = 0$$
$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle - \left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right\rangle > 0.$$

This implies $\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle + \left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right\rangle = 0$ and $\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle = 0$. Hence

$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle = \left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right\rangle = \lambda > 0$$

and F is a conformal immersion of M into $\mathbb{R}^{2,1}$. $\square$

Proposition 3. *A conformal immersion $F = (F_1, F_2, F_3) : M \to \mathbb{R}^{2,1}$ is minimal if and only if $\frac{\partial \phi_-}{\partial z_+} = \frac{\partial \phi_+}{\partial z_-} = 0$, with $\phi_\pm = \frac{\partial F}{\partial z_\pm}$.*

Proof. Let ω_+ and ω_- be the triples of forms of the immersion as defined in Theorem 1. $\omega_+ + \omega_- = \frac{\partial F}{\partial z} \mathrm{d}z$ and consequently $\omega_+ = \frac{\partial F}{\partial z_+} \mathrm{d}z_+$ and $\omega_- = \frac{\partial F}{\partial z_-} \mathrm{d}z_-$. Moreover $\left\langle \frac{\partial F}{\partial z_+}, \frac{\partial F}{\partial z_-} \right\rangle = \frac{1}{4} \left(\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle - \left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right\rangle \right) = \frac{1}{2}\lambda$. Then we have

$$H = \frac{2\nu \cdot \partial^+ \omega_-}{\langle \omega_+, \omega_- \rangle}.$$

With Eqs. (3.4) and (3.5) F is minimal if and only if

$$0 = \frac{1}{2} H \nu = \triangle F = \frac{2\partial^2 F}{\lambda \partial z_+ \partial z_-} = \frac{2\partial^2 F}{\lambda \partial z_- \partial z_+},$$

which yields the result. $\square$

Remark 1. Condition 2(iii) of Theorem 1 is equivalent to the local condition that the forms $\omega_{i\pm}$ are closed and $\partial_- \omega_{i+} = -\partial_+ \omega_{i-}$, moreover it implies that the one-form $\omega_{i+} + \omega_{i-}$ is exact.

From this real Weierstraß representation we can derive a para-complex Weierstrass representation in the following way:

Theorem 2. *Let M be a Lorentzian surface. Then the following two conditions are equivalent:*

(1) *The map $F : M \to \mathbb{R}^{2,1}$ is a conformal immersion.*
(2) *There exists a triple $\omega = (\omega_1, \omega_2, \omega_3)$ of $(1, 0)$-forms on M satisfying the equation*

$$F(q) = \Re \left(\int_p^q (\omega_1, \omega_2, \omega_3) \right) + Constant,$$

such that

$$\omega_1^2 + \omega_2^2 - \omega_3^2 = 0, \tag{3.8}$$
$$\omega_1 \bar{\omega}_1 + \omega_2 \bar{\omega}_2 - \omega_3 \bar{\omega}_3 > 0, \tag{3.9}$$

the 1-forms $\Re(\omega_i)$ are exact. (3.10)

Proof. Considering para-complex $(1, 0)$-forms ω_i, we have $\omega_i = \tilde{\omega}_i + eJ\tilde{\omega}_i$, with $\tilde{\omega}_i \in \Gamma(TM^*)$. Using now the real splitting (2.1), $\tilde{\omega}_i = \omega_{i+} + \omega_{i-}$ holds, where ω_{i+} and ω_{i-} are (1+,0-)- resp. (0+,1-)-forms. Consequently

$$\omega_i{}^2 = ((\omega_{i+} + \omega_{i-}) + e(\omega_{i+} - \omega_{i-}))^2 = 2(\omega_{i+}^2 + \omega_{i-}^2) + 2e(\omega_{i+}^2 - \omega_{i-}^2),$$

and

$$\omega_i \bar{\omega}_i = (\omega_{i+} + \omega_{i-})^2 - (\omega_{i+} - \omega_{i-})^2 = 4\omega_{i+}\omega_{i-}.$$

Simple calculations show that the conditions (3.6) resp. (3.7) of Theorem 1 are equivalent to the conditions (3.8) resp. (3.9).

Moreover $\Re(\omega_i) = \tilde{\omega}_i = \omega_{i+} + \omega_{i-}$. Remark 1 then yields clearly the equivalence between (3.10) and part (iii) of Theorem 1. $\square$

This is a generalization of a result of Konderak (see [13]) for minimal surfaces immersed into $\mathbb{R}^{2,1}$. We remark that the minimality of the immersion is just given by the condition on the $(1, 0)$-forms ω_i to be para-holomorphic (i.e locally $\omega_i = \phi_i \mathrm{d}z$, ϕ_i para-holomorphic).

3.3. A Veronese map

Let $\mathbb{R}P^n = P(\mathbb{R}^{n,1})$ be the real projective space of the pseudo-Euclidean vector space $\mathbb{R}^{n,1}$. We introduce the tautological line bundle of $\mathbb{R}P^n$:

$$\tau_{\mathbb{R}P^n} = \{(\lambda, v) \in \mathbb{R}P^n \times \mathbb{R}^{n,1} | v \in \lambda\}.$$

Obviously this is a subbundle of the trivial $(n+1)$-dimensional bundle $\mathcal{T}^{n+1} = \mathbb{R}P^n \times \mathbb{R}^{n,1}$.

We now consider the quadric

$$Q = \{(x_1, x_2, x_3) \in \mathbb{R}^{2,1} | x_1^2 - x_2^2 + x_3^2 = 0\}$$

and the maps

$$\mathcal{W}_\pm : \mathbb{R}^{1,1} \to \mathbb{R}^{2,1}.$$
$$(x_1, x_2) \mapsto (x_1^2 - x_2^2, \pm(x_1^2 + x_2^2), 2x_1x_2).$$

Then $\mathcal{W}_\pm$ can be seen as maps into the affine quadric Q. Obviously $\mathcal{W}_\pm(x) = \mathcal{W}_\pm(x')$ is equivalent to $x' = \pm x$.

We now define Veronese embeddings by

$$\mathcal{V}_\pm : \mathbb{R}P^1 \to \mathbb{R}P^2$$
$$[x_1, x_2] \mapsto [\mathcal{W}_\pm(x_1, x_2)] = [x_1^2 - x_2^2, \pm(x_1^2 + x_2^2), 2x_1x_2].$$

Proposition 4. *The Veronese embeddings $\mathcal{W}_\pm$ induce diffeomorphisms*

$$\mathcal{V}_\pm : \mathbb{R}P^1 \tilde{\to} [Q]$$

between the projective space $\mathbb{R}P^1$ and the projective quadric

$$[Q] = \{[x_1, x_2, x_3] \in \mathbb{R}P^2 | \, x_1^2 - x_2^2 + x_3^2 = 0\}.$$

Proof. Let $[y_1, y_2, y_3]$ be a point of the projective quadric. Taking affine charts of $\mathbb{R}P^1$ and assuming that $y_3 \neq 0$, we seek for $[x_1, x_2]$, with $x_1, x_2 \neq 0$, such that $[\frac{y_1}{y_3}, \frac{y_2}{y_3}, 1] = [\frac{x_1^2 - x_2^2}{2x_1x_2}, \frac{\pm(x_1^2 + x_2^2)}{2x_1x_2}, 1]$. Summing up the first and second components gives $\frac{x_1}{x_2}$ and consequently the surjectivity. $\square$

Lemma 3. *The following canonical isomorphism holds:*

$$\tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1} \cong \mathcal{V}_\pm^* \tau_{\mathbb{R}P^2}. \tag{3.11}$$

Proof. We have

$$\tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1} = \{([z], v \otimes w) \in \mathbb{R}P^1 \times (\mathbb{R}^{2,1})^{\otimes^2} | v, w \in [z]\}.$$

Moreover

$$\mathcal{V}_\pm^* \tau_{\mathbb{R}P^2} = \{([z], v) \in \mathbb{R}P^1 \times Q | v \in \mathcal{V}_\pm([z]) = [\mathcal{W}_\pm(z)]\}.$$

Using the isomorphism $s \otimes s \to \mathcal{W}_\pm(s)$ we obtain the result. $\square$

Remark that if $k_{\alpha\beta}$ are the transition functions of $\tau_{\mathbb{R}P^1}$ for the covering $\{U_\alpha\}$, then $\tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1}$ and $\mathcal{V}_\pm^* \tau_{\mathbb{R}P^2}$ have the same transition functions $k_{\alpha\beta}^2$ for this covering.

We now define the map

$$\widetilde{\mathcal{V}} : \mathbb{R}P^1 \times \mathbb{R}P^1 \to \mathbb{R}P^2 \times \mathbb{R}P^2, \qquad ([x_1, x_2], [x_1', x_2']) \mapsto (\mathcal{V}_+([x_1, x_2]), \mathcal{V}_-([x_1', x_2'])).$$

Let $\tau_{\mathbb{R}P^n} \boxplus \tau_{\mathbb{R}P^n}$ be the vector bundle defined over $\mathbb{R}P^n \times \mathbb{R}P^n$ such that the fibers are

$$(\tau_{\mathbb{R}P^n} \boxplus \tau_{\mathbb{R}P^n})_{(p^+,p^-)} := (\tau_{\mathbb{R}P^n})_{p^+} \oplus (\tau_{\mathbb{R}P^n})_{p^-},$$

with $(p^+, p^-) \in \mathbb{R}P^n \times \mathbb{R}P^n$.

As it is the Cartesian product of two smooth manifolds, $\mathbb{R}P^n \times \mathbb{R}P^n$ is a para-complex manifold. In fact, using the identification $T_{(p^+,p^-)}(\mathbb{R}P^n \times \mathbb{R}P^n) = T_{p^+}\mathbb{R}P^n \oplus T_{p^-}\mathbb{R}P^n$, we can define a para-complex structure by $J|_{T_{p^\pm}\mathbb{R}P^n} = \pm Id$. We refer to [8] for more details. Then $\tau_{\mathbb{R}P^n} \boxplus \tau_{\mathbb{R}P^n}$ has the structure of a para-complex vector bundle over $\mathbb{R}P^n \times \mathbb{R}P^n$ by defining a para-complex structure which has eigenvalue 1 on the first and -1 on the second summand.

Corollary 1. *The following canonical isomorphism of para-complex vector spaces holds:*

$$(\tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1}) \otimes_C (\tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1}) \cong \widetilde{\mathcal{V}}^*(\tau_{\mathbb{R}P^2} \boxplus \tau_{\mathbb{R}P^2}). \tag{3.12}$$

Proof. Let $k_{\alpha\beta}$ be the transition functions of the bundle $\tau_{\mathbb{R}P^1}$ with respect to an open covering $\{U_\alpha\}$. Then by definition the transition functions of $(\tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1})$ are given by $K_{\alpha\beta}(p^+, p^-) = \begin{pmatrix} k_{\alpha\beta}(p^+) & 0 \\ 0 & k_{\alpha\beta}(p^-) \end{pmatrix}$, for $(p^+, p^-) \in \mathbb{R}P^1 \times \mathbb{R}P^1$. Moreover from Lemma 3 we obtain:

$$\widetilde{\mathcal{V}}^*(\tau_{\mathbb{R}P^2} \boxplus \tau_{\mathbb{R}P^2}) \cong \mathcal{V}_+^* \tau_{\mathbb{R}P^2} \boxplus \mathcal{V}_-^* \tau_{\mathbb{R}P^2} \cong \tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1}.$$

Obviously the transition functions $\widetilde{K}_{\alpha\beta}$ of the bundles $\tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1}$ and $(\tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1}) \otimes_C (\tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1})$ are the same, i.e. $\widetilde{K}_{\alpha\beta} = \begin{pmatrix} k^2_{\alpha\beta}(p^+) & 0 \\ 0 & k^2_{\alpha\beta}(p^-) \end{pmatrix} = K^2_{\alpha\beta}$, which proves the lemma. □

3.4. The spinor representation

Using Theorem 1 and the Veronese map introduced in the last paragraph, we now generalize the results of [14] to Lorentzian surfaces.

Let $\omega_\pm \in \Gamma(T^*M^\pm)$. Locally one can write $\omega_\pm = \phi_\pm \mathrm{d}z_\pm$ where $\phi_\pm \in C^\infty(M)$ and the pair (z_+, z_-) is some adapted local coordinate system on the para-complex surface M. This yields immediately a local identification of $C^\infty(M) = \Omega^{(1+,0-)}(M) = \Gamma(T^*M^+) = \Omega^{(0+,1-)}(M) = \Gamma(T^*M^-)$. Let M be a Lorentzian surface which is conformally immersed in $\mathbb{R}^{2,1}$. The condition (3.7) of Theorem 1 on the isotropic one-forms $\omega_{i\pm}$ implies that

$$\mathcal{M}_\pm := \{x \in M | \phi_{i\pm}(x) = 0, \forall i \in \{1, 2, 3\}\} = \emptyset.$$

Therefore we can consider the map

$$\begin{aligned} &h : M \to \mathbb{R}P^2 \times \mathbb{R}P^2 \\ &x \mapsto (h_+(x), h_-(x)) := ([\phi_{1+}(x), \phi_{2+}(x), \phi_{3+}(x)], [\phi_{1-}(x), \phi_{2-}(x), \phi_{3-}(x)]). \end{aligned}$$

Moreover h can then be considered by condition (3.6) as a map into the product of projective quadrics $[Q] \times [Q] \underset{\widetilde{\mathcal{V}}}{\cong} \mathbb{R}P^1 \times \mathbb{R}P^1$. This allows us to define maps $f : M \to \mathbb{R}P^1 \times \mathbb{R}P^1$, such that $h = \widetilde{\mathcal{V}} \circ f$ and $f_\pm : M \to \mathbb{R}P^1$, such that $h_\pm = \widetilde{\mathcal{V}}_\pm \circ f_\pm$.

Let us now define the maps

$$\begin{aligned} &k^\pm : T^*M^\pm \to h_\pm{}^*(\tau^*_{\mathbb{R}P^2}) \\ &\sum_i^3 a_i \omega_{i\pm}(x) =: \alpha \mapsto l_a^\pm, \end{aligned}$$

where $l_a^{\pm}$ is the linear functional given by $l_a^{\pm}(\phi_+(x)) = a \cdot \phi_+(x) = \sum_i^3 a_i \phi_{i\pm}(x) \in \mathbb{R}$, with $a = (a_1, a_2, a_3)$ and $\phi(x) = (\phi_{1+}(x), \phi_{2+}(x), \phi_{3+}(x))$. We remark that $l_a^{\pm}$ does not depend on the choice of $\mathrm{d}z_{\pm}$. We show that $k^{\pm}$ is an isomorphism: Let $\alpha = \sum_i^3 b_i \omega_{i\pm}(x)$, for another triple $b = (b_1, b_2, b_3) \neq a$, then we have

$$0 = \sum_i^3 (a_i - b_i)\omega_{i\pm}(x) = \sum_i^3 (a_i - b_i)\phi_{i\pm}(x)\mathrm{d}z_{\pm}$$

and consequently $(l_a^{\pm} - l_b^{\pm})(\phi^{\pm}(x)) = 0$, which leads to $l_a = l_b$.

Hence we have the isomorphism

$$T^*M^{\pm} \cong h^{\pm *}(\tau^*_{\mathbb{R}P^2}) \cong f_{\pm}^* \mathcal{V}_{\pm}{}^*(\tau^*_{\mathbb{R}P^2}) \tag{3.13}$$

and finally with Lemma 3 we find the isomorphisms:

$$\kappa^{\pm} : T^*M^{\pm} \cong f_{\pm}^*(\tau^*_{\mathbb{R}P^1}) \otimes_{\mathbb{R}} f_{\pm}^*(\tau^*_{\mathbb{R}P^1}). \tag{3.14}$$

By Proposition 2 the above construction gives explicitly two half spinor bundles

$$\Sigma^{\pm}M := f_{\pm}^*(\tau^*_{\mathbb{R}P^1})$$

on M and, as $f_+^*(\tau^*_{\mathbb{R}P^1}) \oplus f_-^*(\tau^*_{\mathbb{R}P^1}) = f^*(\tau^*_{\mathbb{R}P^1} \boxplus \tau^*_{\mathbb{R}P^1})$, we have

$$T^*M \cong f^*(\tau^*_{\mathbb{R}P^1} \boxplus \tau^*_{\mathbb{R}P^1}) \otimes_C f^*(\tau^*_{\mathbb{R}P^1} \boxplus \tau^*_{\mathbb{R}P^1}). \tag{3.15}$$

Hence

$$\Sigma M := f^*(\tau^*_{\mathbb{R}P^1} \boxplus \tau^*_{\mathbb{R}P^1})$$

is a spin bundle on M in the sense of Definition 7.

The following commutative diagram illustrates the above objects:

$$\begin{tikzcd}
TM^* \arrow[r, "\sim"] & h^*(\tau^*_{\mathbb{R}P^2} \boxplus \tau^*_{\mathbb{R}P^2}) \arrow[r] \arrow[d] & \tau^*_{\mathbb{R}P^2} \boxplus \tau^*_{\mathbb{R}P^2} \arrow[d] \\
f^*(\tau^*_{\mathbb{R}P^1} \boxplus \tau^*_{\mathbb{R}P^1}) \arrow[r] \arrow[ur] \arrow[d] & M \arrow[r, "h"] \arrow[d, "f"'] & {[Q] \times [Q] \subset \mathbb{R}P^2 \times \mathbb{R}P^2} \\
\tau^*_{\mathbb{R}P^1} \boxplus \tau^*_{\mathbb{R}P^1} \arrow[r] & \mathbb{R}P^1 \times \mathbb{R}P^1 \arrow[ur, "\tilde{\mathcal{V}}"'] &
\end{tikzcd}$$

We then have

Theorem 4. *Let M be a strongly oriented Lorentzian surface. Then the following conditions are equivalent.*

(1) *There exists a conformal immersion $M \to \mathbb{R}^{2,1}$ with mean curvature H.*

(2) *There exists a solution $\psi = (\psi_1, \psi_2)$ of the Dirac-type equation*

$$\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \langle \psi_1, \psi_2 \rangle,$$

for some real-valued function H, necessarily the mean curvature of the surface.

Proof. For pairs of sections (s_{1+}, s_{2+}) and (s_{1-}, s_{2-}) of $f^*(\tau_{\mathbb{R}P^1})$ we can write

$$\omega_+ = (\omega_{+1}, \omega_{+2}, \omega_{+3}) = ({s_{1+}}^2 - {s_{2+}}^2, {s_{1+}}^2 + {s_{2+}}^2, 2s_{1+}s_{2+}),$$
$$\omega_- = (\omega_{-1}, \omega_{-2}, \omega_{-3}) = ({s_{1-}}^2 - {s_{2-}}^2, -{s_{1-}}^2 - {s_{2-}}^2, 2s_{1-}s_{2-}).$$

With ${s_{i\pm}}^2 = f_{i\pm}^2 \mathrm{d}z_{\pm}$, we have

$$\partial_-\omega_+ = 2(-f_{1+}\partial_{z^-} f_{1+} + f_{2+}\partial_{z^-} f_{2+}, -f_{1+}\partial_{z^-} f_{1+} - f_{2+}\partial_{z^-} f_{2+}, -f_{2+}\partial_{z^-} f_{1+} - f_{2+}\partial_{z^-} f_{1+})\mathrm{d}z_+ \wedge \mathrm{d}z_-,$$
$$\partial_+\omega_- = 2(f_{1-}\partial_{z^+} f_{1-} - f_{2-}\partial_{z^+} f_{2-}, -f_{1-}\partial_{z^+} f_{1-} - f_{2-}\partial_{z^+} f_{2-}, -f_{2-}\partial_{z^+} f_{1-} - f_{2-}\partial_{z^+} f_{1-})\mathrm{d}z_+ \wedge \mathrm{d}z_-.$$

Then a simple calculation shows that the integrability conditions of Theorem 1 for the pair (ω_+, ω_-) are equivalent to the following conditions on $s_i^{\pm}$:

$$s_{1+}\partial_- s_{1+} = -s_{2-}\partial_+ s_{2-}, \qquad s_{2+}\partial_- s_{2+} = -s_{1-}\partial_+ s_{1-}, \tag{3.16}$$

$$s_{1+}\partial_- s_{2+} = s_{1-}\partial_+ s_{2-}, \qquad s_{2+}\partial_- s_{1+} = s_{2-}\partial_+ s_{1-}. \tag{3.17}$$

We now calculate the mean curvature with respect to $s_i^{\pm}$. The unit normal vector field is given by

$$\nu = \frac{\omega_+ \times \omega_-}{\|\omega_+ \times \omega_-\|},$$

where $\cdot \times \cdot$ is the natural pseudo-vector product in $\mathbb{R}^{2,1}$ (see [20]). We have

$$\omega_+ \times \omega_- = -2(s_{1+}s_{1-} + s_{2+}s_{2-})(s_{1+}s_{2-} + s_{1-}s_{2+},\ s_{1-}s_{2+} - s_{2-}s_{1+},\ s_{1+}s_{1-} - s_{2+}s_{2-}),$$
$$\|\omega_+ \times \omega_-\| = 2(s_{1+}s_{1-} + s_{2+}s_{2-})^2 = -\langle \omega_+, \omega_- \rangle.$$

Then $\nu = -\frac{(s_{1+}s_{2-}+s_{1-}s_{2+}, s_{1-}s_{2+}-s_{2-}s_{1+}, s_{1+}s_{1-}-s_{2+}s_{2-})}{s_{1+}s_{1-}+s_{2+}s_{2-}}$ and consequently

$$\begin{aligned} H &= \frac{2\langle \nu, \partial_+\omega_-\rangle}{\langle \omega_+, \omega_-\rangle} \\ &= -\frac{2(s_{1+}s_{2-} + s_{1-}s_{2+},\ s_{1-}s_{2+} - s_{2-}s_{1+}, s_{1+}s_{1-} - s_{2+}s_{2-})}{-2(s_{1+}s_{1-} + s_{2+}s_{2-})^3} \cdot 2 \begin{pmatrix} s_{1-}\partial_+ s_{1-} - s_{2-}\partial_+ s_{2-} \\ -s_{1-}\partial_+ s_{1-} - s_{2-}\partial_+ s_{2-} \\ s_{1-}\partial_+ s_{2-} + s_{2-}\partial_+ s_{1-} \end{pmatrix} \\ &= \frac{1}{(s_{1+}s_{1-} + s_{2+}s_{2-})^2}(s_{2-}\partial_+ s_{1-} - s_{1-}\partial_+ s_{2-}). \end{aligned}$$

Consider now the spinors $\psi_1 := (s_{1+}, s_{2-})$ and $\psi_2 := (s_{2+}, s_{1-})$. Using the equalities (3.16) and (3.17) we compute

$$H\psi_1 = \frac{2}{(s_{1+}s_{1-} + s_{2+}s_{2-})}(-\partial_+ s_{2-}, \partial_- s_{1+}),$$
$$H\psi_2 = \frac{2}{(s_{1+}s_{1-} + s_{2+}s_{2-})}(-\partial_- s_{2+}, \partial_+ s_{1-}),$$

which is equivalent to the Dirac-type equation

$$\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \langle \psi_1, \psi_2 \rangle,$$

where D is the Dirac operator in the sense of Eq. (3.2). □

3.5. A geometrically invariant spinor representation

Let M^n be an oriented pseudo-Riemannian manifold of signature (p, q), with $p + q = n$, immersed into a pseudo-Riemannian spin manifold N of signature $(p + 1, q)$. Let $\nabla^{\Sigma N}$ be the spin connection on ΣN. Considering the spin connection $\nabla^{\Sigma M}$ induced on the hypersurface, we recall that for the restriction $\varphi = \Phi|_M$ of a spinor $\Phi \in \Gamma(\Sigma N)$ we have (see [9,6]):

$$\nabla^{\Sigma N}_X \varphi = \nabla^{\Sigma M}_X \varphi - \frac{1}{2}A(X) \cdot \varphi, \tag{3.18}$$

for all $X \in \Gamma(TM)$, where "·" denotes the Clifford multiplication on M and A is the Weingarten tensor of the immersion. These considerations lead to the following

Proposition 5 (*[6]*). *If $\Phi \in \Gamma(\Sigma N)$ is a parallel spinor on N, i.e if*

$$\nabla^{\Sigma N}_X \Phi = 0,$$

for all $X \in \Gamma(TM)$, then its restriction $\varphi = \Phi|_M$ to M is a solution of the equation

$$\nabla_X^{\Sigma M}\varphi = \frac{1}{2}A(X)\cdot\varphi, \tag{3.19}$$

where A is the Weingarten tensor of the immersion. Eq. (3.19) is called the generalized Killing equation.

The aim of this section is now to give a geometrically invariant representation of Lorentzian surfaces in $\mathbb{R}^{2,1}$ by solutions of a coupled Dirac equation (resp. coupled generalized Killing equations), similarly to the result of Friedrich [9].

Theorem 5. *Let (M, g) be a strongly oriented pseudo-Riemannian surface of signature $(1,1)$, $H : M \longrightarrow \mathbb{R}$ be a real-valued function. Then the following three statements are equivalent:*

(1) *φ_1 and φ_2 are non-vanishing non-isotropic solutions of the coupled Dirac equations*

$$D\varphi_1 = H\varphi_1, \qquad D\varphi_2 = -H\varphi_2, \tag{3.20}$$

with $\langle\varphi_1,\varphi_2\rangle = 1$,

(2) *φ_1 and φ_2 are non-vanishing non-isotropic solutions of the generalized Killing equations*

$$\nabla_X^{\Sigma M}\varphi_1 = \frac{1}{2}A(X)\cdot\varphi_1, \qquad \nabla_X^{\Sigma M}\varphi_2 = -\frac{1}{2}A(X)\cdot\varphi_2, \tag{3.21}$$

with $\langle\varphi_1,\varphi_2\rangle = 1$ and where A is a g-symmetric endomorphism field with $\frac{1}{2}tr A = H$.

(3) *If M is simply connected, there exists a global isometric spacelike immersion $M \hookrightarrow \mathbb{R}^{2,1}$ with mean curvature H and second fundamental form A.*

Proof. “3 ⇒ 2” Let Φ_1 be a parallel spinor on $\mathbb{R}^{2,1}$ and $\varphi_1 = \Phi_1|_M$ its restriction to M. From Proposition 5 we have that φ_1 is a solution of the generalized Killing equation $\nabla_X^{\Sigma M}\varphi_1 = \frac{1}{2}A(X)\cdot\varphi_1$, where A is the Weingarten tensor of the immersion.

Claim. The spinor $\varphi_2 := \nu \,\tilde{\cdot}\, \varphi_1$ is the solution of the generalized Killing equation

$$\nabla_X^{\Sigma M}\varphi_2 = -\frac{1}{2}A(X)\cdot\varphi_2,$$

where $\tilde{\cdot}$ denotes the Clifford multiplication on $\mathbb{R}^{2,1}$.

Proof. We have by Eq. (3.18)

$$\begin{aligned}\nabla_X^{\Sigma M}\varphi_2 = \nabla_X^{\Sigma M}(\nu \,\tilde{\cdot}\, \varphi_1) &= \left(\nabla_X^{\Sigma\mathbb{R}^{2,1}} + \frac{1}{2}A(X)\,\tilde{\cdot}\,\nu\right)\tilde{\cdot}\,\nu\,\tilde{\cdot}\,\varphi_1\\ &= (\nabla_X^{\Sigma\mathbb{R}^{2,1}}\nu)\,\tilde{\cdot}\,\varphi_1 + \nu\,\tilde{\cdot}\,\nabla_X^{\Sigma\mathbb{R}^{2,1}}\varphi_1 - \frac{1}{2}A(X)\,\tilde{\cdot}\,\varphi_1\\ &= \nu\,\tilde{\cdot}\left(\nabla_X^{\Sigma\mathbb{R}^{2,1}}\varphi_1 + \frac{1}{2}A(X)\,\tilde{\cdot}\,\nu\,\tilde{\cdot}\,\varphi_1\right) = \nu\,\tilde{\cdot}\,\nabla_X^{\Sigma M}\varphi_1.\end{aligned}$$

Hence, as Φ_1 is parallel, we have

$$\nabla_X^{\Sigma M}\varphi_2 = \frac{1}{2}\nu\,\tilde{\cdot}\,A(X)\,\tilde{\cdot}\,\nu\,\tilde{\cdot}\,\varphi_1 = -\frac{1}{2}A(X)\,\tilde{\cdot}\,\nu\,\tilde{\cdot}\,\varphi_2 = -\frac{1}{2}A(X)\cdot\varphi_2. \quad \square$$

Moreover we remark that

$$X\langle\varphi_1,\varphi_2\rangle = \langle\nabla_X^{\Sigma M}\varphi_1,\varphi_2\rangle + \langle\varphi_1,\nabla_X^{\Sigma M}\varphi_2\rangle = \langle\varphi_1,\nu\,\tilde{\cdot}\,\nabla_X^{\Sigma M}\varphi_1\rangle = 0,$$

hence $\langle\varphi_1,\varphi_2\rangle = Const.$ $\square$

“2 ⇒ 1”

$$D\varphi = \sum_{i=1}^{p+q}\varepsilon_i e_i\cdot\nabla^{\Sigma M}\varphi = \sum_{i=1}^{p+q}\varepsilon_i e_i\cdot\frac{1}{2}A_i^j e_j\cdot\varphi,$$

where $A_i^j := \varepsilon_j g(A(e_i), e_j))$ and $\varepsilon_j A_i^j$ is symmetric. Then, as $e_i \cdot e_j$ is anti-symmetric, we have

$$D\varphi = \sum_{i=1}^{p+q} \varepsilon_i \frac{1}{2} A_i^i e_i \cdot e_i \varphi = -\frac{1}{2}\mathrm{tr}(A) \cdot \varphi. \quad \square$$

"1 ⇒ 2" Let φ_1, φ_2 be two solutions of the system of equation (3.20). We define

$$\beta_{\varphi_1}(e_i, e_j) = \langle \nabla_{e_i}^{\Sigma M} \varphi_1, e_j \cdot \varphi_1 \rangle, \qquad \beta_{\varphi_2}(e_i, e_j) = \langle \nabla_{e_i}^{\Sigma M} \varphi_2, e_j \cdot \varphi_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the pseudo-hermitian symmetric $\mathrm{Spin}^+(p, q)$-invariant bilinear form such that $\langle X \cdot \varphi, \psi \rangle = -(-1)^q \langle \varphi, X \cdot \psi \rangle$, for all $X \in \Gamma(TM)$, $\varphi, \psi \in \Gamma(\Sigma M)$ (see [4]).

$$\begin{aligned}
\beta_{\varphi_i}(e_1, e_2) &= \langle \nabla_{e_1}^{\Sigma M} \varphi_i, e_2 \cdot \varphi_i \rangle = -\langle \nabla_{e_1}^{\Sigma M} \varphi_i, e_1^2 \cdot e_2 \cdot \varphi_i \rangle \\
&= -\langle e_1 \cdot \nabla_{e_1}^{\Sigma M} \varphi_i, e_1 \cdot e_2 \cdot \varphi_i \rangle = -\langle D\varphi_i + e_2 \cdot \nabla_{e_2}^{\Sigma M} \varphi_i, e_1 \cdot e_2 \cdot \varphi_i \rangle \\
&= -H \langle \varphi_i, e_1 \cdot e_2 \cdot \varphi_i \rangle - \langle e_2 \cdot \nabla_{e_2}^{\Sigma M} \varphi_i, e_1 \cdot e_2 \varphi_i \rangle.
\end{aligned}$$

Moreover $\langle \varphi_i, e_1 \cdot e_2 \cdot \varphi_i \rangle = \langle e_2 \cdot e_1 \cdot \varphi_i, \varphi_i \rangle = -\langle e_1 \cdot e_2 \cdot \varphi_i, \varphi_i \rangle = -\langle \varphi_i, e_1 \cdot e_2 \cdot \varphi_i \rangle = 0$.

Consequently

$$\begin{aligned}
\beta_{\varphi_i}(e_1, e_2) &= -\langle e_2 \cdot \nabla_{e_2}^{\Sigma M} \varphi_i, e_1 \cdot e_2 \cdot \varphi_i \rangle = \langle e_2 \cdot \nabla_{e_2}^{\Sigma M} \cdot \varphi_i, e_2 \cdot e_1 \cdot \varphi_i \rangle \\
&= \langle \nabla_{e_2}^{\Sigma M} \cdot \varphi_i, e_2^2 \cdot e_1 \cdot \varphi_i \rangle = \beta_\phi(e_2, e_1),
\end{aligned}$$

and β_{φ_i} is symmetric.

Let us define the g-symmetric endomorphisms

$$(B_{\varphi_1})_i^j = g(B_{\varphi_1}(e_i), e_j) := \beta_{\varphi_1}(e_i, e_j) \quad \text{and} \quad (B_{\varphi_2})_i^j = g(B_{\varphi_2}(e_i), e_j) := \beta_{\varphi_2}(e_i, e_j).$$

Clearly $\frac{\mathrm{tr}(B_{\varphi_1})}{|\varphi_1|^2} = g^{ij}(B_{\varphi_1})_{ij} = -\frac{\mathrm{tr}(B_{\varphi_2})}{|\varphi_2|^2} = H$.

Moreover let

$$b_{\varphi_i}^{\pm}(X, Y) = \langle \nabla_X^{\Sigma M} \varphi_i^{\pm}, Y \cdot \varphi_i^{\pm} \rangle$$

and

$$(B_{\varphi_i}^{\pm})_i^j = g(B_{\varphi_i}^{\pm}(e_i), e_j) := \beta_{\varphi_i}^{\pm}(e_i, e_j).$$

With the same calculation as above and with $D\varphi_i^{\pm} = H\varphi_i^{\mp}$, we obtain $\mathrm{tr}(B^{\pm}) = H\langle \varphi_i^{\mp}, \varphi_i^{\pm} \rangle$.

Claim.

$$\langle B_{\varphi_i}^{\pm}(X) \cdot \varphi_i^{\pm}, e_i \cdot \varphi_i^{\mp} \rangle = -3\langle B_{\varphi_i}^{\pm}(X) \cdot \varphi_i^{\mp}, e_i \cdot \varphi_i^{\pm} \rangle. \tag{3.22}$$

Proof. Obviously we can suppose that $\varphi_i^{\pm}(p) \neq 0$ in an open neighborhood of p as $\langle \varphi_i^{+}, \varphi_i^{-} \rangle \neq 0$.

We remark that $\frac{e_i \cdot \varphi_i^{\pm}}{\langle \varphi_i^{+}, \varphi_i^{-} \rangle}$ is a normalized dual frame of $\Sigma^{\mp} M$.

Consequently as $\langle \nabla_X^{\Sigma M} \varphi_i^{\pm}, e_i \cdot \varphi_i^{\mp} \rangle = 0$, because of the isotropy of $\varphi_i^{\pm}$, we have:

$$\begin{aligned}
\nabla_X^{\Sigma M}\varphi_i &= \sum_1^2 \varepsilon_i \left(\langle \nabla_X^{\Sigma M}\varphi_i^+, e_i\varphi_i^+\rangle \frac{e_i\cdot\varphi_i^-}{\langle\varphi_i^+,\varphi_i^-\rangle} + \langle \nabla_X^{\Sigma M}\varphi_i^-, e_i\varphi_i^-\rangle \frac{e_i\cdot\varphi_i^+}{\langle\varphi_i^+,\varphi_i^-\rangle} \right) \\
&= \frac{1}{\langle\varphi_i^+,\varphi_i^-\rangle} \sum_1^2 \varepsilon_i \left(b_{\varphi_i^+}(X,e_i)e_i\cdot\varphi_i^- + b_{\varphi_i^-}(X,e_i)e_i\cdot\varphi_i^+ \right) \\
&= \frac{1}{\langle\varphi_i^+,\varphi_i^-\rangle} (B_{\varphi_i}^+(X)\cdot\varphi_i^- + B_{\varphi_i}^-(X)\cdot\varphi_i^+).
\end{aligned}$$

Comparing degrees, this yields

$$\nabla_X^{\Sigma M}\varphi_i^\pm = \frac{1}{\langle\varphi_i^+,\varphi_i^-\rangle} B_{\varphi_i}^\pm(X)\cdot\varphi_i^\mp.$$

Moreover

$$\langle B_{\varphi_i}^\pm(X)\cdot\varphi_i^\pm, e_i\cdot\varphi_i^\mp\rangle = -2g(B_{\varphi_i}^\pm(X), e_i)\langle\varphi_i^+,\varphi_i^-\rangle - \langle B_{\varphi_i}^\pm(X)\cdot\varphi_i^\mp, e_i\cdot\varphi_i^\mp\rangle,$$

but

$$g(B_{\varphi_i}^\pm(X), e_i) = b_{\varphi_i}^\pm(X,e_i) = \langle\nabla_X^{\Sigma M}\varphi_i^\pm, e_i\varphi_i^\pm\rangle = \frac{1}{\langle\varphi_i^+,\varphi_i^-\rangle}\langle B_{\varphi_i}^\pm(X)\cdot\varphi_i^\mp, e_i\cdot\varphi_i^\pm\rangle$$

$$\langle B_{\varphi_i}^\pm(X)\cdot\varphi_i^\pm, e_i\cdot\varphi_i^\mp\rangle = -3\langle B_{\varphi_i}^\pm(X)\cdot\varphi_i^\mp, e_i\cdot\varphi_i^\mp\rangle. \quad \square$$

Moreover we have:

$$\langle\nabla_X^{\Sigma M}\varphi_i, e_i\varphi_i^\pm\rangle = \langle\nabla_X^{\Sigma M}\varphi_i^+ + \nabla_X^{\Sigma M}\varphi_i^-, e_i\varphi_i^\pm\rangle = \langle\nabla_X^{\Sigma M}\varphi_i^\pm, e_i\varphi_i^\pm\rangle = \frac{1}{\langle\varphi_i^+,\varphi_i^-\rangle}\langle B_{\varphi_i}^\pm(X)\cdot\varphi_i^\mp, e_i\cdot\varphi_i^\pm\rangle$$

and

$$\begin{aligned}
\langle B_{\varphi_i}(X)\cdot\varphi_i, e_i\cdot\varphi_i^\pm\rangle &= \langle B_{\varphi_i}^+(X)\cdot(\varphi_i^+ + \varphi_i^-), e_i\cdot\varphi_i^\pm\rangle + \langle B_{\varphi_i}^-(X)\cdot(\varphi_i^+ + \varphi_i^-), e_i\cdot\varphi_i^\pm\rangle \\
&= \langle B_{\varphi_i}^+(X)\cdot\varphi_i^\mp, e_i\cdot\varphi_i^\pm\rangle + \langle B_{\varphi_i}^-(X)\cdot\varphi_i^\mp, e_i\cdot\varphi_i^\pm\rangle.
\end{aligned}$$

Then with (3.22) we have

$$\begin{aligned}
\langle B_{\varphi_i}(X)\cdot\varphi_i, e_i\cdot\varphi_i^\pm\rangle &= \langle\varphi_i^+,\varphi_i^-\rangle\langle\nabla_X^{\Sigma M}\varphi_i^\pm, e_i\varphi_i^\pm\rangle + \langle B_{\varphi_i}^\pm(X)\cdot\varphi_i^\pm, e_i\cdot\varphi_i^\mp\rangle \\
&= \langle\varphi_i^+,\varphi_i^-\rangle\langle\nabla_X^{\Sigma M}\varphi_i, e_i\varphi_i^\pm\rangle - 3\langle B_{\varphi_i}^\pm(X)\cdot\varphi_i^\mp, e_i\cdot\varphi_i^\pm\rangle
\end{aligned}$$

and finally

$$\langle\nabla_X^{\Sigma M}\varphi_i, e_i\cdot\varphi_i^\pm\rangle = -\frac{1}{2\langle\varphi_i^+,\varphi_i^-\rangle}\langle B_{\varphi_i}^\pm(X)\cdot\varphi_i^\pm, e_i\cdot\varphi_i^\mp\rangle. \tag{3.23}$$

As $e_i\cdot\varphi_i^\pm$ is a dual frame of $\Sigma^\pm M$ it shows that, for all $X\in TM$,

$$\nabla_X^{\Sigma M}\varphi_1 = -\frac{1}{2|\varphi_1|^2}B_{\varphi_1}(X)\cdot\varphi_1, \qquad \nabla_X^{\Sigma M}\varphi_2 = -\frac{1}{2|\varphi_2|^2}B_{\varphi_2}(X)\cdot\varphi_2.$$

As $\langle\varphi_1,\varphi_2\rangle = Const$, we have

$$0 = X\langle\varphi_1,\varphi_2\rangle = \langle\nabla_X^{\Sigma M}\varphi_1,\varphi_2\rangle + \langle\varphi_1,\nabla_X^{\Sigma M}\varphi_2\rangle = \left\langle -\frac{B_{\varphi_1}(X)}{2|\varphi_1|^2} - \frac{B_{\varphi_2}(X)}{2|\varphi_2|^2}\cdot\varphi_1, \varphi_2\right\rangle.$$

Let

$$B(X) := \frac{B_{\varphi_1}(X)}{|\varphi_1|^2} + \frac{B_{\varphi_2}(X)}{|\varphi_2|^2}.$$

It is well-defined as the spinors φ_1, φ_2 are non-trivial at any point. $B : T(M^{1,1}) \to T(M^{1,1})$ is obviously g-symmetric, and $\operatorname{tr}(B(X)) = H - H = 0$, i.e. in matrix form $B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, with $a, b \in \mathbb{R}$.

This yields

$$0 = \langle B(e_1) \cdot \varphi_1, \varphi_2 \rangle = a\langle e_1 \cdot \varphi_1, \varphi_2 \rangle - b\langle e_2 \cdot \varphi_1, \varphi_2 \rangle$$

and

$$0 = \langle B(e_2) \cdot \varphi_1, \varphi_2 \rangle = b\langle e_1 \cdot \varphi_1, \varphi_2 \rangle + a\langle e_2 \cdot \varphi_1, \varphi_2 \rangle.$$

If $a \neq 0$ and $b \neq 0$ we get with a simple calculation that $\langle e_1 \cdot \varphi_1, \varphi_2 \rangle = \langle e_2 \cdot \varphi_1, \varphi_2 \rangle = 0$.

We remark that $e_i \cdot \varphi_1$ is a basis of ΣM, then we have

$$\varphi_2 = \langle \varphi_2, e_1 \cdot \varphi_1 \rangle \frac{e_1 \cdot \varphi_1}{|\varphi_1|^2} + \langle \varphi_2, e_2 \cdot \varphi_1 \rangle \frac{e_2 \cdot \varphi_1}{|\varphi_2|^2} = 0.$$

Consequently $B = 0$ and $\frac{B_{\varphi_1}(X)}{|\varphi_1|^2} = -\frac{B_{\varphi_2}(X)}{|\varphi_2|^2} =: -A(X)$, which conclude the proof. □

“$2 \Rightarrow 3$”: Recall that the spin curvature is defined by

$$R^{\Sigma M}(X, Y) = \nabla_X^{\Sigma M} \nabla_Y^{\Sigma M} \varphi - \nabla_Y^{\Sigma M} \nabla_X^{\Sigma M} - \nabla_{[X,Y]}^{\Sigma M} \varphi,$$

and can be computed in terms of the curvature tensor R^M in the following way:

$$R^{\Sigma M}(e_k, e_l) \cdot \varphi = \frac{1}{2} \sum_{i \leq j} \varepsilon_i \varepsilon_j \langle R(e_k, e_l)e_i, e_j \rangle e_i \cdot e_j \cdot \varphi. \tag{3.24}$$

Then a simple calculation shows that the integrability conditions for the generalized Killing equations (3.21) are given by:

$$R^{\Sigma M}(X, Y) \cdot \varphi_1 = d^\nabla A(X, Y)\varphi_1 + (A(Y) \cdot A(X) - A(X) \cdot A(Y)) \cdot \varphi_1 \tag{3.25}$$

$$R^{\Sigma M}(X, Y) \cdot \varphi_2 = -d^\nabla A(X, Y)\varphi_2 + (A(Y) \cdot A(X) - A(X) \cdot A(Y)) \cdot \varphi_2. \tag{3.26}$$

With the equation we calculate in dimension 2:

$$R^{\Sigma M}(e_1, e_2) \cdot \varphi_i = \frac{1}{2} \varepsilon_1 \varepsilon_2 R_{1221} e_1 \cdot e_2 \cdot \varphi_i = \frac{1}{2} R_{1212} e_1 \cdot e_2 \cdot \varphi_i.$$

Consequently, using the fact that

$$A(e_2)A(e_1) - A(e_1)A(e_2) = -2\det(A)e_1 \cdot e_2,$$

the integrability conditions (3.25) can be expressed by

$$R_{1212} e_1 \cdot e_2 \cdot \varphi_i = -\det(A)e_1 \cdot e_2 \cdot \varphi_i + ((\nabla_{e_2}^{\Sigma M} A)(e_1) - (\nabla_{e_1}^{\Sigma M} A)(e_2)) \cdot \varphi_i.$$

Let us now define the vector field

$$B = (\nabla_{e_2}^{\Sigma M} A)(e_1) - (\nabla_{e_1}^{\Sigma M} A)(e_2)$$

and the function

$$f = R_{1212} + \det(A).$$

Then we obtain the system of equations

$$B \cdot \varphi_1 = f e_1 \cdot e_2 \cdot \varphi_1, \qquad B \cdot \varphi_2 = -f e_1 \cdot e_2 \cdot \varphi_2. \tag{3.27}$$

We recall that the spinor bundle decomposes under the action of the real volume form $\omega_{1,1}$ into the direct sum

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M,$$

where $\Sigma^+ M$, respectively $\Sigma^- M$, are the eigenspaces to the eigenvalues 1, respectively -1. Then, for any spinor $\varphi_i \in \Gamma(\Sigma)$ we have $\varphi_i = \varphi_i^+ + \varphi_i^-$. Consequently we obtain the following system of equations:

$$B \cdot \varphi_1^{\pm} = \mp f \varphi_1^{\mp}, \qquad B \cdot \varphi_2^{\pm} = \pm f \varphi_1^{\mp}. \tag{3.28}$$

This yields

$$\|B\|^2 \varphi_i^{\pm} = f^2 \varphi_i^{\pm}, \quad i = 1, 2 \tag{3.29}$$

and consequently $\|B\|^2 \geq 0$. Moreover we have

$$\langle B \cdot \varphi_1, B \cdot \varphi_2 \rangle = -f^2 \langle e_1 \cdot e_2 \cdot \varphi_1, e_1 \cdot e_2 \cdot \varphi_2 \rangle = f^2 \langle \varphi_1, \varphi_2 \rangle$$

and

$$\langle B \cdot \varphi_1, B \cdot \varphi_2 \rangle = \langle \varphi_1, B \cdot B \cdot \varphi_2 \rangle = -\|B\|^2 \langle \varphi_1, \varphi_2 \rangle.$$

Then $\|B\|^2 \leq 0$ holds and finally $B = 0$, as B is non-isotropic. In fact if $\|B\| = 0$, then $B \cdot \varphi = 0$: writing $B = B_1 e_1 + B_2 e_2$, we have

$$B_1 e_1 \cdot \varphi = -B_2 e_2 \cdot \varphi \Leftrightarrow B_1 \varphi = B_2 \omega_{1,1} \cdot \varphi \Leftrightarrow B_1 \varphi^{\pm} = \pm B_2 \varphi^{\pm},$$

which yields, as φ^+ and φ^- are linearly independent, that $B_1 = B_2 = 0$. □

Remark 2. If $M^{1,1}$ is immersed into a pseudo-Riemannian manifold $N^{2,1}$ admitting a Killing spinor $\Phi \in \Gamma(\Sigma N)$ (i.e $\nabla^{\Sigma N} \Phi = \lambda X \cdot_N \Phi$, $\lambda \in \mathbb{C}$), the restriction $\varphi = \Phi_M$ is a solution of the equation

$$\nabla^{\Sigma M} \varphi = \frac{1}{2} A(X) \cdot \varphi + \mu X \cdot \omega_{p,q}^{\mathbb{C}} \cdot \varphi, \tag{3.30}$$

with $\mu = -\mathrm{i}\lambda$. Moreover we recall that the model spaces $\mathbb{M}_\kappa^{p,q}$ with curvature κ admit the maximal number of linearly independent Killing spinors, where $\lambda = \pm \frac{\kappa}{2}$, if $\kappa \geq 0$ and $\lambda = \pm \frac{\mathrm{i}\kappa}{2}$, if $\kappa < 0$. Using this fact and replacing the parallel spinor by Killing spinors on $\mathbb{S}^{2,1}$ (resp. $H^{2,1}$) the same result as in Theorem 5 can be computed for surfaces in the Lorentzian 3-space forms using analogous calculations, where the two generalized Killing equations are exactly given by (3.30), changing the sign of the right terms in the second one. The supplementary term gives exactly the curvature term in the Gauss and Codazzi equations. This is a generalization of the case of Riemannian surfaces immersed in Riemannian 3-space forms proven in [17].

Acknowledgments

The author thanks her thesis advisor Vicente Cortés for his support and Lars Schäfer for helpful discussions.

References

[1] U. Abresch, Spinor representation of CMC surfaces, Lectures at Luminy, 1989.
[2] M.F. Atiyah, Riemannian surfaces and spin structures, Ann. Sci. É.N.S 4^e Sér. 4 (1) (1971) 47–62.
[3] C. Bär, Extrinsic bounds for eigenvalues of the Dirac operator, Ann. Global Anal. Geom. 16 (1998) 573–596.
[4] H. Baum, Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltikeit, Teubner Text der Mathematik, Band 41, Leipzig, 1981.
[5] H. Baum, Th. Friedrich, R. Grunewald, I. Kath, Twistors and Killing spinors on Riemannian manifolds, Teubner-Verlag, Leipzig/Stuttgart, 1991.
[6] C. Bär, P. Gauduchon, A. Moroianu, Generalized cylinders in semi-riemannian and spin geometry, Math. Z. 249 (2005) 545–580.
[7] V. Cruceanu, P. Fortuny, P.M. Gadea, A survey on paracomplex geometry, Rocky Mountain J. Math. 26 (1996) 83–115.
[8] V. Cortés, C. Mayer, T. Mohaupt, F. Saueressig, Special geometry of euclidean supersymmetry I: Vectormultiplets, J. High Energy Phys. JHEP 03 (2004) 028. hep-th/0312001.
[9] Th. Friedrich, On the spinor representation of surfaces in euclidean 3-Space, J. Geom. Phys. 28 (1–2) (1998) 143–157.
[10] Th. Friedrich, Dirac-operatoren in der Riemannschen Geometrie, Vieweg, Braunschweig Wiesbaden, 1997.
[11] P. Ginsparg, G. Moore, Lectures on 2-D gravity and 2-D string theory. hep-th/9304011, 1993.
[12] L. Inoguchi, M. Toda, Timelike minimal surfaces via loop groups, Acta Appl. Math. 83, 3 (II) (2004) 313–355.

[13] J.J. Konderak, A Weierstraß representation theorem for Lorentz surfaces, Complex Variables 50 (5) (2005) 112–319.
[14] R. Kusner, N. Schmidt, The spinor representation of surfaces in space, preprint arXiv:dg-ga/9610005 v1, 1996.
[15] H.B. Lawson, M.-L. Michelson, Spin geometry, Princeton University Press, Princeton, 1989.
[16] J.W. Milnor, Spin structures on manifolds, Enseign. Math 9 (1963) 198–203.
[17] B. Morel, Surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$ via spinors, preprint arXiv:math.DG/0204090, 2002.
[18] I.A. Taimanov, The Weierstrass representation of closed surfaces in $\mathbb{R}^3$, Funct. Anal. Appl. 32 (4) (1998) 49–62.
[19] L. Voss, Eigenwerte des Dirac-Operators auf Hyperflächen, Diplomarbeit, Humboldt-Universität zu Berlin, 1999.
[20] T. Weinstein, An Introduction to Lorentz Surfaces, in: De Gruyter Expositions in Math., vol. 22, Berlin, New York, 1996.