Journal of Geometry and Physics 59 (2009) 79–103

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Curve flows in Lagrange–Finsler geometry, bi-Hamiltonian structures and solitons

Stephen C. Anco [a], Sergiu I. Vacaru [b,*]

[a] Department of Mathematics, Brock University, St. Catharines, Ontario L2S 3A1, Canada

[b] Faculty of Mathematics, University "Al. I. Cuza" Iaşi, 700506, Iaşi, Romania

ARTICLE INFO

Article history:
Received 5 April 2007
Received in revised form 6 October 2008
Accepted 13 October 2008
Available online 19 October 2008

MSC:
37K05
37K10
37K25
35Q53
53B20
53B40
53C21
53C60

Keywords:
Curve flow
(semi) Riemannian spaces
Nonholonomic manifold
Nonlinear connection
Lagrange and Finsler geometry
Bi-Hamiltonian
Soliton equation

ABSTRACT

Methods in Riemann–Finsler geometry are applied to investigate bi-Hamiltonian structures and related mKdV hierarchies of soliton equations derived geometrically from regular Lagrangians and flows of non-stretching curves in tangent bundles. The total space geometry and nonholonomic flows of curves are defined by Lagrangian semisprays inducing canonical nonlinear connections (N-connections), Sasaki type metrics and linear connections. The simplest examples of such geometries are given by tangent bundles on Riemannian symmetric spaces $G/SO(n)$ provided with an N-connection structure and an adapted metric, for which we elaborate a complete classification, and by generalized Lagrange spaces with constant Hessian. In this approach, bi-Hamiltonian structures are derived for geometric mechanical models and (pseudo) Riemannian metrics in gravity. The results yield horizontal/vertical pairs of vector sine-Gordon equations and vector mKdV equations, with the corresponding geometric curve flows in the hierarchies described in an explicit form by nonholonomic wave maps and mKdV analogs of nonholonomic Schrödinger maps on a tangent bundle.

1. Introduction

Some interesting studies in geometric mechanics and field theory and theory of partial differential equations are related to nonholonomic structures characterizing integrability of certain nonlinear physical systems, their global and local symmetries [1–5]. In parallel, the differential geometry of plane and space curves has received considerable attention in the theory of integrable nonlinear partial differential equations and applications to modern physics [6–12]. More particularly, it is well known that both the modified Korteweg–de Vries (mKdV) equation and the sine-Gordon (SG) equation can be encoded as flows of the curvature invariant of plane curves in Euclidean plane geometry. Similarly, curve flows in Riemannian manifolds of constant curvature give rise to a vector generalization of the mKdV equation and encode its bi-Hamiltonian structure in a natural geometric way [13,14]. This approach provides an elegant geometric origin for previous results on the Hamiltonian structure of multi-component mKdV equations (see, e.g. Refs. [15,16]).

* Corresponding author.
E-mail addresses: sanco@brocku.ca (S.C. Anco), Sergiu.Vacaru@gmail.com (S.I. Vacaru).

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2008.10.006

In recent work [17], geometric flows of curves were studied in Riemannian symmetric spaces $M = G/SO(n)$ which provide the simplest generalization of n-dimensional constant-curvature Riemannian geometries. The isometry groups G of these spaces are exhausted by the Lie groups $SO(n+1)$ and $SU(n)$, as known from Cartan's classification [19]. The main results of [17] were to show that, firstly, the Cartan structure equations for torsion and curvature of a moving parallel frame and its associated frame connection 1-form encode $O(n-1)$-invariant bi-Hamiltonian operators. Secondly, this bi-Hamiltonian structure generates a hierarchy of integrable flows of curves in which the frame components of the principal normal along the curve (analogous to curvature invariants) satisfy $O(n-1)$-invariant multi-component soliton equations that include vector mKdV equations and vector sine-Gordon equations. The two groups $G = SO(n+1), SU(n)$ give different soliton hierarchies and account precisely for the two known integrable versions [18,20,21] of vector mKdV equations and vector sine-Gordon equations. Thirdly, the curve flows corresponding to such vector soliton equations were shown to be described geometrically by wave maps and mKdV analogs of Schrödinger maps into the curved manifolds $M = G/SO(n)$.

A crucial condition [14,17] behind such constructions is the fact that the frame curvature matrix is constant on these spaces $M = G/SO(n)$. This approach can be developed into a geometric formalism for mapping arbitrary (semi) Riemannian metrics [22] and regular Lagrange mechanical systems into bi-Hamiltonian structures and related solitonic equations following certain methods elaborated in the geometry of generalized Finsler and Lagrange spaces [2,3,23] and nonholonomic manifolds with applications in modern gravity [24–26].

The first aim of this paper is to prove that solitonic hierarchies can be generated by (semi) Riemannian metrics g_{ij} on a manifold V of dimension $\dim V = n \geq 2$ if the geometrical objects are lifted into the total space of the tangent bundle TV, or of a vector bundle $\mathcal{E} = (M, \pi, E)$, $\dim E = m \geq n$, by a moving parallel frame formulation of geometric curve flows when V has constant matrix curvature as defined canonically with respect to certain preferred frames. The second purpose is to elaborate applications in geometric mechanics, in particular, that the dynamics defined by any regular Lagrangian can be encoded in terms of bi-Hamiltonian structures and related solitonic hierarchies. It will be emphasized that, in a similar way, any solution of the Einstein equations given by a generically off-diagonal metric can be mapped into solitonic equations.

The paper is organized as follows:

In Section 2 we outline the geometry of vector bundles equipped with a nonlinear connection. We emphasize the possibility to define fundamental geometric objects induced by a (semi) Riemannian metric on the base space when the Riemannian curvature tensor has constant coefficients with respect to a preferred nonholonomic framing.

In Section 3 we consider curve flows on nonholonomic vector bundles. We sketch an approach to classification of such spaces defined by conventional horizontal and vertical symmetric (semi) Riemannian subspaces and equipped with nonholonomic distributions defined by a nonlinear connection structure. Bi-Hamiltonian operators are then derived for a canonical distinguished connection, adapted to the nonlinear connection structure, for which the distinguished curvature coefficients are constant.

Section 4 is devoted to the formalism of distinguished bi-Hamiltonian operators and vector soliton equations for arbitrary (semi) Riemannian spaces admitting nonholonomic deformations to symmetric Riemannian spaces. We define the basic equations for nonholonomic curve flows. Then we consider the properties of cosymplectic and symplectic operators adapted to the nonlinear connection structure. Finally, we construct solitonic hierarchies of bi-Hamiltonian anholonomic curve flows.

Section 5 contains some applications of the formalism in modern mechanics and gravity.

We conclude further with some remarks in Section 6. The Appendix contains necessary definitions and formulas from the geometry of nonholonomic manifolds.

2. Nonholonomic structures on manifolds

In this section, we prove that for any (semi) Riemannian metric g_{ij} on a manifold V it is possible to define lifts to the tangent bundle TV provided with canonical nonlinear connection (in brief, N-connection), Sasaki type metric and canonical linear connection structure. The geometric constructions will be elaborated in general form for vector bundles.

2.1. N-connections induced by Riemannian metrics

Let $\mathcal{E} = (E, \pi, F, M)$ be a (smooth) vector bundle of over base manifold M, with dimensions $\dim M = n$ and $\dim E = (n+m)$, for $n \geq 2$, and with $m \geq n$ being the dimension of typical fiber F. Here $\pi : E \to M$ defines a surjective submersion. At any point $u \in E$, the total space E splits into "horizontal", M_u, and "vertical", F_u, subspaces. We denote the local coordinates in the form $u = (x, y)$, or $u^\alpha = \left(x^i, y^a\right)$, with horizontal indices i, j, k, etc. $= 1, 2, \ldots, n$ and vertical indices a, b, c, etc. $= n+1, n+2, \ldots, n+m$.[1] The summation rule on repeated "upper" and "lower" indices will be applied.

Let the base manifold M be equipped with a (semi) Riemannian metric, namely a second rank tensor of fixed signature,[2] $h\underline{g} = \underline{g}_{ij}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j$. It is possible to introduce a vertical metric structure $v\underline{g} = \underline{g}_{ab}(x)\mathrm{d}y^a \otimes \mathrm{d}y^b$ by completing the matrix $\underline{g}_{ij}(x)$

[1] In the particular case when we have a tangent bundle $E = TM$, both type of indices run over the same values since $n = m$ but it is convenient to distinguish the horizontal and vertical ones by using different groups of Latin indices.

[2] In physics literature, one uses the term (pseudo) Riemannian/Euclidean space.

diagonally with ± 1 till any nondegenerate second rank tensor $\underline{g}_{ab}(x)$ if $m > n$. This defines a metric structure $\mathbf{g} = [h\underline{g}, v\underline{g}]$ (we shall also use the notation $\underline{g}_{\alpha\beta} = [\underline{g}_{ij}, \underline{g}_{ab}]$) on $\mathcal{E}$. We can deform the metric structure, $\underline{g}_{\alpha\beta} \to g_{\alpha\beta} = [g_{ij}, g_{ab}]$, by considering a frame (vielbein) transformation,

$$g_{\alpha\beta}(x, y) = e_{\alpha}^{\ \underline{\alpha}}(x, y)\, e_{\beta}^{\ \underline{\beta}}(x, y) g_{\underline{\alpha}\underline{\beta}}(x), \tag{1}$$

with coefficients $\underline{g}_{\alpha\beta}(x) = g_{\underline{\alpha}\underline{\beta}}(x)$. The coefficients $e_{\alpha}^{\ \underline{\alpha}}(x, y)$ will be defined below (see formula (18)) from the condition of generating curvature tensors with constant coefficients with respect to certain preferred frames.

For any g_{ab} obtained from $g_{\alpha\beta}$, we consider a generating function

$$\mathcal{L}(x, y) = g_{ab}(x, y) y^a y^b$$

inducing a vertical metric

$$\tilde{g}_{ab} = \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial y^a \partial y^b} \tag{2}$$

which is "weakly" regular if $\det |\tilde{g}_{ab}| \neq 0$.[3]

By straightforward calculations we can prove the following[4]:

Theorem 2.1. *The Euler–Lagrange equations on TM,*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0,$$

for the Lagrangian $L = \sqrt{|\mathcal{L}|}$, where $y^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau}$ for a path curve $x^i(\tau)$ on M, depending on parameter τ, are equivalent to the "nonlinear" geodesic equations

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} + 2\widetilde{G}^i\left(x^k, \frac{\mathrm{d}x^j}{\mathrm{d}\tau}\right) = 0$$

defining path curves of a canonical semispray $S = y^i \frac{\partial}{\partial x^i} - 2\widetilde{G}^i(x, y) \frac{\partial}{\partial y^i}$, where

$$2\widetilde{G}^i(x, y) = \frac{1}{2}\, \tilde{g}^{ij}\left(\frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i}\right)$$

with $\tilde{g}^{ij}$ being inverse to the vertical metric (2).

This theorem has an important geometric mechanical interpretation.

Corollary 2.1. *For any (semi) Riemannian metric $\underline{g}_{ij}(x)$ on M, we can associate canonically an effective regular Lagrange mechanics on TM with the Euler–Lagrange equations transformed into nonlinear (semispray) geodesic equations.*

The differential of map $\pi : E \to M$ is defined by fiber preserving morphisms of the tangent bundles TE and TM. Its kernel is just the vertical subspace vE with a related inclusion mapping $i : vE \to TE$.

Definition 2.1. A nonlinear connection (N-connection) $\mathbf{N}$ on a vector bundle $\mathcal{E}$ is defined by the splitting on the left of an exact sequence

$$0 \to vE \xrightarrow{i} TE \to TE/vE \to 0,$$

i.e. by a morphism of submanifolds $\mathbf{N} : TE \to vE$ such that $\mathbf{N} \circ i$ is the identity map in vE.

Equivalently, an N-connection is defined by a Whitney sum of conventional horizontal (h) subspace, (hE), and vertical (v) subspace, (vE),

$$TE = hE \oplus vE. \tag{3}$$

This sum defines a nonholonomic (alternatively, anholonomic, or non-integrable) distribution of horizontal (h) and vertical (v) subspaces on TE. Locally, an N-connection is determined by its coefficients $N_i^a(u)$,

$$\mathbf{N} = N_i^a(u) \mathrm{d}x^i \otimes \frac{\partial}{\partial y^a}.$$

The well-known class of linear connections consists of a particular subclass with the coefficients being linear in y^a, i.e., $N_i^a(u) = \Gamma^a_{bj}(x) y^b$.

3 Similar values, for $e_{\alpha}^{\ \underline{\alpha}} = \delta_{\alpha}^{\ \underline{\alpha}}$, where $\delta_{\alpha}^{\ \underline{\alpha}}$ is the Kronecker symbol, were introduced for the so-called generalized Lagrange spaces when $\mathcal{L}$ was called the "absolute energy" [2].

4 See Refs. [2,3] for details of a similar proof; here we note that in our case, in general, $e_{\alpha}^{\ \underline{\alpha}} \neq \delta_{\alpha}^{\ \underline{\alpha}}$.

Remark 2.1. A manifold (or a bundle space) is called nonholonomic if it is provided with a nonholonomic distribution (see historical details and summary of results in [24]). In a particular case, when the nonholonomic distribution is of type (3), such spaces are called N-anholonomic [26].

Any N-connection $\mathbf{N} = \{N_i^a(u)\}$ may be characterized by an N-adapted frame (vielbein) structure $\mathbf{e}_\nu = (e_i, e_a)$, where

$$\mathbf{e}_i = \frac{\partial}{\partial x^i} - N_i^a(u)\frac{\partial}{\partial y^a} \quad \text{and} \quad e_a = \frac{\partial}{\partial y^a}, \tag{4}$$

and the dual frame (coframe) structure $\mathbf{e}^\mu = (e^i, \mathbf{e}^a)$, where

$$e^i = \mathrm{d}x^i \quad \text{and} \quad \mathbf{e}^a = \mathrm{d}y^a + N_i^a(u)\mathrm{d}x^i. \tag{5}$$

We remark that $\mathbf{e}_\nu = (\mathbf{e}_i, e_a)$ and $\mathbf{e}^\mu = (e^i, \mathbf{e}^a)$ are, respectively, the former "N-elongated" partial derivatives $\delta_\nu = \delta/\partial u^\nu = (\delta_i, \partial_a)$ and N-elongated differentials $\delta^\mu = \delta u^\mu = (d^i, \delta^a)$ which emphasize that operators (4) and (5) define, correspondingly, certain "N-elongated" partial derivatives and differentials which are more convenient for tensor and integral calculations on such nonholonomic manifolds.[5]

For any N-connection, we can introduce its N-connection curvature

$$\mathbf{\Omega} = \frac{1}{2}\Omega_{ij}^a \, \mathrm{d}^i \wedge \mathrm{d}^j \otimes \partial_a,$$

with the coefficients defined as the Nijenhuis tensor,

$$\Omega_{ij}^a = \mathbf{e}_{[j}N_{i]}^a = \mathbf{e}_j N_i^a - \mathbf{e}_i N_j^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}. \tag{6}$$

The vielbeins (5) satisfy the nonholonomy (equivalently, anholonomy) relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma \tag{7}$$

with (antisymmetric) nontrivial anholonomy coefficients $W_{ia}^b = \partial_a N_i^b$ and $W_{ji}^a = \Omega_{ij}^a$.

These geometric objects can be defined in a form adapted to an N-connection structure via decompositions being invariant under parallel transport preserving the splitting (3). In this case we call them "distinguished" (by the N-connection structure), i.e. d-objects. For instance, a vector field $\mathbf{X} \in T\mathbf{V}$ is expressed

$$\mathbf{X} = (hX,\ vX), \quad \text{or} \quad \mathbf{X} = X^\alpha \mathbf{e}_\alpha = X^i \mathbf{e}_i + X^a e_a,$$

where $hX = X^i \mathbf{e}_i$ and $vX = X^a e_a$ state, respectively, the N-adapted horizontal (h) and vertical (v) components of the vector (which following Refs. [2,3] is called a distinguished vector, in brief, d-vector). In a similar fashion, the geometric objects on $\mathbf{V}$, for instance, tensors, spinors, connections, etc. can be defined and called respectively d-tensors, d-spinors, d-connections if they are adapted to the N-connection splitting (3).

Theorem 2.2. *Any (semi) Riemannian metric $\underline{g}_{ij}(x)$ on M induces a canonical N-connection on TM.*

Proof. We sketch a proof by defining the coefficients of N-connection

$$\tilde{N}^i_{\ j}(x, y) = \frac{\partial \tilde{G}^i}{\partial y^j} \tag{8}$$

where

$$\begin{aligned} \tilde{G}^i &= \frac{1}{4}\tilde{g}^{ij}\left(\frac{\partial^2 \mathcal{L}}{\partial y^i \partial x^k} y^k - \frac{\partial \mathcal{L}}{\partial x^j}\right) = \frac{1}{4}\tilde{g}^{ij} g_{\,jk}\gamma^k_{lm} y^l y^m, \\ \gamma^i_{lm} &= \frac{1}{2}g^{ih}(\partial_m g_{lh} + \partial_l g_{mh} - \partial_h g_{lm}), \quad \partial_h = \partial/\partial x^h, \end{aligned} \tag{9}$$

with g_{ah} and $\tilde{g}_{ij}$ defined respectively by formulas (1) and (2). □

The N-adapted operators (4) and (5) defined by the N-connection coefficients (8) are denoted respectively $\tilde{\mathbf{e}}_\nu = (\tilde{\mathbf{e}}_i, e_a)$ and $\tilde{\mathbf{e}}^\mu = (e^i, \tilde{\mathbf{e}}^a)$.

2.2. Canonical linear connection and metric structures

The constructions will be performed on a vector bundle $\mathbf{E}$ provided with N-connection structure. We shall emphasize the special properties of a tangent bundle (TM, π, M) when the linear connection and metric are induced by a (semi) Riemannian metric on M.

[5] We shall use "boldface" symbols whenever necessary as emphasis for any space and/or geometrical objects equipped with/adapted to an N-connection structure, or for the coefficients computed with respect to N-adapted frames.

Definition 2.2. A distinguished connection (i.e., d-connection) $\mathbf{D} = (h\mathbf{D}, v\mathbf{D})$ is a linear connection preserving under parallel transport the nonholonomic decomposition (3).

The N-adapted components $\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}$ of a d-connection $\mathbf{D}_\alpha = (\mathbf{e}_\alpha \rfloor \mathbf{D})$ are defined by equations

$$\mathbf{D}_\alpha \mathbf{e}_\beta = \mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}\mathbf{e}_\gamma, \quad \text{or} \quad \mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}(u) = \left(\mathbf{D}_\alpha \mathbf{e}_\beta\right)\rfloor \mathbf{e}^\gamma. \tag{10}$$

The N-adapted splitting into h- and v-covariant derivatives is stated by

$$h\mathbf{D} = \left\{\mathbf{D}_k = \left(L^i_{\ jk}, L^a_{bk}\right)\right\}, \quad \text{and} \quad v\mathbf{D} = \left\{\mathbf{D}_c = \left(C^i_{\ jk}, C^a_{\ bc}\right)\right\},$$

where, by definition, $L^i_{\ jk} = \left(\mathbf{D}_k \mathbf{e}_j\right)\rfloor e^i$, $L^a_{bk} = (\mathbf{D}_k e_b)\rfloor \mathbf{e}^a$, $C^i_{jc} = \left(\mathbf{D}_c \mathbf{e}_j\right)\rfloor e^i$, $C^a_{bc} = (\mathbf{D}_c e_b)\rfloor \mathbf{e}^a$. The components $\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta} = \left(L^i_{\ jk}, L^a_{bk}, C^i_{jc}, C^a_{\ bc}\right)$ completely define a d-connection $\mathbf{D}$ on $\mathbf{E}$.

The simplest way to perform N-adapted computations is to use differential forms. For instance, starting with the d-connection 1-form,

$$\mathbf{\Gamma}^{\alpha}_{\ \beta} = \mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}\mathbf{e}^\gamma, \tag{11}$$

with the coefficients defined with respect to N-elongated frames (5) and (4), the torsion of a d-connection,

$$\mathcal{T}^\alpha \doteqdot \mathbf{D}\mathbf{e}^\alpha = d\mathbf{e}^\alpha + \Gamma^{\alpha}_{\ \beta} \wedge \mathbf{e}^\beta, \tag{12}$$

is characterized by (N-adapted) d-torsion components,

$$\begin{aligned} &T^i_{\ jk} = L^i_{\ jk} - L^i_{\ kj}, \qquad T^i_{\ ja} = -T^i_{\ aj} = C^i_{\ ja}, \qquad T^a_{\ ji} = \Omega^a_{\ ji}, \\ &T^a_{\ bi} = -T^a_{\ ib} = \frac{\partial N^a_i}{\partial y^b} - L^a_{\ bi}, \qquad T^a_{\ bc} = C^a_{\ bc} - C^a_{\ cb}. \end{aligned} \tag{13}$$

For d-connection structures on TM, we have to identify indices in the form $i \leftrightarrows a, j \leftrightarrows b, \ldots$ and the components of N- and d-connections, for instance, $N^a_i \leftrightarrows N^j_i$ and $L^i_{\ jk} \leftrightarrows L^a_{\ bk}, C^i_{\ ja} \leftrightarrows C^b_{\ ca} \leftrightarrows C^i_{\ jk}$.

Definition 2.3. A distinguished metric (i.e., d-metric) on a vector bundle $\mathbf{E}$ is a nondegenerate second rank metric tensor $\mathbf{g} = g \oplus_{\mathbf{N}} h$, equivalently

$$\mathbf{g} = g_{ij}(x, y)\, e^i \otimes e^j + \ h_{ab}(x, y)\, \mathbf{e}^a \otimes \mathbf{e}^b, \tag{14}$$

adapted to the N-connection decomposition (3).

From the class of arbitrary d-connections $\mathbf{D}$ on $\mathbf{V}$, one distinguishes those which are metric compatible (metrical) satisfying the condition

$$\mathbf{D}\mathbf{g} = \mathbf{0} \tag{15}$$

including all h- and v-projections $D_j g_{kl} = 0$, $D_a g_{kl} = 0$, $D_j h_{ab} = 0$, $D_a h_{\ bc} = 0$. For d-metric structures on $\mathbf{V} \simeq TM$, with $g_{ij} = h_{ab}$, the condition of vanishing “nonmetricity” (15) transforms into

$$h\mathbf{D}(g) = 0 \quad \text{and} \quad v\mathbf{D}(h) = 0, \tag{16}$$

i.e. $D_j g_{kl} = 0$ and $D_a g_{kl} = 0$.

For any metric structure $\mathbf{g}$ on a manifold, there is the unique metric compatible and torsionless Levi-Civita connection ∇ for which $\nabla\mathbf{g} = \mathbf{0}$. This connection is not a d-connection because it does not preserve under parallelism the N-connection splitting (3). One has to consider less constrained cases, admitting nonzero torsion coefficients, when a d-connection is constructed canonically for a d-metric structure. A simple minimal metric compatible extension of ∇ is that of canonical d-connection with $T^i_{\ jk} = 0$ and $T^a_{\ bc} = 0$ but $T^i_{\ ja}$, $T^a_{\ ji}$ and $T^a_{\ bi}$ are not zero, see (13). The coefficient formulas for such connections are given in Appendix, see (69) and related discussion.

Lemma 2.1. *Any (semi) Riemannian metric $\underline{g}_{ij}(x)$ on a manifold M induces a canonical d-metric structure on TM,*

$$\tilde{\mathbf{g}} = \tilde{g}_{ij}(x, y)\, e^i \otimes e^j + \ \tilde{g}_{ij}(x, y)\, \tilde{\mathbf{e}}^i \otimes \tilde{\mathbf{e}}^j, \tag{17}$$

where $\tilde{\mathbf{e}}^i$ are elongated as in (5), but with $\tilde{N}^i_{\ j}$ given by (8).

Proof. This construction is trivial by lifting to the so-called Sasaki metric [27] but in our case using the coefficients $\tilde{g}_{ij}$ (2). □

Proposition 2.1. *There exist canonical d-connections on TM induced by a (semi) Riemannian metric $\underline{g}_{ij}(x)$ on M.*

Proof. We can construct an example in explicit form by introducing $\tilde{g}_{ij}$ and $\tilde{g}_{ab}$ in formulas (70), see Appendix, in order to compute the coefficients $\tilde{\Gamma}^{\alpha}_{\ \beta\gamma} = (\tilde{L}^i_{\ jk}, \tilde{C}^a_{\ bc})$. □

The above Lemma and Proposition establish the following result.

Theorem 2.3. *Any (semi) Riemannian metric $\underline{g}_{ij}(x)$ on M induces a nonholonomic (semi) Riemannian structure on TM.*

We note that the induced Riemannian structure is nonholonomic because on TM there is a non-integrable distribution (3) defining $\tilde{N}^i_j$. The corresponding curvature curvature tensor $\widetilde{R}^{\alpha}_{\beta\gamma\tau} = \{\widetilde{R}^i{}_{hjk}, \tilde{P}^i{}_{jka}, \tilde{S}^a{}_{bcd}\}$ is computed by substituting $\tilde{g}_{ij}, \tilde{N}^i_j$ and $\widetilde{\mathbf{e}}_k$ into formulas (75), from Appendix, for $\tilde{\Gamma}^{\alpha}_{\beta\gamma} = (\tilde{L}^i{}_{jk}, \tilde{C}^a{}_{bc})$. Here one should be noted that the constructions on TM depend on arbitrary vielbein coefficients $e^{\underline{\alpha}}_{\alpha}(x, y)$ in (1). We can restrict such coefficients in order to generate various particular classes of (semi) Riemannian geometries on TM, for instance, in order to generate symmetric Riemannian spaces with constant curvature, see Refs. [19,28,29].

Corollary 2.2. *There are lifts of a (semi) Riemannian metric $\underline{g}_{ij}(x)$ on M, $\dim M = n$, generating a Riemannian structure on TM with the curvature coefficients of the canonical d-connection coinciding (with respect to N-adapted bases) with those for a Riemannian space of constant curvature of dimension $n + n$.*

Proof. For a given metric $\underline{g}_{ij}(x)$ on M, we chose such coefficients $e^{\underline{\alpha}}_{\alpha}(x, y) = \left\{e^{\underline{a}}_{a}(x, y)\right\}$ in (1) that

$$g_{ab}(x, y) = e^{\underline{a}}_{a}(x, y)\, e^{\underline{b}}_{b}(x, y) g_{\underline{ab}}(x)$$

produces a vertical metric (2) of type

$$\tilde{g}_{ef} = \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial y^e \partial y^f} = \frac{1}{2}\frac{\partial^2 (e^{\underline{a}}_{a}\, e^{\underline{b}}_{b} y^a y^b)}{\partial y^e \partial y^f} g_{\underline{ab}}(x) = \mathring{g}_{ef}, \tag{18}$$

where $\mathring{g}_{ab}$ is the metric of a symmetric Riemannian space (of constant curvature). Considering a prescribed $\mathring{g}_{ab}$, we have to integrate two times on y^e in order to find any solution for $e^{\underline{a}}_{a}$ defining a frame structure in the vertical subspace. The next step is to construct the d-metric $\mathring{g}_{\alpha\beta} = [\, \mathring{g}_{ij}, \mathring{g}_{ab}]$ of type (17), in our case, with respect to a nonholonomic base elongated by $\tilde{\mathring{N}}^i_j$, generated by $\underline{g}_{ij}(x)$ and $\tilde{g}_{ef} = \mathring{g}_{ab}$, like in (8) and (9). This defines a constant-curvature Riemannian space of dimension $n + n$. The coefficients of the canonical d-connection, which in this case coincide with those for the Levi-Civita connection, and the coefficients of the Riemannian curvature can be computed respectively by putting $\tilde{g}_{ef} = \mathring{g}_{ab}$ in formulas (70) and (75), see Appendix. Finally, we note that the induced symmetric Riemannian space contains additional geometric structures like the N-connection and anholonomy coefficients $W^{\gamma}_{\alpha\beta}$, see (7). □

There are various possibilities to generate on TM nonholonomic Riemannian structures from a given metric $\underline{g}_{ij}(x)$ on M. They result in different geometrical and physical models. In this work, we emphasize the possibility of generating spaces with constant curvature because for such symmetric spaces one can derive a bi-Hamiltonian hierarchy of curve flows and associated solitonic equations.

Example 2.1. The simplest example where a Riemannian structure with constant matrix curvature coefficients is generated on TM comes from a d-metric induced by $\tilde{g}_{ij} = \delta_{ij}$, i.e.

$$\tilde{\mathbf{g}}_{[E]} = \delta_{ij} e^i \otimes e^j + \delta_{ij}\, \tilde{\mathbf{e}}^i \otimes \tilde{\mathbf{e}}^j, \tag{19}$$

with $\tilde{\mathbf{e}}^i$ given by $\tilde{N}^i_j$ as defined by $\underline{g}_{ij}(x)$ on M.

It should be noted that the metric (19) is generically off-diagonal with respect to a coordinate basis because, in general, the anholonomy coefficients from (7) are not zero. This way, we model on TM a nonholonomic Euclidean space with vanishing curvature coefficients of the canonical d-connection (it can be verified by putting respectively the constant coefficients of metric (19) into formulas (70) and (75)). We note that the conditions of Theorem 2.1 are not satisfied by the d-metric (19) (the coefficients $\tilde{g}_{ij} = \delta_{ij}$ are not defined as in (2)), so we cannot associate a geometrical mechanics model for such constructions.

There is an important generalization:

Example 2.2. We can consider $\mathcal{L}$ as a hypersurface in TM for which the matrix $\partial^2\mathcal{L}/\partial y^a \partial y^b$ (i.e. the Hessian, following the analogy with Lagrange mechanics and field theory) is constant and nondegenerate. This states that $\tilde{g}_{ij} = const$, which produces vanishing curvature coefficients for the canonical d-connection induced by $\underline{g}_{ij}(x)$ on M.

Finally, we note that a number of geometric ideas and methods applied in this section were considered in the approaches to the geometry of nonholonomic spaces and generalized Finsler–Lagrange geometry elaborated by the schools of Vranceanu and Miron and by Bejancu in Romania [30,31,2,3,23,24], see also Section 5. In these approaches it is possible to construct geometric models with metric compatible linear connections, which is important for developments connected with modern (non)commutative gravity and string theory [25,26]. For Finsler spaces with nontrivial nonmetricity, for instance, those defined by the the Berwald and Chern connections [32], the resulting physical theories with local anisotropy fall outside of the class of standard models.

3. Curve flows and anholonomic constraints

We now formulate the geometry of curve flows adapted to the nonlinear connection structure.

3.1. N-adapted curve flows

Let us consider a vector bundle $\mathcal{E} = (E, \pi, F, M)$, $\dim E = n + m$ (in the particular case, $E = TM$, we have $m = n$) equipped with a d-metric $\mathbf{g} = [g, h]$ (14) and N-connection N_i^a (3) structures. A non-stretching curve $\gamma(\tau, l)$ on $\mathbf{V}$, where τ is a parameter and l is the arclength of the curve on $\mathbf{V}$, is defined by an evolution d-vector $\mathbf{Y} = \gamma_\tau$ and tangent d-vector $\mathbf{X} = \gamma_l$ such that $\mathbf{g}(\mathbf{X}, \mathbf{X}) = 1$. Such curves $\gamma(\tau, l)$ sweep out a two-dimensional surface in $T_{\gamma(\tau,l)}\mathbf{V} \subset T\mathbf{V}$.

We shall work with N-adapted bases (4) and (5) and the connection 1-form $\mathbf{\Gamma}^\alpha_{\ \beta} = \mathbf{\Gamma}^\alpha_{\ \beta\gamma}\mathbf{e}^\gamma$ with the coefficients $\mathbf{\Gamma}^\alpha_{\ \beta\gamma}$ for the canonical d-connection operator $\mathbf{D}$ (69) (see Appendix), acting in the form

$$\mathbf{D}_\mathbf{X}\mathbf{e}_\alpha = (\mathbf{X}\rfloor\mathbf{\Gamma}^{\ \gamma}_\alpha)\mathbf{e}_\gamma \quad \text{and} \quad \mathbf{D}_\mathbf{Y}\mathbf{e}_\alpha = (\mathbf{Y}\rfloor\mathbf{\Gamma}^{\ \gamma}_\alpha)\mathbf{e}_\gamma, \tag{20}$$

where "$\rfloor$" denotes the interior product and the indices are lowered and raised respectively by the d-metric $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ and its inverse $\mathbf{g}^{\alpha\beta} = [g^{ij}, h^{ab}]$. We note that $\mathbf{D}_\mathbf{X} = \mathbf{X}^\alpha\mathbf{D}_\alpha$ is the covariant derivation operator along curve $\gamma(\tau, l)$. It is convenient to fix the N-adapted frame to be parallel to curve $\gamma(l)$ adapted in the form

$$\begin{aligned} &e^1 \doteqdot h\mathbf{X}, \quad \text{for } i = 1, \text{ and } e^{\widehat{i}}, \text{ where } h\mathbf{g}(h\mathbf{X}, e^{\widehat{i}}) = 0, \\ &\mathbf{e}^{n+1} \doteqdot v\mathbf{X}, \quad \text{for } a = n+1, \text{ and } \mathbf{e}^{\widehat{a}}, \text{ where } v\mathbf{g}(v\mathbf{X}, \mathbf{e}^{\widehat{a}}) = 0, \end{aligned} \tag{21}$$

for $\widehat{i} = 2, 3, \ldots, n$ and $\widehat{a} = n+2, n+3, \ldots, n+m$. For such frames, the covariant derivative of each "normal" d-vector $\mathbf{e}^{\widehat{\alpha}}$ is parallel to the d-vectors adapted to $\gamma(\tau, l)$,

$$\begin{aligned} &\mathbf{D}_\mathbf{X}e^{\widehat{i}} = -\rho^{\widehat{i}}(u)X \quad \text{and} \quad \mathbf{D}_{h\mathbf{X}}h\mathbf{X} = \rho^{\widehat{i}}(u)\mathbf{e}_{\widehat{i}}, \\ &\mathbf{D}_\mathbf{X}\mathbf{e}^{\widehat{a}} = -\rho^{\widehat{a}}(u)X \quad \text{and} \quad \mathbf{D}_{v\mathbf{X}}v\mathbf{X} = \rho^{\widehat{a}}(u)e_{\widehat{a}}, \end{aligned} \tag{22}$$

in terms of some coefficient functions $\rho^{\widehat{i}}(u)$ and $\rho^{\widehat{a}}(u)$. The formulas (20) and (22) are distinguished into h- and v-components for $\mathbf{X} = h\mathbf{X} + v\mathbf{X}$ and $\mathbf{D} = (h\mathbf{D}, v\mathbf{D})$ for $\mathbf{D} = \{\Gamma^{\gamma}_{\ \alpha\beta}\}$, $h\mathbf{D} = \{L^i_{\ jk}, L^a_{bk}\}$ and $v\mathbf{D} = \{C^i_{jc}, C^a_{\ bc}\}$. Along $\gamma(l)$, we can pull back differential forms in a parallel N-adapted form. For instance, $\mathbf{\Gamma}^{\alpha\beta}_\mathbf{X} \doteqdot \mathbf{X}\rfloor\mathbf{\Gamma}^{\alpha\beta}$.

An algebraic characterization of parallel frames can be obtained if we perform a frame transformation preserving the decomposition (3) to an orthonormal basis $\mathbf{e}_{\alpha'}$,

$$\mathbf{e}_\alpha \to A^{\ \alpha'}_\alpha(u)\, \mathbf{e}_{\alpha'}, \tag{23}$$

called an orthonormal d-basis, where the coefficients of the d-metric (14) are transformed into the Euclidean ones $\mathbf{g}_{\alpha'\beta'} = \delta_{\alpha'\beta'}$. In distinguished form, we obtain two skew matrices

$$\mathbf{\Gamma}^{i'j'}_{h\mathbf{X}} \doteqdot h\mathbf{X}\rfloor\mathbf{\Gamma}^{i'j'} = 2e^{[i'}_{h\mathbf{X}}\rho^{j']} \quad \text{and} \quad \mathbf{\Gamma}^{a'b'}_{v\mathbf{X}} \doteqdot v\mathbf{X}\rfloor\mathbf{\Gamma}^{a'b'} = 2\mathbf{e}^{[a'}_{v\mathbf{X}}\rho^{b']}$$

where

$$e^{i'}_{h\mathbf{X}} \doteqdot g(h\mathbf{X}, e^{i'}) = [1, \underbrace{0, \ldots, 0}_{n-1}] \quad \text{and} \quad e^{a'}_{v\mathbf{X}} \doteqdot h(v\mathbf{X}, e^{a'}) = [1, \underbrace{0, \ldots, 0}_{m-1}]$$

and

$$\mathbf{\Gamma}_{h\mathbf{X}i'}^{\quad\ j'} = \begin{bmatrix} 0 & \rho^{j'} \\ -\rho_{i'} & \mathbf{0}_{[h]} \end{bmatrix} \quad \text{and} \quad \mathbf{\Gamma}_{v\mathbf{X}a'}^{\quad\ b'} = \begin{bmatrix} 0 & \rho^{b'} \\ -\rho_{a'} & \mathbf{0}_{[v]} \end{bmatrix}$$

with $\mathbf{0}_{[h]}$ and $\mathbf{0}_{[v]}$ being respectively $(n-1) \times (n-1)$ and $(m-1) \times (m-1)$ matrices. The above presented row-matrices and skew-matrices show that locally an N-anholonomic manifold $\mathbf{V}$ of dimension $n + m$, with respect to distinguished orthonormal frames, are characterized algebraically by pairs of unit vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$ preserved respectively by the $SO(n-1)$ and $SO(m-1)$ rotation subgroups of the local N-adapted frame structure group $SO(n) \oplus SO(m)$. The connection matrices $\mathbf{\Gamma}_{h\mathbf{X}i'}^{\quad\ j'}$ and $\mathbf{\Gamma}_{v\mathbf{X}a'}^{\quad\ b'}$ belong to the orthogonal complements of the corresponding Lie subalgebras and algebras, $\mathfrak{so}(n-1) \subset \mathfrak{so}(n)$ and $\mathfrak{so}(m-1) \subset \mathfrak{so}(m)$. The torsion (12) and curvature (71) (see Appendix) tensors in orthonormal component form with respect to (21) can be mapped into a distinguished orthonormal dual frame (23),

$$\mathcal{T}^{\alpha'} \doteqdot \mathbf{D}_\mathbf{X}\mathbf{e}^{\alpha'}_\mathbf{Y} - \mathbf{D}_\mathbf{Y}\mathbf{e}^{\alpha'}_\mathbf{X} + \mathbf{e}^{\beta'}_\mathbf{Y}\Gamma^{\ \alpha'}_{\mathbf{X}\beta'} - \mathbf{e}^{\beta'}_\mathbf{X}\Gamma^{\ \alpha'}_{\mathbf{Y}\beta'}, \tag{24}$$

and

$$\mathcal{R}^{\alpha'}_{\beta'}(\mathbf{X}, \mathbf{Y}) = \mathbf{D}_\mathbf{Y}\Gamma^{\ \alpha'}_{\mathbf{X}\beta'} - \mathbf{D}_\mathbf{X}\Gamma^{\ \alpha'}_{\mathbf{Y}\beta'} + \Gamma^{\ \gamma'}_{\mathbf{Y}\beta'}\Gamma^{\ \alpha'}_{\mathbf{X}\gamma'} - \Gamma^{\ \gamma'}_{\mathbf{X}\beta'}\Gamma^{\ \alpha'}_{\mathbf{Y}\gamma'}, \tag{25}$$

where $\mathbf{e}^{\alpha'}_{\mathbf{Y}} \doteqdot \mathbf{g}(\mathbf{Y}, \mathbf{e}^{\alpha'})$ and $\Gamma^{\ \alpha'}_{\mathbf{Y}\beta'} \doteqdot \mathbf{Y}\rfloor\Gamma^{\ \alpha'}_{\beta'} = \mathbf{g}(\mathbf{e}^{\alpha'}, \mathbf{D}_{\mathbf{Y}}\mathbf{e}_{\beta'})$ define respectively the N-adapted orthonormal frame row-matrix and the canonical d-connection skew-matrix in the flow directions, where $\mathcal{R}^{\ \alpha'}_{\beta'}(\mathbf{X},\mathbf{Y}) \doteqdot \mathbf{g}(\mathbf{e}^{\alpha'}, [\mathbf{D}_{\mathbf{X}}, \mathbf{D}_{\mathbf{Y}}]\mathbf{e}_{\beta'})$ is the curvature matrix. Both torsion and curvature components can be distinguished into h- and v-components like (13) and (72), by considering N-adapted decompositions of type

$$\mathbf{g} = [g, h], \qquad \mathbf{e}_{\beta'} = (\mathbf{e}_{j'}, e_{b'}), \qquad \mathbf{e}^{\alpha'} = (e^{i'}, e^{a'}), \qquad \mathbf{X} = h\mathbf{X} + v\mathbf{X}, \qquad \mathbf{D} = (h\mathbf{D}, v\mathbf{D}).$$

Finally, we note that the matrices for torsion (24) and curvature (25) can be computed for any metric compatible linear connection like the Levi-Civita and the canonical d-connection. For our purposes, we are interested to define a frame in which the curvature tensor has constant coefficients and the torsion tensor vanishes.

3.2. Anholonomic bundles with constant matrix curvature

For vanishing N-connection curvature and torsion, we get a holonomic Riemannian manifold with constant curvature, wherein the equations (24) and (25) directly encode a bi-Hamiltonian structure, see details in Refs. [13,14]. A larger class of Riemannian manifolds for which the frame curvature matrix is constant consists of the symmetric spaces $M = G/H$ for compact semisimple Lie groups $G \supset H$ (where H is required to be invariant under an involutive automorphism of G). A complete classification and summary of main results for such spaces is given in Refs. [19,28]. Constancy of the frame curvature matrix is a consequence of the fact that the Riemannian curvature and the metric tensors on the curved manifold $M = G/H$ are covariantly constant and G-invariant. In Ref. [17], a bi-Hamiltonian structure was shown to be encoded in the frame equations analogous to (24) and (25) for the symmetric spaces $M = G/SO(n)$ with $H = SO(n) \supset O(n-1)$. All such spaces are exhausted by $G = SO(n+1), SU(n)$. The derivation of the bi-Hamiltonian structure exploited the canonical soldering of the spaces $G/SO(n)$ onto Klein geometries [29]. A similar derivation [33] holds for the Lie groups $G = SO(n+1), SU(n)$ themselves when they are viewed as symmetric spaces in the standard manner [28]. A broad generalization of this derivation to arbitrary Riemannian symmetric spaces, including arbitrary compact semisimple Lie groups, has been obtained in Ref. [34].

3.2.1. Symmetric nonholonomic tangent bundles

We suppose that the base manifold is a constant-curvature symmetric space of the form $M = hG/SO(n)$ with the isotropy subgroup $hH = SO(n) \supset O(n)$ and that the typical fiber space is likewise of the constant-curvature symmetric form $F = vG/SO(m)$ with the isotropy subgroup $vH = SO(m) \supset O(m)$. This means (according to Cartan's classification of symmetric spaces [19]) that we have $hG = SO(n+1)$ and $vG = SO(m+1)$, which is general enough for a study of real holonomic and nonholonomic manifolds and geometric mechanics models.[6]

Our aim is to solder in a canonical way (like in the N-connection geometry) the horizontal and vertical symmetric Riemannian spaces of dimension n and m with a (total) symmetric Riemannian space V of dimension $n+m$, when $V = G/SO(n+m)$ with the isotropy group $H = SO(n+m) \supset O(n+m)$ and $G = SO(n+m+1)$. First, we note that the just mentioned horizontal, vertical and total symmetric Riemannian spaces can be identified with respective Klein geometries. For instance, the metric tensor $hg = \{\mathring{g}_{ij}\}$ on M is defined by the Cartan–Killing inner product $\langle\cdot,\cdot\rangle_h$ on $T_x hG \simeq h\mathfrak{g}$ restricted to the Lie algebra quotient spaces $h\mathfrak{p} = h\mathfrak{g}/h\mathfrak{h}$, with $T_x hH \simeq h\mathfrak{h}$, where $h\mathfrak{g} = h\mathfrak{h} \oplus h\mathfrak{p}$ is stated such that there is an involutive automorphism of hG leaving hH fixed, i.e. $[h\mathfrak{h}, h\mathfrak{p}] \subseteq h\mathfrak{p}$ and $[h\mathfrak{p}, h\mathfrak{p}] \subseteq h\mathfrak{h}$. In a similar form, we can define the inner products and Lie algebras: $vg = \{\mathring{h}_{ab}\}$ is given by restriction of $\langle\cdot,\cdot\rangle_v$ on $T_y vG \simeq v\mathfrak{g}$ to $v\mathfrak{p} = v\mathfrak{g}/v\mathfrak{h}$ with $T_y vH \simeq v\mathfrak{h}$, $v\mathfrak{g} = v\mathfrak{h} \oplus v\mathfrak{p}$, where

$$[v\mathfrak{h}, v\mathfrak{p}] \subseteq v\mathfrak{p}, \qquad [v\mathfrak{p}, v\mathfrak{p}] \subseteq v\mathfrak{h}; \tag{26}$$

likewise $\mathbf{g} = \{\mathring{g}_{\alpha\beta}\}$ is given by restriction of $\langle\cdot,\cdot\rangle_{\mathbf{g}}$ on $T_{(x,y)}G \simeq \mathfrak{g}$ to $\mathfrak{p} = \mathfrak{g}/\mathfrak{h}$, with $T_{(x,y)}H \simeq \mathfrak{h}$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ where

$$[\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}.$$

We parametrize the metric structure with constant coefficients on $V = G/SO(n+m)$ in the form

$$\mathring{g} = \mathring{g}_{\alpha\beta} du^\alpha \otimes du^\beta,$$

where u^α are local coordinates and

$$\mathring{g}_{\alpha\beta} = \begin{bmatrix} \mathring{g}_{ij} + \mathring{N}^a_i \mathring{N}^b_j \mathring{h}_{ab} & \mathring{N}^e_j \mathring{h}_{ae} \\ \mathring{N}^e_i \mathring{h}_{be} & \mathring{h}_{ab} \end{bmatrix} \tag{27}$$

where trivial, constant, N-connection coefficients $\mathring{N}^e_j = \mathring{h}^{eb}\mathring{g}_{jb}$ are defined in terms of any given coefficients $\mathring{h}^{eb}$ and $\mathring{g}_{jb}$, i.e. from the inverse metrics coefficients defined respectively by the metric on $hG = SO(n+1)$ and by the metric coefficients out of $(n\times n)$- and $(m\times m)$-terms of the metric $\mathring{g}_{\alpha\beta}$. As a result, we define an equivalent d-metric structure of type (14)

$$\begin{aligned} \mathring{\mathbf{g}} &= \mathring{g}_{ij}\, e^i \otimes e^j + \mathring{h}_{ab} \mathring{\mathbf{e}}^a \otimes \mathring{\mathbf{e}}^b, \\ e^i &= dx^i, \qquad \mathring{\mathbf{e}}^a = dy^a + \mathring{N}^e_i dx^i, \end{aligned} \tag{28}$$

[6] It is necessary to consider $hG = SU(n)$ and $vG = SU(m)$ for geometric models involving spinor and gauge fields.

defining a trivial $(n+m)$-splitting $\mathring{\mathbf{g}} = \mathring{g} \oplus_{\mathring{\mathbf{N}}} \mathring{h}$ because all nonholonomy coefficients $\mathring{W}^{\gamma}_{\alpha\beta}$ and N-connection curvature coefficients $\mathring{\Omega}^{a}_{ij}$ are zero. In a more general form, we can consider any covariant coordinate transforms of (28) preserving the $(n+m)$-splitting resulting in any $W^{\gamma}_{\alpha\beta} = 0$ (7) and $\Omega^{a}_{ij} = 0$ (6). We denote such trivial N-anholonomic Riemannian spaces as $\mathring{\mathbf{V}} = [hG = SO(n+1), vG = SO(m+1), \mathring{N}^{e}_{i}]$. It can be considered that such trivially N-anholonomic group spaces possess a Lie d-algebra symmetry $\mathfrak{so}_{\mathring{N}}(n+m) \doteqdot \mathfrak{so}(n) \oplus \mathfrak{so}(m)$.

The simplest generalization for a vector bundle $\mathring{\mathbf{E}}$ is to consider nonholonomic distributions on $V = G/SO(n+m)$ defined locally by arbitrary N-connection coefficients $N^{a}_{i}(x,y)$ with nonvanishing $W^{\gamma}_{\alpha\beta}$ and Ω^{a}_{ij} but with constant d-metric coefficients when

$$\begin{aligned} \mathbf{g} &= \mathring{g}_{ij}\, e^{i} \otimes e^{j} + \mathring{h}_{ab}\, \mathbf{e}^{a} \otimes \mathbf{e}^{b}, \\ e^{i} &= \mathrm{d}x^{i}, \qquad \mathbf{e}^{a} = \mathrm{d}y^{a} + N^{a}_{i}(x,y)\mathrm{d}x^{i}. \end{aligned} \tag{29}$$

This metric is very similar to (19) but with the coefficients $\mathring{g}_{ij}$ and $\mathring{h}_{ab}$ induced by the corresponding Lie d-algebra structure $\mathfrak{so}_{\mathring{N}}(n+m)$. Such spaces transform into N-anholonomic Riemann–Cartan manifolds $\mathring{\mathbf{V}}_{\mathbf{N}} = [hG = SO(n+1), vG = SO(m+1), N^{e}_{i}]$ with nontrivial N-connection curvature and induced d-torsion coefficients of the canonical d-connection (see formulas (13) computed for constant d-metric coefficients and the canonical d-connection coefficients in (69)). One has zero curvature for the canonical d-connection (in general, such spaces are curved ones with generically off-diagonal metric (29) and nonzero curvature tensor for the Levi-Civita connection).[7] This allows us to classify the N-anholonomic manifolds (and vector bundles) as having the same group and algebraic properties as pairs of symmetric Riemannian spaces of dimension n and m but nonholonomically soldered to the symmetric Riemannian space of dimension $n+m$. With respect to N-adapted orthonormal bases (23), with distinguished h- and v-subspaces, we obtain the same inner products and group and Lie algebra spaces as in (26).

The classification of N-anholonomic vector bundles is almost similar to that for symmetric Riemannian spaces if we consider that $n = m$ and try to model tangent bundles of such spaces, provided with N-connection structure. For instance, we can take a (semi) Riemannian structure with the N-connection induced by a absolute energy structure like in (8) and with the canonical d-connection structure (69), for $\tilde{g}_{ef} = \mathring{g}_{ab}$ (18). A straightforward computation of the canonical d-connection coefficients[8] and of d-curvatures for ${}^{\circ}\tilde{g}_{ij}$ and ${}^{\circ}\tilde{N}^{i}{}_{j}$ proves that the nonholonomic Riemannian manifold $(M = SO(n+1)/SO(n), {}^{\circ}\mathcal{L})$ possess constant both zero canonical d-connection curvature and torsion but with induced nontrivial N-connection curvature ${}^{\circ}\tilde{\Omega}^{i}{}_{jk}$. Such spaces, being tangent to symmetric Riemannian spaces, are classified similarly to the Riemannian ones with constant matrix curvature, see (26) for $n = m$, but possessing a nonholonomic structure induced by generating function ${}^{\circ}\mathcal{L}$.

3.2.2. *N-anholonomic Klein spaces*

The bi-Hamiltonian and solitonic constructions in Refs. [34,17,14,21] are based on soldering Riemannian symmetric-space geometries onto Klein geometries [29]. For the N-anholonomic spaces of dimension $n+n$, with constant d-curvatures, similar constructions will hold but we have to adapt them to the N-connection structure.

There are two Hamiltonian variables given by the principal normals ${}^{h}\nu$ and ${}^{v}\nu$, respectively, in the horizontal and vertical subspaces, defined by the canonical d-connection $\mathbf{D} = (h\mathbf{D}, v\mathbf{D})$, see formulas (21) and (22),

$${}^{h}\nu \doteqdot \mathbf{D}_{h\mathbf{X}} h\mathbf{X} = \nu^{\widehat{i}} \mathbf{e}_{\widehat{i}} \quad \text{and} \quad {}^{v}\nu \doteqdot \mathbf{D}_{v\mathbf{X}} v\mathbf{X} = \nu^{\widehat{a}} e_{\widehat{a}}.$$

This normal d-vector $\mathbf{v} = ({}^{h}\nu, {}^{v}\nu)$, with components of type $\nu^{\alpha} = (\nu^{i}, \nu^{a}) = (\nu^{1}, \nu^{\widehat{i}}, \nu^{n+1}, \nu^{\widehat{a}})$, is defined with respect to the tangent direction of curve γ. There is also the principal normal d-vector $\varpi = ({}^{h}\varpi, {}^{v}\varpi)$ with components of type $\varpi^{\alpha} = (\varpi^{i}, \varpi^{a}) = (\varpi^{1}, \varpi^{\widehat{i}}, \varpi^{n+1}, \varpi^{\widehat{a}})$ defined with respect to the flow direction, with

$${}^{h}\varpi \doteqdot \mathbf{D}_{h\mathbf{Y}} h\mathbf{X} = \varpi^{\widehat{i}} \mathbf{e}_{\widehat{i}}, \qquad {}^{v}\varpi \doteqdot \mathbf{D}_{v\mathbf{Y}} v\mathbf{X} = \varpi^{\widehat{a}} e_{\widehat{a}},$$

representing a Hamiltonian d-covector field. We can consider that the normal part of the flow d-vector

$$\mathbf{h}_{\perp} \doteqdot \mathbf{Y}_{\perp} = h^{\widehat{i}} \mathbf{e}_{\widehat{i}} + h^{\widehat{a}} e_{\widehat{a}}$$

represents a Hamiltonian d-vector field. For such configurations, we can consider parallel N-adapted frames $\mathbf{e}_{\alpha'} = (\mathbf{e}_{i'}, e_{a'})$ when the h-variables $\nu^{\widehat{i'}}, \varpi^{\widehat{i'}}, h^{\widehat{i'}}$ are respectively encoded in the top row of the horizontal canonical d-connection matrices $\mathbf{\Gamma}_{h\mathbf{X}i'}{}^{j'}$ and $\mathbf{\Gamma}_{h\mathbf{Y}i'}{}^{j'}$ and in the row matrix $\left(\mathbf{e}^{i'}_{\mathbf{Y}}\right)_{\perp} \doteqdot \mathbf{e}^{i'}_{\mathbf{Y}} - g_{\parallel} \mathbf{e}^{i'}_{\mathbf{X}}$ where $g_{\parallel} \doteqdot g(h\mathbf{Y}, h\mathbf{X})$ is the tangential h-part of the flow d-vector. A similar encoding holds for v-variables $\nu^{\widehat{a'}}, \varpi^{\widehat{a'}}, h^{\widehat{a'}}$ in the top row of the vertical canonical d-connection matrices $\mathbf{\Gamma}^{\ b'}_{v\mathbf{X}a'}$ and $\mathbf{\Gamma}^{\ b'}_{v\mathbf{Y}a'}$ and in the row matrix $\left(\mathbf{e}^{a'}_{\mathbf{Y}}\right)_{\perp} \doteqdot \mathbf{e}^{a'}_{\mathbf{Y}} - h_{\parallel} \mathbf{e}^{a'}_{\mathbf{X}}$ where $h_{\parallel} \doteqdot h(v\mathbf{Y}, v\mathbf{X})$ is the tangential v-part of the flow d-vector. In an abbreviated notation, we shall write $\mathbf{v}^{\alpha'}$ and $\varpi^{\alpha'}$ where the primed Greek indices $\alpha', \beta', \ldots$ will denote both

[7] With constant values for the d-metric coefficients we get zero coefficients for the canonical d-connection which then yields vanishing curvature coefficients (72).

[8] On tangent bundles, such d-connections can be defined to be torsionless.

N-adapted and then orthonormalized components of geometric objects (like d-vectors, d-covectors, d-tensors, d-groups, d-algebras, d-matrices) admitting further decompositions into h- and v-components defined as non-integrable distributions of such objects.

With respect to N-adapted orthonormal frames, the geometry of N-anholonomic manifolds is defined algebraically, on their tangent bundles, by pairs of horizontal and vertical Klein spaces (see [29] for a summary of Klein geometry, and for bi-Hamiltonian soliton constructions, see [34,14]). The N-connection structure induces an N-anholonomic Klein geometry stated by two left–invariant $h\mathfrak{g}$- and $v\mathfrak{g}$-valued Maurer–Cartan forms on the Lie d-group $\mathbf{G} = (h\mathbf{G}, v\mathbf{G})$ that are identified with the zero–curvature canonical d-connection 1-form ${}^{\mathbf{G}}\mathbf{\Gamma} = \{{}^{\mathbf{G}}\mathbf{\Gamma}^{\alpha'}_{\ \beta'}\}$, via

$$^{\mathbf{G}}\mathbf{\Gamma}^{\alpha'}_{\ \beta'} = {}^{\mathbf{G}}\mathbf{\Gamma}^{\alpha'}_{\ \beta'\gamma'}\mathbf{e}^{\gamma'} = {}^{h\mathbf{G}}L^{i'}_{\ j'k'}\mathbf{e}^{k'} + {}^{v\mathbf{G}}C^{i'}_{\ j'k'}e^{k'}.$$

For the case of a trivial N-connection structure in vector bundles with the base and typical fiber spaces being symmetric Riemannian spaces, we identify ${}^{h\mathbf{G}}L^{i'}_{\ j'k'}$ and ${}^{v\mathbf{G}}C^{i'}_{\ j'k'}$ with the coefficients of the Cartan connections ${}^{h\mathbf{G}}L$ and ${}^{v\mathbf{G}}C$, respectively, for the $h\mathbf{G}$ and $v\mathbf{G}$, both with vanishing curvatures. We have

$$d\,{}^{\mathbf{G}}\mathbf{\Gamma} + \frac{1}{2}[{}^{\mathbf{G}}\mathbf{\Gamma},, {}^{\mathbf{G}}\mathbf{\Gamma}] = 0$$

and, respectively, for h- and v-components, $d^{h\mathbf{G}}\mathbf{L} + \frac{1}{2}[{}^{h\mathbf{G}}L, {}^{h\mathbf{G}}L] = 0$ and $d^{v\mathbf{G}}\mathbf{C} + \frac{1}{2}[{}^{v\mathbf{G}}\mathbf{C}, {}^{v\mathbf{G}}\mathbf{C}] = 0$, where d denotes the total derivative on the d-group manifold $\mathbf{G} = h\mathbf{G} \oplus v\mathbf{G}$ or its restrictions on $h\mathbf{G}$ or $v\mathbf{G}$. We can consider that ${}^{\mathbf{G}}\mathbf{\Gamma}$ defines the so-called Cartan d-connection for non-integrable N-connection structures (see details and supersymmetric/noncommutative developments in [25,26]).

The Cartan d-connection determines an N-anholonomic Riemannian geometric structure on the nonholonomic bundle

$$\mathring{\mathbf{E}} = [hG = SO(n+1), vG = SO(m+1), N^e_i],$$

derived through the following decomposition of the Lie d-algebra $\mathfrak{g} = h\mathfrak{g} \oplus v\mathfrak{g}$. For the horizontal splitting, we have $h\mathfrak{g} = \mathfrak{so}(n) \oplus h\mathfrak{p}$, with $[h\mathfrak{p}, h\mathfrak{p}] \subset \mathfrak{so}(n)$ and, for the vertical splitting, we have $[\mathfrak{so}(n), h\mathfrak{p}] \subset h\mathfrak{p}$; for the vertical splitting, $v\mathfrak{g} = \mathfrak{so}(m) \oplus v\mathfrak{p}$, with $[v\mathfrak{p}, v\mathfrak{p}] \subset \mathfrak{so}(m)$ and $[\mathfrak{so}(m), v\mathfrak{p}] \subset v\mathfrak{p}$. When $n = m$, any canonical d-objects (N-connection, d-metric, d-connection, etc.) derived from (29) using the Cartan d-connection agree with ones determined by the canonical d-connection (70) on a tangent bundle.

It is useful to consider a quotient space with distinguished structure group $\mathbf{V_N} = \mathbf{G}/SO(n) \oplus SO(m)$ regarding $\mathbf{G}$ as a principal $(SO(n) \oplus SO(m))$-bundle over $\mathring{\mathbf{E}}$, which is an N-anholonomic bundle. In this case, we can always fix a local section of this bundle and pull-back ${}^{\mathbf{G}}\mathbf{\Gamma}$ to give a $(h\mathfrak{g} \oplus v\mathfrak{g})$-valued 1-form ${}^{\mathfrak{g}}\mathbf{\Gamma}$ in a point $u \in \mathring{\mathbf{E}}$. Any change of local sections define $SO(n) \oplus SO(m)$ gauge transformations of the canonical d-connection ${}^{\mathfrak{g}}\mathbf{\Gamma}$ all preserving the nonholonomic decomposition (3).

There are involutive automorphisms $h\sigma = \pm 1$ and $v\sigma = \pm 1$, respectively, of $h\mathfrak{g}$ and $v\mathfrak{g}$, defined that $\mathfrak{so}(n)$ (or $\mathfrak{so}(m)$) is the eigenspace $h\sigma = +1$ (or $v\sigma = +1$) and $h\mathfrak{p}$ (or $v\mathfrak{p}$) is the eigenspace $h\sigma = -1$ (or $v\sigma = -1$). By means of an N-adapted decomposition relative to these eigenspaces, the symmetric part of the canonical d-connection

$$\mathbf{\Gamma} \doteqdot \frac{1}{2}\left({}^{\mathfrak{g}}\mathbf{\Gamma} + \sigma\left({}^{\mathfrak{g}}\mathbf{\Gamma}\right)\right),$$

defines a $(\mathfrak{so}(n) \oplus \mathfrak{so}(m))$-valued d-connection 1-form, with respective h- and v-splitting $\boldsymbol{L} \doteqdot \frac{1}{2}\left({}^{h\mathfrak{g}}\mathbf{L} + h\sigma\left({}^{h\mathfrak{g}}\mathbf{L}\right)\right)$ and $\boldsymbol{C} \doteqdot \frac{1}{2}({}^{v\mathfrak{g}}\mathbf{C} + h\sigma({}^{v\mathfrak{g}}\mathbf{C}))$. Likewise the antisymmetric part

$$\mathbf{e} \doteqdot \frac{1}{2}\left({}^{\mathfrak{g}}\mathbf{\Gamma} - \sigma\left({}^{\mathfrak{g}}\mathbf{\Gamma}\right)\right),$$

defines a $(h\mathfrak{p} \oplus v\mathfrak{p})$-valued N-adapted coframe for the Cartan–Killing inner product $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$ on $T_u\mathbf{G} \simeq h\mathfrak{g} \oplus v\mathfrak{g}$ restricted to $T_u\mathbf{V_N} \simeq \mathfrak{p}$, with respective h- and v-splitting $h\mathbf{e} \doteqdot \frac{1}{2}\left({}^{h\mathfrak{g}}\mathbf{L} - h\sigma\left({}^{h\mathfrak{g}}\mathbf{L}\right)\right)$ and $v\boldsymbol{e} \doteqdot \frac{1}{2}({}^{v\mathfrak{g}}\mathbf{C} - h\sigma({}^{v\mathfrak{g}}\mathbf{C}))$. This inner product, distinguished into h- and v-components, provides a d-metric structure of type $\mathbf{g} = [g, h]$ (14), where $g = \langle h\mathbf{e} \otimes h\mathbf{e}\rangle_{h\mathfrak{p}}$ and $h = \langle v\mathbf{e} \otimes v\mathbf{e}\rangle_{v\mathfrak{p}}$ on $\mathbf{V_N} = \mathbf{G}/SO(n) \oplus SO(m)$.

We generate a $\mathbf{G}$-invariant d-derivative $\mathbf{D}$ whose restriction to the tangent space $T\mathbf{V_N}$ for any N-anholonomic curve flow $\gamma(\tau, l)$ in $\mathbf{V_N} = \mathbf{G}/SO(n) \oplus SO(m)$ is defined via

$$\mathbf{D_X}\mathbf{e} = [\mathbf{e}, \gamma_l \rfloor \mathbf{\Gamma}] \quad \text{and} \quad \mathbf{D_Y}\mathbf{e} = [\mathbf{e}, \gamma_\tau \rfloor \mathbf{\Gamma}], \tag{30}$$

which have h- and v-decompositions. The derivatives $\mathbf{D_X}$ and $\mathbf{D_Y}$ are equivalent to those considered in (20) and obey the Cartan structure equations (24) and (25). For canonical d-connections, a large class of N-anholonomic spaces of dimension $n = m$, the d-torsions are zero and the d-curvatures have constant coefficients.

Let $\mathbf{e}^{\alpha'} = (e^{i'}, \mathbf{e}^{a'})$ be an N-adapted orthonormal coframe being identified with the $(h\mathfrak{p} \oplus v\mathfrak{p})$-valued coframe $\mathbf{e}$ in a fixed orthonormal basis for $\mathfrak{p} = h\mathfrak{p} \oplus v\mathfrak{p} \subset h\mathfrak{g} \oplus v\mathfrak{g}$. Considering the kernel/ cokernel of Lie algebra multiplications in the h- and v-subspaces, respectively, $[\mathbf{e}_{h\mathbf{X}}, \cdot]_{h\mathfrak{g}}$ and $[\mathbf{e}_{v\mathbf{X}}, \cdot]_{v\mathfrak{g}}$, we can decompose the coframes into parallel and perpendicular parts with respect to $\mathbf{e_X}$. We write

$$\mathbf{e} = (\mathbf{e}_C = h\mathbf{e}_C + v\mathbf{e}_C, \mathbf{e}_{C^\perp} = h\mathbf{e}_{C^\perp} + v\mathbf{e}_{C^\perp}),$$

for $\mathfrak{p}(=h\mathfrak{p}\oplus v\mathfrak{p})$-valued mutually orthogonal d-vectors $\mathbf{e}_C$ and $\mathbf{e}_{C^\perp}$, satisfying the conditions $[\mathbf{e}_\mathbf{X},\mathbf{e}_C]_\mathfrak{g}=0$ and $\left[\mathbf{e}_\mathbf{X},\mathbf{e}_{C^\perp}\right]_\mathfrak{g}\neq 0$; such conditions can be stated in h- and v-component form, respectively, $[h\mathbf{e}_\mathbf{X},h\mathbf{e}_C]_{h\mathfrak{g}}=0$, $\left[h\mathbf{e}_\mathbf{X},h\mathbf{e}_{C^\perp}\right]_{h\mathfrak{g}}\neq 0$ and $[v\mathbf{e}_\mathbf{X},v\mathbf{e}_C]_{v\mathfrak{g}}=0$, $\left[v\mathbf{e}_\mathbf{X},v\mathbf{e}_{C^\perp}\right]_{v\mathfrak{g}}\neq 0$. Then we have the algebraic decompositions

$$T_u\mathbf{V_N}\simeq\mathfrak{p}=h\mathfrak{p}\oplus v\mathfrak{p}=\mathfrak{g}=h\mathfrak{g}\oplus v\mathfrak{g}/\mathfrak{so}(n)\oplus\mathfrak{so}(m)$$

and

$$\mathfrak{p}=\mathfrak{p}_C\oplus\mathfrak{p}_{C^\perp}=(h\mathfrak{p}_C\oplus v\mathfrak{p}_C)\oplus\left(h\mathfrak{p}_{C^\perp}\oplus v\mathfrak{p}_{C^\perp}\right),$$

with $\mathfrak{p}_\parallel\subseteq\mathfrak{p}_C$ and $\mathfrak{p}_{C^\perp}\subseteq\mathfrak{p}_\perp$, where $\left[\mathfrak{p}_\parallel,\mathfrak{p}_C\right]=0$, $\langle\mathfrak{p}_{C^\perp},\mathfrak{p}_C\rangle=0$, but $\left[\mathfrak{p}_\parallel,\mathfrak{p}_{C^\perp}\right]\neq 0$ (i.e. $\mathfrak{p}_C$ is the centralizer of $\mathbf{e}_\mathbf{X}$ in $\mathfrak{p}=h\mathfrak{p}\oplus v\mathfrak{p}\subset h\mathfrak{g}\oplus v\mathfrak{g}$); in h- and v-components, thus $h\mathfrak{p}_\parallel\subseteq h\mathfrak{p}_C$ and $h\mathfrak{p}_{C^\perp}\subseteq h\mathfrak{p}_\perp$, where $\left[h\mathfrak{p}_\parallel,h\mathfrak{p}_C\right]=0$, $\langle h\mathfrak{p}_{C^\perp},h\mathfrak{p}_C\rangle=0$, but $\left[h\mathfrak{p}_\parallel,h\mathfrak{p}_{C^\perp}\right]\neq 0$ (i.e. $h\mathfrak{p}_C$ is the centralizer of $\mathbf{e}_{h\mathbf{X}}$ in $h\mathfrak{p}\subset h\mathfrak{g}$) and $v\mathfrak{p}_\parallel\subseteq v\mathfrak{p}_C$ and $v\mathfrak{p}_{C^\perp}\subseteq v\mathfrak{p}_\perp$, where $\left[v\mathfrak{p}_\parallel,v\mathfrak{p}_C\right]=0$, $\langle v\mathfrak{p}_{C^\perp},v\mathfrak{p}_C\rangle=0$, but $\left[v\mathfrak{p}_\parallel,v\mathfrak{p}_{C^\perp}\right]\neq 0$ (i.e. $v\mathfrak{p}_C$ is the centralizer of $\mathbf{e}_{v\mathbf{X}}$ in $v\mathfrak{p}\subset v\mathfrak{g}$). Using the canonical d-connection derivative $\mathbf{D_X}$ of a d-covector algebraically perpendicular (or parallel) to $\mathbf{e_X}$, we get a new d-vector which is algebraically parallel (or perpendicular) to $\mathbf{e_X}$, i.e. $\mathbf{D_X}\mathbf{e}_C\in\mathfrak{p}_{C^\perp}$ (or $\mathbf{D_X}\mathbf{e}_{C^\perp}\in\mathfrak{p}_C$) in h- and v-components such formulas are written $\mathbf{D}_{h\mathbf{X}}h\mathbf{e}_C\in h\mathfrak{p}_{C^\perp}$ (or $\mathbf{D}_{h\mathbf{X}}h\mathbf{e}_{C^\perp}\in h\mathfrak{p}_C$) and $\mathbf{D}_{v\mathbf{X}}v\mathbf{e}_C\in v\mathfrak{p}_{C^\perp}$ (or $\mathbf{D}_{v\mathbf{X}}v\mathbf{e}_{C^\perp}\in v\mathfrak{p}_C$). All such d-algebraic relations can be written in N-anholonomic manifolds and canonical d-connection settings, for instance, using certain relations of type

$$\mathbf{D_X}(\mathbf{e}^{\alpha'})_C=\mathbf{v}^{\alpha'}_{\ \beta'}(\mathbf{e}^{\beta'})_{C^\perp}\quad\text{and}\quad\mathbf{D_X}(\mathbf{e}^{\alpha'})_{C^\perp}=-\mathbf{v}^{\alpha'}_{\ \beta'}(\mathbf{e}^{\beta'})_C,$$

for some antisymmetric d-tensors $\mathbf{v}^{\alpha'\beta'}=-\mathbf{v}^{\beta'\alpha'}$. Note we will get an N-adapted $(SO(n)\oplus SO(m))$-parallel frame defining a generalization of the concept of Riemannian parallel frame [14] on N-adapted manifolds whenever $\mathfrak{p}_C$ is larger than $\mathfrak{p}_\parallel$. Substituting $\mathbf{e}^{\alpha'}=(e^{i'},\mathbf{e}^{a'})$ into the last formulas and considering h- and v-components, we define $SO(n)$-parallel and $SO(m)$-parallel frames (for simplicity we omit these formulas when the Greek letter indices are split into Latin letter h- and v-indices).

The final conclusion of this section is that the Cartan structure equations on hypersurfaces swept out by nonholonomic curve flows on N-anholonomic spaces with constant matrix curvature for the canonical d-connection geometrically encode separate $O(n-1)$- and $O(m-1)$-invariant (respectively, horizontal and vertical) bi-Hamiltonian operators, whenever the N-connection freedom of the d-group action $SO(n)\oplus SO(m)$ on $\mathbf{e}$ and $\mathbf{\Gamma}$ is used to fix them to be an N-adapted parallel coframe and its associated canonical d-connection 1-form, related to the canonical covariant derivative on N-anholonomic manifolds.

4. Anholonomic bi-Hamiltonian structures and vector soliton equations

Introducing N-adapted orthonormal bases, for N-anholonomic spaces of dimension $n+n$, with constant curvatures of the canonical d-connection, we will be able to derive bi-Hamiltonian and vector soliton structures similarly to the case of symmetric Riemannian spaces (see Refs. [21,17,34]).

4.1. Basic equations for N-anholonomic curve flows

In this section, we shall prove the results for the h-components of certain N-anholonomic manifolds with constant d-curvature and then duplicate the formulas for the v-components omitting similar details.

There is an isomorphism between the real space $\mathfrak{so}(n)$ and the Lie algebra of $n\times n$ skew-symmetric matrices. This yields an isomorphism between $h\mathfrak{p}\simeq\mathbb{R}^n$ and the tangent spaces $T_xM=\mathfrak{so}(n+1)/\mathfrak{so}(n)$ of the Riemannian manifold $M=SO(n+1)/SO(n)$ as described by the following canonical decomposition

$$h\mathfrak{g}=\mathfrak{so}(n+1)\supset h\mathfrak{p}\in\begin{bmatrix}0 & h\mathbf{p}\\ -h\mathbf{p}^{\mathrm{T}} & h\mathbf{0}\end{bmatrix}\quad\text{for } h\mathbf{0}\in h\mathfrak{h}=\mathfrak{so}(n)$$

with $h\mathbf{p}=\{p^{i'}\}\in\mathbb{R}^n$ being the h-component of the d-vector $\mathbf{p}=(p^{i'},p^{a'})$ and $h\mathbf{p}^{\mathrm{T}}$ denoting the transposition of the row $h\mathbf{p}$. The Cartan–Killing inner product on $h\mathfrak{p}$ is given by

$$\begin{aligned}h\mathbf{p}\cdot h\mathbf{p}&=\left\langle\begin{bmatrix}0 & h\mathbf{p}\\ -h\mathbf{p}^{\mathrm{T}} & h\mathbf{0}\end{bmatrix},\begin{bmatrix}0 & h\mathbf{p}\\ -h\mathbf{p}^{\mathrm{T}} & h\mathbf{0}\end{bmatrix}\right\rangle\\ &\doteqdot\frac{1}{2}\mathrm{tr}\left\{\begin{bmatrix}0 & h\mathbf{p}\\ -h\mathbf{p}^{\mathrm{T}} & h\mathbf{0}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}0 & h\mathbf{p}\\ -h\mathbf{p}^{\mathrm{T}} & h\mathbf{0}\end{bmatrix}\right\},\end{aligned}$$

where tr denotes the trace of the corresponding product of matrices. This product identifies canonically $h\mathfrak{p}\simeq\mathbb{R}^n$ with its dual $h\mathfrak{p}^*\simeq\mathbb{R}^n$. In a similar fashion, we can define the Cartan–Killing inner product $v\mathbf{p}\cdot v\mathbf{p}$ where

$$v\mathfrak{g}=\mathfrak{so}(m+1)\supset v\mathfrak{p}\in\begin{bmatrix}0 & v\mathbf{p}\\ -v\mathbf{p}^{\mathrm{T}} & v\mathbf{0}\end{bmatrix}\quad\text{for } v\mathbf{0}\in v\mathfrak{h}=\mathfrak{so}(m)$$

with $v\mathbf{p}=\{p^{a'}\}\in\mathbb{R}^m$ being the v-component of the d-vector $\mathbf{p}=(p^{i'}p^{a'})$. In general, in the tangent bundle of an N-anholonomic manifold, we can consider the Cartan–Killing N-adapted inner product $\mathbf{p}\cdot\mathbf{p}=h\mathbf{p}\cdot h\mathbf{p}+v\mathbf{p}\cdot v\mathbf{p}$.

Following the introduced Cartan–Killing parametrizations, we analyze the flow $\gamma(\tau, l)$ of a non-stretching curve in $\mathbf{V_N} = \mathbf{G}/SO(n) \oplus SO(m)$. Let us introduce a coframe $\mathbf{e} \in T^*_\gamma \mathbf{V_N} \otimes (h\mathfrak{p} \oplus v\mathfrak{p})$, which is an N-adapted $(SO(n) \oplus SO(m))$-parallel covector frame along γ, and its associated canonical d-connection 1-form $\mathbf{\Gamma} \in T^*_\gamma \mathbf{V_N} \otimes (\mathfrak{so}(n) \oplus \mathfrak{so}(m))$. Such d-objects are respectively parametrized:

$$\mathbf{e_X} = \mathbf{e}_{h\mathbf{X}} + \mathbf{e}_{v\mathbf{X}},$$

for

$$\mathbf{e}_{h\mathbf{X}} = \gamma_{h\mathbf{X}} \rfloor h\mathbf{e} = \begin{bmatrix} 0 & (1, \overrightarrow{0}) \\ -(1, \overrightarrow{0})^{\mathrm{T}} & h\mathbf{0} \end{bmatrix}$$

and

$$\mathbf{e}_{v\mathbf{X}} = \gamma_{v\mathbf{X}} \rfloor v\mathbf{e} = \begin{bmatrix} 0 & (1, \overleftarrow{0}) \\ -(1, \overleftarrow{0})^{\mathrm{T}} & v\mathbf{0} \end{bmatrix},$$

where we write $(1, \overrightarrow{0}) \in \mathbb{R}^n$, $\overrightarrow{0} \in \mathbb{R}^{n-1}$ and $(1, \overleftarrow{0}) \in \mathbb{R}^m$, $\overleftarrow{0} \in \mathbb{R}^{m-1}$;

$$\mathbf{\Gamma} = [\mathbf{\Gamma}_{h\mathbf{X}}, \mathbf{\Gamma}_{v\mathbf{X}}],$$

with

$$\mathbf{\Gamma}_{h\mathbf{X}} = \gamma_{h\mathbf{X}} \rfloor \mathbf{L} = \begin{bmatrix} 0 & (0, \overrightarrow{0}) \\ -(0, \overrightarrow{0})^{\mathrm{T}} & \mathbf{L} \end{bmatrix} \in \mathfrak{so}(n+1),$$

where

$$\mathbf{L} = \begin{bmatrix} 0 & \overrightarrow{v} \\ -\overrightarrow{v}^{\mathrm{T}} & h\mathbf{0} \end{bmatrix} \in \mathfrak{so}(n), \quad \overrightarrow{v} \in \mathbb{R}^{n-1}, h\mathbf{0} \in \mathfrak{so}(n-1),$$

and also

$$\mathbf{\Gamma}_{v\mathbf{X}} = \gamma_{v\mathbf{X}} \rfloor \mathbf{C} = \begin{bmatrix} 0 & (0, \overleftarrow{0}) \\ -(0, \overleftarrow{0})^{\mathrm{T}} & \mathbf{C} \end{bmatrix} \in \mathfrak{so}(m+1),$$

where

$$\mathbf{C} = \begin{bmatrix} 0 & \overleftarrow{v} \\ -\overleftarrow{v}^{\mathrm{T}} & v\mathbf{0} \end{bmatrix} \in \mathfrak{so}(m), \quad \overleftarrow{v} \in \mathbb{R}^{m-1}, v\mathbf{0} \in \mathfrak{so}(m-1).$$

The above parametrizations involve no loss of generality, because they can be achieved by the available gauge freedom using $SO(n)$ and $SO(m)$ rotations on the N-adapted coframe and canonical d-connection 1-form, distinguished in h- and v-components.

We introduce decompositions of horizontal $SO(n+1)/SO(n)$ matrices

$$h\mathfrak{p} \ni \begin{bmatrix} 0 & h\mathbf{p} \\ -h\mathbf{p}^{\mathrm{T}} & h\mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & \left(h\mathbf{p}_{\parallel}, \overrightarrow{0}\right) \\ -\left(h\mathbf{p}_{\parallel}, \overrightarrow{0}\right)^{\mathrm{T}} & h\mathbf{0} \end{bmatrix} + \begin{bmatrix} 0 & \left(0, h\overrightarrow{\mathbf{p}}_{\perp}\right) \\ -\left(0, h\overrightarrow{\mathbf{p}}_{\perp}\right)^{\mathrm{T}} & h\mathbf{0} \end{bmatrix},$$

into tangential and normal parts relative to $\mathbf{e}_{h\mathbf{X}}$ via corresponding decompositions of h-vectors $h\mathbf{p} = (h\mathbf{p}_{\parallel}, h\overrightarrow{\mathbf{p}}_{\perp}) \in \mathbb{R}^n$ relative to $\left(1, \overrightarrow{0}\right)$, when $h\mathbf{p}_{\parallel}$ is identified with $h\mathfrak{p}_C$ and $h\overrightarrow{\mathbf{p}}_{\perp}$ is identified with $h\mathfrak{p}_{\perp} = h\mathfrak{p}_{C^{\perp}}$. In a similar form, we decompose vertical $SO(m+1)/SO(m)$ matrices

$$v\mathfrak{p} \ni \begin{bmatrix} 0 & v\mathbf{p} \\ -v\mathbf{p}^{\mathrm{T}} & v\mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & \left(v\mathbf{p}_{\parallel}, \overleftarrow{0}\right) \\ -\left(v\mathbf{p}_{\parallel}, \overleftarrow{0}\right)^{\mathrm{T}} & v\mathbf{0} \end{bmatrix} + \begin{bmatrix} 0 & \left(0, v\overleftarrow{\mathbf{p}}_{\perp}\right) \\ -\left(0, v\overleftarrow{\mathbf{p}}_{\perp}\right)^{\mathrm{T}} & v\mathbf{0} \end{bmatrix},$$

into tangential and normal parts relative to $\mathbf{e}_{v\mathbf{X}}$ via corresponding decompositions of h-vectors $v\mathbf{p} = (v\mathbf{p}_{\parallel}, v\overleftarrow{\mathbf{p}}_{\perp}) \in \mathbb{R}^m$ relative to $\left(1, \overleftarrow{0}\right)$, when $v\mathbf{p}_{\parallel}$ is identified with $v\mathfrak{p}_C$ and $v\overleftarrow{\mathbf{p}}_{\perp}$ is identified with $v\mathfrak{p}_{\perp} = v\mathfrak{p}_{C^{\perp}}$.

There are analogous matrix decompositions relative to the flow direction. In the h-direction, we parametrize

$$\mathbf{e}_{h\mathbf{Y}} = \gamma_{\tau} \rfloor h\mathbf{e} = \begin{bmatrix} 0 & \left(h\mathbf{e}_{\parallel}, h\overrightarrow{\mathbf{e}}_{\perp}\right) \\ -\left(h\mathbf{e}_{\parallel}, h\overrightarrow{\mathbf{e}}_{\perp}\right)^{\mathrm{T}} & h\mathbf{0} \end{bmatrix},$$

when $\mathbf{e}_{h\mathbf{Y}} \in h\mathfrak{p}$, $\left(h\mathbf{e}_{\parallel}, h\overrightarrow{\mathbf{e}}_{\perp}\right) \in \mathbb{R}^n$ and $h\overrightarrow{\mathbf{e}}_{\perp} \in \mathbb{R}^{n-1}$, and

$$\boldsymbol{\Gamma}_{h\mathbf{Y}} = \gamma_{h\mathbf{Y}} \rfloor \mathbf{L} = \begin{bmatrix} 0 & (0, \overrightarrow{0}) \\ -(0, \overrightarrow{0})^{\mathrm{T}} & h\varpi_{\tau} \end{bmatrix} \in \mathfrak{so}(n+1), \tag{31}$$

where

$$h\varpi_{\tau} = \begin{bmatrix} 0 & \overrightarrow{\varpi} \\ -\overrightarrow{\varpi}^{\mathrm{T}} & h\boldsymbol{\Theta} \end{bmatrix} \in \mathfrak{so}(n), \quad \overrightarrow{\varpi} \in \mathbb{R}^{n-1}, h\boldsymbol{\Theta} \in \mathfrak{so}(n-1).$$

In the v-direction, we parametrize

$$\mathbf{e}_{v\mathbf{Y}} = \gamma_{\tau} \rfloor v\mathbf{e} = \begin{bmatrix} 0 & \left(v\mathbf{e}_{\parallel}, v\overleftarrow{\mathbf{e}}_{\perp}\right) \\ -\left(v\mathbf{e}_{\parallel}, v\overleftarrow{\mathbf{e}}_{\perp}\right)^{\mathrm{T}} & v\mathbf{0} \end{bmatrix},$$

when $\mathbf{e}_{v\mathbf{Y}} \in v\mathfrak{p}$, $\left(v\mathbf{e}_{\parallel}, v\overleftarrow{\mathbf{e}}_{\perp}\right) \in \mathbb{R}^m$ and $v\overleftarrow{\mathbf{e}}_{\perp} \in \mathbb{R}^{m-1}$, and

$$\boldsymbol{\Gamma}_{v\mathbf{Y}} = \gamma_{v\mathbf{Y}} \rfloor \mathbf{C} = \begin{bmatrix} 0 & (0, \overleftarrow{0}) \\ -(0, \overleftarrow{0})^{\mathrm{T}} & v\varpi_{\tau} \end{bmatrix} \in \mathfrak{so}(m+1),$$

where

$$v\varpi_{\tau} = \begin{bmatrix} 0 & \overleftarrow{\varpi} \\ -\overleftarrow{\varpi}^{\mathrm{T}} & v\boldsymbol{\Theta} \end{bmatrix} \in \mathfrak{so}(m), \quad \overleftarrow{\varpi} \in \mathbb{R}^{m-1}, v\boldsymbol{\Theta} \in \mathfrak{so}(m-1).$$

The components $h\mathbf{e}_{\parallel}$ and $h\overrightarrow{\mathbf{e}}_{\perp}$ correspond to the decomposition

$$\mathbf{e}_{h\mathbf{Y}} = h\mathbf{g}(\gamma_{\tau}, \gamma_{l}) e_{h\mathbf{X}} + (\gamma_{\tau})_{\perp} \rfloor h\mathbf{e}_{\perp}$$

into tangential and normal parts relative to $\mathbf{e}_{h\mathbf{X}}$. In a similar form, one considers $v\mathbf{e}_{\parallel}$ and $v\overleftarrow{\mathbf{e}}_{\perp}$ corresponding to the decomposition

$$\mathbf{e}_{v\mathbf{Y}} = v\mathbf{g}(\gamma_{\tau}, \gamma_{l}) e_{v\mathbf{X}} + (\gamma_{\tau})_{\perp} \rfloor v\mathbf{e}_{\perp}.$$

Using the above stated matrix parametrizations, we get

$$\begin{aligned}
&[\mathbf{e}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{Y}}] = -\begin{bmatrix} 0 & 0 \\ 0 & h\mathbf{e}_{\perp} \end{bmatrix} \in \mathfrak{so}(n+1), \quad \text{with} \\
&h\mathbf{e}_{\perp} = \begin{bmatrix} 0 & h\overrightarrow{\mathbf{e}}_{\perp} \\ -(h\overrightarrow{\mathbf{e}}_{\perp})^{\mathrm{T}} & h\mathbf{0} \end{bmatrix} \in \mathfrak{so}(n); \\
&[\boldsymbol{\Gamma}_{h\mathbf{Y}}, \mathbf{e}_{h\mathbf{Y}}] = -\begin{bmatrix} 0 & \left(0, \overrightarrow{\varpi}\right) \\ -\left(0, \overrightarrow{\varpi}\right)^{\mathrm{T}} & 0 \end{bmatrix} \in h\mathfrak{p}_{\perp}; \\
&[\boldsymbol{\Gamma}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{Y}}] = -\begin{bmatrix} 0 & \left(-\overrightarrow{v} \cdot h\overrightarrow{\mathbf{e}}_{\perp}, h\mathbf{e}_{\parallel}\overrightarrow{v}\right) \\ -\left(-\overrightarrow{v} \cdot h\overrightarrow{\mathbf{e}}_{\perp}, h\mathbf{e}_{\parallel}\overrightarrow{v}\right)^{\mathrm{T}} & h\mathbf{0} \end{bmatrix} \in h\mathfrak{p};
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
&[\mathbf{e}_{v\mathbf{X}}, \mathbf{e}_{v\mathbf{Y}}] = -\begin{bmatrix} 0 & 0 \\ 0 & v\mathbf{e}_{\perp} \end{bmatrix} \in \mathfrak{so}(m+1), \quad \text{with} \\
&v\mathbf{e}_{\perp} = \begin{bmatrix} 0 & v\overleftarrow{\mathbf{e}}_{\perp} \\ -(v\overleftarrow{\mathbf{e}}_{\perp})^{\mathrm{T}} & v\mathbf{0} \end{bmatrix} \in \mathfrak{so}(m); \\
&[\boldsymbol{\Gamma}_{v\mathbf{Y}}, \mathbf{e}_{v\mathbf{Y}}] = -\begin{bmatrix} 0 & \left(0, \overleftarrow{\varpi}\right) \\ -\left(0, \overleftarrow{\varpi}\right)^{\mathrm{T}} & 0 \end{bmatrix} \in v\mathfrak{p}_{\perp}; \\
&[\boldsymbol{\Gamma}_{v\mathbf{X}}, \mathbf{e}_{v\mathbf{Y}}] = -\begin{bmatrix} 0 & \left(-\overleftarrow{v} \cdot v\overleftarrow{\mathbf{e}}_{\perp}, v\mathbf{e}_{\parallel}\overleftarrow{v}\right) \\ -\left(-\overleftarrow{v} \cdot v\overleftarrow{\mathbf{e}}_{\perp}, v\mathbf{e}_{\parallel}\overleftarrow{v}\right)^{\mathrm{T}} & v\mathbf{0} \end{bmatrix} \in v\mathfrak{p}.
\end{aligned} \tag{33}$$

We can use formulas (32) and (33) in order to write the structure equations (24) and (25) in terms of N-adapted frames and connection 1-forms soldered to the Klein geometry of N-anholonomic spaces using the relations (30). One obtains respectively the $\mathbf{G}$-invariant N-adapted torsion and curvature generated by the canonical d-connection,

$$\mathbf{T}(\gamma_{\tau}, \gamma_{l}) = (\mathbf{D}_{\mathbf{X}}\gamma_{\tau} - \mathbf{D}_{\mathbf{Y}}\gamma_{l}) \rfloor \mathbf{e} = \mathbf{D}_{\mathbf{X}}\mathbf{e}_{\mathbf{Y}} - \mathbf{D}_{\mathbf{Y}}\mathbf{e}_{\mathbf{X}} + [\boldsymbol{\Gamma}_{\mathbf{X}}, \mathbf{e}_{\mathbf{Y}}] - [\boldsymbol{\Gamma}_{\mathbf{Y}}, \mathbf{e}_{\mathbf{X}}] \tag{34}$$

and

$$\mathbf{R}(\gamma_{\tau}, \gamma_{l})\mathbf{e} = [\mathbf{D}_{\mathbf{X}}, \mathbf{D}_{\mathbf{Y}}]\,\mathbf{e} = \mathbf{D}_{\mathbf{X}}\boldsymbol{\Gamma}_{\mathbf{Y}} - \mathbf{D}_{\mathbf{Y}}\boldsymbol{\Gamma}_{\mathbf{X}} + [\boldsymbol{\Gamma}_{\mathbf{X}}, \boldsymbol{\Gamma}_{\mathbf{Y}}] \tag{35}$$

where $\mathbf{e_X} \doteqdot \gamma_l \rfloor \mathbf{e}, \mathbf{e_Y} \doteqdot \gamma_\tau \rfloor \mathbf{e}, \mathbf{\Gamma_X} \doteqdot \gamma_l \rfloor \mathbf{\Gamma}$ and $\mathbf{\Gamma_Y} \doteqdot \gamma_\tau \rfloor \mathbf{\Gamma}$. The formulas (34) and (35)are respectively equivalent to (13) and (72). In general, $\mathbf{T}(\gamma_\tau, \gamma_l) \neq 0$ and $\mathbf{R}(\gamma_\tau, \gamma_l)\mathbf{e}$ cannot be defined to have constant matrix coefficients with respect to an N-adapted basis. For N-anholonomic spaces with dimensions $n = m$, we have $\mathbf{T}(\gamma_\tau, \gamma_l) = 0$ and $\mathbf{R}(\gamma_\tau, \gamma_l)\mathbf{e}$ defined by constant, or vanishing, d-curvature coefficients (see discussions related to formulas (75) and (70)). For such cases, we can consider the h- and v-components of (34) and (35) in a similar manner as for symmetric Riemannian spaces but with the canonical d-connection instead of the Levi-Civita one. One obtains, respectively,

$$\begin{aligned} 0 &= (\mathbf{D}_{h\mathbf{X}}\gamma_\tau - \mathbf{D}_{h\mathbf{Y}}\gamma_l)\rfloor h\mathbf{e} \\ &= \mathbf{D}_{h\mathbf{X}}\mathbf{e}_{h\mathbf{Y}} - \mathbf{D}_{h\mathbf{Y}}\mathbf{e}_{h\mathbf{X}} + [\mathbf{L}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{Y}}] - [\mathbf{L}_{h\mathbf{Y}}, \mathbf{e}_{h\mathbf{X}}]\,; \\ 0 &= (\mathbf{D}_{v\mathbf{X}}\gamma_\tau - \mathbf{D}_{v\mathbf{Y}}\gamma_l)\rfloor v\mathbf{e} \\ &= \mathbf{D}_{v\mathbf{X}}\mathbf{e}_{v\mathbf{Y}} - \mathbf{D}_{v\mathbf{Y}}\mathbf{e}_{v\mathbf{X}} + [\mathbf{C}_{v\mathbf{X}}, \mathbf{e}_{v\mathbf{Y}}] - [\mathbf{C}_{v\mathbf{Y}}, \mathbf{e}_{v\mathbf{X}}]\,, \end{aligned} \tag{36}$$

and

$$\begin{aligned} &h\mathbf{R}(\gamma_\tau, \gamma_l)h\mathbf{e} = [\mathbf{D}_{h\mathbf{X}}, \mathbf{D}_{h\mathbf{Y}}]\, h\mathbf{e} = \mathbf{D}_{h\mathbf{X}}\mathbf{L}_{h\mathbf{Y}} - \mathbf{D}_{h\mathbf{Y}}\mathbf{L}_{h\mathbf{X}} + [\mathbf{L}_{h\mathbf{X}}, \mathbf{L}_{h\mathbf{Y}}] \\ &v\mathbf{R}(\gamma_\tau, \gamma_l)v\mathbf{e} = [\mathbf{D}_{v\mathbf{X}}, \mathbf{D}_{v\mathbf{Y}}]\, v\mathbf{e} = \mathbf{D}_{v\mathbf{X}}\mathbf{C}_{v\mathbf{Y}} - \mathbf{D}_{v\mathbf{Y}}\mathbf{C}_{v\mathbf{X}} + [\mathbf{C}_{v\mathbf{X}}, \mathbf{C}_{v\mathbf{Y}}]\,. \end{aligned} \tag{37}$$

Following N-adapted curve flow parametrizations (32) and (33), the Eqs. (36) and (37) are written

$$\begin{aligned} &0 = \mathbf{D}_{h\mathbf{X}} h\mathbf{e}_\parallel + \overrightarrow{v} \cdot h\overrightarrow{\mathbf{e}}_\perp, \qquad 0 = \mathbf{D}_{v\mathbf{X}} v\mathbf{e}_\parallel + \overleftarrow{v} \cdot v\overleftarrow{\mathbf{e}}_\perp, ; \\ &0 = \overrightarrow{\varpi} - h\mathbf{e}_\parallel \overrightarrow{v} + \mathbf{D}_{h\mathbf{X}} h\overrightarrow{\mathbf{e}}_\perp, \qquad 0 = \overleftarrow{\varpi} - v\mathbf{e}_\parallel \overleftarrow{v} + \mathbf{D}_{v\mathbf{X}} v\overleftarrow{\mathbf{e}}_\perp; \end{aligned} \tag{38}$$

and

$$\begin{aligned} &\mathbf{D}_{h\mathbf{X}}\overrightarrow{\varpi} - \mathbf{D}_{h\mathbf{Y}}\overrightarrow{v} + \overrightarrow{v}\rfloor h\mathbf{\Theta} = h\overrightarrow{\mathbf{e}}_\perp, \qquad \mathbf{D}_{v\mathbf{X}}\overleftarrow{\varpi} - \mathbf{D}_{v\mathbf{Y}}\overleftarrow{v} + \overleftarrow{v}\rfloor v\mathbf{\Theta} = v\overleftarrow{\mathbf{e}}_\perp; \\ &\mathbf{D}_{h\mathbf{X}} h\mathbf{\Theta} - \overrightarrow{v}\otimes\overrightarrow{\varpi} + \overrightarrow{\varpi}\otimes\overrightarrow{v} = 0, \qquad \mathbf{D}_{v\mathbf{X}} v\mathbf{\Theta} - \overleftarrow{v}\otimes\overleftarrow{\varpi} + \overleftarrow{\varpi}\otimes\overleftarrow{v} = 0. \end{aligned} \tag{39}$$

The tensor products and interior products are defined in the form: for the h-components, $\otimes$ denotes the outer product of pairs of vectors ($1 \times n$ row matrices), producing $n \times n$ matrices $\overrightarrow{A} \otimes \overrightarrow{B} = \overrightarrow{A}^{\mathrm{T}}\overrightarrow{B}$, and $\rfloor$ denotes multiplication of $n \times n$ matrices on vectors ($1 \times n$ row matrices), such that $\overrightarrow{A}\rfloor\left(\overrightarrow{B}\otimes\overrightarrow{C}\right) = \left(\overrightarrow{A}\cdot\overrightarrow{B}\right)\overrightarrow{C}$ which is the transpose of the standard matrix product on column vectors. Likewise, for the v-components, we just change $n \to m$ and $\overrightarrow{A} \to \overleftarrow{A}$. For the sequel, $\wedge$ will denote the skew product of vectors $\overrightarrow{A}\wedge\overrightarrow{B} = \overrightarrow{A}\otimes\overrightarrow{B} - \overrightarrow{B}\otimes\overrightarrow{A}$.

The variables $\mathbf{e}_\parallel$ and $\mathbf{\Theta}$, written in h- and v-components, can be expressed in terms of the variables $\overrightarrow{v}, \overrightarrow{\varpi}, h\overrightarrow{\mathbf{e}}_\perp$ and $\overleftarrow{v}, \overleftarrow{\varpi}, v\overleftarrow{\mathbf{e}}_\perp$ (see respectively the first two equations in (38) and the last two equations in (39)) by

$$h\mathbf{e}_\parallel = -\mathbf{D}_{h\mathbf{X}}^{-1}(\overrightarrow{v}\cdot h\overrightarrow{\mathbf{e}}_\perp), \qquad v\mathbf{e}_\parallel = -\mathbf{D}_{v\mathbf{X}}^{-1}(\overleftarrow{v}\cdot v\overleftarrow{\mathbf{e}}_\perp),$$

and

$$h\mathbf{\Theta} = \mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v}\otimes\overrightarrow{\varpi} - \overrightarrow{\varpi}\otimes\overrightarrow{v}\right), \qquad v\mathbf{\Theta} = \mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v}\otimes\overleftarrow{\varpi} - \overleftarrow{\varpi}\otimes\overleftarrow{v}\right).$$

Substituting these expressions, correspondingly, in the last two equations in (38) and in the first two equations in (39), we express

$$\overrightarrow{\varpi} = -\mathbf{D}_{h\mathbf{X}} h\overrightarrow{\mathbf{e}}_\perp - \mathbf{D}_{h\mathbf{X}}^{-1}(\overrightarrow{v}\cdot h\overrightarrow{\mathbf{e}}_\perp)\overrightarrow{v}, \qquad \overleftarrow{\varpi} = -\mathbf{D}_{v\mathbf{X}} v\overleftarrow{\mathbf{e}}_\perp - \mathbf{D}_{v\mathbf{X}}^{-1}(\overleftarrow{v}\cdot v\overleftarrow{\mathbf{e}}_\perp)\overleftarrow{v},$$

contained in the h- and v-flow equations respectively on $\overrightarrow{v}$ and $\overleftarrow{v}$, considered as evolution equations via $\mathbf{D}_{h\mathbf{Y}}\overrightarrow{v} = \overrightarrow{v}_\tau$ and $\mathbf{D}_{h\mathbf{Y}}\overleftarrow{v} = \overleftarrow{v}_\tau$,

$$\begin{aligned} &\overrightarrow{v}_\tau = \mathbf{D}_{h\mathbf{X}}\overrightarrow{\varpi} - \overrightarrow{v}\rfloor\mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v}\otimes\overrightarrow{\varpi} - \overrightarrow{\varpi}\otimes\overrightarrow{v}\right) - \overrightarrow{R}\, h\overrightarrow{\mathbf{e}}_\perp, \\ &\overleftarrow{v}_\tau = \mathbf{D}_{v\mathbf{X}}\overleftarrow{\varpi} - \overleftarrow{v}\rfloor\mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v}\otimes\overleftarrow{\varpi} - \overleftarrow{\varpi}\otimes\overleftarrow{v}\right) - \overleftarrow{S}\, v\overleftarrow{\mathbf{e}}_\perp, \end{aligned} \tag{40}$$

where the scalar curvatures of the canonical d-connection, $\overrightarrow{R}$ and $\overleftarrow{S}$ are defined by formulas (74) in Appendix. For symmetric Riemannian spaces like $SO(n+1)/SO(n) \simeq S^n$, $\overrightarrow{R}$ is just the scalar curvature $\chi = 1$, see [17]. On N-anholonomic manifolds under consideration, the values $\overrightarrow{R}$ and $\overleftarrow{S}$ are constrained by be certain zero or nonzero constants.

The above presented considerations lead to a proof of the following main results.

Lemma 4.1. *On N-anholonomic spaces with constant-curvature matrix coefficients for the canonical d-connection, there are N-adapted Hamiltonian symplectic operators,*

$$h\mathcal{J} = \mathbf{D}_{h\mathbf{X}} + \mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v}\cdot\right)\overrightarrow{v} \quad \textit{and} \quad v\mathcal{J} = \mathbf{D}_{v\mathbf{X}} + \mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v}\cdot\right)\overleftarrow{v}, \tag{41}$$

and cosymplectic operators

$$h\mathcal{H} \doteqdot \mathbf{D}_{h\mathbf{X}} + \overrightarrow{v}\rfloor\mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v}\wedge\right) \quad \textit{and} \quad v\mathcal{H} \doteqdot \mathbf{D}_{v\mathbf{X}} + \overleftarrow{v}\rfloor\mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v}\wedge\right). \tag{42}$$

The properties of operators (41) and (42) are defined by

Theorem 4.1. *The d-operators $\mathcal{J} = (h\mathcal{J}, v\mathcal{J})$ and $\mathcal{H} = (h\mathcal{H}, v\mathcal{H})$ are respectively $(O(n-1), O(m-1))$-invariant Hamiltonian symplectic and cosymplectic operators with respect to the flow variables $(\overrightarrow{v}, \overleftarrow{v})$. Consequently the curve flow equations on N-anholonomic manifolds with constant d-connection curvature have a Hamiltonian form: the h –flows are given by*

$$\overrightarrow{v}_\tau = h\mathcal{H}\left(\overrightarrow{\varpi}\right) - \overrightarrow{R}\, h\overrightarrow{\mathbf{e}}_\perp = h\mathfrak{R}\left(h\overrightarrow{\mathbf{e}}_\perp\right) - \overrightarrow{R}\, h\overrightarrow{\mathbf{e}}_\perp, \qquad \overrightarrow{\varpi} = h\mathcal{J}\left(h\overrightarrow{\mathbf{e}}_\perp\right); \tag{43}$$

the v-flows are given by

$$\overleftarrow{v}_\tau = v\mathcal{H}\left(\overleftarrow{\varpi}\right) - \overleftarrow{S}\, v\overleftarrow{\mathbf{e}}_\perp = v\mathfrak{R}\left(v\overleftarrow{\mathbf{e}}_\perp\right) - \overleftarrow{S}\, v\overleftarrow{\mathbf{e}}_\perp, \qquad \overleftarrow{\varpi} = v\mathcal{J}\left(v\overleftarrow{\mathbf{e}}_\perp\right), \tag{44}$$

where the hereditary recursion d-operator has the respective h- and v-components

$$h\mathfrak{R} = h\mathcal{H} \circ h\mathcal{J} \quad \text{and} \quad v\mathfrak{R} = v\mathcal{H} \circ v\mathcal{J}. \tag{45}$$

Proof. Details in the case of holonomic structures are given in Ref. [13] and generalized to arbitrary gauge groups in [34]. For the present case, the main additional considerations consist of soldering certain classes of generalized Lagrange spaces with $(O(n-1), O(m-1))$-gauge symmetry onto the Klein geometry of N-anholonomic spaces. □

4.2. bi-Hamiltonian anholonomic curve flows and solitonic hierarchies

Following the usual solitonic techniques, see details in Refs. [17,35], the recursion h-operator from (45),

$$\begin{aligned} h\mathfrak{R} &= \mathbf{D}_{h\mathbf{X}}\left(\mathbf{D}_{h\mathbf{X}} + \mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v}\cdot\right)\overrightarrow{v}\right) + \overrightarrow{v}\rfloor\mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v}\wedge\mathbf{D}_{h\mathbf{X}}\right) \\ &= \mathbf{D}_{h\mathbf{X}}^2 + |\mathbf{D}_{h\mathbf{X}}|^2 + \mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v}\cdot\right)\overrightarrow{v}_l - \overrightarrow{v}\rfloor\mathbf{D}_{h\mathbf{X}}^{-1}(\overrightarrow{v}_l\wedge), \end{aligned} \tag{46}$$

generates a horizontal hierarchy of commuting Hamiltonian vector fields $h\overrightarrow{\mathbf{e}}_\perp^{(k)}$ starting from $h\overrightarrow{\mathbf{e}}_\perp^{(0)} = \overrightarrow{v}_l$ given by the infinitesimal generator of l-translations in terms of arclength l along the curve.

A vertical hierarchy of commuting vector fields $v\overleftarrow{\mathbf{e}}_\perp^{(k)}$ starting from $v\overleftarrow{\mathbf{e}}_\perp^{(0)} = \overleftarrow{v}_l$ is generated by the recursion v-operator

$$\begin{aligned} v\mathfrak{R} &= \mathbf{D}_{v\mathbf{X}}\left(\mathbf{D}_{v\mathbf{X}} + \mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v}\cdot\right)\overleftarrow{v}\right) + \overleftarrow{v}\rfloor\mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v}\wedge\mathbf{D}_{v\mathbf{X}}\right) \\ &= \mathbf{D}_{v\mathbf{X}}^2 + |\mathbf{D}_{v\mathbf{X}}|^2 + \mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v}\cdot\right)\overleftarrow{v}_l - \overleftarrow{v}\rfloor\mathbf{D}_{v\mathbf{X}}^{-1}(\overleftarrow{v}_l\wedge). \end{aligned} \tag{47}$$

There are related hierarchies, generated by adjoint operators $\mathfrak{R}^* = (h\mathfrak{R}^*, v\mathfrak{R}^*)$, of involutive variational h-covector fields $\overrightarrow{\varpi}^{(k)} = \delta\left(hH^{(k)}\right)/\delta\overrightarrow{v}$ in terms of Hamiltonians $hH = hH^{(k)}(\overrightarrow{v}, \overrightarrow{v}_l, \overrightarrow{v}_{2l}, \ldots)$ starting from $\overrightarrow{\varpi}^{(0)} = \overrightarrow{v}$, $hH^{(0)} = \frac{1}{2}|\overrightarrow{v}|^2$ and of involutive variational v-covector fields $\overleftarrow{\varpi}^{(k)} = \delta\left(vH^{(k)}\right)/\delta\overleftarrow{v}$ in terms of Hamiltonians $vH = vH^{(k)}(\overleftarrow{v}, \overleftarrow{v}_l, \overleftarrow{v}_{2l}, \ldots)$ starting from $\overleftarrow{\varpi}^{(0)} = \overleftarrow{v}$, $vH^{(0)} = \frac{1}{2}|\overleftarrow{v}|^2$. The relations between hierarchies are established correspondingly by formulas

$$h\overrightarrow{\mathbf{e}}_\perp^{(k)} = h\mathcal{H}\left(\overrightarrow{\varpi}^{(k)}, \overrightarrow{\varpi}^{(k+1)}\right) = h\mathcal{J}\left(h\overrightarrow{\mathbf{e}}_\perp^{(k)}\right)$$

and

$$v\overleftarrow{\mathbf{e}}_\perp^{(k)} = v\mathcal{H}\left(\overleftarrow{\varpi}^{(k)}, \overleftarrow{\varpi}^{(k+1)}\right) = v\mathcal{J}\left(v\overleftarrow{\mathbf{e}}_\perp^{(k)}\right),$$

where $k = 0, 1, 2, \ldots$. All hierarchies (horizontal, vertical and their adjoint ones) have a typical mKdV scaling symmetry, for instance, $l \to \lambda l$ and $\overrightarrow{v} \to \lambda^{-1}\overrightarrow{v}$ under which the values $h\overrightarrow{\mathbf{e}}_\perp^{(k)}$ and $hH^{(k)}$ have scaling weight $2+2k$, while $\overrightarrow{\varpi}^{(k)}$ has scaling weight $1+2k$.

The above presented considerations prove

Corollary 4.1. *The recursion d-operator (45) gives rise to N-adapted hierarchies of distinguished horizontal and vertical commuting bi-Hamiltonian flows, respectively, on $\overrightarrow{v}$ and $\overleftarrow{v}$ that are given by $O(n-1)$- and $O(m-1)$-invariant d-vector evolution equations,*

$$\begin{aligned} \overrightarrow{v}_\tau &= h\overrightarrow{\mathbf{e}}_\perp^{(k+1)} - \overrightarrow{R}\, h\overrightarrow{\mathbf{e}}_\perp^{(k)} = h\mathcal{H}\left(\delta\left(hH^{(k,\overrightarrow{R})}\right)/\delta\overrightarrow{v}\right) \\ &= (h\mathcal{J})^{-1}\left(\delta\left(hH^{(k+1,\overrightarrow{R})}\right)/\delta\overrightarrow{v}\right) \end{aligned}$$

with horizontal Hamiltonians $hH^{(k+1,\overrightarrow{R})} = hH^{(k+1,\overrightarrow{R})} - \overrightarrow{R}\, hH^{(k,\overrightarrow{R})}$ *and*

$$\begin{aligned}\overleftarrow{v}_{\tau} &= v\,\overleftarrow{\mathbf{e}}_{\perp}^{(k+1)} - \overleftarrow{S}\, v\,\overleftarrow{\mathbf{e}}_{\perp}^{(k)} = v\mathcal{H}\left(\delta\left(vH^{(k,\overleftarrow{S})}\right)/\delta\overleftarrow{v}\right)\\ &= (v\mathcal{J})^{-1}\left(\delta\left(vH^{(k+1,\overleftarrow{S})}\right)/\delta\overleftarrow{v}\right)\end{aligned}$$

with vertical Hamiltonians $vH^{(k+1,\overleftarrow{S})} = vH^{(k+1,\overleftarrow{S})} - \overleftarrow{S}\, vH^{(k,\overleftarrow{S})}$, *for* $k = 0, 1, 2, \ldots$. *The d-operators* $\mathcal{H}$ *and* $\mathcal{J}$ *are N-adapted and mutually compatible, from which one can construct an alternative (explicit) Hamiltonian d-operator* $\mathcal{Q} = \mathcal{H}\circ\mathcal{J}\circ\mathcal{H} = \mathfrak{R}\circ\mathcal{H}$.

4.2.1. Formulation of the main theorem

The main goal of this paper is to prove that for any regular Lagrange system we can define naturally an N-adapted bi-Hamiltonian hierarchy of flows inducing anholonomic solitonic configurations.

Theorem 4.2. *For any anholonomic vector bundle with prescribed d-metric structure, there is a hierarchy of bi-Hamiltonian N-adapted flows of curves* $\gamma(\tau, l) = [h\gamma(\tau, l), v\gamma(\tau, l)]$ *described by geometric nonholonomic map equations.* 0 *flows are defined as convective (traveling wave) maps*

$$(h\gamma)_{\tau} = (h\gamma)_{h\mathbf{X}} \quad \text{and} \quad (v\gamma)_{\tau} = (v\gamma)_{v\mathbf{X}}. \tag{48}$$

The $+1$ *flows are defined by non-stretching mKdV maps*

$$\begin{aligned} -(h\gamma)_{\tau} &= \mathbf{D}^2_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}} + \frac{3}{2}\left|\mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}\right|^2_{h\mathbf{g}}(h\gamma)_{h\mathbf{X}},\\ -(v\gamma)_{\tau} &= \mathbf{D}^2_{v\mathbf{X}}(v\gamma)_{v\mathbf{X}} + \frac{3}{2}\left|\mathbf{D}_{v\mathbf{X}}(v\gamma)_{v\mathbf{X}}\right|^2_{v\mathbf{g}}(v\gamma)_{v\mathbf{X}},\end{aligned} \tag{49}$$

and the $+2, \ldots$ *flows as higher order analogs. Finally, there are* -1 *flows are defined by the kernels of recursion operators* (46) *and* (47) *inducing non-stretching wave maps*

$$\mathbf{D}_{h\mathbf{Y}}(h\gamma)_{h\mathbf{X}} = 0 \quad \text{and} \quad \mathbf{D}_{v\mathbf{Y}}(v\gamma)_{v\mathbf{X}} = 0. \tag{50}$$

We carry out the proof in the next Section 4.2.2.

Remark 4.1. Counterparts of N-adapted bi-Hamiltonian hierarchies and related solitonic equations in Theorem 4.2 can be constructed for $SU(n) \oplus SU(m)/SO(n) \oplus SO(m)$ as done in Ref. [14] for Riemannian symmetric spaces $SU(n)/SO(n)$. Such results may be very relevant in modern quantum/(non)commutative gravity.

Indeed, constructions similar to Theorem 4.2 can be carried out in gravity models with nontrivial torsion and nonholonomic structure and related geometry of noncommutative/ superspaces and anholonomic spinors. Finally, it should be emphasized that a number of exact solutions in gravity [25,26] can be nonholonomically deformed in order to generate nonholonomic hierarchies of gravitational solitons of type (48) and (49) or (50), which will be considered in further publications.

4.2.2. Proof of the main theorem

We provide a proof of Theorem 4.2 for the horizontal flows. The approach is based on the method provided in Refs. [14,17], but in this proof the Levi-Civita connection on symmetric Riemannian spaces is replaced by the horizontal components of the canonical d-connection in a generalized Lagrange space with constant d-curvature coefficients. The vertical constructions are similar.

One obtains a vector mKdV equation up to a convective term (which can be absorbed by redefinition of coordinates) defining the $+1$ flow for $h\overrightarrow{\mathbf{e}}_{\perp} = \overrightarrow{v}_{l}$,

$$\overrightarrow{v}_{\tau} = \overrightarrow{v}_{3l} + \frac{3}{2}|\overrightarrow{v}|^2 - \overrightarrow{R}\,\overrightarrow{v}_{l},$$

when the $+(k+1)$ flow gives a vector mKdV equation of higher order $3+2k$ on $\overrightarrow{v}$ and there is a 0 h-flow $\overrightarrow{v}_{\tau} = \overrightarrow{v}_{l}$ arising from $h\overrightarrow{\mathbf{e}}_{\perp} = 0$ and $h\overrightarrow{\mathbf{e}}_{\parallel} = 1$ belonging outside the hierarchy generated by $h\mathfrak{R}$. Such flows correspond to N-adapted horizontal motions of the curve $\gamma(\tau, l) = [h\gamma(\tau, l), v\gamma(\tau, l)]$ given by

$$(h\gamma)_{\tau} = f\left((h\gamma)_{h\mathbf{X}}, \mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}, \mathbf{D}^2_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}, \ldots\right)$$

subject to the non-stretching condition $|\,(h\gamma)_{h\mathbf{X}}\,|_{h\mathbf{g}} = 1$, where the equation of motion is derived from the identifications

$$(h\gamma)_{\tau} \longleftrightarrow \mathbf{e}_{h\mathbf{Y}}, \qquad \mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}} \longleftrightarrow \mathcal{D}_{h\mathbf{X}}\mathbf{e}_{h\mathbf{X}} = [\mathbf{L}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{X}}]$$

and so on, which maps the constructions from the tangent space of the curve to the space $h\mathfrak{p}$. For such identifications we have

$$[\mathbf{L}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{X}}] = -\begin{bmatrix} 0 & \left(0, \overrightarrow{v}\right) \\ -\left(0, \overrightarrow{v}\right)^{\mathrm{T}} & h\mathbf{0} \end{bmatrix} \in h\mathfrak{p},$$

$$[\mathbf{L}_{h\mathbf{X}}, [\mathbf{L}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{X}}]] = -\begin{bmatrix} 0 & \left(|\overrightarrow{v}|^2, \overrightarrow{0}\right) \\ -\left(|\overrightarrow{v}|^2, \overrightarrow{0}\right)^{\mathrm{T}} & h\mathbf{0} \end{bmatrix}$$

and so on, see similar calculus in (32). At the next step, noting for the $+1$ h-flow that

$$h\overrightarrow{\mathbf{e}}_{\perp} = \overrightarrow{v}_l \quad \text{and} \quad h\overrightarrow{\mathbf{e}}_{\parallel} = -\mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v}\cdot\overrightarrow{v}_l\right) = -\frac{1}{2}|\overrightarrow{v}|^2,$$

we compute

$$\begin{aligned}
\mathbf{e}_{h\mathbf{Y}} &= \begin{bmatrix} 0 & \left(h\mathbf{e}_{\parallel}, h\overrightarrow{\mathbf{e}}_{\perp}\right) \\ -\left(h\mathbf{e}_{\parallel}, h\overrightarrow{\mathbf{e}}_{\perp}\right)^{\mathrm{T}} & h\mathbf{0} \end{bmatrix} \\
&= -\frac{1}{2}|\overrightarrow{v}|^2 \begin{bmatrix} 0 & \left(1, \overrightarrow{\mathbf{0}}\right) \\ -\left(0, \overrightarrow{\mathbf{0}}\right)^{\mathrm{T}} & h\mathbf{0} \end{bmatrix} + \begin{bmatrix} 0 & \left(0, \overrightarrow{v}_{h\mathbf{X}}\right) \\ -\left(0, \overrightarrow{v}_{h\mathbf{X}}\right)^{\mathrm{T}} & h\mathbf{0} \end{bmatrix} \\
&= \mathbf{D}_{h\mathbf{X}}\,[\mathbf{L}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{X}}] + \frac{1}{2}\,[\mathbf{L}_{h\mathbf{X}}, [\mathbf{L}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{X}}]] \\
&= -\mathcal{D}_{h\mathbf{X}}\,[\mathbf{L}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{X}}] - \frac{3}{2}|\overrightarrow{v}|^2\mathbf{e}_{h\mathbf{X}}.
\end{aligned}$$

From the above presented identifications related to the first and second terms, it follows that

$$\begin{aligned}
|\overrightarrow{v}|^2 &= \langle[\mathbf{L}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{X}}], [\mathbf{L}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{X}}]\rangle_{h\mathfrak{p}} \longleftrightarrow h\mathbf{g}\left(\mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}, \mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}\right) \\
&= \left|\mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}\right|^2_{h\mathbf{g}},
\end{aligned}$$

and we can identify $\mathcal{D}_{h\mathbf{X}}\,[\mathbf{L}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{X}}]$ with $\mathbf{D}^2_{h\mathbf{X}}\,(h\gamma)_{h\mathbf{X}}$ and write

$$-\mathbf{e}_{h\mathbf{Y}} \longleftrightarrow \mathbf{D}^2_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}} + \frac{3}{2}\left|\mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}\right|^2_{h\mathbf{g}}(h\gamma)_{h\mathbf{X}}$$

which is just the first Eq. (49) in the Theorem 4.2 defining a non-stretching mKdV map h-equation induced by the h-part of the canonical d-connection.

Using the adjoint representation $ad\,(\cdot)$ acting in the Lie algebra $h\mathfrak{g} = h\mathfrak{p} \oplus \mathfrak{so}(n)$, with

$$ad\,([\mathbf{L}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{X}}])\,\mathbf{e}_{h\mathbf{X}} = \begin{bmatrix} 0 & \left(0, \overrightarrow{\mathbf{0}}\right) \\ -\left(0, \overrightarrow{\mathbf{0}}\right)^{\mathrm{T}} & \overrightarrow{\mathbf{v}} \end{bmatrix} \in \mathfrak{so}(n+1),$$

where

$$\overrightarrow{\mathbf{v}} = -\begin{bmatrix} 0 & \overrightarrow{v} \\ -\overrightarrow{v}^{\mathrm{T}} & h\mathbf{0} \end{bmatrix} \in \mathfrak{so}(n),$$

we note (applying $ad\,([\mathbf{L}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{X}}])$ again)

$$ad\,([\mathbf{L}_{h\mathbf{X}}, \mathbf{e}_{h\mathbf{X}}])^2\,\mathbf{e}_{h\mathbf{X}} = -|\overrightarrow{v}|^2 \begin{bmatrix} 0 & \left(1, \overrightarrow{\mathbf{0}}\right) \\ -\left(1, \overrightarrow{\mathbf{0}}\right)^{\mathrm{T}} & \mathbf{0} \end{bmatrix} = -|\overrightarrow{v}|^2\mathbf{e}_{h\mathbf{X}}.$$

Hence Eq. (49) can be represented in alternative form

$$-(h\gamma)_{\tau} = \mathbf{D}^2_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}} - \frac{3}{2}\overrightarrow{R}^{-1} ad\left(\mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}\right)^2 (h\gamma)_{h\mathbf{X}},$$

which is more convenient for analysis of higher order flows on $\overrightarrow{v}$ subjected to higher-order geometric partial differential equations. Here we note that the 0 flow on $\overrightarrow{v}$ corresponds to just a convective (linear traveling h-wave but subjected to certain nonholonomic constraints) map equation (48).

Now we consider a -1 flow contained in the h-hierarchy derived from the property that $h\overrightarrow{\mathbf{e}}_{\perp}$ is annihilated by the h-operator $h\mathcal{J}$ and mapped into $h\mathfrak{R}(h\overrightarrow{\mathbf{e}}_{\perp}) = 0$. This mean that $h\mathcal{J}(h\overrightarrow{\mathbf{e}}_{\perp}) = \overrightarrow{\varpi} = 0$. Such properties together with (31) and

Eqs. (40) imply $\mathbf{L}_\tau = 0$ and hence $h\mathcal{D}_\tau \mathbf{e}_{h\mathbf{X}} = [\mathbf{L}_\tau, \mathbf{e}_{h\mathbf{X}}] = 0$ for $h\mathcal{D}_\tau = h\mathbf{D}_\tau + [\mathbf{L}_\tau, \cdot]$. We obtain the equation of motion for the h-component of curve, $h\gamma(\tau, l)$, following the correspondences $\mathbf{D}_{h\mathbf{Y}} \longleftrightarrow h\mathcal{D}_\tau$ and $h\gamma_l \longleftrightarrow \mathbf{e}_{h\mathbf{X}}$,

$$\mathbf{D}_{h\mathbf{Y}}\,(h\gamma(\tau, l)) = 0,$$

which is just the first equation in (50).

Finally, we note that the formulas for the v-components, stated by Theorem 4.2 can be derived in a similar form by respective substitution in the above proof of the h-operators and h-variables into v-ones, for instance, $h\gamma \to v\gamma$, $h\overrightarrow{\mathbf{e}}_\perp \to v\overleftarrow{\mathbf{e}}_\perp$, $\overrightarrow{v} \to \overleftarrow{v}$, $\overrightarrow{\varpi} \to \overleftarrow{\varpi}$, $\mathbf{D}_{h\mathbf{X}} \to \mathbf{D}_{v\mathbf{X}}$, $\mathbf{D}_{h\mathbf{Y}} \to \mathbf{D}_{v\mathbf{Y}}$, $\mathbf{L} \to \mathbf{C}$, $\overrightarrow{R} \to \overleftarrow{S}$, $h\mathcal{D} \to v\mathcal{D}$, $h\mathfrak{R} \to v\mathfrak{R}$, $h\mathcal{J} \to v\mathcal{J}$.

4.3. Nonholonomic mKdV and SG hierarchies

We present explicit constructions when solitonic hierarchies are derived following the conditions of Theorem 4.2.

The h-flow and v-flow equations resulting from (50) are

$$\overrightarrow{v}_\tau = -\overrightarrow{R}\, h\overrightarrow{\mathbf{e}}_\perp \quad \text{and} \quad \overleftarrow{v}_\tau = -\overleftarrow{S}\, v\overleftarrow{\mathbf{e}}_\perp, \tag{51}$$

when, respectively,

$$0 = \overrightarrow{\varpi} = -\mathbf{D}_{h\mathbf{X}} h\overrightarrow{\mathbf{e}}_\perp + h\mathbf{e}_\parallel \overrightarrow{v}, \qquad \mathbf{D}_{h\mathbf{X}} h\mathbf{e}_\parallel = h\overrightarrow{\mathbf{e}}_\perp \cdot \overrightarrow{v}$$

and

$$0 = \overleftarrow{\varpi} = -\mathbf{D}_{v\mathbf{X}} v\overleftarrow{\mathbf{e}}_\perp + v\mathbf{e}_\parallel \overleftarrow{v}, \qquad \mathbf{D}_{v\mathbf{X}} v\mathbf{e}_\parallel = v\overleftarrow{\mathbf{e}}_\perp \cdot \overleftarrow{v}.$$

The d-flow equations possess horizontal and vertical conservation laws

$$\mathbf{D}_{h\mathbf{X}}\left((h\mathbf{e}_\parallel)^2 + |h\overrightarrow{\mathbf{e}}_\perp|^2\right) = 0,$$

for $(h\mathbf{e}_\parallel)^2 + |h\overrightarrow{\mathbf{e}}_\perp|^2 = \langle h\mathbf{e}_\tau, h\mathbf{e}_\tau\rangle_{h\mathfrak{p}} = |\,(h\gamma)_\tau\,|^2_{h\mathbf{g}}$, and

$$\mathbf{D}_{v\mathbf{Y}}\left((v\mathbf{e}_\parallel)^2 + |v\overleftarrow{\mathbf{e}}_\perp|^2\right) = 0,$$

for $(v\mathbf{e}_\parallel)^2 + |v\overleftarrow{\mathbf{e}}_\perp|^2 = \langle v\mathbf{e}_\tau, v\mathbf{e}_\tau\rangle_{v\mathfrak{p}} = |\,(v\gamma)_\tau\,|^2_{v\mathbf{g}}$. This corresponds to

$$\mathbf{D}_{h\mathbf{X}}|\,(h\gamma)_\tau\,|^2_{h\mathbf{g}} = 0 \quad \text{and} \quad \mathbf{D}_{v\mathbf{X}}|\,(v\gamma)_\tau\,|^2_{v\mathbf{g}} = 0.$$

(The problem of formulating conservation laws on N-anholonomic spaces – in particular, on nonholonomic vector bundles – is analyzed in Ref. [26]. In general, such laws are more sophisticated than those on (semi) Riemannian spaces because of nonholonomic constraints resulting in non-symmetric Ricci tensors and different types of identities. But for the geometries modeled for dimensions $n = m$ with canonical d-connections, we get similar h- and v-components of the conservation law equations as on symmetric Riemannian spaces.)

Without loss of generality we can rescale conformally the variable τ in order to get $|\,(h\gamma)_\tau\,|^2_{h\mathbf{g}} = 1$ and (perhaps by a different rescaling) $|\,(v\gamma)_\tau\,|^2_{v\mathbf{g}} = 1$, so that

$$(h\mathbf{e}_\parallel)^2 + |h\overrightarrow{\mathbf{e}}_\perp|^2 = 1 \quad \text{and} \quad (v\mathbf{e}_\parallel)^2 + |v\overleftarrow{\mathbf{e}}_\perp|^2 = 1.$$

In this case, we can express $h\mathbf{e}_\parallel$ and $h\overrightarrow{\mathbf{e}}_\perp$ in terms of $\overrightarrow{v}$ and its derivatives and, similarly, we can express $v\mathbf{e}_\parallel$ and $v\overleftarrow{\mathbf{e}}_\perp$ in terms of $\overleftarrow{v}$ and its derivatives, which follows from (51). The N-adapted wave map equations describing the -1 flows reduce to a system of two independent nonlocal evolution equations for the h- and v-components,

$$\overrightarrow{v}_\tau = -\mathbf{D}^{-1}_{h\mathbf{X}}\left(\sqrt{\overrightarrow{R}^2 - |\overrightarrow{v}_\tau|^2}\,\overrightarrow{v}\right) \quad \text{and} \quad \overleftarrow{v}_\tau = -\mathbf{D}^{-1}_{v\mathbf{X}}\left(\sqrt{\overleftarrow{S}^2 - |\overleftarrow{v}_\tau|^2}\,\overleftarrow{v}\right).$$

For N-anholonomic spaces of constant scalar d-curvatures, we can rescale the equations on τ to the case when the terms $\overrightarrow{R}^2, \overleftarrow{S}^2 = 1$, and the evolution equations transform into a system of hyperbolic d-vector equations,

$$\mathbf{D}_{h\mathbf{X}}(\overrightarrow{v}_\tau) = -\sqrt{1 - |\overrightarrow{v}_\tau|^2}\,\overrightarrow{v} \quad \text{and} \quad \mathbf{D}_{v\mathbf{X}}(\overleftarrow{v}_\tau) = -\sqrt{1 - |\overleftarrow{v}_\tau|^2}\,\overleftarrow{v}, \tag{52}$$

where $\mathbf{D}_{h\mathbf{X}} = \partial_{hl}$ and $\mathbf{D}_{v\mathbf{X}} = \partial_{vl}$ are usual partial derivatives, respectively, along hl and vl with $\overrightarrow{v}_\tau$ and $\overleftarrow{v}_\tau$ considered as scalar functions for the covariant derivatives $\mathbf{D}_{h\mathbf{X}}$ and $\mathbf{D}_{v\mathbf{X}}$ defined by the canonical d-connection. It also follows that $h\overrightarrow{\mathbf{e}}_\perp$ and $v\overleftarrow{\mathbf{e}}_\perp$ obey corresponding vector sine-Gordon (SG) equations

$$\left(\sqrt{(1 - |h\overrightarrow{\mathbf{e}}_\perp|^2)^{-1}}\,\partial_{h\mathbf{l}}(h\overrightarrow{\mathbf{e}}_\perp)\right)_\tau = -h\overrightarrow{\mathbf{e}}_\perp \tag{53}$$

and

$$\left(\sqrt{(1 - |v\overleftarrow{\mathbf{e}}_\perp|^2)^{-1}}\,\partial_{v\mathbf{l}}(v\overleftarrow{\mathbf{e}}_\perp)\right)_\tau = -v\overleftarrow{\mathbf{e}}_\perp. \tag{54}$$

The above presented formulas and Corollary 4.1 imply

Theorem 4.3. *The recursion d-operator $\mathfrak{R} = (h\mathfrak{R}, h\mathfrak{R})$ (45) generates two hierarchies of vector mKdV symmetries: the first one is horizontal (see (46)),*

$$\begin{aligned}
&\overrightarrow{v}^{(0)}_{\tau} = \overrightarrow{v}_{hl}, \qquad \overrightarrow{v}^{(1)}_{\tau} = h\mathfrak{R}(\overrightarrow{v}_{hl}) = \overrightarrow{v}_{3hl} + \frac{3}{2}|\overrightarrow{v}|^2\overrightarrow{v}_{hl},\\
&\overrightarrow{v}^{(2)}_{\tau} = h\mathfrak{R}^2(\overrightarrow{v}_{hl}) = \overrightarrow{v}_{5hl} + \frac{5}{2}\left(|\overrightarrow{v}|^2\overrightarrow{v}_{2hl}\right)_{hl} + \frac{5}{2}\left((|\overrightarrow{v}|^2)_{hlhl} + |\overrightarrow{v}_{h\mathbf{l}}|^2 + \frac{3}{4}|\overrightarrow{v}|^4\right)\overrightarrow{v}_{hl} - \frac{1}{2}|\overrightarrow{v}_{hl}|^2\overrightarrow{v},
\end{aligned} \tag{55}$$

and so on,

with all such flows commuting with the -1 flow

$$(\overrightarrow{v}_{\tau})^{-1} = h\overrightarrow{\mathbf{e}}_{\perp} \tag{56}$$

associated to the vector SG equation (53); the second one is vertical (see (47)),

$$\begin{aligned}
&\overleftarrow{v}^{(0)}_{\tau} = \overleftarrow{v}_{v\mathbf{l}}, \qquad \overleftarrow{v}^{(1)}_{\tau} = v\mathfrak{R}(\overleftarrow{v}_{vl}) = \overleftarrow{v}_{3vl} + \frac{3}{2}|\overleftarrow{v}|^2\overleftarrow{v}_{vl},\\
&\overleftarrow{v}^{(2)}_{\tau} = v\mathfrak{R}^2(\overleftarrow{v}_{vl}) = \overleftarrow{v}_{5vl} + \frac{5}{2}\left(|\overleftarrow{v}|^2\overleftarrow{v}_{2vl}\right)_{vl} + \frac{5}{2}\left((|\overleftarrow{v}|^2)_{vl\,vl} + |\overleftarrow{v}_{vl}|^2 + \frac{3}{4}|\overleftarrow{v}|^4\right)\overleftarrow{v}_{v\mathbf{l}} - \frac{1}{2}|\overleftarrow{v}_{vl}|^2\overleftarrow{v},
\end{aligned} \tag{57}$$

and so on,

with all such flows commuting with the -1 flow

$$(\overleftarrow{v}_{\tau})^{-1} = v\overleftarrow{\mathbf{e}}_{\perp} \tag{58}$$

associated to the vector SG equation (54).

Furthermore, the adjoint d-operator $\mathfrak{R}^* = \mathcal{J} \circ \mathcal{H}$ generates a horizontal hierarchy of Hamiltonians,

$$\begin{aligned}
&hH^{(0)} = \frac{1}{2}|\overrightarrow{v}|^2, \qquad hH^{(1)} = -\frac{1}{2}|\overrightarrow{v}_{hl}|^2 + \frac{1}{8}|\overrightarrow{v}|^4,\\
&hH^{(2)} = \frac{1}{2}|\overrightarrow{v}_{2hl}|^2 - \frac{3}{4}|\overrightarrow{v}|^2|\overrightarrow{v}_{hl}|^2 - \frac{1}{2}\left(\overrightarrow{v}\cdot\overrightarrow{v}_{hl}\right) + \frac{1}{16}|\overrightarrow{v}|^6,
\end{aligned} \tag{59}$$

and so on,

and vertical hierarchy of Hamiltonians

$$\begin{aligned}
&vH^{(0)} = \frac{1}{2}|\overleftarrow{v}|^2, \qquad vH^{(1)} = -\frac{1}{2}|\overleftarrow{v}_{vl}|^2 + \frac{1}{8}|\overleftarrow{v}|^4,\\
&vH^{(2)} = \frac{1}{2}|\overleftarrow{v}_{2vl}|^2 - \frac{3}{4}|\overleftarrow{v}|^2|\overleftarrow{v}_{vl}|^2 - \frac{1}{2}\left(\overleftarrow{v}\cdot\overleftarrow{v}_{vl}\right) + \frac{1}{16}|\overleftarrow{v}|^6,
\end{aligned} \tag{60}$$

and so on,

all of which are conserved densities for respective horizontal and vertical -1 flows and determining higher conservation laws for the corresponding hyperbolic equations (53) and (54).

The above presented horizontal equations (53), (55), (56) and (59) and of vertical equations (54), (57), (58) and (60) have similar mKdV scaling symmetries but with distinct parameters λ_h and λ_v because, in general, there are two independent scalar curvatures $\overrightarrow{R}$ and $\overleftarrow{S}$, see (74). The horizontal scaling symmetries are $hl \to \lambda_h hl$, $\overrightarrow{v} \to (\lambda_h)^{-1}\overrightarrow{v}$ and $\tau \to (\lambda_h)^{1+2k}$, for $k = -1, 0, 1, 2, \ldots$., and likewise for the vertical scaling symmetries.

Finally, we consider again Remark 4.1 stating that similar results (proved in Section 4) can be derived for unitary groups with complex variables. The generated bi-Hamiltonian horizontal and vertical hierarchies and solitonic equations will be different from those defined for real orthogonal groups; for holonomic spaces this is demonstrated in Ref. [14]. This distinguishes substantially the models of gauge gravity with structure groups like the unitary one from those with orthogonal groups.

5. Applications in geometric mechanics, Finsler geometry and gravity

Here we consider some interesting examples when the data defining fundamental geometric structures in mechanics and Finsler geometry, or exact solutions in gravity, can be transformed into solitonic hierarchies. The possibility to describe gravitational and electromagnetic interactions by mechanical models, and vice versa, will be investigated.

5.1. Geometric mechanics and Finsler geometry

A differentiable Lagrangian $L(x, y)$, i.e. a fundamental Lagrange function, is defined by a map $L : (x, y) \in TM \to L(x, y) \in \mathbb{R}$ of class $\mathcal{C}^\infty$ on $\widetilde{TM} = TM \setminus \{0\}$ and continuous on the null section $0 : M \to TM$ of π. A regular Lagrangian has nondegenerate Hessian

$$ {}^{(L)}g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 L(x, y)}{\partial y^i \partial y^j} \quad (61) $$

when rank $|g_{ij}| = n$ on $\widetilde{TM}$.

Definition 5.1. A Lagrange space is a pair $L^n = [M, L(x, y)]$ with ${}^{(L)}g_{ij}$ being of fixed signature over $\mathbf{V} = \widetilde{TM}$.

The notion of Lagrange space was introduced by Kern [1] and elaborated in details by R. Miron's school, see Refs. [2,3,37], as a natural extension of Finsler geometry [38,23,39,23] (see also Refs. [26,40], on Lagrange–Finsler super/noncommutative geometry). Straightforward calculations (on nonholonomic manifolds they are reviewed in Refs. [26]) establish the following results:

1. The Euler–Lagrange equations
$$ \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0 $$
where $y^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau}$ for $x^i(\tau)$ depending on parameter τ, are equivalent to the "nonlinear" geodesic equations
$$ \frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} + 2G^i\left(x^k, \frac{\mathrm{d}x^j}{\mathrm{d}\tau}\right) = 0 $$
defining paths of a canonical semispray
$$ S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i} $$
where
$$ 2G^i(x, y) = \frac{1}{2}{}^{(L)}g^{ij}\left(\frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i}\right) $$
with ${}^{(L)}g^{ij}$ being inverse to (61).
2. There exists on $\mathbf{V} \simeq \widetilde{TM}$ a canonical N-connection
$$ {}^{(L)}N^i_j = \frac{\partial G^i(x, y)}{\partial y^i} \quad (62) $$
defined by the fundamental Lagrange function $L(x, y)$, which prescribes nonholonomic frame structures of type (4) and (5), ${}^{(L)}\mathbf{e}_\nu = (\mathbf{e}_i, e_a)$ and ${}^{(L)}\mathbf{e}^\mu = (e^i, \mathbf{e}^a)$.
3. There is a canonical metric structure
$$ {}^{(L)}\mathbf{g} = g_{ij}(x, y)e^i \otimes e^j + g_{ij}(x, y)\mathbf{e}^i \otimes \mathbf{e}^j \quad (63) $$
constructed as a Sasaki type lift from M for $g_{ij}(x, y)$.
4. There is a unique metrical and, in this case, torsionless canonical d-connection ${}^{(L)}\mathbf{D} = (hD, vD)$ with the nontrivial coefficients with respect to ${}^{(L)}\mathbf{e}_\nu$ and ${}^{(L)}\mathbf{e}^\mu$ parametrized respectively $\Gamma^\alpha_{\beta\gamma} = (L^i{}_{jk}, C^a{}_{bc})$, for
$$ \begin{aligned} L^i{}_{jk} &= \frac{1}{2}g^{ih}(\mathbf{e}_k g_{jh} + \mathbf{e}_j g_{kh} - \mathbf{e}_h g_{\ jk}), \\ C^i{}_{jk} &= \frac{1}{2}g^{ih}(e_k g_{jh} + e_b g_{kh} - e_e g_{\ bc}) \end{aligned} \quad (64) $$
defining the generalized Christoffel symbols, where (for simplicity, we omitted the left up labels (L) for N-adapted bases). The connection ${}^{(L)}\mathbf{D}$ is metric compatible and torsionless, see an outline of main definitions and formulas in the next subsection and Appendix.

We conclude that any regular Lagrange mechanics can be geometrized as a nonholonomic Riemann manifold $\mathbf{V}$ equipped with canonical N-connection (62) and adapted d-connection (64) and d-metric structures (63) all induced by a $L(x, y)$. In some approaches to Finsler geometry and generalizations [32], one consider nontrivial non-metric structures. For instance, the so-called Chern d-connection ${}^{Ch}\mathbf{D}$ is also a minimal extension of the Levi-Civita connection when $h^{Ch}\mathbf{D}(g) = 0$ but $v^{Ch}\mathbf{D}(h) \neq 0$, see formulas (16). Such generalized Riemann–Finsler spaces can be modeled on nonholonomic metric-affine manifolds, with nonmetricity, see detailed discussions in [26,2,3,24]. We note that the N-connection is induced by the semispray configurations subjected to generalized nonlinear geodesic equations equivalent to the Euler–Lagrange equations.

An N-connection structure transforms a Riemannian space into a nonholonomic one with preferred honholonomic frame structure of type (4) and (5).

Remark 5.1. Any Finsler geometry with a fundamental Finsler function $F(x, y)$, being homogeneous of type $F(x, \lambda y) = \lambda F(x, y)$, for nonzero $\lambda \in \mathbb{R}$, may be considered as a particular case of Lagrange geometry when $L = F^2$. We shall apply the methods of Finsler geometry in this work.[9]

For applications in optics of nonhomogeneous media and gravity (see, for instance, Refs. [3,26,41,25]) one considers metric forms of type $g_{ij} \sim e^{\lambda(x,y)\,(L)}g_{ij}(x, y)$ which cannot be derived explicitly from a mechanical Lagrangian. In the so-called generalized Lagrange geometry one considers Sasaki type metrics (63) with certain general coefficients both for the d-metric and N-connection and canonical d-connection, i.e. when ${}^{(L)}g_{ij} \to \left[g_{ij}(x, y), h_{ab}(x, y)\right]$, and ${}^{(L)}N^i_j \to N^i_j(x, y)$. We shall use the term (generalized) Lagrange–Finsler geometry for all such geometries modeled on tangent bundles or on arbitrary N-anholonomic manifold $\mathbf{V}$. In original form, such spaces were called generalized Lagrange spaces and denoted $GL^n = (M, g_{ij}(x, y))$, see [2,3].

5.2. Analogous models and Lagrange–Finsler geometry

For a regular Lagrangian $\mathsf{L}(x, y)$ and corresponding Euler–Lagrange equations it is possible to construct canonically a generalized Finsler geometry on tangent bundle [1–3]. In our approach, such constructions follow from the Theorem 2.1 if instead of $g_{\underline{\alpha}\underline{\beta}}(x)$ there are considered the metric components

$$g_{\underline{ij}}(x, y) = \frac{1}{2}\frac{\partial^2 \mathsf{L}(x, y)}{\partial y^{\underline{i}}\partial y^{\underline{j}}}, \tag{65}$$

introduced in (1) for $e^{\underline{\alpha}}_{\alpha}(x, y) = \delta^{\underline{\alpha}}_{\alpha}$. For arbitrary given $g_{\underline{ij}}(x, y)$ and $e^{\underline{\alpha}}_{\alpha}(x, y)$ on TM, defining a d-metric (17), we can compute the corresponding N-connection (8) and canonical d-connection coefficients (69). In general, the coefficients of d-curvatures (75) are not constant and we are not able to generate solitonic hierarchies. In order to solve the problem we have to choose $e^{\underline{\alpha}}_{\alpha}(x, y)$ so that the equations

$$\frac{\partial^2 \mathsf{L}(x, y)}{\partial y^{\underline{a}}\partial y^{\underline{b}}} = 2\mathring{g}_{ab}e^{a}_{\underline{a}}(x, y)e^{b}_{\underline{b}}(x, y)$$

are satisfied (similarly to (18), for given L and $\mathring{g}_{ab}$), resulting in constant-curvature coefficients with respect to N-adapted bases. This allows us to define $\mathbf{D}^{\mathsf{L}}_{h\mathbf{X}}$ and $\mathbf{D}^{\mathsf{L}}_{v\mathbf{X}}$ from formulas (48), (49) or (50), as pairs of h- and v-operators with "nonholonomic mixture" (cf. Theorem 4.2).

Corollary 5.1. *Any regular Lagrangian $\mathsf{L}(x, y)$ induces a hierarchy of bi-Hamiltonian N-adapted flows of curves described by geometric nonholonomic (solitonic) map equations.*

We can describe equivalently any regular Lagrange mechanics in terms of solitonic equations, their solutions and symmetries.

Remark 5.2. Considering $\mathsf{L}(x, y) = F^2(x, y)$, where the homogeneous $F(x, \lambda y) = \lambda F(x, y)$ is a fundamental Finsler metric function (see, for instance, [3]), we can encode in solitonic constructions all data for a Finsler geometry.

Here it should be noted that there were elaborated alternative approaches when gravitational effects are modeled in continuous and discrete media, see review [36], but they do not allow one to generate the Einstein equations starting from a Lagrangian in a mechanical models like in [1–3], or from a distribution on a nonholonomic manifold [30,31,24,26]. In our approach, for a d-metric (17) induced by (65) (introduced in (1)), we can prove a geometric mechanical analog of Theorem 2.3:

Theorem 5.1. *Any regular Lagrangian $\mathsf{L}(x, y)$ and frame structure $e^{\underline{\alpha}}_{\alpha}(x, y)$ define a nonholonomic (semi) Riemannian geometry on TM.*

This way we can model various gravitational effects by certain mechanical configurations.

Corollary 5.2. *A regular Lagrangian $\mathsf{L}(x, y)$ and frame structure $e^{\underline{\alpha}}_{\alpha}(x, y)$ on TM model a vacuum gravity configuration with effective metric $\tilde{g}_{\alpha\beta} = [\tilde{g}_{ij}, \tilde{g}_{ij}]$ (17) if the corresponding canonical d-connection (69) has vanishing Ricci d-tensor (73).*

We conclude that for a fixed Lagrangian $\mathsf{L}(x, y)$ we can define frame structures $e^{\underline{\alpha}}_{\alpha}(x, y)$ on TM inducing Einstein spaces or curved spaces with constant-curvature coefficients (computed with respect to certain N-adapted basis).

[9] In another direction, there is a proof [32] that any Lagrange fundamental function L can be modeled as a singular case in a certain class of Finsler geometries of extra dimension. Nevertheless the concept of Lagrangian is a very important geometrical and physical one and we shall distinguish the cases when we model a Lagrange or a Finsler geometry: A physical or mechanical model with a Lagrangian is not only a "singular" case for a Finsler geometry but reflects a proper set of geometric objects and structures with possible new concepts in physical theories.

Definition 5.2. A geometric model defined by data $\mathbf{E}^{n+m} = [g_{ij}(x^k, y^c), g_{ab}(x^k, y^c), N_i^a(x^k, y^c)]$ is N-anholonomically equivalent to another one given by data $\widetilde{\mathbf{E}}^{n+m} = [\widetilde{g}_{ij}(x^k, y^c), \widetilde{g}_{ab}(x^k, y^c), \widetilde{N}_i^a(x^k, y^c)]$ if for the same splitting $n+m$ with $\widetilde{\mathbf{e}}_\alpha \to \mathbf{e}_\alpha$ there are nontrivial polarizations $[\eta_{ij}(x^k, y^c), \eta_{ab}(x^k, y^c), \eta_i^a(x^k, y^c)]$ for which $g_{\alpha\beta} = \eta_{\alpha\beta}\widetilde{g}_{\alpha\beta}$ and $N_i^a = \eta_i^a \widetilde{N}_i^a$.

In general, the physical and geometric properties of two such N-anholonomically related spaces $\mathbf{E}^{n+m}$ and $\widetilde{\mathbf{E}}^{n+m}$ are very different. Nevertheless, we are able to compute the coefficients of geometrical and physical objects, define symmetries, conservation laws and fundamental equations on a space from similar values of another one if the polarizations are stated in explicit form. In some particular cases, we can consider that $\mathbf{E}^{n+m}$ is defined by the data with constant-curvature coefficients with respect to an N-adapted basis (or an exact solution of the Einstein equations) but $\widetilde{\mathbf{E}}^{n+m}$ is related to a regular Lagrange/Finsler geometry.

5.3. Modeling field interactions in Lagrange–Finsler geometry

Let us consider a regular Lagrangian

$$\mathsf{L}(x, y) = m_0 \underline{a}_{ij}(x) y^i y^j + e_0 \underline{A}_i(x) y^i, \tag{66}$$

where m_0, e_0 are constants, $\underline{A}_i(x)$ is a vector field and $\underline{a}_{ij}(x) = a_{\underline{ij}}(x)$ is a second rank symmetric tensor. The metric coefficients (65) are $g_{\underline{ij}} = m_0 a_{\underline{ij}}(x)$ and their frame transforms (1) are given by

$$\widetilde{g}_{ab}(x, y) = e_a^{\ \underline{a}}(x, y) e_b^{\ \underline{b}}(x, y) g_{\underline{ab}}(x) = m_0 a_{ab}(x, y), \tag{67}$$

where $a_{ab} = e_a^{\ \underline{a}} e_b^{\ \underline{b}} a_{\underline{ab}}$. For simplicity, we consider frame transforms to local bases $e_\alpha = (e_i, e_a)$ when $e_a = e_a^{\ \underline{a}} \partial/\partial y^{\underline{a}}$ are holonomic vectors but $e_i = e_i^{\ \underline{i}} \partial/\partial x^{\underline{i}}$ will be defined by an N-connection like in (4). We compute (see Theorem 2.1 and formula (8))

$$\widetilde{G}^i = \frac{1}{2} {}^g\Gamma^i_{jk} y^j y^k + \frac{1}{m_0} a^{ij}\, {}^gF_{jk} y^k,$$

where ${}^g\Gamma^i_{jk}$ are Christoffel symbols of the tensor $\widetilde{g}_{ab}$ constructed by using derivatives $\partial/\partial x^i$, ${}^gF_{jk} = \frac{e_0}{4}({}^gD_k A_j - {}^gD_j A_k)$ with the covariant derivative ${}^gD_j$ defined by ${}^g\Gamma^i_{jk}$, and

$$\tilde{N}^i_{\ j}(x, y) = {}^g\Gamma^i_{jk} y^k - {}^gF_{jk}. \tag{68}$$

The N-connection curvature (6) is

$$\Omega^a_{ik} = y^b\, {}^gR^a_{bjk} - 2\, {}^gD_{[k}\, {}^gF^a_{i]},$$

where ${}^gR^a_{bjk}$ is the curvature tensor of the Levi-Civita connection for $g_{\underline{ab}}(x)$, computed with respect to a usual coordinate base and then nonholonomically transformed by $e_a^{\ \underline{a}}(x, y)$.

For $m_0 = mc$ and $e_0 = 2e/m$, where m and e are respectively the mass and electric charge of a point particle and c is the light speed, the Lagrangian (66) describes the dynamics of a point electrically charged particle in a curved background space M with metric $\underline{a}_{ij}(x)$. The formula (67) states a class of nonholonomic deformations of the metric on M to TM. We can treat ${}^gF_{jk}$ as an effective electro-magnetic tensor field on TM. The auto-parallel curves $u^\alpha(\tau) = [x^i(\tau), y^i(\tau) = \mathrm{d}x^i/\tau]$, parametrized by a scalar variable τ, adapted to $\tilde{N}^i_{\ j}(x, y)$ (68), are

$$\frac{\mathrm{d}y^a}{\mathrm{d}\tau} + {}^g\Gamma^a_{bc}(x, y) y^b y^c = {}^gF^a_b(x, y) y^b.$$

The nonholonomic Riemannian mechanical model of $\mathsf{L}(x, y)$ is completely defined by a d-metric (17) with coefficients (67),

$$\tilde{\mathbf{g}} = m_0 a_{ij}(x, y)\, e^i \otimes e^j + m_0 a_{ij}(x, y) \tilde{\mathbf{e}}^i \otimes \tilde{\mathbf{e}}^j,$$

where $\tilde{\mathbf{e}}^i$ are elongated by $\tilde{N}^i_{\ j}$ from (68). From Corollary 5.2, we have:

Conclusion 5.1. *Any solution of the Einstein–Maxwell equations given by gravitational field $\underline{a}_{ij}(x)$ and electromagnetic potential $\underline{A}_i(x)$ is N-anholonomically equivalent to a mechanical model with regular Lagrangian (66) and associated nonholonomic geometry on TM with induced d-metric $\tilde{\mathbf{g}}$ and N-connection structure $\tilde{N}^i_{\ j}$.*

If the frame coefficients $e_a^{\ \underline{a}}$ are chosen to yield constant d-curvatures coefficients, we get a particular statement of the main Theorem 4.2:

Corollary 5.3. *The Einstein–Maxwell gravity equations contain solitonic solutions that can be modeled as a hierarchy of bi-Hamiltonian N-adapted flows of curves.*

In a similar form, we can derive spinor–solitonic nonholonomic hierarchies for Einstein–Dirac equations if we consider unitary groups and distinguished Clifford structures [26].

6. Conclusion

In this paper, it has been shown that the geometry of regular Lagrange mechanics and generalized Lagrange–Finsler spaces can be encoded in nonholonomic hierarchies of bi-Hamiltonian structures and related solitonic equations derived for curve flows. Although the constructions are performed in explicit form for a special class of generalized Lagrange spaces with constant matrix curvature coefficients computed with respect to canonical nonholonomic frames (induced by Lagrange or Finsler metric fundamental functions), the importance of the resulting geometric nonlinear analysis of such physical systems is beyond doubt for further applications in classical and quantum field theory. This remarkable generality appears naturally for all types of models of gravitational, gauge and spinor interactions, geometrized in terms of vector/spinor bundles, in supersymmetric and/or (non) commutative variants when nonholonomic frames and generalized linear and nonlinear connections are introduced into consideration.

If the gauge group structure of vector bundles and nonholonomic manifolds is defined by an orthogonal group $\mathbf{H} = SO(n) \oplus SO(m)$ acting on the base/horizontal and typical fiber/vertical subspaces with, say, respective dimensions of n and m, then these subspaces will be Riemannian symmetric spaces $\mathbf{G}/\mathbf{H}$ whose geometric properties are determined by the Lie groups $\mathbf{G} = SO(n) \oplus SO(m)$ and $\mathbf{G} = SU(n) \oplus SU(m)$. This structure can be formulated equivalently in terms of geometric objects defined on pairs of Klein geometries. The bi-Hamiltonian solitonic hierarchies are generated naturally by recursion operators associated with the horizontal and vertical flows of curves on such spaces, with these flows (of SG and mKdV type) being geometrically described by wave maps and mKdV analogs of Schrödinger maps.

In the case of holonomic manifolds, the construction of bi-Hamiltonian curve flows, soliton equations, and geometric PDE maps has recently been generalized in Ref. [34] to all Riemannian symmetric manifolds $M = G/H$ including all compact semisimple Lie groups $K = G/H$ as given by $G = K \times K$ and $H = \mathrm{diag}\, G$. This generalization makes it possible to study more general field models, such as quaternionic and octonian systems.

Considering arbitrary curved spaces and mechanical or field systems, it is not clear if any bi-Hamiltonian structures and solitonic equations can be derived from curve flows. The answer seems to be negative because arbitrary curved spaces do not possess constant matrix curvatures with respect to certain orthonormalized frames which is crucial for encoding recursion operators and associated solitonic hierarchies.[10] Nevertheless, the results proved in our work provide a new geometric method of solitonic encoding of data for quite general types of curved spaces and nonlinear physical theories. This follows from the fact that we can always nonholonomically deform a curved space having nonconstant curvature given by the distinguished linear connection into a similar space for which this curvature becomes constant.

Our approach employs certain methods elaborated in Finsler and Lagrange geometry when the geometric objects are adapted to nonholonomic distributions on vector/tangent bundles, or (in general) on nonholonomic manifolds. This accounts for existence of the canonical nonlinear connection (defined as non-integral distributions into conventional horizontal and vertical subspaces, and associated nonholonomic frames), metric and linear connection structures all derived from a regular Lagrange (in particular, Finsler) metric function and/or by moving frames and generically off-diagonal metrics in gravity theories. Of course, such spaces are not generally defined to have constant linear connection curvatures and vanishing torsions to which the curve-flow solitonic generation techniques cannot applied.

We proved that for a very large class of regular Lagrangians and the so-called generalized Lagrange space metrics, there can be nonholonomic deformations of geometric objects to equivalent ones on generalized Lagrange configurations, for instance, with constant Hessian for the so-called absolute energy function. The curvature matrix, with respect to the correspondingly adapted (to the canonical connection structure) frames, can be defined with constant, even vanishing, coefficients. For such configurations, we can apply the former methods elaborated for symmetric Riemannian spaces in order to generate curve-flow solitonic hierarchies. Here should be noted that such special classes of nonholonomic manifolds are equipped with generic off-diagonal metrics and possess nontrivial curvature for the Levi-Civita connection. They are characterized additionally by nontrivial nonlinear connection curvature and nonzero anholonomy coefficients for preferred frame structures, and anholonomically induced torsion even their curvature of the canonical distinguished connection is zero.

The utility of the method of anholonomic frames with associated nonlinear connection structures is that we can work with respect to such frames and correspondingly canonical metrics as defined on horizontal/ vertical pairs of symmetric Riemannian spaces whose metric and linear connection structure is soldered from an underlying Klein geometry. In this regard, the bi-Hamiltonian solitonic hierarchies are generated as nonholonomic distributions of horizontal and vertical moving equations and conservation laws, containing all information for a regular Lagrangian and corresponding Euler–Lagrange equations.

We note that curve-flow solitonic hierarchies can be constructed in a similar manner, for instance, for Einstein–Yang–Mills–Dirac equations, derived following the anholonomic frame method, in noncommutative generalizations of gravity and geometry and possible quantum models based on nonholonomic Lagrange–Fedosov manifolds. During a long term review of this paper, a series of new important curve-flow solitonic results were obtained. For instance, the results of [22] were generalized in [42] for arbitrary Einstein and (pseudo) Riemannian metrics for which alternative d-connections

[10] A series of recent works [42–47] provide a number of generalizations and examples when solitonic hierarchies can be derived for arbitrary Einstein, Riemann–Cartan and Larange–Finsler spaces, their Ricci flow evolutions and/or quantum deformations.

(to the Levi-Civita one) were defined. It was proven that such metric compatible d-connections with constant-curvature and Ricci d-tensor coefficients, with respect to certain N-adapted basis, can be constructed. All constructions can be redefined equivalently for the Levi-Civita connection (because all connections under consideration are completely defined by a metric structure).

Finally, we conclude that solitonic hierarchies and the bi-Hamilton and N-connection formalisms happen to very efficient in the theory of nonholonomic Ricci flows and evolution of physically valuable nonlinear gravitational wave solutions [43–45] and Fedosov quantization of Einstein and Lagrange–Finsler spaces [46,47]. So, the results of this paper can be naturally generalized for arbitrary classical and quantum (commutative and noncommutative of type [25]) nonholonomic gravitational interactions, generalized Lagrange–Finsler systems and their Ricci flow evolutions (see also review [48]). In general, we can encode the information on such field/mechanical/evolution models into corresponding solitonic hierarchies and, inversely, to extract certain nonlinear interaction and/or evolution models from systems of solitonic equations.

Acknowledgements

The second author (S.V.) is grateful to A. Bejancu for very important references on the geometry of nonholonomic manifolds and to T. Wolf for hosting a visit to the Mathematics Department at Brock University. He also thanks M. Anastasiei for kind support.

Appendix. Some local formulas

This Appendix outlines some local results from geometry of nonlinear connections (see Refs. [2,3,25,26] for proofs and details). There are two types of preferred linear connections uniquely determined by a generic off-diagonal metric structure with $n+m$ splitting, see $\mathbf{g} = g \oplus_N h$ (14):

1. The Levi-Civita connection $\nabla = \{\Gamma^{\alpha}_{\beta\gamma}\}$ is by definition torsionless, $\mathcal{T} = 0$, and satisfies the metric compatibility condition, $\nabla \mathbf{g} = 0$.
2. The canonical d-connection $\widehat{\mathbf{\Gamma}}^{\gamma}_{\alpha\beta} = \left(\widehat{L}^{i}_{\ jk}, \widehat{L}^{a}_{bk}, \widehat{C}^{i}_{jc}, \widehat{C}^{a}_{\ bc}\right)$ is also metric compatible, i.e. $\widehat{\mathbf{D}}\mathbf{g} = 0$, but the torsion vanishes only on h- and v-subspaces, i.e. $\widehat{T}^{i}_{\ jk} = 0$ and $\widehat{T}^{a}_{\ bc} = 0$, for certain nontrivial values of $\widehat{T}^{i}_{\ ja}, \widehat{T}^{a}_{\ bi}, \widehat{T}^{a}_{\ ji}$. For simplicity, we omit hats on symbols and write $L^{i}_{\ jk}$ instead of $\widehat{L}^{i}_{\ jk}$, $T^{i}_{\ ja}$ instead of $\widehat{T}^{i}_{\ ja}$ and so on, but preserve the general symbols $\widehat{\mathbf{D}}$ and $\widehat{\mathbf{\Gamma}}^{\gamma}_{\alpha\beta}$.

By a straightforward calculus with respect to N-adapted frames (4) and (5), one can verify that the requested properties for $\widehat{\mathbf{D}}$ on $\mathbf{E}$ are satisfied if

$$
\begin{aligned}
L^{i}_{\ jk} &= \frac{1}{2} g^{ir} (\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk}),\\
L^{a}_{bk} &= e_b(N^{a}_{k}) + \frac{1}{2} h^{ac} (\mathbf{e}_k h_{bc} - h_{dc}\, e_b N^{d}_{k} - h_{db}\, e_c N^{d}_{k}),\\
C^{i}_{jc} &= \frac{1}{2} g^{ik} e_c g_{jk}, \qquad C^{a}_{\ bc} = \frac{1}{2} h^{ad} (e_c h_{bd} + e_c h_{cd} - e_d h_{bc}).
\end{aligned}
\tag{69}
$$

For $\mathbf{E} = TM$, the canonical d-connection $\tilde{\mathbf{D}} = (h\tilde{D}, v\tilde{D})$ can be defined in a torsionless form[11] with the coefficients $\Gamma^{\alpha}_{\ \beta\gamma} = (L^{i}_{\ jk}, L^{a}_{\ bc})$,

$$
\begin{aligned}
L^{i}_{\ jk} &= \frac{1}{2} g^{ih} (\mathbf{e}_k g_{jh} + \mathbf{e}_j g_{kh} - \mathbf{e}_h g_{jk}),\\
C^{a}_{\ bc} &= \frac{1}{2} h^{ae} (e_c h_{be} + e_b h_{ce} - e_e h_{bc}).
\end{aligned}
\tag{70}
$$

The curvature of a d-connection $\mathbf{D}$,

$$
\mathcal{R}^{\alpha}_{\ \beta} \doteqdot \boldsymbol{D}\boldsymbol{\Gamma}^{\alpha}_{\ \beta} = d\mathbf{\Gamma}^{\alpha}_{\ \beta} - \mathbf{\Gamma}^{\gamma}_{\ \beta} \wedge \mathbf{\Gamma}^{\alpha}_{\ \gamma},
\tag{71}
$$

splits into six types of N-adapted components with respect to (4) and (5),

$$
\begin{aligned}
\mathbf{R}^{\alpha}_{\ \beta\gamma\delta} &= \left(R^{i}_{\ hjk}, R^{a}_{\ bjk}, P^{i}_{\ hja}, P^{c}_{\ bja}, S^{i}_{\ jbc}, S^{a}_{\ bdc}\right),\\
R^{i}_{\ hjk} &= \mathbf{e}_k L^{i}_{\ hj} - \mathbf{e}_j L^{i}_{\ hk} + L^{m}_{\ hj} L^{i}_{\ mk} - L^{m}_{\ hk} L^{i}_{\ mj} - C^{i}_{\ ha} \Omega^{a}_{\ kj},\\
R^{a}_{\ bjk} &= \mathbf{e}_k L^{a}_{\ bj} - \mathbf{e}_j L^{a}_{\ bk} + L^{c}_{\ bj} L^{a}_{\ ck} - L^{c}_{\ bk} L^{a}_{\ cj} - C^{a}_{\ bc} \Omega^{c}_{\ kj},\\
P^{i}_{\ jka} &= e_a L^{i}_{\ jk} - D_k C^{i}_{\ ja} + C^{i}_{\ jb} T^{b}_{\ ka}, \qquad P^{c}_{\ bka} = e_a L^{c}_{\ bk} - D_k C^{c}_{\ ba} + C^{c}_{\ bd} T^{c}_{\ ka},\\
S^{i}_{\ jbc} &= e_c C^{i}_{\ jb} - e_b C^{i}_{\ jc} + C^{h}_{\ jb} C^{i}_{\ hc} - C^{h}_{\ jc} C^{i}_{\ hb},\\
S^{a}_{\ bcd} &= e_d C^{a}_{\ bc} - e_c C^{a}_{\ bd} + C^{e}_{\ bc} C^{a}_{\ ed} - C^{e}_{\ bd} C^{a}_{\ ec}.
\end{aligned}
\tag{72}
$$

[11] Namely where the d-connection has the same coefficients as the Levi-Civita connection with respect to N-elongated bases (4) and (5).

Contracting respectively the components, $\mathbf{R}_{\alpha\beta} \doteqdot \mathbf{R}^{\tau}_{\ \alpha\beta\tau}$, one computes the h–v-components of the Ricci d-tensor (there are four N-adapted components)

$$R_{ij} \doteqdot R^{k}_{\ ijk}, \qquad R_{ia} \doteqdot -P^{k}_{\ ika}, \qquad R_{ai} \doteqdot P^{b}_{\ aib}, \qquad S_{ab} \doteqdot S^{c}_{\ abc}. \tag{73}$$

The scalar curvature is defined by contracting the Ricci d-tensor with the inverse metric $\mathbf{g}^{\alpha\beta}$,

$$\overleftrightarrow{\mathbf{R}} \doteqdot \mathbf{g}^{\alpha\beta}\mathbf{R}_{\alpha\beta} = g^{ij}R_{ij} + h^{ab}S_{ab} = \overrightarrow{R} + \overleftarrow{S}. \tag{74}$$

If $\mathbf{E} = TM$, there are only three classes of d-curvatures,

$$\begin{aligned} R^{i}_{\ hjk} &= \mathbf{e}_k L^{i}_{\ hj} - \mathbf{e}_j L^{i}_{\ hk} + L^{m}_{\ hj} L^{i}_{\ mk} - L^{m}_{\ hk} L^{i}_{\ mj} - C^{i}_{\ ha}\Omega^{a}_{\ kj}, \\ P^{i}_{\ jka} &= e_a L^{i}_{\ jk} - \mathbf{D}_k C^{i}_{\ ja} + C^{i}_{\ jb} T^{b}_{\ ka}, \\ S^{a}_{\ bcd} &= e_d C^{a}_{\ bc} - e_c C^{a}_{\ bd} + C^{e}_{\ bc} C^{a}_{\ ed} - C^{e}_{\ bd} C^{a}_{\ ec}, \end{aligned} \tag{75}$$

where all indices $a, b, \ldots, i, j, \ldots$ run over the same values and, for instance, $C^{e}_{\ bc} \to C^{i}_{\ jk}$, etc.

References

[1] J. Kern, Lagrange geometry, Arch. Math. 25 (1974) 438–443.
[2] R. Miron, M. Anastasiei, Vector Bundles and Lagrange Spaces with Applications to Relativity, Geometry Balkan Press, Bukharest, 1997, (translation from Romanian of (Editura Academiei Romane, 1984)).
[3] R. Miron, M. Anastasiei, The Geometry of Lagrange paces: Theory and Applications FTPH no. 59, Kluwer Academic Publishers, Dordrecht, Boston, London, 1994.
[4] M. de León, P.R. Rodrigues, Methods of Differential Geometry in Analytical Mechanics, in: Ser. 152, North-Holland Math, Amsterdam, 1989.
[5] M. de León, D. Martin de Diego, A. Santamaria–Merino, Symmetries in classical field theory, Int. J. Geom. Methods Mod. Phys. 1 (2004) 651–710.
[6] G.L. Lamb Jr., Solitons on moving space curves, J. Math. Phys. 18 (1977) 1654–1661.
[7] R.E. Goldstein, D.M. Petrich, The Korteweg-de Vries hierarchy as dynamics of closed curves in the plane, Phys. Rev. Lett. 67 (1991) 3203–3206.
[8] K. Nakayama, H. Segur, M. Wadati, Integrability and the motion of curves, Phys. Rev. Lett. 69 (1992) 2603–2606.
[9] J. Langer, R. Perline, Curve motion inducing modified Korteweg-de Vries systems, Phys. Lett. A 239 (1998) 36–40.
[10] K.-S. Chou, C. Qu, The KdV equation and motion of plane curves, J. Phys. Soc. Japan 70 (2001) 1912–1916.
[11] K.-S. Chou, C. Qu, Integrable equations arising from motions of plane curves, Phys. D 162 (2002) 9–33. Integrable equations arising from motions of plane curves II, J. Nonlinear Sci. 13, 2003, 487–517.
[12] G. Mari Beffa, J. Sanders, J.-P. Wang, Integrable systems in three–dimensional Riemannian geometry, J. Nonlinear Sci. 12 (2002) 143–167.
[13] J. Sanders, J.-P. Wang, Integrable systems in n dimensional Riemannian geometry, Mosc. Math. J. 3 (2003) 1369–1393.
[14] S.C. Anco, bi-Hamiltonian operators, integrable flows of curves using moving frames, and geometric map equations, J. Phys. A. 39 (2006) 2043–2072.
[15] C. Athorne, Some integrable equations associated with symmetric spaces, in: Nonlinear Evolutions [Balaruc-les-Bains, 198], World Sci. Publ, Teaneck, NJ, 1988, pp. 191–198.
[16] C. Athorne, Local Hamiltonian structures of multicomponent KdV equations, J. Phys. A. 21 (1988) 4549–4556.
[17] S.C. Anco, Hamiltonian flows of curves in $G/SO(n)$ and vector soliton equations of mKdV and sine-Gordon Type, Symmetry, Integrability and Geometry: Methods and Applications 2 (2006) 044.
[18] J.-P. Wang, Generalized Hasimoto Transformation and Vector Sine–Gordon Equation, in: S. Abenda, G. Gaeta, S. Walcher (Eds.), SPT 2002: Symmetry and Perturbation Theory (Cala Gonone), WorldScientific, River Edge, 2002, pp. 276–283.
[19] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Providence, Amer. Math. Soc, 2001.
[20] V.V. Sokolov, T. Wolf, Classification of integrable vector polynomial evolution equation, J. Phys. A. 34 (2001) 11139–11148.
[21] S.C. Anco, T. Wolf, Some symmetry classifications of hyperbolic vector evolution equations, J. Nonlinear Math. Phys. 12 (suppl. 1) (2005) 13–31; J. Nonlinear Math. Phys. 12 (2005) 607–608 (erratum).
[22] S. Vacaru, Curve flows and solitonic hierarchies generated by (semi) Riemannian metrics..
[23] A. Bejancu, Finsler Geometry and Applications, Ellis Horwood, Chichester, England, 1990.
[24] A. Bejancu, H.R. Farran, Foliations and Geometric Structures, Springer, 2005.
[25] S. Vacaru, Exact solutions with noncommutative symmetries in Einstein and gauge Gravity, J. Math. Phys. 46 (2005) 042503.
[26] S. Vacaru, P. Stavrinos, E. Gaburov, D. Gonţa, Clifford and Riemann–Finsler Structures in Geometric Mechanics and Gravity. Selected Works, in: Differential Geometry—Dynamical Systems, Monograph, vol. 7, Geometry Balkan Press, Bucharest, 2006 www.mathem.pub.ro/dgds/mono/va-t.pdfgr-qc/0508023.
[27] K. Yano, S. Ishihara, Tangent and Cotangent Bundles, M. Dekker, Inc., New York, 1978.
[28] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Vols. I and II, Wiley, 1969.
[29] R.W. Sharpe, Differential Geometry, Springer–Verlag, New York, 1997.
[30] G. Vranceanu, Sur les espaces non holonomes, C. R. Acad. Paris 103 (1926) 852–854.
[31] G. Vranceanu, Leçons de Geometrie Differentielle, Vol II (Edition de l'Academie de la Republique Populaire de Roumanie, 1957).
[32] D. Bao, S.-S. Chern, Z. Shen, An Introduction to Riemann–Finsler Geometry, in: Graduate Texts in Math, vol. 200, Springer–Verlag, 2000.
[33] S.C. Anco, Hamiltonian curve flows in Lie groups $G \subset U(N)$ and vector NLS, mKdV, sine-Gordon soliton equations, in: IMA Volumes in Mathematics and its Applications, in: Symmetries and Overdetermined Systems of Partial Differential Equations, vol. 144, AMS, 2007, pp. 223–250.
[34] S.C. Anco, Group-invariant soliton equations and bi-Hamiltonian geometric curve flows in Riemannian symmetric spaces, J. Geom. Phys. 58 (2008) 1–37.
[35] A. Sergyeyev, Why nonlocal recursion operators produce local symmetries: New results and applications, J. Phys. A. 38 (2005) 3397–3407.
[36] C. Barcelo, S. Liberati, M. Visser, Analogue gravity, Living Rev. Rel. 8 (2005) 12.
[37] R. Miron, D. Hrimiuc, S. Shimada, V.S. Sabau, The Geometry of Hamilton and Lagrange Spaces, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.
[38] E. Cartan, Les Espaces de Finsler, Hermann, Paris, 1935.
[39] M. Matsumoto, Foundations of Finsler Geometry and Special Finsler Spaces, Kaisisha, Shigaken, 1986.
[40] S. Vacaru, Interactions, Strings and Isotopies in Higher Order Anisotropic Superspaces, Hadronic Press, Palm Harbor, FL, USA, 1998.
[41] F. Etayo, R. Santamariá, S. Vacaru, Lagrange–Fedosov nonholonomic manifolds, J. Math. Phys. 46 (2005) 032901.
[42] S. Vacaru, Curve flows and solitonic hierarchies generated by Einstein metrics. arXiv: 0810.0707 [math-ph].
[43] S. Vacaru, Ricci flows and solitonic pp–waves, Internat. J. Modern Phys. A 21 (2006) 4899–4912.
[44] S. Vacaru, Nonholonomic Ricci flows: II. Evolution equations and dynamics, J. Math. Phys. 49 (2008) 043504.
[45] S. Vacaru, Nonholonomic Ricci flows: III. Curve flows and solitonic hierarchies. arXiv: 0704.2062 [math.DG].
[46] S. Vacaru, Deformation quantization of almost Kähler models and Lagrange–Finsler spaces, J. Math. Phys. 48 (2007) 123509.
[47] S. Vacaru, Deformation quantization of nonholonomic almost Kähler models and Einstein gravity, Phys. Lett. A 372 (2008) 2949–2955.
[48] S. Vacaru, Finsler and Lagrange geometries in Einstein and string gravity, Int. J. Geom. Methods Mod. Phys. (IJGMMP) 5 (2008) 473–511.