



# A new characterization of the $n$ -dimensional Einstein static spacetime



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## ABSTRACT

The Einstein static spacetime is characterized as the unique geodesically complete and simply connected Lorentzian manifold such that the geodesic flow acts by isometries of the Sasaki metric on any null congruence associated to a conformal timelike vector field.

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## 1. Introduction

The Einstein static spacetime was introduced by Einstein in 1917 as the first cosmological model attending to the General Relativity Theory. This model has positive spatial sectional curvature, therefore it is a closed, finite spatial universe. In those days, the idea of a static universe was generally accepted, because the observations by Hubble had not yet been realized. Once Hubble's expansion seemed to have been established as an empirical fact, Einstein was forced to abandon his static universe model. The static spacetime again raised the interest from the physical point of view in the seventies of the last century (see [1] for a historical survey). From a geometrical point of view, let us recall that the Einstein static universe plays a central role in the study of conformal Lorentzian geometry (see [2] and references therein).

The  $n$ -dimensional Einstein static spacetime may be mathematically described as the manifold  $\mathbb{R} \times S^{n-1}(1/\sqrt{a})$  endowed with the Lorentzian product metric  $-dt^2 + h$ , where  $S^{n-1}(1/\sqrt{a})$  denotes the usual round sphere of radius  $1/\sqrt{a}$ . The main purpose of this short note is, roughly speaking, to characterize the Einstein static spacetime as the unique spacetime in which the geodesic flow acts isometrically on the space of null directions. In order to face this aim, first we have to establish rigorously what such behavior of the geodesic flow means.

Attending to the causal character of the tangent vectors of a Lorentzian manifold  $M$ , we will consider three fiber subbundles of the tangent fiber bundle  $TM$ ; namely, the unit timelike fiber bundle  $S_{-1}(M)$ , the unit spacelike fiber bundle  $S_1(M)$  and the null fiber bundle  $S_0(M)$ . The geodesic flow  $\{\Phi_t\}$  leaves every fiber bundle  $S_\epsilon(M)$  invariant for  $\epsilon = -1, 0, +1$ . This allows us to consider timelike, spacelike and null geodesic flows depending on the fiber bundle  $S_\epsilon(M)$  we are considering. The Lorentzian metric  $g$  on  $M$  can be lifted to a semi-Riemannian metric  $\hat{g}$  (Sasaki metric) of index 2 on  $TM$  (Section 2) and  $\hat{g}$

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provides a semi-Riemannian metric when is restricted to  $S_\epsilon(M)$  for  $\epsilon = -1, +1$  (Section 4). Thus, the question of when the timelike and spacelike geodesic flows act isometrically can be rigorously stated. This question can be answered in similar way to [3, Proposition 1.104] as follows. For  $\epsilon = -1, +1$ , the geodesic vector field is Killing for the corresponding restriction of the Sasaki metric to  $S_\epsilon(M)$  if and only if  $M$  has constant sectional curvature  $\epsilon$  (Theorem 4.2). These results for timelike and spacelike fiber bundles are included for the sake of completeness (Section 4).

Our proof for the case of the unit timelike fiber bundle  $S_{-1}(M)$  strongly depends on the fact that the sectional curvature of a Lorentzian manifold behaves quite differently from that one in Riemannian manifolds. Recall that in Riemannian geometry, the concepts of a manifold being positively curved or negative curved based on the sign of the sectional curvature have been extensively studied. On the contrary, roughly speaking, specification of bounds on sectional curvature forces to a Lorentzian manifold to have constant sectional curvature. As far as we know, the first result of this type was obtained by Wolf in the context of symmetric spaces with indefinite metrics [4]. This result was widely extended by Kulkarni without any hypothesis of homogeneity as follows: if the sectional curvature function of an  $n(\geq 3)$ -dimensional Lorentzian manifold  $M$  is either bounded from above or bounded from below, then  $M$  has constant sectional curvature [5]. This statement is still true if it is just assumed that the sectional curvature is bounded either on all timelike (Lorentzian) or on all spacelike linear tangent planes of  $M$  [6]. The particular case of Lorentzian linear tangent planes is a key fact to end the proof of Theorem 4.2. These results on the tendency to be bounded of the sectional curvature of a Lorentzian manifold are summarized in [7, Proposition 8.28].

Several technical difficulties arise in the case of the null geodesic flow. On one hand,  $\widehat{g}$  is always degenerate on  $S_0(M)$ . Thus the Sasaki metric does not provide a semi-Riemannian metric on  $S_0(M)$  (Remark 4.1). In order to avoid this difficulty, we replace  $S_0(M)$  with the null congruence  $C_K M$  associated with a timelike vector field  $K$  (Section 1). The manifold  $C_K M$  is a codimension two submanifold of the tangent fiber bundle  $TM$  and it can be seen as the manifold of null directions. For every timelike vector field  $K$ , the associated null congruence  $C_K M$  inherits a Lorentzian metric from  $\widehat{g}$ . This manifold of null directions was used to characterize Robertson–Walker spaces [8,9] and to study conjugate points along null geodesics [10,11]. Also, in an appropriate sense, the null congruence associated with a timelike vector field allows to describe the null sectional curvature as a smooth function on the set of degenerate tangent planes [12]. On the other hand, for a timelike vector field  $K$  the null congruence  $C_K M$  is not invariant by the geodesic flow in general. In fact,  $C_K M$  is invariant by the geodesic flow if and only if the timelike vector field  $K$  is assumed to be conformal (Section 2). Thus, in order to state with precision the meaning of the sentence *the geodesic flow acts isometrically on the space of null directions*, we will take a timelike conformal vector field  $K$  and we will consider its associated null congruence  $C_K M$  as a Lorentz manifold with  $\widehat{g}_K = \widehat{g}|_{C_K M}$ .

*When does the geodesic flow restricted to a such  $C_K M$  act by isometries of  $\widehat{g}_K$ ?*

The main result of this paper can be stated as follows.

*The geodesic vector field is Killing on the null congruence  $C_K M$  if and only if the vector field  $K$  is parallel and all linear tangent planes on  $M$  belonging to the (integrable) distribution  $K^\perp$  have constant sectional curvature  $a > 0$ . If, in addition,  $M$  is geodesically complete and simply connected, then it is globally isometric to the  $n$ -dimensional Einstein static spacetime  $\mathbb{R} \times \mathbb{S}^{n-1}(1/\sqrt{a})$  (Theorem 3.2).*

The paper is organized as follows. In Section 2 we review standard definitions and results and summarize without proof several useful properties of the null congruence. Section 3 contains the proof of the main result and in Section 4 we briefly sketch the case of timelike and spacelike unit fiber bundles.

## 2. Preliminaries

Let  $(M, g)$  be an  $n(\geq 3)$ -dimensional Lorentzian manifold, that is, a (connected) smooth manifold  $M$  endowed with a nondegenerate metric  $g$  with signature  $(-, +, \dots, +)$ . We shall write  $\nabla$  for its Levi-Civita connection and  $R$  for its Riemann–Christoffel curvature tensor. Our convention on the curvature tensor is  $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$  for  $X, Y, Z \in \mathfrak{X}(M)$ . A tangent vector  $u \in T_p M$  is said to be timelike if  $g(u, u) < 0$ , null if  $g(u, u) = 0$  and  $u \neq 0$ , and spacelike if  $g(u, u) > 0$  or  $u = 0$ . A vector field  $K \in \mathfrak{X}(M)$  is said to be timelike if  $K_p$  is timelike for all  $p \in M$ . From now on, the Lorentzian manifold  $(M, g)$  is assumed to be time-oriented, that is, a global timelike vector field  $K$  has been fixed. So a timelike or null tangent vector  $v \in T_p M$  is said to be future (resp. past) with respect to  $K$  if  $g(v, K_p) < 0$  (resp.  $g(v, K_p) > 0$ ).

As usual  $\pi: TM \rightarrow M$  will denote the natural projection on  $M$  from its tangent bundle  $TM$ . Now, for  $v \in T_p M$ , the connection map  $c$  of the Levi-Civita connection  $\nabla$  [13, Chapter 2] is the linear map  $c: T_v TM \rightarrow T_p M$  given as follows. For  $\xi \in T_v TM$ , let us take  $\alpha$  any curve in  $TM$  with initial data  $\alpha(0) = v$  and  $\frac{d\alpha}{dt}|_0 = \xi$ . Then  $c(\xi) = \frac{\nabla \alpha}{dt}|_0$ , where  $\frac{\nabla \alpha}{dt}$  denotes the covariant derivative of  $\alpha$  as a vector field along the curve  $\pi \circ \alpha$ . The connection map  $c$  permits to construct the following isomorphism of vector spaces,

$$T_v TM \rightarrow T_p M \oplus T_p M, \quad \xi \rightarrow (\pi_*(\xi), c(\xi)).$$

A semi-Riemannian metric  $\widehat{g}$  of index two on  $TM$  may be introduced as follows,

$$\widehat{g}(\zeta, \xi) = g(\pi_*(\zeta), \pi_*(\xi)) + g(c(\zeta), c(\xi)),$$

for all  $\zeta, \xi \in T_v TM$ . The metric  $\widehat{g}$  is called the Sasaki metric on  $TM$ .

Let us consider the geodesic vector field  $\mathbf{Z}_g \in \mathfrak{X}(TM)$  induced from the metric  $g$ . That is, at every  $v \in T_pM$ , take the geodesic  $\gamma_v$  with initial data  $\gamma_v(0) = p$  and  $\frac{d\gamma_v}{dt}|_0 = v$ . Thus,  $\mathbf{Z}_g(v) = \frac{d\gamma'_v}{dt}|_{t=0}$  where  $\gamma'_v$  is now seen as a curve in  $TM$ . The collection of 1-parameter group (local) transformations  $\{\Phi_t\}$  of  $\mathbf{Z}_g$  is called the geodesic flow of  $(M, g)$ . Note that  $\Phi_t(v) = \gamma'_v(t)$  for every  $v \in TM \cap \text{Dom}(\Phi_t)$ . The differential of every transformation  $\Phi_t$  can be described in the following terms [14, Lemma 3.1.17]. For every  $\xi \in T_vTM$ , consider the unique Jacobi vector field  $J \in \mathfrak{X}(\gamma_v)$  which satisfies  $J(0) = \pi_*(\xi)$  and  $\frac{\nabla J}{dt}|_0 = c(\xi)$ . Take  $\mathcal{J} \in \mathfrak{X}(\gamma'_v)$  such that  $\pi_*(\mathcal{J}(s)) = J(s)$  and  $c(\mathcal{J}(s)) = \frac{\nabla J}{dt}|_s$  at every  $s$ . Thus, we have  $(\Phi_t)_*(\xi) = \mathcal{J}(t)$ . Therefore,

$$\widehat{g}((\Phi_t)_*(\xi), (\Phi_t)_*(\xi)) = g(J(t), J(t)) + g\left(\frac{\nabla J}{dt}\Big|_t, \frac{\nabla J}{dt}\Big|_t\right). \quad (1)$$

The null congruence  $C_KM$  associated with a timelike vector field  $K \in \mathfrak{X}(M)$  is defined as the following submanifold of the tangent bundle  $TM$ ,

$$C_KM = \{v \in TM \mid g(v, v) = 0 \text{ and } g(v, K_{\pi(v)}) = -1\}.$$

We take the null congruence in the future with respect to  $K$ , in contrast to the definition used in [10,8,9]. The tangent space  $T_vC_KM$  at  $v \in C_KM$  is proven [10] to be equal to

$$T_vC_KM = \left\{ \xi \in T_vTM : g(c(\xi), v) = g(K_{\pi(v)}, c(\xi)) + g(v, \nabla_{\pi_*(\xi)}K) = 0 \right\}. \quad (2)$$

Every null congruence  $C_KM$  inherits a Lorentzian metric from the Sasaki metric  $\widehat{g}$ . Let us denote by  $\widehat{g}_K$  the restriction  $\widehat{g}|_{C_KM}$ . Thus,  $\pi : (C_KM, \widehat{g}_K) \rightarrow (M, g)$  is a semi-Riemannian submersion with spacelike fibers [10]. According to [15], the horizontal distribution at  $v \in C_KM$  satisfies

$$\mathcal{H}_v = \left\{ \xi \in T_vTM : c(\xi) = g(v, \nabla_{\pi_*(\xi)}K)v \right\}. \quad (3)$$

The vector field  $\mathbf{Z}_g|_{C_KM}$  is not tangent to  $C_KM$  for a general timelike vector field  $K \in \mathfrak{X}(M)$ . Let us recall that a vector field  $K$  is said to be conformal when  $\mathcal{L}_Kg = 2\rho g$ , where  $\rho$  is a (necessarily smooth) function on  $M$ . This condition can be written in an equivalent way as follows,

$$g(\nabla_XK, Y) + g(X, \nabla_YK) = 2\rho g(X, Y),$$

for every  $X, Y \in \mathfrak{X}(M)$ . When  $\rho = 0$ , the conformal vector field  $K$  is said to be Killing. The vector field  $\mathbf{Z}_g|_{C_KM}$  is tangent to  $C_KM$  if and only if the timelike vector field  $K$  is conformal [10]. In this case, we denote  $\mathbf{Z}_g|_{C_KM}$  by  $\mathbf{Z}_g$ .

### 3. Main result

**Lemma 3.1.** *Let  $(M, g)$  be a Lorentzian manifold which admits a timelike conformal vector field  $K$ . Then  $\mathbf{Z}_g \in \mathfrak{X}(C_KM)$  is a Killing vector field for  $\widehat{g}_K$  if and only if for all  $v \in C_KM$  and  $\xi \in T_vC_KM$ ,*

$$g\left(c(\xi), \pi_*(\xi)\right) = g\left(R(\pi_*(\xi), v)v, c(\xi)\right). \quad (4)$$

**Proof.** Let  $J \in \mathfrak{X}(\gamma_v)$  be the Jacobi vector field which satisfies  $J(0) = \pi_*(\xi)$  and  $\frac{\nabla J}{dt}|_0 = c(\xi)$ . From (1) and the Jacobi equation, it can be deduced that  $\mathbf{Z}_g$  is a Killing vector field for  $\widehat{g}_K$  if and only if

$$g\left(\frac{\nabla J}{dt}, J\right) = g\left(R(J, \gamma'_v)\gamma'_v, \frac{\nabla J}{dt}\right). \quad (5)$$

Thus, the result is easily followed from (5).  $\square$

**Theorem 3.2.** *Let  $(M, g)$  be an  $n(\geq 3)$ -dimensional Lorentzian manifold which admits a timelike conformal vector field  $K$ . Then  $\mathbf{Z}_g \in \mathfrak{X}(C_KM)$  is a Killing vector field for  $\widehat{g}_K$  if and only if the vector field  $K$  is parallel and every (spacelike) linear tangent plane  $\Pi \subset K^\perp$  has sectional curvature  $\mathcal{K}(\Pi) = -g(K, K) = a > 0$ . If, in addition,  $(M, g)$  is geodesically complete and simply connected, then it is globally isometric to the  $n$ -dimensional Einstein static space  $\mathbb{R} \times \mathbb{S}^{n-1}(1/\sqrt{a})$  where  $\mathbb{S}^{n-1}(1/\sqrt{a})$  is the standard round sphere with radius  $1/\sqrt{a}$ .*

**Proof.** Fix  $p \in M$  and  $v \in C_KM \cap T_pM$ . Let us consider  $\hat{x} \in T_vC_KM$  the horizontal lift of  $x \in T_pM$ . From (3) and Lemma 3.1, we get  $g(c(\hat{x}), x) = g(v, \nabla_xK)g(v, x) = 0$ . Now it is not difficult to obtain  $\nabla K = 0$ . Let  $\Pi = \text{Span}\{x, y\} \subset K^\perp_p$  be a linear tangent plane with  $\{x, y\}$  an orthonormal basis. Consider  $u = \frac{1}{a}K_p + \frac{1}{\sqrt{a}}y \in C_KM$  for  $a = -g(K, K)$ . The sectional curvature  $\mathcal{K}(\Pi)$  of  $\Pi$  satisfies,

$$\mathcal{K}(\Pi) = g(R(x, y)y, x) = ag(R(x, u)u, x).$$

Taking into account that  $K$  is a parallel vector field and (2), there exists  $\xi \in T_uC_KM$  such that  $\pi_*(\xi) = c(\xi) = x$ . Then, Lemma 3.1 implies  $\mathcal{K}(\Pi) = a$ . Conversely, a standard argument shows that  $R(x, y)z = a[g(z, y)x - g(z, x)y]$  for every

$x, y, z \in K^\perp$ . For  $v \in C_K M \cap T_p M$  we have  $v = \frac{1}{a}K_p + \frac{1}{\sqrt{a}}z \in C_K M \cap T_p M$  with  $z \in K_p^\perp$ . Consider  $\xi \in T_v C_K M$  and let us compute

$$g\left(R(\pi_*(\xi), v)v, c(\xi)\right) = \frac{1}{a}g\left(R(c(\xi), z)z, \pi_*(\xi)\right) = g(c(\xi), \pi_*(\xi)). \quad (6)$$

Thus, Lemma 3.1 shows that  $Z_g$  is Killing. For  $(M, g)$  geodesically complete and simply connected, the de Rham–Wu decomposition theorem [16] gives that  $(M, g)$  is globally isometric to  $\mathbb{R} \times \mathbb{S}^{n-1}(1/\sqrt{a})$ .  $\square$

#### 4. Further results

As it was mentioned in Introduction, for the cases of timelike and spacelike unit tangent fiber bundles, we can proceed analogously to the proof of the unit tangent bundle of a Riemannian manifold [3, Chapter 1]. We add these results for the sake of completeness and can be summarized as follows. Let us introduce the following two fiber bundles over  $M$ ,

$$S_\epsilon(M) = \left\{v \in TM : g(v, v) = \epsilon\right\},$$

for  $\epsilon = -1, +1$ . Thus,  $S_{-1}(M)$  (resp.  $S_{+1}(M)$ ) is the unit timelike (resp. spacelike) fiber bundle over  $M$ . The tangent space  $T_x S_\epsilon(M)$  satisfies

$$T_x S_\epsilon(M) = \left\{\xi \in T_x TM : g(c(\xi), x) = 0\right\}.$$

The corresponding restrictions of the Sasaki metric  $\widehat{g}_- = \widehat{g}|_{S_{-1}(M)}$  and  $\widehat{g}_+ = \widehat{g}|_{S_{+1}(M)}$  provide a Lorentz metric and a semi-Riemannian metric with index 2, respectively. The geodesic vector field  $Z_g$  is always tangent to  $S_\epsilon(M)$  for  $\epsilon = -1, +1$ .

**Remark 4.1.** The restriction of the Sasaki metric on the null fiber bundle  $S_0(M) = \{v \in TM : g(v, v) = 0, v \neq 0\}$  does not provide a semi-Riemannian metric. In fact, consider on  $S_0(M)$  the vector field  $A$  generating the flow of positive dilations. At every  $v \in S_0(M)$ , we have  $A(v) = \frac{d\alpha}{dt}|_{t=0}$  where  $\alpha(t) = v + tv$  and therefore,  $\pi_*(A(v)) = 0$  and  $c(A(v)) = v$  hold for all  $v \in S_0(M)$ . For every  $\xi \in \mathfrak{X}(S_0(M))$ , a direct computation shows that

$$\widehat{g}(A(v), \xi(v)) = g\left(\mu(0), \frac{\nabla \mu}{dt}\bigg|_0\right) = \frac{1}{2} \frac{d}{dt}\bigg|_0 g(\mu, \mu) = 0,$$

where  $\mu$  is any curve in  $S_0(M)$  with initial data  $\mu(0) = v$  and  $\frac{d\mu}{dt}\big|_0 = \xi(v)$ .

**Theorem 4.2.** Let  $(M, g)$  be a Lorentzian manifold. The following assertions are equivalent.

1.  $Z_g$  is Killing on  $S_\epsilon(M)$  with respect to  $\widehat{g}_\epsilon$ .
2.  $(M, g)$  has constant sectional curvature  $\epsilon$ .

**Proof.** In a similar way to Lemma 3.1, it can be stated that  $Z_g$  is Killing on  $S_\epsilon(M)$  if and only if

$$g\left(c(\xi), \pi_*(\xi)\right) = g\left(R(\pi_*(\xi), x)x, c(\xi)\right), \quad (7)$$

for all  $\xi \in T_x S_\epsilon(M)$  and  $x \in S_\epsilon(M)$ . For a Lorentzian manifold with constant sectional curvature  $\epsilon = \pm 1$ , the formula (7) holds. Assume now  $\epsilon = +1$  and let  $\Pi \subset T_p M$  be a nondegenerate tangent plane. There exists an orthonormal basis  $\{x, y\}$  of  $\Pi$  with  $x \in \Pi \cap S_{+1}(M)$ . Let us take  $\xi \in T_x S_{+1}(M)$  such that  $\pi_*(\xi) = c(\xi) = y$ . Then we obtain  $K(\Pi) = +1$  from (7). For  $\epsilon = -1$ , in the same manner we can see that every Lorentzian linear tangent plane  $\Pi$  has sectional curvature  $-1$ . Finally, [7, Proposition 8.28] implies that  $(M, g)$  has constant sectional curvature.  $\square$

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