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Novikov algebras carrying an invariant Lorentzian symmetric bilinear form[☆]

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ABSTRACT

In this note, we shall classify Novikov algebras that admit an invariant Lorentzian symmetric bilinear form.

1. Introduction

A finite-dimensional algebra $(A, \cdot)$ over a field $\mathbb{F}$ is called *left-symmetric* if it satisfies the identity

$$(x, y, z) = (y, x, z), \quad \text{for all } x, y, z \in A, \tag{1}$$

where (x, y, z) denotes the associator $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$. In this case, the commutator $[x, y] = x \cdot y - y \cdot x$ defines a bracket that makes A into a Lie algebra. We denote by $\mathcal{G}_A$ this Lie algebra and call it *the associated Lie algebra to A*. Conversely, if $\mathcal{G}$ is a Lie algebra endowed with a left-symmetric product satisfying the condition $[x, y] = x \cdot y - y \cdot x$, then we say that this left-symmetric product is *compatible* with the Lie structure of $\mathcal{G}$.

Let A be a left-symmetric algebra over a field $\mathbb{F}$, and let L_x and R_x denote the left and right multiplications by the element $x \in A$, respectively.

The identity (1) is now equivalent to the formula

$$\left[L_x, L_y\right] = L_{[x,y]}, \quad \text{for all } x, y \in A, \tag{2}$$

or, in other words, the linear map $L : \mathcal{G}_A \to \mathrm{End}(A)$ is a representation of Lie algebras.

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We notice that (1) is also equivalent to the formula

$$\left[L_x, R_y\right] = R_{x\cdot y} - R_y \circ R_x, \quad \text{for all } x, y \in A. \tag{3}$$

We say that A is *a Novikov algebra* if it satisfies the identity

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y, \quad \text{for all } x, y, z \in A. \tag{4}$$

In terms of left and right multiplications, (4) is equivalent to each of the following identities

$$\left[R_x, R_y\right] = 0, \quad \text{for all } x, y \in A, \tag{5}$$

$$L_{x\cdot y} = R_y \circ L_x, \quad \text{for all } x, y \in A. \tag{6}$$

Definition 1. A nondegenerate symmetric bilinear form $\langle , \rangle$ on a Novikov algebra A is said to be invariant if $\langle R_x y, z\rangle = \langle y, R_x z\rangle$, for all $x, y, z \in A$.

For example, if we let $A_{k,0}$ denote the real vector space $\mathbb{R}^k$ with zero multiplication and $F_{k+1,0} = \langle e_0, e_1, \ldots, e_k : e_i \cdot e_0 = e_i,\ i = 0, \ldots, k\rangle$, then it is easy to verify that these are (associative) Novikov algebras, and that any invariant positive definite symmetric bilinear form on $A_{k,0}$ or $F_{k+1,0}$ is invariant.

Notice here that $F_{k+1,0}$ is nothing but the left-symmetric algebra appearing in Example 4 of [1]. In particular, $F_{1,0}$ is nothing but the field $\mathbb{R}$.

The following theorem which is a result of Zelmanov (see [1, Proposition 4]) classifies the Novikov algebras provided with an invariant positive definite symmetric bilinear form.

Theorem 2 ([1]). *Let A be a real Novikov algebra provided with an invariant positive definite symmetric bilinear form. Then A is an orthogonal direct sum of the form $A = \bigoplus_i^{\perp} A_i$, where each A_i is isomorphic to either the algebra $A_{k,0}$ or the algebra $F_{k+1,0}$, for some integer $k \geq 1$. In particular, A is associative.*

In light of this theorem, a natural question arises: What kind of Novikov algebras do we obtain if we replace the condition "positive definite" with "Lorentzian" in the statement of the above theorem?

The purpose of this note is to answer this question by proving the following result.

Theorem 3. *Let A be a real n-dimensional Novikov algebra provided with an invariant Lorentzian symmetric bilinear form. Then A is isomorphic to an orthogonal direct sum of the form $A = A_1 \oplus A_2$, where A_1 is an algebra in Table 1 and A_2 is a direct sum of the algebras $A_{k,0}$ and $F_{k+1,0}$.*

Remark 4. It should be mentioned that, in Theorem 3, A_2 is a direct sum of the algebras $A_{k,0}$ and $F_{k+1,0}$ in the following precise sense: in the case A_1 is nontrivial, A_2 is the orthogonal direct sum of the algebras $A_{k,0}$ and $F_{k+1,0}$. In that case, the restriction of the Lorentzian symmetric bilinear form to any nontrivial factor $A_{k,0}$ or $F_{k+1,0}$ is positive definite. In the case A_1 is trivial, A_2 is an orthogonal direct sum of the form $A_2 = B_1 \oplus B_2$, where B_1 is a pseudo-orthogonal direct sum of two degenerate factors of the algebras $A_{k,0}$ and $F_{k+1,0}$ and A_2 is a direct sum of the algebras $A_{k,0}$ and $F_{k+1,0}$.

Remark 5. By Theorem 2, a Novikov algebra equipped with an invariant positive definite symmetric bilinear form is necessarily associative. On the contrary, by Theorem 3, a Novikov algebra equipped with an invariant Lorentzian symmetric bilinear form need not be associative. We observe for instance that the algebras L_{k+1} and L^{λ}_{k+1} are not associative.

Table 1

LSA	Basis	Non-zero products	Lie algebra
$A_{k,0}$	$e_1, \ldots, e_k$		$\mathbb{R}^k$
$A_{2,1}$	e_1, e_2	$e_1 \cdot e_1 = e_2$	$\mathbb{R}^2$
$S_{2,1}$	e_1, e_2	$e_1 \cdot e_2 = e_2$	$\mathcal{G}_2$
$N_{3,1}$	e_1, e_2, e_3	$e_1 \cdot e_2 = e_3$	$\mathcal{H}_3$
$N_{3,2}$	e_1, e_2, e_3	$e_2 \cdot e_2 = e_1,\ e_1 \cdot e_2 = e_3$	$\mathcal{H}_3$
$A_{2,2}$	e_1, e_2	$e_1 \cdot e_1 = e_1, e_1 \cdot e_2 = e_2$ $e_2 \cdot e_1 = e_2,\ e_2 \cdot e_2 = -e_1$	$\mathbb{R}^2$
$F_{k+1,0}$ $k \geq 0$	$e_0, \ldots, e_k$	$e_i \cdot e_0 = e_i,\ 0 \leq i \leq k$	$\mathbb{R}$, if $k = 0$ $\mathcal{G}_{k+1}$, if $k \geq 1$
$A_{k+1,1}$ $k \geq 2$	$e_0, \ldots, e_k$	$e_1 \cdot e_1 = e_2$, $e_i \cdot e_0 = e_i,\ 0 \leq i \leq k$	$\mathcal{G}_{k+1}$

(*continued on next page*)

Table 1 (*continued*)

LSA	Basis	Non-zero products	Lie algebra
$A_{k+1,m}$ $3 \le m \le k$	$e_0, \ldots, e_k$	$e_0 \cdot e_0 = e_0,\ e_1 \cdot e_1 = e_2,$ $e_i \cdot e_0 = e_i,\ m \le i \le k$	$\mathcal{G}_{k-m+1} \oplus \mathbb{R}^m$
$N_{k+1,1}$ $k \ge 3$	$e_0, \ldots, e_k$	$e_1 \cdot e_3 = e_2,$ $e_i \cdot e_0 = e_i,\ 0 \le i \le k$	$\mathcal{N}_{k+1}$
$N_{k+1,m}$ $4 \le m \le k$	$e_0, \ldots, e_k$	$e_0 \cdot e_0 = e_0,\ e_1 \cdot e_3 = e_2,$ $e_i \cdot e_0 = e_i,\ m \le i \le k$	$\mathcal{N}_{k-m+1} \oplus \mathbb{R}^m$
$S_{k+1,2}$ $k \ge 2$	$e_0, \ldots, e_k$	$e_1 \cdot e_2 = e_2,$ $e_i \cdot e_0 = e_i,\ 0 \le i \le k$	$\mathcal{S}_{k+1}$
$S_{k+1,m}$ $3 \le m \le k$	$e_0, \ldots, e_k$	$e_0 \cdot e_0 = e_0,\ e_1 \cdot e_2 = e_2,$ $e_i \cdot e_0 = e_i,\ m \le i \le k$	$\mathcal{S}_{k-m+1} \oplus \mathbb{R}^m$
$N_{k+1,2}$ $k \ge 3$	$e_0, \ldots, e_k$	$e_2 \cdot e_2 = e_3,\ e_3 \cdot e_2 = e_1,$ $e_i \cdot e_0 = e_i,\ 0 \le i \le k$	$\mathcal{N}_{k+1}$
$\widetilde{N}_{k+1,m}$ $4 \le m \le k$	$e_0, \ldots, e_k$	$e_0 \cdot e_0 = e_0,$ $e_2 \cdot e_2 = e_3,\ e_3 \cdot e_2 = e_1,$ $e_i \cdot e_0 = e_i,\ m \le i \le k$	$\mathcal{N}_{k-m+1} \oplus \mathbb{R}^m$
L_{k+1}	$e_0, \ldots, e_k$	$e_0 \cdot e_0 = e_0,\ e_0 \cdot e_1 = e_2,$ $e_i \cdot e_0 = e_i,\ 1 \le i \le k$	$\mathcal{L}_{k+1}$
L^{λ}_{k+1}	$e_0, \ldots, e_k$	$e_0 \cdot e_0 = e_0, e_0 \cdot e_1 = \lambda e_1,$ $e_i \cdot e_0 = e_i,\ 1 \le i \le k$	$\mathcal{L}^{\lambda}_{k+1}$

The structures of the Lie algebras appearing in Table 1 are explicitly described in Table 2.

Table 2

Lie algebra	Basis	Non-zero brackets
$\mathbb{R}^k$	$e_1, \ldots, e_k$	
$\mathcal{H}_3$	e_1, e_2, e_3	$[e_1, e_2] = e_3$
$\mathcal{G}_{k+1},\ k \ge 1$	$e_0, \ldots, e_k$	$[e_i, e_0] = e_i,\ 1 \le i \le k$
$\mathcal{N}_{k+1},\ k \ge 3$	$e_0, \ldots, e_k$	$[e_1, e_2] = e_3,$ $[e_i, e_0] = e_i,\ 1 \le i \le k$
$\mathcal{S}_{k+1},\ k \ge 2$	$e_0, \ldots, e_k$	$[e_1, e_2] = e_2,$ $[e_i, e_0] = e_i,\ 1 \le i \le k$
$\mathcal{L}_{k+1},\ k \ge 2$	$e_0, \ldots, e_k$	$[e_1, e_0] = e_1 - e_2,$ $[e_i, e_0] = e_i,\ 2 \le i \le k$
$\mathcal{L}^{\lambda}_{k+1},\ k \ge 2$	$e_0, \ldots, e_k$	$[e_1, e_0] = (1-\lambda)\, e_1,\ \lambda \ne 0,$ $[e_i, e_0] = e_i,\ 2 \le i \le k$

1.1. *Self-adjoint operators of Lorentzian vector spaces*

A Lorentzian vector space $(V, \langle , \rangle)$ is n-dimensional real vector space V endowed with a Lorentzian scalar product $\langle , \rangle$, that is, a nondegenerate symmetric bilinear form of index 1. This means one can find a basis $\{e_1, \ldots, e_n\}$ of V such that $\langle e_1, e_1 \rangle = -1$, $\langle e_i, e_i \rangle = 1$ for $2 \le i \le n$, and $\langle e_i, e_j \rangle = 0$ otherwise. The simplest example for a Lorentzian vector space is Minkowski space $\mathbb{R}^n_1$: $\mathbb{R}^n$ with scalar product $\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + \cdots x_n y_n$.

Let $(V, \langle , \rangle)$ be an n-dimensional real vector space. A non-zero vector X is called spacelike, timelike, or null if $\langle X, X \rangle > 0$, < 0, or $= 0$, respectively. The zero vector is supposed to be spacelike. A subspace $W \subseteq V$ is called nondegenerate, degenerate, spacelike, or Lorentzian if the restriction $\langle , \rangle_{|W}$ of $\langle , \rangle$ to W is nondegenerate, degenerate, positive definite, or indefinite, respectively.

An endomorphism A of $(V, \langle , \rangle)$ is said to be self-adjoint if it satisfies $\langle AX, Y \rangle = \langle X, AY \rangle$ for all $X, Y \in V$. Self-adjoint endomorphisms of a Lorentzian vector space are classified according to the following well known result (see for instance [2, pp. 261–262]).

Lemma 6. *Let A be a self-adjoint endomorphism of an n-dimensional Lorentzian vector space $(V, \langle , \rangle)$. Then, A has a matrix of one of the following four forms:*

(i) $A = D_n$,

(ii) $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \oplus D_{n-2}, \quad b \neq 0,$

(iii) $A = \begin{pmatrix} \lambda & 0 \\ \pm 1 & \lambda \end{pmatrix} \oplus D_{n-2},$

(iv) $A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 1 & 0 & \lambda \end{pmatrix} \oplus D_{n-3},$

where D_k denotes a diagonal matrix diag $\{\lambda_1, \ldots, \lambda_k\}$*. In cases* (i) *and* (ii)*, A is represented with respect to an orthonormal basis $\{e_1, \ldots, e_n\}$ such that $\langle e_1, e_1\rangle = -1$, $\langle e_i, e_i\rangle = 1$, for $2 \leq i \leq n$, and all other products are zeros. In cases* (iii) *and* (iv)*, A is represented with respect to a pseudo-orthonormal basis $\{e_1, \ldots, e_n\}$ such that $\langle e_1, e_2\rangle = \langle e_i, e_i\rangle = 1$, for $3 \leq i \leq n$, and all other products are zeros.*

1.2. Radicals of a Novikov algebra

The notion of radical of a left-symmetric algebra was firstly introduced by J.L. Koszul (see [3]). Given a left-symmetric algebra A over a field $\mathbb{F}$, one defines *the radical $R(A)$* of A to be the largest left ideal contained in the subset

$$I(A) = \{a \in A : \mathrm{tr}\,(R_a) = 0\}.$$

It turns out that $R(A)$ is nothing but the largest complete left ideal of A. In general, the radical of an arbitrary left-symmetric algebra is not a two-sided ideal (cf. [3]). As we will see below (see Proposition 7), the radical of a Novikov algebra over a field of characteristic zero is a two-sided ideal.

Another important notion of radical is that of right radical. Let A be a left-symmetric algebra over a field $\mathbb{F}$ of characteristic zero, and let I be a two-sided ideal of A. We say that I is *right-nilpotent* if there exists some fixed integer $n \geq 1$ such that $R_{a_1} \cdots R_{a_n} = 0$ for all $a_i \in I$. It turns out that the largest right-nilpotent ideal need not exist for an arbitrary left-symmetric algebra, because the sum of any two right-nilpotent ideals need not be right-nilpotent. In the special case of Novikov algebras, it was shown in [1] that a Novikov algebra A has always a unique maximal right-nilpotent two-sided ideal $N(A)$, called the *right radical* of A. In the same paper, it was also shown that $I(A)$ is a two-sided ideal that is right-nilpotent. From these two facts, we can deduce the following:

Proposition 7 (*[4]*)**.** *Let A be a Novikov algebra over a field $\mathbb{F}$ of characteristic zero. Then, we have $N(A) = R(A) = I(A)$.*

We say that A is *complete* if R_x is a nilpotent operator, for all $x \in A$. In this context, if G is an n-dimensional simply connected Lie group with Lie algebra $\mathcal{G}$, then giving a compatible complete left-symmetric product on $\mathcal{G}$ can be interpreted as giving a complete left-invariant affine connection on G or, equivalently, as giving a simply transitive affine action of G on an n-dimensional vector space E. The following important fact is a direct consequence of Proposition 7.

Corollary 8. *A Novikov algebra is right-nilpotent if and only if it is complete.*

1.3. Extensions of left-symmetric algebras

In this section, we shall briefly discuss the problem of extension of a left-symmetric algebra by another left-symmetric algebra. Suppose we are given a vector space A as an extension of a left-symmetric algebra K by another left-symmetric algebra E. We want to define a left-symmetric structure on A in terms of the left-symmetric structures given on K and E. In other words, we want to define a left-symmetric product on A for which E becomes a two-sided ideal in A such that $A/E \cong K$. This is equivalent to let

$$0 \to E \to A \to K \to 0$$

become a short exact sequence of left-symmetric algebras.

Theorem 9 (*[5]*)**.** *There exists a left-symmetric structure on A extending a left-symmetric algebra K by a left-symmetric algebra E if and only if there exist two linear maps $\lambda, \rho : K \to \mathrm{End}(E)$ and a bilinear map $\omega : K \times K \to E$ such that, for all $x, y, z \in K$ and $a, b \in E$, the following conditions are satisfied.*

(i) $\lambda_x (a \cdot b) = \lambda_x(a) \cdot b + a \cdot \lambda_x(b) - \rho_x(a) \cdot b$.
(ii) $\rho_x ([a, b]) = a \cdot \rho_x(b) - b \cdot \rho_x(a)$.
(iii) $\left[\lambda_x, \lambda_y\right] = \lambda_{[x,y]} + L_{\omega(x,y)-\omega(y,x)}$, *where $L_{\omega(x,y)-\omega(y,x)}$ denotes the left multiplication in E by $\omega (x, y) - \omega (y, x)$.*
(iv) $\left[\lambda_x, \rho_y\right] = \rho_{x\cdot y} - \rho_y \circ \rho_x + R_{\omega(x,y)}$, *where $R_{\omega(x,y)}$ denotes the right multiplication in E by $\omega (x, y)$.*
(v) $\omega (x, y \cdot z) - \omega (y, x \cdot z) + \lambda_x (\omega (y, z)) - \lambda_y (\omega (x, z)) - \omega ([x, y], z) - \rho_z (\omega (x, y) - \omega (y, x)) = 0$.

If the conditions of Theorem 9 are fulfilled, then the extended left-symmetric product on $A \cong E \times K$ is given by

$$(a, x) \cdot (b, y) = \left(a \cdot b + \lambda_x(b) + \rho_y(a) + \omega (x, y), x \cdot y\right). \tag{7}$$

It is remarkable that if the left-symmetric product of E is trivial, then the conditions of Theorem 9 simplify to the following three conditions:

(i) $\left[\lambda_x, \lambda_y\right] = \lambda_{[x,y]}$, i.e. λ is a representation of Lie algebras,
(ii) $\left[\lambda_x, \rho_y\right] = \rho_{x\cdot y} - \rho_y \circ \rho_x$.
(iii) $\omega(x, y\cdot z) - \omega(y, x\cdot z) + \lambda_x(\omega(y,z)) - \lambda_y(\omega(x,z)) - \omega([x,y], z) - \rho_z(\omega(x,y) - \omega(y,x)) = 0$.

In this case, E becomes a K-bimodule and the extended product given in (7) simplifies too.

Recall that if K is a left-symmetric algebra and V is a vector space, then we say that V is a K-bimodule if there exist two linear maps $\lambda, \rho : K \to \mathrm{End}(V)$ which satisfy the conditions (i) and (ii) stated above.

Let K be a left-symmetric algebra, and let V be a K-bimodule. Let $L^p(K, V)$ be the space of all p-linear maps from K to V, and define two coboundary operators $\delta_1 : L^1(K, V) \to L^2(K, V)$ and $\delta_2 : L^2(K, V) \to L^3(K, V)$ as follows: for a linear map $h \in L^1(K, V)$ we set

$$\delta_1 h(x, y) = \rho_y(h(x)) + \lambda_x(h(y)) - h(x\cdot y),$$

and for a bilinear map $\omega \in L^2(K, V)$ we set

$$\delta_2\omega(x, y, z) = \omega(x, y\cdot z) - \omega(y, x\cdot z) + \lambda_x(\omega(y,z)) - \lambda_y(\omega(x,z)) - \omega([x,y], z) - \rho_z(\omega(x,y) - \omega(y,x)).$$

It is straightforward to check that $\delta_2 \circ \delta_1 = 0$. Therefore, if we set $Z^2_{\lambda,\rho}(K, V) = \ker\delta_2$ and $B^2_{\lambda,\rho}(K, V) = \mathrm{Im}\,\delta_1$, we can define a notion of second cohomology for the actions λ and ρ by simply setting $H^2_{\lambda,\rho}(K, V) = Z^2_{\lambda,\rho}(K, V)/B^2_{\lambda,\rho}(K, V)$.

As in the case of extensions of Lie algebras, we can prove that for given linear maps $\lambda, \rho : K \to \mathrm{End}(V)$, the equivalence classes of extensions $0 \to V \to A \to K \to 0$ of K by V are in one-to-one correspondence with the elements of the second cohomology group $H^2_{\lambda,\rho}(K, V)$.

1.4. Simple left-symmetric algebras

An algebra A over a field $\mathbb{F}$ is called simple if it has no proper two-sided ideal and A is not the zero algebra of dimension 1. Therefore, since $A^2 = A\cdot A$ is a two-sided ideal of A, we have $A^2 = A$ in case A is simple.

In [1], the following results were proved.

Theorem 10 ([1])**.** *A simple Novikov algebra A over a field $\mathbb{F}$ of characteristic zero is isomorphic to $\mathbb{F}$.*

Proposition 11 ([1])**.** *Let A be a Novikov algebra over a field of characteristic zero. Then, A can be decomposed into a direct sum of ideals $A = \bigoplus_i A_i$, where each ideal A_i is either right-nilpotent or $A_i/N(A_i)$ is a field.*

It is worth mentioning that when the field $\mathbb{F}$ is not algebraically closed, then simple Novikov algebras over $\mathbb{F}$ of dimension ≥ 2 can exist. Here is an example of a two-dimensional simple Novikov algebra over $\mathbb{R}$.

Example 12 (*A Two-Dimensional Simple Novikov Algebra Over* $\mathbb{R}$)**.** Over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let us consider the two-dimensional commutative associative algebra $A_{2,\mathbb{F}}$ defined by the following multiplication table: $e_1\cdot e_1 = e_1, e_1\cdot e_2 = e_2\cdot e_1 = e_2, e_2\cdot e_2 = -e_1$. Being commutative, $A_{2,\mathbb{F}}$ is a Novikov algebra; and by setting $e_1' = \frac{1}{2}(e_1 + ie_2), e_2' = \frac{1}{2}(e_1 - ie_2)$, we can easily see that $A_{2,\mathbb{C}}$ is a direct sum of fields, that is $A_{2,\mathbb{C}} \cong \mathbb{C} \oplus \mathbb{C}$. However, it is not difficult to show that $A_{2,\mathbb{R}}$ is simple.

Remark 13. It is worth pointing out that, in the example above, $A_{2,\mathbb{C}}$ is nothing but the complexification of $A_{2,\mathbb{R}}$. It follows that the complexification of a simple left-symmetric algebra (even Novikov) need not be simple.

Anyway, the following theorem shows that $A_{2,\mathbb{R}}$ is the only simple Novikov algebra over $\mathbb{R}$ of dimension ≥ 2.

Theorem 14 ([4])**.** *A real simple Novikov algebra is isomorphic to either $A_{2,\mathbb{R}}$ or the field $\mathbb{R}$.*

As a consequence of this result, Proposition 11 can be stated in the following more precise form.

Proposition 15. *A Novikov algebra over a field $\mathbb{R}$ can be decomposed into a direct sum of ideals $A = \bigoplus_i A_i$, where each ideal A_i is either right-nilpotent or $A_i/N(A_i)$ is isomorphic to $A_{2,\mathbb{R}}$ or the field $\mathbb{R}$.*

Throughout this paper, we will use the notation $A_{2,2}$ in place of $A_{2,\mathbb{R}}$.

It is clear that a right-nilpotent algebra is necessarily complete. In the case of Novikov algebras, we have the following

Proposition 16 ([4])**.** *A complete Novikov algebra over a field of characteristic zero cannot be simple.*

2. Proof of Theorem 3

From now on, A will be a Novikov real algebra equipped with an invariant Lorentzian symmetric bilinear form $\langle , \rangle$. According to Proposition 15, A can be decomposed into a direct sum of ideals $A = \bigoplus_i A_i$, where each ideal A_i is either complete (i.e., right-nilpotent) or $A_i/N(A_i)$ is isomorphic to $A_{2,\mathbb{R}}$ or the field $\mathbb{R}$. In what follows, for a fixed i, let $\langle , \rangle_{|A_i}$ denote the restriction of $\langle , \rangle$ to A_i.

2.1. The case of a complete ideal A_i

Assume that A_i is complete and $n = \dim A_i \geq 2$. Then, we have $A_i^2 = 0$ by completeness of A_i. We have to consider three cases.

- *Case* 1. $\langle , \rangle_{|A_i}$ is positive definite.

In this case, being self-adjoint with respect to the positive definite scalar product $\langle , \rangle_{|A_i}$, a right multiplication R_x is necessarily diagonalizable relative to an orthonormal basis of A_i. Since R_x is nilpotent, it follows that $R_x = 0$. This means that $A_i^2 = 0$, that is, A_i is isomorphic to $A_{n,0}$.

- *Case* 2. $\langle , \rangle_{|A_i}$ is degenerate.

In this case, write $A_i = \mathbb{R}e_0 \oplus \overline{A}_i$, where e_0 is a null vector (i.e., $\langle e_0, e_0 \rangle = 0$) and $\overline{A}_i$ is spacelike, and let $\overline{A}_i^{\perp}$ denote the orthogonal to $\overline{A}_i$ in A. Note here that $e_1 \in \bigoplus_{j\neq i} A_j$ and

$$\overline{A}_i^{\perp} \subseteq \mathbb{R}e_0 \oplus \left(\bigoplus_{j\neq i} A_j \right).$$

Lemma 17. *With the notation above, for all $x, y \in A_i$ and $z \in \bigoplus_{j\neq i} A_j$, we have*

(i) $R_x z = 0$.
(ii) $R_x e_0 = 0$.
(iii) $R_x y \in \overline{A}_i$.

Proof. Assertion (i) follows from the fact that $A_i \cdot A_j = 0$ for all $i \neq j$, given that A is a direct sum of ideals $A = \bigoplus_i A_i$.

To prove (ii), we note that since A_i is an ideal we have $R_x y \in A_i$ for all $x \in A_i$ and $y \in A$. It follows that $\langle R_x e_0, y \rangle = \langle e_0, R_x y \rangle = 0$, that is, $R_x e_0 = 0$ for all $x \in A_i$.

To prove (iii), we note that since $\overline{A}_i$ is spacelike, then we may choose another null vector e_1 in $\overline{A}_i^{\perp}$ such that $\langle e_0, e_1 \rangle = 1$. Since $e_1 \in \bigoplus_{j\neq i} A_j$, then by (i), we have $R_x e_1 = 0$. It follows that $\langle R_x y, e_1 \rangle = \langle y, R_x e_1 \rangle = 0$, for all $x, y \in A_i$. Since $R_x y \in A_i = \mathbb{R}e_0 \oplus \overline{A}_i$, we deduce that $R_x y \in \overline{A}_i$. ■

For $x \in A_i$, let $R_{x|_{\overline{A}_i}}$ denote the restriction of R_x to $\overline{A}_i$. By assertion (iii) of Lemma 17, we have $R_{x|_{\overline{A}_i}} : \overline{A}_i \to \overline{A}_i$, and since R_x is a nilpotent self-adjoint operator and $\langle , \rangle_{|_{\overline{A}_i}}$ is positive definite, we deduce that $R_{x|_{\overline{A}_i}} = 0$. It follows from this and assertion (ii) of Lemma 17 that $A_i^2 = 0$, that is, A_i is isomorphic to $A_{n,0}$.

- *Case* 3. $\langle , \rangle_{|A_i}$ is indefinite.

In this case, being self-adjoint, the operator $R_x : A_i \to A_i$ can take one of the four possible forms of Lemma 6, for all $x \in A_i$.

Let $x \in A_i$. Since R_x is nilpotent, we deduce two thins. First, R_x cannot admit complex eigenvalues, that is, it cannot take form (ii) of Lemma 6. Second, $R_x = 0$ if R_x is diagonalizable, that is, it has the form (i) of Lemma 6.

If R_x has the form (iii) of Lemma 6, then we easily deduce that R_x simplifies to

$$R_x = \begin{pmatrix} 0 & 0 \\ \pm 1 & 0 \end{pmatrix} \oplus 0_{n-2}.$$

If R_x has the form (iv) of Lemma 6, then we easily deduce that R_x simplifies to

$$R_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \oplus 0_{n-3}.$$

Since A is Novikov, the operators R_x should commute with each other. Since, as we can easily verify, the above matrices do not commute, it follows that the operators R_x are either all of the form

$$R_x = \begin{pmatrix} 0 & 0 \\ \lambda(x) & 0 \end{pmatrix} \oplus 0_{n-2}, \tag{8}$$

or all of the form

$$R_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda(x) \\ \lambda(x) & 0 & 0 \end{pmatrix} \oplus 0_{n-3}, \tag{9}$$

for some real function $\lambda(x)$.

Assume first that R_x is of the form (8) for all $x \in A_i$. In this case, we get $e_1 \cdot x = \lambda(x)e_2$ and all other products are zeros. It follows that there exists a basis $e_1, \ldots, e_n$ of A_i such that $e_1 \cdot e_i = \lambda_i e_2$, with $\lambda_i \in \mathbb{R}$, and all other products are zeros. We distinguish three cases.

If $\lambda_2 = \cdots = \lambda_n = 0$, then the associated Lie algebra $\mathcal{G}_i$ is isomorphic to $\mathbb{R}^n$, and the structure of A_i is given by

$$e_1 \cdot e_1 = \lambda e_2, \quad \lambda \in \mathbb{R},$$

and all other products are zeros. We deduce that A_i is isomorphic to either $A_{n,0}$ or $A_{2,1} \oplus A_{n-2,0}$.

If $\lambda_2 \neq 0$, then by setting $e_i' = e_i - \frac{\lambda_i}{\lambda_2} e_2$ for all $i \geq 4$, we get $e_1 \cdot e_i' = 0$ for all $i \geq 4$. It follows that the structure of A_i is given by

$$e_1 \cdot e_1 = \lambda_1 e_2, \qquad e_1 \cdot e_2 = \lambda_2 e_2, \quad \text{with } \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } \lambda_2 \neq 0,$$

and all other products are zeros. Again, by setting $e_1' = \frac{1}{\lambda_2}\left(e_1 - \frac{\lambda_i}{\lambda_2} e_2\right)$, we get $e_1' \cdot e_1' = 0$. It follows that A_i is isomorphic to

$$S_2 \oplus A_{n-2,0} = \langle e_1, \ldots, e_n : e_1 \cdot e_2 = e_2 \rangle,$$

and the associated Lie algebra $\mathcal{G}_i$ is isomorphic to $\mathcal{G}_2 \oplus \mathbb{R}^{n-2}$.

If $\lambda_2 = 0$ and $\lambda_{i_0} \neq 0$ for some $i_0 \geq 3$, say $i_0 = 3$ to simplify, then by setting $e_i' = e_i - \frac{\lambda_i}{\lambda_3} e_3$ for all $i \geq 4$, we get $e_1 \cdot e_i' = 0$ for all $i \geq 4$. It follows that the structure of A_i is given by

$$e_1 \cdot e_1 = \lambda_1 e_2, \qquad e_1 \cdot e_3 = \lambda_2 e_2, \quad \text{with } \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } \lambda_2 \neq 0,$$

and all other products are zeros. Again, by setting $e_1' = e_1 - \lambda_1 e_2$, we see that A_i is isomorphic to $N_{3,1} \oplus A_{n-3,0}$. The associated Lie algebra $\mathcal{G}_i$ is isomorphic to $\mathcal{H}_3 \oplus \mathbb{R}^{n-3}$.

Assume now that R_x is of the form (9) for all $x \in A_i$. In this case, we get $e_2 \cdot x = \lambda(x)e_3$, $e_3 \cdot x = \lambda(x)e_1$, and all other products are zeros. It follows that there exists a basis $e_1, \ldots, e_n$ of A_i such that $e_2 \cdot e_i = \lambda_i e_2$, $e_3 \cdot e_i = \lambda_i e_1$, with $\lambda_i \in \mathbb{R}$, and all other products are zeros. By left-symmetry we have

$$(e_2 \cdot e_i) \cdot e_i - e_2 \cdot (e_i \cdot e_i) = (e_i \cdot e_2) \cdot e_i - e_i \cdot (e_2 \cdot e_i),$$

which yields

$$\lambda_i (\lambda_i e_1 + e_i \cdot e_3) = e_2 \cdot (e_i \cdot e_i) + (e_i \cdot e_2) \cdot e_i.$$

For $i = 3$, the above equality yields $\lambda_3 = 0$, and for $i = 1$ or $i \geq 4$ it yields $\lambda_i = 0$. It follows that there exists a basis $e_1, \ldots, e_n$ of A_i such that

$$e_2 \cdot e_2 = \lambda e_3, \qquad e_3 \cdot e_2 = \lambda e_1, \quad \text{with } \lambda \in \mathbb{R},$$

and all other products are zeros. We deduce that A_i is isomorphic to either $A_{n,0}$ or $N_{3,2} \oplus A_{n-3,0}$, and the associated Lie algebra $\mathcal{G}_i$ is isomorphic to either $\mathbb{R}^n$ or $\mathcal{H}_3 \oplus \mathbb{R}^{n-3}$.

2.2. The case of a non-complete ideal A_i

For a fixed i, assume that A_i is non-complete and let $\langle , \rangle_{|_{N(A_i)}}$ denote the restriction of $\langle , \rangle$ to $N(A_i)$, respectively. We can analyze as we did in the complete case, but we prefer to proceed as follows. We know, by Proposition 15, that $A_i/N(A_i)$ is isomorphic to $A_{2,2}$ or the field $\mathbb{R}$. Accordingly, we must consider two cases.

2.2.1. The case when $A_i/N(A_i)$ is isomorphic to $A_{2,2}$

In this case, we first claim that A_i is necessarily Lorentzian (hence nondegenerate). This is a direct consequence of the following lemma.

Lemma 18. *Any invariant nondegenerate symmetric bilinear form on $A_{2,2}$ is necessarily Lorentzian.*

Proof. Recall that $A_{2,2}$ has a basis e_1, e_2 such that $e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = e_2 \cdot e_1 = e_2$, $e_2 \cdot e_2 = -e_1$, and let $\langle , \rangle$ be an arbitrary invariant nondegenerate symmetric bilinear form on $A_{2,2}$. By invariance of $\langle , \rangle$, we have $\langle R_{e_2} e_2, e_1 \rangle = \langle e_2, R_{e_2} e_1 \rangle$, which yields $\langle e_2, e_2 \rangle = -\langle e_1, e_1 \rangle$. It follows that the symmetric matrix of $\langle , \rangle$ has the form

$$M_{\langle , \rangle} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

where $a = \langle e_1, e_1 \rangle$ and $b = \langle e_1, e_2 \rangle$. Its discriminant is $\det M_{\langle , \rangle} = -\left(a^2 + b^2\right) < 0$, and therefore $\langle , \rangle$ is Lorentzian. ∎

Since $N(A_i)$ is a two-sided ideal of A_i, we get a short exact sequence of left-symmetric algebras

$$0 \to N(A_i) \to A_i \to A_{2,2} \to 0.$$

Setting $A_i = N(A_i) \oplus A_{2,2}$ as a vector space, there exist two linear maps $\lambda, \rho : A_{2,2} \to \mathrm{End}(N(A_i))$ and a bilinear map $\omega : A_{2,2} \times A_{2,2} \to N(A_i)$ such that the left-symmetric product of A_i can be written as

$$(x, u) \cdot (y, v) = (x \cdot y + \lambda_u(y) + \rho_v(x) + \omega(u, v), u \cdot v), \tag{10}$$

for all $x, y \in N(A_i)$ and $u, v \in A_{2,2}$.

By the invariance of $\langle , \rangle$, we have $\langle (x, u) \cdot (y, v), (z, w) \rangle = \langle (x, u), (z, w) \cdot (y, v) \rangle$. Since the restriction of $\langle , \rangle$ to both $N(A_i)$ and $A_{2,2}$ is nondegenerate, we can, without loss of generality, write $\langle (x, u), (y, v) \rangle = \langle x, y \rangle + \langle u, v \rangle$. Thus, the above equality yields

$$\langle \lambda_u(y) + \rho_v(x) + \omega(u, v), z \rangle = \langle x, \lambda_w(y) + \rho_v(z) + \omega(w, v) \rangle. \tag{11}$$

By substituting $z = x$ and $y = 0$ into Eq. (11), it follows that $\omega(u, v) = \omega(w, v)$ for all $u, v, w \in A_{2,2}$, from which we deduce that $\omega = 0$. Again, by taking $z = x$ in (11), we get $\lambda_u(y) = \lambda_w(y)$ for all $y \in N(A_i)$ and $u, v \in A_{2,2}$, from which we deduce that $\lambda = 0$. Thus, the left-symmetric product (10) simplifies as follows:

$$(x, u) \cdot (y, v) = (x \cdot y + \rho_v(x), u \cdot v), \tag{12}$$

for all $x, y \in N(A_i)$ and $u, v \in A_{2,2}$.

Since $N(A_i)$ is complete (i.e., right nilpotent) and $\langle , \rangle_{|N(A_i)}$ is positive definite, we deduce from what we have did previously in the complete case that $N(A_i)^2 = 0$, that is, $N(A_i)$ is isomorphic to $A_{n,0}$.

Since $\lambda = 0$ and $\omega = 0$, it follows that (iv) of Theorem 9 simplifies to

$$\rho_{u \cdot v} = \rho_v \circ \rho_u,$$

for all $u, v \in A_{2,2}$. In particular, we have

$$\rho_{e_1}^2 = \rho_{e_1}, \qquad \rho_{e_2}^2 = -\rho_{e_1}, \qquad \rho_{e_1} \circ \rho_{e_2} = \rho_{e_2} \circ \rho_{e_1} = \rho_{e_2}. \tag{13}$$

By the last identity of (13), ρ_{e_1} and ρ_{e_2} commute. It follows that they are simultaneously diagonalizable. Since they are self-adjoint with respect to the positive definite scalar product $\langle , \rangle_{|N(A_i)}$, there exists an orthonormal basis $e_3, \ldots, e_n$ of $N(A_i)$ relative to which $\rho_{e_1} = \mathrm{diag}\{\lambda_3, \ldots, \lambda_n\}$ and $\rho_{e_2} = \mathrm{diag}\{\mu_3, \ldots, \mu_n\}$, where here $n = \dim A_i$.

From the first two identities of (13), we get $\lambda_j^2 + \mu_j^2 = 0$ for all $j = 3, \ldots, n$. Consequently, we have $\rho_{e_1} = \rho_{e_2} = 0$, that is, $\rho = 0$. We deduce, using (12), that

$$A_i = A_{2,2} \overset{\perp}{\oplus} A_{n-2,0},$$

that is, A_i is an orthogonal direct sum of the algebras $A_{2,2}$ and $A_{n-2,0}$.

2.2.2. *The case when $A_i/N(A_i)$ is isomorphic to the field $\mathbb{R}$*

Unless otherwise stated, we will use the following notation throughout this subsubsection. We consider the field $\mathbb{R}$ as a one-dimensional algebra over itself, and we let $n = \dim N(A_i)$. In this case, we have a short exact sequence of left-symmetric algebras

$$0 \to N(A_i) \to A_i \to \mathbb{R} \to 0.$$

Setting $A_i = N(A_i) \oplus \mathbb{R}$ as a vector space, and letting 1 denote the unit element of the field $\mathbb{R}$, we will adopt the notation (x, a) to denote elements of A_i. In particular, setting $e_0 = (0, 1)$, we have $e_0 = e_0 \cdot e_0$. According to the above short exact sequence, there exist two endomorphisms $\lambda, \rho : N(A_i) \to N(A_i)$ and a bilinear map $\omega : \mathbb{R}^2 \to N(A_i)$ such that the left-symmetric product of A_i can be written as

$$(x, a) \cdot (y, b) = (x \cdot y + a\lambda(y) + b\rho(x) + ab\omega(1, 1), ab), \tag{14}$$

for all $x, y \in N(A_i)$ and $a, b \in \mathbb{R}$. Consequently, we have $e_0 = e_0 \cdot e_0 = (\omega(1, 1), 0) + e_0$, and we conclude that $\omega = 0$.

Now, to handle the present case, we have to consider three cases depending on the signature of $\langle , \rangle_{|A_i}$.

The case when $\langle , \rangle_{|A_i}$ is positive definite. In this case, $\langle , \rangle_{|N(A_i)}$ is positive definite as well, and we can write $\langle (x, a), (y, b) \rangle = \langle x, y \rangle + ab$ for all $x, y \in N(A_i)$ and $a, b \in \mathbb{R}$.

By the invariance of $\langle , \rangle$, we have $\langle (x, a) \cdot (y, b), (z, c) \rangle = \langle (x, a), (z, c) \cdot (y, b) \rangle$. This latter equation, by virtue of (14), yields the following equation

$$\langle a\lambda(y) + b\rho(x), z \rangle = \langle x, c\lambda(y) + b\rho(x) \rangle.$$

Now, we can proceed in the same way as we did in the case that $A_i/N(A_i)$ is isomorphic to $A_{2,2}$ to show that $\lambda = 0$ and ρ is self-adjoint. As a consequence, we conclude, by virtue of (iv) of Theorem 9, that $\rho^2 = \rho$.

On the other hand, since $N(A_i)$ is complete and the R_x are self-adjoint with respect to the positive definite scalar product $\langle , \rangle_{|_{N(A_i)}}$, we deduce as in the complete case that $N(A_i)^2 = 0$, that is, $N(A_i)$ is isomorphic to $A_{n,0}$. We also deduce that the left-symmetric product of A_i given by (14) reduces to

$$(x, a) \cdot (y, b) = (b\rho(x), ab) . \tag{15}$$

Since ρ is self-adjoint with respect to the positive definite scalar product $\langle , \rangle_{|_{N(A_i)}}$, then we may assume without loss that there exists an orthonormal basis $\bar{e}_1, \ldots, \bar{e}_n$ of $N(A_i)$ with respect to which $\rho = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$. From the equation $\rho^2 = \rho$, we deduce that $\lambda_i^2 = \lambda_i$ (i.e., $\lambda_i = 0$ or 1) for all $i = 1, \ldots, n$. Thus, without loss of generality, we may assume that $\rho = Id_m \oplus 0_{n-m}$, for some integer $m \leq n$, where Id_m is the $m \times m$ identity matrix.

Setting $e_0 = (0, 1)$ and $e_i = (\bar{e}_i, 0)$ for $i = 1, \ldots, n$, and using (15), we get

$$e_i \cdot e_0 = e_i, \quad 0 \leq i \leq m,$$

and all other products are zeros. Thus, A_i is isomorphic to $F_{m+1,0} \oplus A_{n-m-1,0}$.

We deduce that the Lie algebra $\mathcal{G}_{A_i}$ associated to A_i has the property that $[x, y] = ax + bx$, for all x, y, where a and b are real constants. According to [6], $\mathcal{G}_{A_i}$ contains a codimension one commutative ideal E and an element $e_0 \notin E$ such that $[e_0, x] = x$ for all $x \in E$.

The case when $\langle , \rangle_{|_{A_i}}$ is Lorentzian and $\langle , \rangle_{|_{N(A_i)}}$ is non-degenerate. In this case, we can write $\langle (x, a), (y, b)\rangle = \langle x, y\rangle + \varepsilon ab$ for all $x, y \in N(A_i)$ and $a, b \in \mathbb{R}$, where $\varepsilon = 1$ or -1 according to whether the restriction of $\langle , \rangle$ to $N(A_i)$ is indefinite or positive definite, respectively.

On the other hand, in a similar fashion as we did in Case 1 of Section 2.1, we can deduce by the invariance of $\langle , \rangle$ that $\lambda = 0$ and ρ is self-adjoint. As a consequence, we conclude, by virtue of (iv) of Theorem 9, that

$$\rho^2 = \rho. \tag{16}$$

We have to consider three subcases.

- *Subcase* 2.1. $\langle , \rangle_{|_{N(A_i)}}$ is positive definite.

In this subcase, since $N(A_i)$ is complete and all R_x are self-adjoint with respect to the positive definite scalar product $\langle , \rangle_{|_{N(A_i)}}$, we deduce that $N(A_i)^2 = 0$. Now, as we did in the case when $\langle , \rangle_{|_{A_i}}$ is positive definite, we can use (16) and the fact that ρ is self-adjoint with respect to the positive definite scalar product $\langle , \rangle_{|_{N(A_i)}}$ to prove that A_i is isomorphic to $F_{m+1,0} \oplus A_{n-m-1,0}$.

- *Subcase* 2.2. $\langle , \rangle_{|_{N(A_i)}}$ is indefinite.

In this subcase, since $\langle , \rangle_{|_{N(A_i)}}$ is indefinite, the operator ρ may not be diagonalizable although it is self-adjoint with respect to $\langle , \rangle_{|_{N(A_i)}}$. However, with Eq. (16) in hand, we can show that ρ is indeed diagonalizable.

Claim 19. *The operator ρ is diagonalizable.*

Proof. According to Lemma 6, there are four possible forms for ρ.

If ρ is not diagonalizable and takes the form (ii) of Lemma 6 (i.e., ρ has a complex eigenvalue), then with the notation of that lemma we get from Eq. (16) that $a^2 - b^2 = a$, $2ab = b$, and $\lambda_i^2 = \lambda_i$ for all $i = 1, \ldots, n-2$. Obviously, the first two equations are incompatible. Thus, this case cannot occur.

Similarly, we can show that the case when ρ is not diagonalizable and takes one of the forms (iii) or (iv) of Lemma 6 cannot occur too. ■

By the above claim, ρ is diagonalizable. As in the case when $\langle , \rangle_{|_{A_i}}$ is positive definite, and by using (16), we may assume without of generality that $\rho = Id_m \oplus 0_{n-m}$, for some integer $m \leq n$. Setting $N(A_i) = V_i \oplus \ker \rho$, where $V_i = \ker(\rho - \mathrm{Id})$, we have $\dim V_i = m$ and $\rho_{|_{V_i}} = Id_m$.

With the notation as at the beginning of Section 2.2.2, the left-symmetric product of A_i given by (14) reduces to

$$(x, a) \cdot (y, b) = (x \cdot y + b\rho(x), ab) , \tag{17}$$

where ρ is here represented by some matrix of the form $Id_m \oplus 0_{n-m}$. Of course, the reduced form $\rho = Id_m \oplus 0_{n-m}$ is not necessarily with respect to the basis $\bar{e}_1, \ldots, \bar{e}_n$ above.

On the other hand, since $N(A_i)$ is complete and $\langle , \rangle_{|_{N(A_i)}}$ is Lorentzian, we deduce from what we obtained in Case 3 of Section 2.1 that $N(A_i)$ is isomorphic to either $A_{n,0}$, $A_{2,1} \oplus A_{n-2,0}$, $S_2 \oplus A_{n-2,0}$, $N_{3,1} \oplus A_{n-3,0}$, or $N_{3,2} \oplus A_{n-3,0}$. Accordingly, we have five subsubcases to consider.

- *Subsubcase* 2.2.1. $N(A_i)$ is isomorphic to $A_{n,0}$.

In a similar manner to the case when $\langle , \rangle_{|_{A_i}}$ is positive definite, we can prove that A_i is isomorphic to $F_{m+1,0} \oplus A_{n-m-1,0}$.

- *Subsubcase* 2.2.2. $N(A_i)$ is isomorphic to $A_{2,1} \oplus A_{n-2,0}$.

Recall, from Case 3 of Section 2.1, that the structure of $A_{2,1} \oplus A_{n-2,0}$ is initially given as follows. There exists a pseudo-orthonormal basis $\bar{e}_1, \ldots, \bar{e}_n$ of $N(A_i)$, with $\bar{e}_1$ and $\bar{e}_2$ null satisfying $\langle \bar{e}_1, \bar{e}_2 \rangle = 1$, and such that $\bar{e}_1 \cdot \bar{e}_1 = \bar{e}_2$ is the only nonzero product. With notation as above, we claim that we have either $\bar{e}_1, \bar{e}_2 \in V_i$ or $\bar{e}_1, \bar{e}_2 \notin V_i$; that is $\varepsilon_1 = \varepsilon_2$. For indeed, if for example $\bar{e}_1 \in V_i$ and $\bar{e}_2 \notin V_i$, then since ρ is self-adjoint and $\langle \bar{e}_1, \bar{e}_2 \rangle = 1$ we get $1 = \langle \rho(\bar{e}_1), \bar{e}_2 \rangle = \langle \bar{e}_1, \rho(\bar{e}_2) \rangle = 0$, a contradiction.

If we set $e_0 = (0, 1)$ and $e_i = (\bar{e}_i, 0)$ for $i = 1, \ldots, n$, then by using (17), we see that the structure of A_i is given by

$$e_0 \cdot e_0 = e_0, \qquad e_1 \cdot e_1 = e_2, \qquad e_i \cdot e_0 = \varepsilon_i e_i,$$

where $\varepsilon_i = \rho(e_i) = 0$ or 1. Taking into account the above claim, we conclude that there are exactly two possible non-isomorphic structures for A_i which are given as follows.

1. There exists an integer $m \geq 2$ such that the multiplication table of A_i is given by

$$e_1 \cdot e_1 = e_2, \qquad e_i \cdot e_0 = e_i, \quad 0 \leq i \leq m.$$

2. There exists an integer $m \geq 3$ such that the multiplication table of A_i is given by

$$e_0 \cdot e_0 = e_0, \qquad e_1 \cdot e_1 = e_2, \qquad e_i \cdot e_0 = e_i, \quad m \leq i \leq n-1.$$

In the first case, A_i is isomorphic to $N_{m+1,2} \oplus A_{n-m-1,0}$. In the second case, A_i is isomorphic to $\widetilde{N}_{n,m}$. For both structures, the associated Lie algebra $\mathcal{G}_{A_i}$ is isomorphic to a direct product of the form $E \oplus \mathbb{R}e_0 \oplus \mathbb{R}^k$, where E is a commutative ideal such that $[x, e_0] = x$ for all $x \in E$.

- *Subsubcase* 2.2.3. $N(A_i)$ is isomorphic to $S_2 \oplus A_{n-2,0}$.

Recall, from Case 3 of Section 2.1, that the structure of $S_2 \oplus A_{n-2,0}$ is initially given as follows. There exist $\lambda_1, \lambda_2 \in \mathbb{R}$, with $\lambda_2 \neq 0$, and a pseudo-orthonormal basis $\bar{e}_1, \ldots, \bar{e}_n$ of $N(A_i)$, with $\bar{e}_1$ and $\bar{e}_2$ null and satisfying $\langle \bar{e}_1, \bar{e}_2 \rangle = 1$, such that $\bar{e}_1 \cdot \bar{e}_1 = \lambda_1 \bar{e}_2$ and $\bar{e}_1 \cdot \bar{e}_2 = \lambda_2 \bar{e}_2$ are the only nonzero products.

If we set $e_0 = (0, 1)$ and $e_i = (\bar{e}_i, 0)$ for $i = 1, \ldots, n$, then by using (17), we see that the structure of A_i is given by

$$e_0 \cdot e_0 = e_0, \qquad e_1 \cdot e_1 = \lambda_1 e_2, \qquad e_1 \cdot e_2 = \lambda_2 e_2, \qquad e_i \cdot e_0 = \varepsilon_i e_i,$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_2 \neq 0$, and $\varepsilon_i = \rho(e_i) = 0$ or 1.

By replacing e_1 with $e_1' = \frac{1}{\lambda_2}\left(e_1 - \frac{\lambda_1}{\lambda_2} e_2\right)$, we get $e_1' \cdot e_1' = 0$, $e_1' \cdot e_2 = e_2$, and $e_1' \cdot e_0 = \frac{1}{\lambda_2}\left(\varepsilon_1 e_1 - \frac{\lambda_1}{\lambda_2} \varepsilon_2 e_2\right) = \frac{\varepsilon_1}{\lambda_2}\left(e_1 - \frac{\lambda_1}{\lambda_2} e_2\right) = \varepsilon_1 e_1'$ since, similar to Section 2.2.2 we can easily claim that we have either $\bar{e}_1, \bar{e}_2 \in V_i$ or $\bar{e}_1, \bar{e}_2 \notin V_i$; that is $\varepsilon_1 = \varepsilon_2$. Accordingly, we conclude that there are exactly two possible non-isomorphic structures for A_i which are given as follows.

1. There exists an integer $m \geq 2$ such that the multiplication table of A_i is given by

$$e_1 \cdot e_2 = e_2, \qquad e_i \cdot e_0 = e_i, \quad 0 \leq i \leq m.$$

2. There exists an integer $m \geq 3$ such that the multiplication table of A_i is given by

$$e_0 \cdot e_0 = e_0, \qquad e_1 \cdot e_2 = e_2, \qquad e_i \cdot e_0 = e_i, \quad m \leq i \leq n.$$

In the first case, A_i is isomorphic to $S_{m+1,2} \oplus A_{n-m-1,0}$. In the second case, A_i is isomorphic to $\widetilde{N}_{m+1,n}$. In both cases, the associated Lie algebra $\mathcal{G}_{A_i}$ is isomorphic to a direct product of the form $\mathcal{G}_2 \oplus E \oplus \mathbb{R}e_0 \oplus \mathbb{R}^k$, where E is a commutative ideal such that $[x, e_0] = x$ for all $x \in \mathcal{G}_2 \oplus E$ in the case of the first structure (resp. $x \in E$ in the case of the second structure).

- *Subsubcase* 2.2.4. $N(A_i)$ is isomorphic to $N_{3,1} \oplus A_{n-3,0}$.

Recall, from Case 3 of Section 2.1, that the structure of $N_{3,1} \oplus A_{n-3,0}$ is initially given as follows. There exist $\lambda_1, \lambda_2 \in \mathbb{R}$, with $\lambda_2 \neq 0$, and a pseudo-orthonormal basis $\bar{e}_1, \ldots, \bar{e}_n$ of $N(A_i)$, with $\bar{e}_1$ and $\bar{e}_2$ null and satisfying $\langle \bar{e}_1, \bar{e}_2 \rangle = 1$, such that $\bar{e}_1 \cdot \bar{e}_1 = \lambda_1 \bar{e}_2$ and $\bar{e}_1 \cdot \bar{e}_3 = \lambda_2 \bar{e}_2$ are the only nonzero products.

If we set $e_0 = (0, 1)$ and $e_i = (\bar{e}_i, 0)$ for $i = 1, \ldots, n$, then by using (17), we see that the structure of A_i is given by

$$e_0 \cdot e_0 = e_0, \qquad e_1 \cdot e_1 = \lambda_1 e_2, \qquad e_1 \cdot e_3 = \lambda_2 e_2, \qquad e_i \cdot e_0 = \varepsilon_i e_i,$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_2 \neq 0$, and $\varepsilon_i = 0$ or 1.

Claim 20. *We have* $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$.

Proof. Similar to Section 2.2.2 we can easily claim that we have either $\bar{e}_1, \bar{e}_2 \in V_i$ or $\bar{e}_1, \bar{e}_2 \notin V_i$; that is $\varepsilon_1 = \varepsilon_2$. On the other hand, since the product on A_i is left-symmetry, we have $(e_1 \cdot e_3) \cdot e_0 - e_1 \cdot (e_3 \cdot e_0) = (e_3 \cdot e_1) \cdot e_0 - e_3 \cdot (e_1 \cdot e_0) = 0 \cdot e_0 - \varepsilon_1 e_3 \cdot e_1 = 0$, from which we get $\varepsilon_2 \lambda_2 e_2 = \varepsilon_3 \lambda_2 e_2$. Since $\lambda_2 \neq 0$, we deduce that $\varepsilon_2 = \varepsilon_3$. ■

Now, by replacing e_1 with $e'_1 = e_1 - \frac{\lambda_1}{\lambda_2} e_3$ and using the above claim, we easily verify that $e'_1 \cdot e'_1 = 0$, $e'_1 \cdot e_3 = e_2$, and $e'_1 \cdot e_0 = \varepsilon_1 e_1 - \frac{\lambda_1}{\lambda_2} \varepsilon_3 e_3 = \varepsilon_1 e'_1$. We conclude that there are exactly two possible non-isomorphic structures for A_i which are given as follows.

1. There exists an integer $m \geq 3$ such that the multiplication table of A_i is given by
$$e_1 \cdot e_3 = e_2, \qquad e_i \cdot e_0 = e_i, \quad 0 \leq i \leq m.$$
2. There exists an integer $m \geq 4$ such that the multiplication table of A_i is given by
$$e_0 \cdot e_0 = e_0, \qquad e_1 \cdot e_3 = e_2, \qquad e_i \cdot e_0 = e_i, \quad m \leq i \leq n.$$

For both structures, the associated Lie algebra $\mathcal{G}_{A_i}$ is isomorphic to a direct product of the form $\mathcal{H}_3 \oplus E \oplus \mathbb{R}e_0 \oplus \mathbb{R}^k$, where E is a commutative ideal such that $[x, e_0] = x$ for all $x \in \mathcal{H}_2 \oplus E$ in the case of the first structure (resp. $x \in E$ in the case of the second structure).

- *Subsubcase* 2.2.5. $N(A_i)$ is isomorphic to $N_{3,2} \oplus A_{n-3,0}$.

Recall, from Case 3 of Section 2.1, that the structure of $N_{3,2} \oplus A_{n-3,0}$ is initially given as follows. There exists $0 \neq \lambda \in \mathbb{R}$, and a pseudo-orthonormal basis $\bar{e}_1, \ldots, \bar{e}_n$ of $N(A_i)$, with $\bar{e}_1$ and $\bar{e}_2$ null and satisfying $\langle \bar{e}_1, \bar{e}_2 \rangle = 1$, such that $\bar{e}_2 \cdot \bar{e}_2 = \lambda \bar{e}_3$ and $\bar{e}_3 \cdot \bar{e}_2 = \lambda \bar{e}_1$ are the only nonzero products.

If we set $e_0 = (0, 1)$ and $e_i = (\bar{e}_i, 0)$ for $i = 1, \ldots, n$, then by using (17), we see that the structure of A_i is given by
$$e_0 \cdot e_0 = e_0, \qquad e_2 \cdot e_2 = \lambda e_3, \qquad e_3 \cdot e_2 = \lambda e_1, \qquad e_i \cdot e_0 = \varepsilon_i e_i,$$
with $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $\varepsilon_i = 0$ or 1.

Claim 21. *We have* $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$.

Proof. Similar to Section 2.2.2 we can easily claim that we have either $\bar{e}_1, \bar{e}_2 \in V_i$ or $\bar{e}_1, \bar{e}_2 \notin V_i$; that is $\varepsilon_1 = \varepsilon_2$. On the other hand, since the product on A_i is Novikov, we have $(e_3 \cdot e_2) \cdot e_0 = (e_3 \cdot e_0) \cdot e_2$, from which we get $\varepsilon_1 \lambda e_1 = \varepsilon_3 \lambda e_1$. Since $\lambda \neq 0$, we deduce that $\varepsilon_1 = \varepsilon_3$. ■

Now, by replacing e_i with $\frac{1}{\lambda} e_i$ for $1 \leq i \leq 3$ and using the above claim, we conclude that there are exactly two possible non-isomorphic structures for A_i which are given as follows.

1. There exists an integer $m \geq 3$ such that the multiplication table of A_i is given by
$$e_2 \cdot e_2 = e_3, \qquad e_3 \cdot e_2 = e_1, \qquad e_i \cdot e_0 = e_i, \quad 0 \leq i \leq m.$$
2. There exists an integer $m \geq 4$ such that the multiplication table of A_i is given by
$$e_0 \cdot e_0 = e_0, \qquad e_2 \cdot e_2 = e_3, \qquad e_3 \cdot e_2 = e_1, \qquad e_i \cdot e_0 = e_i, \quad m \leq i \leq n.$$

For both structures, the associated Lie algebra $\mathcal{G}_{A_i}$ is isomorphic to a direct product of the form $\mathcal{H}_3 \oplus E \oplus \mathbb{R}e_0 \oplus \mathbb{R}^k$, where E is a commutative ideal such that $[x, e_0] = x$ for all $x \in \mathcal{H}_2 \oplus E$ in the case of the first structure (resp. $x \in E$ in the case of the second structure).

The case when $\langle , \rangle_{|A_i}$ *is Lorentzian and* $\langle , \rangle_{|N(A_i)}$ *is degenerate*. To avoid confusion, we will omit the subscript i of A_i and simply write A. It is clear that we can write $A = N(A) \oplus \mathbb{R}e_0$ for some null vector e_0 satisfying $e_0 \cdot e_0 = e_0$. Furthermore, there exists another null vector $e_1 \in N(A)$ such that $\langle e_0, e_1 \rangle = 1$. Let V be the orthogonal complement of e_1 in $N(A)$. Obviously, $\langle , \rangle_{|V}$ is positive definite, so that we have the orthogonal decomposition $A = N(A) \oplus \operatorname{span}\{e_0, e_1\}$. It follows that the scalar product on A is given by
$$\langle (x + ae_1 + \alpha e_0), (y + be_1 + \beta e_0) \rangle = \langle x, y \rangle + \alpha b + a\beta,$$
and since $N(A)^2 = 0$ (see Case 2 of Section 2.1), the left-symmetric product on A is given by
$$(x + ae_1 + \alpha e_0) \cdot (y + be_1 + \beta e_0) = \alpha\lambda(y + be_1) + \beta\rho(x + ae_1) + \alpha\beta e_0, \tag{18}$$
where here $x, y \in V$ and $a, b, \alpha, \beta \in \mathbb{R}$.

Let $e_2, \ldots, e_n$ be an orthonormal basis of V, and let us set $\lambda(e_i) = \sum_{j=1}^n \lambda_{ij} e_j$, $\rho(e_i) = \sum_{j=1}^n \rho_{ij} e_j$ for all i, $1 \leq i \leq n$. By the invariance of $\langle , \rangle$, we have
$$\left\langle (e_i + ae_1 + \alpha e_0) \cdot \left(e_j + be_1 + \beta e_0\right), e_k + ce_1 + \gamma e_0 \right\rangle = \left\langle e_i + ae_1 + \alpha e_0, (e_k + ce_1 + \gamma e_0) \cdot \left(e_j + be_1 + \beta e_0\right) \right\rangle,$$
for all real numbers $a, b, c, \alpha, \beta, \gamma$ and integers i, j, k such that $2 \leq i, j, k \leq n$. This yields
$$\begin{aligned} &\alpha\left(\lambda_{jk} + b\lambda_{1k}\right) + \beta(\rho_{ik} + a\rho_{1k}) + \alpha\beta c + \beta\gamma(\rho_{i1} + a\rho_{11}) \\ &\quad = \gamma\left(\lambda_{ji} + b\lambda_{1i}\right) + \beta(\rho_{ki} + c\rho_{1i}) + \beta\gamma a + \alpha\beta(\rho_{k1} + c\rho_{11}). \end{aligned} \tag{19}$$

For $i = k$, $\beta = 0$, and $a = c$, we see that (19) becomes $(\alpha - \gamma)\left(\lambda_{ji} + b\lambda_{1i}\right) = 0$. Since α and γ are arbitraries, we deduce that $\lambda_{ij} = \lambda_{1j} = 0$ for all $i, j \geq 2$. Taking this into account, we see that, for $i = k$ and $a = c$, (19) becomes

$$\alpha\beta a + \beta\gamma\,(\rho_{i1} + a\rho_{11}) = \beta\gamma a + \alpha\beta\,(\rho_{k1} + a\rho_{11})\,.$$

By setting $a = 0$ in the above equation, we deduce that $\rho_{i1} = 0$ for all $i \geq 2$; which when substituted once again in the above equation yields $\rho_{11} = 1$.

Now, by taking into consideration that $\lambda_{ij} = \lambda_{1j} = \rho_{i1} = 0$ for all $i, j \geq 2$, we get from (19) that

$$\beta\,(\rho_{ik} + a\rho_{1k}) = \beta\,(\rho_{ki} + c\rho_{1i})\,,$$

from which we deduce that $\rho_{ik} = \rho_{ki}$ and $\rho_{1k} = 0$ for all $i, k \geq 2$, given that a, c, and β are arbitraries.

To summarize, we have shown that, relative to the basis $e_1, \ldots, e_n$, we have

$$\lambda = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{n1} \\ 0 & & \\ \vdots & & 0 \\ 0 & & \end{pmatrix}, \qquad \rho = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix},$$

with B a symmetric matrix (i.e., $B = B^t$) which we can identify to $\rho_{|V}$. It follows that $\lambda_{|V} = 0$ and $\rho_{|V}$ is self-adjoint with respect to the positive definite scalar product $\langle , \rangle_{|V}$. Furthermore, when applied to $B = \rho_{|V}$, formula (iv) of Theorem 9 implies that $B^2 = B$. Thus, without loss of generality, we can assume that $\rho = Id_m \oplus 0_{n-m}$, for some integer $m \leq n$. Since $N(A)^2 = 0$, then by virtue of (iv) of Theorem 9 we have $[\lambda, \rho] + \rho^2 = \rho$. Applying this formula to e_i, we deduce that $\lambda_{i1} = 0$ for all $i > m$. Consequently, by applying formula (18), we conclude that there exists some integer $m \geq 1$ and real numbers λ_i for which the structure of A is given by

$$e_0 \cdot e_0 = e_0, \qquad e_i \cdot e_0 = e_i, \qquad e_0 \cdot e_i = \lambda_i e_1, \quad 1 \leq i \leq m.$$

Accordingly, we have three cases to consider.

1. If $\lambda_i = 0$ for all i, then A is isomorphic to $F_{m+1,0} \oplus A_{n-m,0}$.
2. If $\lambda_1 = 0$ and $\lambda_{i_0} \neq 0$ for some i_0, then by replacing e_i with $e_i' = e_i - \frac{\lambda_i}{\lambda_{i_0}} e_{i_0}$ for $i \neq i_0$ and $2 \leq i \leq m$, we easily get $e_i' \cdot e_0 = e_i'$ and $e_0 \cdot e_i' = 0$. Setting $e_1' = e_2$ and $e_2' = \lambda_{i_0} e_1$, and assuming without loss of generality that $i_0 = 2$, we conclude that A is isomorphic to the algebra given by

$$e_0 \cdot e_0 = e_0, \qquad e_0 \cdot e_1 = e_2, \qquad e_i \cdot e_0 = e_i, \quad 1 \leq i \leq m.$$

In this case, the associated Lie algebra $\mathcal{G}_A$ is isomorphic to a direct product of the form $E \oplus \operatorname{span}\{e_0, e_1\} \oplus \mathbb{R}^k$, with $[e_1, e_0] = e_1 - e_2$ and $E = \operatorname{span}\{e_2, \ldots, e_m\}$ is a commutative ideal such that $[x, e_0] = x$ for all $x \in E$. In other words, $\mathcal{G}_A$ is isomorphic to $\mathcal{F}_{m+1} \oplus \mathbb{R}^{n-m-1}$.

Remark 22. In dimension 3, we obtain the left-symmetric of type (C12) appearing in Table 4 of the classification of three-dimensional Novikov algebras given in [7]. In this case, the associated Lie algebra is described by Lemma 4.10 of [6].

3. If $\lambda_1 \neq 0$, then by replacing e_i with $e_i' = e_i - \frac{\lambda_i}{\lambda_1} e_1$ for $i \geq 2$, we easily get $e_i' \cdot e_0 = e_i'$ and $e_0 \cdot e_i' = 0$. Thus, there exists some real number $\lambda \neq 0$ for which A is isomorphic to the algebra A_λ given by

$$e_0 \cdot e_0 = e_0, \qquad e_0 \cdot e_1 = \lambda e_1, \qquad e_i \cdot e_0 = e_i, \quad 1 \leq i \leq m.$$

It is noticeable that we can easily check that A_λ is isomorphic to A_μ if and only if $\lambda = \mu$.

The associated Lie algebra $\mathcal{G}_{A_\lambda}$ to A_λ is isomorphic to a direct product of the form $E \oplus \operatorname{span}\{e_0, e_1\} \oplus \mathbb{R}^k$, with $[e_1, e_0] = (1 - \lambda)\,e_1$ and $E = \operatorname{span}\{e_2, \ldots, e_m\}$ is a commutative ideal such that $[x, e_0] = x$ for all $x \in E$. In other words, $\mathcal{G}_{A_\lambda}$ is isomorphic to $\mathcal{L}^\lambda_{m+1} \oplus \mathbb{R}^{n-m-1}$. In particular, in dimension 3, we see that if $\lambda = 2$ then $\mathcal{G}_{A_2}$ is isomorphic to the (unimodular) Lie algebra of $E\,(1, 1)$: the group of rigid motions of Minkowski 2-space.

The case when $\langle , \rangle_{|A_i}$ is degenerate and $\langle , \rangle_{|N(A_i)}$ is positive definite. In this case, we can write $A_i = N\,(A_i) \oplus \mathbb{R}e_0$ (orthogonal direct sum), where e_0 is a null vector satisfying $e_0 \cdot e_0 = e_0$. The left-symmetric product of A_i is given by formula (14), and the scalar product on A_i is given by: $\langle (x, a)\,, (y, b)\rangle = \langle x, y\rangle$ for all $x, y \in N\,(A_i)$ and $a, b \in \mathbb{R}$.

Since the restriction of $\langle , \rangle$ to $N\,(A_i)$ is positive definite, we conclude that $N\,(A_i)^2 = 0, \lambda = 0$ and ρ is self-adjoint. Furthermore, by virtue of (iv) of Theorem 9, we deduce that $\rho^2 = \rho$. In fact, all these can be derived in a fashion similar to what we did in Case 1 of Section 2.1.

Now, we can proceed as in the case when $\langle , \rangle_{|A_i}$ is positive definite. First, without loss of generality, we deduce from the equation $\rho^2 = \rho$ that $\rho = Id_m \oplus 0_{n-m}$, for some integer $m \leq n$. Then, by using (15), we conclude that A_i is isomorphic to $\mathbb{R}^{n-m} \oplus F_{m+1,0}$.

The case when both $\langle , \rangle_{|A_i}$ and $\langle , \rangle_{|N(A_i)}$ are degenerate. As in the case when $\langle , \rangle_{|A_i}$ is Lorentzian and $\langle , \rangle_{|N(A_i)}$ is degenerate, to avoid confusion we will omit the subscript i of A_i and simply write A. It is clear that we can write $A = N(A) \oplus \mathbb{R}e_0$ (orthogonal direct sum), where e_0 is a unit spacelike vector satisfying $e_0 \cdot e_0 = e_0$. Furthermore, we can write $N(A) = V \oplus \mathbb{R}e_1$ (orthogonal direct sum), where e_1 is a null vector in $N(A)$ such that $\langle e_0, e_1 \rangle = 0$ and V is a Euclidean (i.e., $\langle , \rangle_{|V}$ is positive definite) subspace of $N(A)$. Thus, we have the orthogonal decomposition $A = V \oplus \mathbb{R}e_1 \oplus \mathbb{R}e_0$. It follows that the scalar product (which is positive semi-definite) on A is given by

$$\langle (x + ae_1 + \alpha e_0), (y + be_1 + \beta e_0) \rangle = \langle x, y \rangle + a\beta,$$

and since $N(A)^2 = 0$, the left-symmetric product on A is given by

$$(x + ae_1 + \alpha e_0) \cdot (y + be_1 + \beta e_0) = \alpha\lambda (y + be_1) + \beta\rho (x + ae_1) + \alpha\beta e_0, \tag{20}$$

where here $x, y \in V$ and $a, b, \alpha, \beta \in \mathbb{R}$.

Let $e_2, \ldots, e_n$ be an orthonormal basis of V, and let us set nd $\lambda (e_i) = \sum_{j=1}^{n} \lambda_{ij} e_j$, $\rho (e_i) = \sum_{j=1}^{n} \rho_{ij} e_j$ for all i, $1 \leq i \leq n$. By the invariance of $\langle , \rangle$, we have

$$\left\langle (e_i + ae_1 + \alpha e_0) \cdot \left(e_j + be_1 + \beta e_0\right), e_k + ce_1 + \gamma e_0 \right\rangle = \left\langle e_i + ae_1 + \alpha e_0, (e_k + ce_1 + \gamma e_0) \cdot \left(e_j + be_1 + \beta e_0\right) \right\rangle,$$

for all real numbers $a, b, c, \alpha, \beta, \gamma$ and integers i, j, k such that $2 \leq i, j, k \leq n$. This yields

$$\alpha \left(\lambda_{jk} + b\lambda_{1k}\right) + \beta (\rho_{ik} + a\rho_{1k}) = \gamma \left(\lambda_{ji} + b\lambda_{1i}\right) + \beta (\rho_{ki} + c\rho_{1i}). \tag{21}$$

For $i = k$, $\beta = 0$, and $a = c$, we see that (21) becomes $(\alpha - \gamma) \left(\lambda_{ji} + b\lambda_{1i}\right) = 0$. Since α and γ are arbitraries, we deduce that $\lambda_{ij} = \lambda_{1j} = 0$ for all $i, j \geq 2$. Taking this into account, we see that (21) becomes

$$\beta (\rho_{ik} + a\rho_{1k}) = \beta (\rho_{ki} + c\rho_{1i}),$$

from which we deduce that $\rho_{ik} = \rho_{ki}$ and $\rho_{1k} = 0$ for all $i, k \geq 2$, given that a, c, and β are arbitraries. Thus, we have shown that $\lambda (e_i) = \lambda_{i1} e_1$ for all $i \geq 1$, $\rho (e_1) = \rho_{11} e_1$, and $\rho (e_i) = \sum_{j=1}^{n} \rho_{ij} e_j$ for all $i \geq 2$.

Let now $\overline{e} \in \bigoplus_{j \geq 2} A_j$ be a null vector such that $\langle e_1, \overline{e} \rangle = 1$. By the invariance of $\langle , \rangle$ and the fact that $\overline{e} \cdot \left(e_j + be_1 + \beta e_0\right) = 0$, we have

$$\left\langle (e_i + ae_1 + \alpha e_0) \cdot \left(e_j + be_1 + \beta e_0\right), \overline{e} \right\rangle = \left\langle e_i + ae_1 + \alpha e_0, \overline{e} \cdot \left(e_j + be_1 + \beta e_0\right) \right\rangle = 0,$$

for all real numbers a, b, α, β and integers i, j such that $2 \leq i, j \leq n$. This yields

$$\alpha (\lambda_{i1} + b\lambda_{11}) + \beta (\rho_{i1} + a\rho_{11}) = 0,$$

from which we deduce that $\lambda_{i1} = \lambda_{11} = \rho_{i1} = \rho_{11} = 0$. To summarize, we have shown that, relative to the basis $e_1, \ldots, e_n$, we have $\lambda = 0$ and

$$\rho = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & B & \\ 0 & & \end{pmatrix},$$

with B an $(n-1)$-square symmetric matrix (i.e., $B = B^t$) which we can identify to $\rho_{|V}$.

Furthermore, when applied to $B = \rho_{|V}$, formula (iv) of Theorem 9 implies that $B^2 = B$. Thus, without loss of generality, we can assume that

$$\rho = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & Id_{m-1} & \vdots \\ 0 & \cdots & 0_{n-m} \end{pmatrix}$$

for some integer $m \leq n$. Consequently, by applying formula (20), we see that there exists some integer $m \geq 1$ for which the structure of A is given by

$$e_i \cdot e_0 = \varepsilon_i e_i, \quad 0 \leq i \leq n,$$

with $\varepsilon_i = 1$ for $i = 0$ or $2 \leq i \leq m$, and $\varepsilon_i = 0$ for $i = 1$ or $m + 1 \leq i \leq n$. We conclude that A is isomorphic to either $A_{n,0}$ or $F_{m+1,0} \oplus A_{n-m,0}$.

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