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Affine hypersurfaces with parallel difference tensor relative to affine α-connection

Cece Li *

School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, People's Republic of China

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ABSTRACT

Li and Zhang (2014) studied affine hypersurfaces of $\mathbb{R}^{n+1}$ with parallel difference tensor relative to the affine α-connection $\nabla^{(\alpha)}$, and characterized the generalized Cayley hypersurfaces by $K^{n-1} \neq 0$ and $\nabla^{(\alpha)}K = 0$ for some nonzero constant α, where the affine α-connection $\nabla^{(\alpha)}$ of information geometry was introduced on affine hypersurface. In this paper, by a slightly different method we continue to study affine hypersurfaces with $\nabla^{(\alpha)}K = 0$, if $\alpha = 0$ we further assume that the Pick invariant vanishes and affine metric is of constant sectional curvature. It is proved that they are either hyperquadrics or improper affine hypersphere with flat indefinite affine metric, the latter can be locally given as a graph of a polynomial of at most degree $n + 1$ with constant Hessian determinant. In particular, if the affine metric is definite, Lorentzian, or its negative index is 2, we complete the classification of such hypersurfaces.

1. Introduction

In classical affine differential geometry, one of the most attracting results is the Pick–Berwald theorem, stating that the induced affine connection ∇ of a non-degenerate hypersurface coincides with the Levi-Civita connection $\widehat{\nabla}$ of affine metric h, or equivalently cubic form $C := \nabla h$ vanishes, if and only if it is a non-degenerate hyperquadric. This theorem has been generalized in many directions. In particular, affine hypersurfaces with parallel cubic form have been studied in various settings for more than twenty years, and their classification is an important problem in affine differential geometry.

On the one hand, one can consider affine hypersurfaces whose cubic form is parallel relative to the affine connection, i.e., $\nabla C = 0$. The non-degenerate affine hypersurface satisfying $\nabla C = 0$ is either a hyperquadric or a graph immersion of a polynomial of degree 3. In the latter, the immersion must be an improper affine hypersphere, and the graph function is of constant Hessian determinant [1,2]. Such hypersurfaces with dimension $n \leq 5$ have been classified in [3,2,4,5], respectively. Finally, by the so-called "method of algorithmic sequence of coordinate changes" Gigena [6] determined the classification for all dimensions.

On the other hand, parallelism of the cubic form can also be considered with respect to the connection $\widehat{\nabla}$. Affine hypersurfaces satisfying $\widehat{\nabla} C = 0$, or equivalently the difference tensor $K := \nabla - \widehat{\nabla}$ satisfying $\widehat{\nabla} K = 0$, are locally homogeneous affine hypersphere [7,8]. Moreover, the classification of such hypersurfaces for low dimension is obtained in [9,8,10–13]. Recently, Z. Hu, et al. [14,15] classified the hypersurfaces for all dimensions when the affine metric is definite or Lorentzian. Under additional condition such hypersurfaces had been considered in [16], and there also appear similar researches [17,18] in centroaffine differential geometry.

* Tel.: +86 13783169803.
E-mail addresses: ceceli@sina.com, lcc201@163.com.

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Moreover, another class of affine hypersurface is obtained when the condition $\nabla K = 0$ is imposed. Non-degenerate affine hypersurfaces with this property have been studied in [19]. It is proved that they are either hyperquadrics or improper affine hyperspheres. In the latter case the affine metric is flat, the difference tensor is nilpotent, i.e., $K^m = 0$ for some $m > 1$, $[K_X, K_Y] = 0$ for all vector fields X, Y and the graph function is given by a polynomial of degree $m + 1$. In particular, the classifications for $m = 2, n - 1, n$ were obtained.

Recently, the authors in [20] studied the affine hypersurfaces with $\nabla^{(\alpha)}K = 0$, and characterized the generalized Cayley hypersurfaces by $K^{n-1} \neq 0$ and $\nabla^{(\alpha)}K = 0$ for some nonzero constant α. There, for each $\alpha \in \mathbb{R}$ a torsion-free connection $\nabla^{(\alpha)}$ is introduced on affine hypersurface by

$$\nabla^{(\alpha)} := (1-\alpha)\widehat{\nabla} + \alpha\nabla, \tag{1.1}$$

which coincides with the affine α-connection of information geometry (cf. [21]), thus we call $\nabla^{(\alpha)}$ the *affine α-connection* of affine hypersurface.

The generalized Cayley hypersurface, constructed by M. Eastwood and V. Ezhov in [22], is a whole family of homogeneous affine hypersurfaces with a parameter α. It is a graph immersion of a polynomial $x_{n+1} = \Phi(x_1, \ldots, x_n; \alpha)$ of degree $(n + 1)$ in affine space $\mathbb{R}^{n+1}$, where

$$\Phi(x_1, \ldots, x_n; \alpha) = \sum_{k=2}^{n+1} \tfrac{(-1)^k}{k!} \prod_{i=0}^{k-3} [(1-\alpha)i + 2] \sum_{j_1+\cdots+j_k=n+1} x_{j_1} \cdots x_{j_k}, \tag{1.2}$$

and $(x_1, \ldots, x_{n+1})$ are the standard coordinates of $\mathbb{R}^{n+1}$. This is the Cayley surface (cf. [3]) when $n = 2$ and the Cayley hypersurface (cf. (1) of [22]) when $\alpha = 0$. They are improper affine hyperspheres with flat affine metric and $[K_X, K_Y] = 0$ for all X, Y (cf. [20]), and the vanishing of affine invariant $\nabla^{(\alpha)}K$ can be seen as a criteria for distinguishing the generalized Cayley hypersurfaces.

In this paper, by a slightly different method from [20] we continue to study affine hypersurfaces with $\nabla^{(\alpha)}K = 0$. Up to a sign of affine normal, assume that the negative index s of affine metric satisfies $n - 2s \geq 0$, we can state our main results as follows.

Main Theorem. *Let M be a non-degenerate affine hypersurface of $\mathbb{R}^{n+1}$ satisfying $\nabla^{(\alpha)}K = 0$. If $\alpha = 0$ we further assume that the Pick invariant vanishes and affine metric is of constant sectional curvature.*
Then either M is a hyperquadric, i.e., $K = 0$, or $K \neq 0$ and M is an improper affine hypersphere with flat indefinite affine metric. In the latter case, there exists an integer $m : 2 \leq m \leq \min\{2s + 1, n\}$ such that $K^m = 0$ and $K^{m-1} \neq 0$, M is locally given as the graph of a polynomial of degree $m + 1$ with constant Hessian determinant. In particular,

(i) *If M is locally strongly convex, then M is locally affine equivalent to a locally strongly convex hyperquadric.*
(ii) *If the affine metric is Lorentzian, i.e., $s = 1$, then M is locally affine equivalent to either a Lorentzian hyperquadric or one of the graph immersions:*

$$x_{n+1} = x_1x_2 - \tfrac{1}{3}x_1^3 + \tfrac{1}{2}\sum_{i=3}^{n} x_i^2,$$

$$x_{n+1} = \Phi(x_1, x_2, x_3; \alpha) + \tfrac{1}{2}\sum_{i=4}^{n} x_i^2.$$

(iii) *If the negative index of affine metric is 2, i.e., $s = 2$, then $n \geq 4$, M is locally affine equivalent to either a corresponding hyperquadric or one of the graph immersions:*

$$x_{n+1} = -\Phi(x_1, x_2, x_3; \alpha) + \tfrac{1}{2}\sum_{i=4}^{n} x_i^2,$$

$$x_{n+1} = \Phi(x_1, x_2, x_3; \alpha) + \Phi(x_4, x_5, x_6; \alpha) + \tfrac{1}{2}\sum_{i=7}^{n} x_i^2,$$

$$x_{n+1} = \Phi(x_1, \ldots, x_m; \alpha) + \tfrac{1}{2}\sum_{i=m+1}^{n} x_i^2, \quad m \in \{4, 5\},$$

$$x_{n+1} = \Phi(x_1, \ldots, x_m; \alpha) - \tfrac{1}{2}x_{m+1}^2 + \frac{1}{2}\sum_{i=m+2}^{n} x_i^2, \quad m \in \{2, 3\},$$

$$x_{n+1} = \Phi(x_1, \ldots, x_m; \alpha) + x_{m+1}x_{m+2} - \tfrac{1}{3}x_{m+1}^3 + \frac{1}{2}\sum_{i=m+3}^{n} x_i^2, \quad m \in \{2, 3\},$$

$$x_{n+1} = x_1x_2 - \tfrac{1}{3}x_1^3 + \varepsilon x_3x_4 - \varepsilon x_1x_3^2 - \tfrac{a}{3}x_3^3 + \tfrac{1}{2}\sum_{i=5}^{n} x_i^2,$$

$$x_{n+1} = \Phi(x_1, x_2, x_3; \alpha) + \varepsilon x_4 x_5 - \varepsilon x_1 x_4^2 - \tfrac{a}{3} x_4^3 - \beta x_4^2 x_6 + \tfrac{(3-\alpha)\beta^2}{12} x_4^4 + \tfrac{1}{2}\sum_{i=6}^{n} x_i^2,$$

$$\begin{aligned} x_{n+1} = {} & \Phi(x_1, x_2, x_3; \alpha) + x_4 x_6 + \tfrac{1}{2} x_5^2 - 2x_1 x_4 x_5 - x_2 x_4^2 - b x_4^2 x_5 - \gamma x_4^2 x_7 \\ & - \tfrac{a}{3} x_4^3 + \tfrac{3-\alpha}{2} x_1^2 x_4^2 + \tfrac{(3-\alpha)b}{3} x_1 x_4^3 + \tfrac{(3-\alpha)(b^2+\gamma^2)}{12} x_4^4 + \tfrac{1}{2}\sum_{i=7}^{n} x_i^2, \end{aligned}$$

where $\varepsilon = \pm 1$, a, b, β, γ are constant, and $\beta = 0$ if $n = 5$, $\gamma = 0$ if $n = 6$.

Here $(x_1, \ldots, x_{n+1})$ are the standard coordinates of $\mathbb{R}^{n+1}$, and Φ is given by (1.2).

Remark 1.1. $\nabla^{(\alpha)} K = 0$ generalizes the previous parallelism of K. Note that $\widehat{\nabla} K = 0$ and $\nabla K = 0$ correspond to $\alpha = 0$ and $\alpha = 1$, respectively. Hence our results extend and develop the results of [14,19].

Remark 1.2. The hypersurfaces of the Main Theorem are affine hypersphere with constant sectional curvature and zero Pick invariant. Closely related to this, Vrancken [23] has classified affine hyperspheres with constant sectional curvature and non-vanishing Pick invariant, and Vrancken–Li–Simon [24] classified locally strongly convex affine hyperspheres with constant sectional curvature. In this sense, we find some new classes of affine hyperspheres with constant sectional curvature and zero Pick invariant.

Remark 1.3. Note that the compositions of improper affine hyperspheres—revisited (cf. Sect. 5 of [25]), using as building blocks either the generalized Cayley hypersurfaces or hyperquadric, are locally homogeneous and also satisfy $\nabla^{(\alpha)} K = 0$, whose characterization and its application to the present subject will be considered in the forthcoming paper. Furthermore, we conjecture that affine hypersurfaces with $\nabla^{(\alpha)} K = 0$ are locally homogeneous, which based on above, Remark 4.3 and the facts: Affine hypersurfaces with $\widehat{\nabla} K = 0$ are locally homogeneous [8]. From the viewpoint of the complete unimodular Hessian algebra Y. Choi and K. Chang in [26] conjecture that affine hypersurfaces with $\nabla K = 0$ are locally homogeneous.

Remark 1.4. Note that affine hypersurface M equipped with the structure $(M, h, \nabla^{(\alpha)}, \nabla^{(-\alpha)})$ for $\alpha \neq 0$ is a statistical manifold. Moreover, if the affine α-connection $\nabla^{(\alpha)}$ is of constant curvature, it follows from the main results of T. Kurose [27] that there exist two affine immersions $(M, \nabla^{(\alpha)})$ and $(M, \nabla^{(-\alpha)})$ with the same affine metric h, which are related to each other by the Legendre transformation or conormal transformation.

This paper is organized as follows. In Section 2, we introduce the theory of local equiaffine hypersurfaces, and some definitions and properties related to $\nabla^{(\alpha)}$ and K. In Section 3, we construct the canonical basis on the hypersurfaces with commutable difference tensor. In Section 4, by the key Lemma 4.1 and the canonical basis we complete the partial classification. As its corollary, Main Theorem immediately follows.

2. Preliminaries

We briefly recall the theory of local equiaffine hypersurfaces in [28,29]. Let $\mathbb{R}^{n+1}$ be the standard $(n+1)$-dimensional real affine space, i.e., $\mathbb{R}^{n+1}$ endowed with the standard flat connection D and its parallel volume form, given by the determinant. Let $F : M \hookrightarrow \mathbb{R}^{n+1}$ be an oriented hypersurface, and ξ be any transversal vector field on M, i.e., $T_p\mathbb{R}^{n+1} = T_pM \oplus \operatorname{span}\{\xi_p\}$, $\forall\, p \in M$. For any tangent vector fields $X, Y, X_1, \ldots, X_n$, we write

$$D_X F_*(Y) = F_*(\nabla_X Y) + h(X, Y)\xi, \tag{2.1}$$

$$\theta(X_1, \ldots, X_n) = det(F_*(X_1), \ldots, F_*(X_n), \xi), \tag{2.2}$$

thus defining a torsion-free affine connection ∇, a symmetric bilinear form h, and a volume element θ on M. M is said to be non-degenerate if h is non-degenerate (this condition is independent of the choice of the transversal vector field). If M is non-degenerate, up to sign there exists a unique choice of transversal vector field such that $\nabla\theta = 0$ and $\theta = w_h$, where w_h is the metric volume element induced by h. This special transversal vector field ξ, called the *affine normal*, induces the *affine connection* ∇ and a pseudo-Riemannian metric h on M. We call h the *affine metric, or Berwald–Blaschke metric* and $C := \nabla h$ the *cubic form*. Equipped with such affine structure, M is called *affine hypersurface, or Blaschke hypersurface*.

The condition $\nabla\theta = 0$ shows that $D_X\xi$ is tangent to M for all X. Hence we can define a $(1, 1)$-type tensor S on M, called *the affine shape operator*, by

$$D_X\xi = -F_*(SX), \tag{2.3}$$

and the *affine mean curvature* by $H = \frac{1}{n}\text{trace } S$. The hypersurface M is called an *affine hypersphere* if $S = H\, id$, then one easily proves that $H = const$ if $n \geq 2$. M is called a *proper affine hypersphere* if $H \neq 0$ and an *improper affine hypersphere* if $H = 0$. For an improper affine hypersphere the affine normal ξ is constant.

By the notations in the introduction, the difference tensor K, related to cubic form C by

$$C(X, Y, Z) = -2h(K(X, Y), Z), \tag{2.4}$$

is a symmetric $(1, 2)$-tensor, and satisfies the apolarity condition $\operatorname{tr} K_Z = 0$, and $h(K(X, Y), Z)$ is totally symmetric for all X, Y and Z.

Denote by R and $\widehat{R}$ the curvature tensors of ∇ and $\widehat{\nabla}$, respectively. Then we have the Gauss equations

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY,$$
$$\widehat{R}(X,Y)Z = \tfrac{1}{2}\big[h(Y,Z)SX - h(X,Z)SY + h(SY,Z)X - h(SX,Z)Y\big] - [K_X, K_Y]Z, \tag{2.5}$$

and the Codazzi equation

$$(\widehat{\nabla}_X K)(Y,Z) - (\widehat{\nabla}_Y K)(X,Z) = \frac{1}{2}\big[h(Y,Z)SX - h(X,Z)SY - h(SY,Z)X + h(SX,Z)Y\big],$$

where $(\widehat{\nabla}_X K)(Y,Z) = \widehat{\nabla}_X(K(Y,Z)) - K(\widehat{\nabla}_X Y, Z) - K(Y, \widehat{\nabla}_X Z)$. Contracting the Gauss equation (2.5) twice we have

$$\chi = H + J, \tag{2.6}$$

where $J = \frac{1}{n(n-1)}h(K,K)$ is the *Pick invariant* and χ is the *normalized scalar curvature* of h. For an affine hypersphere with constant sectional curvature and $J = 0$, then $\chi = H$ and (2.5) reduces to

$$\widehat{R}(X,Y)Z = H\big[h(Y,Z)X - h(X,Z)Y\big], \tag{2.7}$$
$$[K_X, K_Y]Z = 0. \tag{2.8}$$

We prepare the following definitions and lemmas.

Definition 2.1 (*cf. Definition 2.1 of [20]*)**.** On affine hypersurface M, for each $\alpha \in \mathbb{R}$ we can define a torsion-free connection by $\nabla^{(\alpha)} := (1-\alpha)\widehat{\nabla} + \alpha\nabla$, called the *affine α-connection* of M, and its curvature tensor $R^{(\alpha)}$ by

$$R^{(\alpha)}(X,Y) := [\nabla^{(\alpha)}_X, \nabla^{(\alpha)}_Y] - \nabla^{(\alpha)}_{[X,Y]}.$$

Then, by $K_X = \nabla_X - \widehat{\nabla}_X$ we see that $(1-\alpha)K_X Y = \nabla_X Y - \nabla^{(\alpha)}_X Y$ and

$$\begin{aligned} R^{(\alpha)}(X,Y)Z &= (1-\alpha)\widehat{R}(X,Y)Z + \alpha R(X,Y)Z + \alpha(\alpha-1)[K_X,K_Y]Z \\ &= \tfrac{1+\alpha}{2}[h(Y,Z)SX - h(X,Z)SY] + \tfrac{1-\alpha}{2}[h(SY,Z)X - h(SX,Z)Y] + (\alpha^2-1)[K_X,K_Y]Z. \end{aligned} \tag{2.9}$$

Moreover, (2.4) shows

$$(\nabla^{(\alpha)}_X h)(Y,Z) = -2\alpha h(K(X,Y),Z), \tag{2.10}$$

and (cf. (7) of [7] and (3.1) of [19])

$$h((\widehat{\nabla}_X K)(Y,Z), W) = -\tfrac{1}{2}(\widehat{\nabla}_X C)(Y,Z,W),$$
$$h((\nabla_X K)(Y,Z), W) = -\tfrac{1}{2}(\nabla_X C)(Y,Z,W) + 2h(K_X K_Y Z, W).$$

Combining this with Definition 2.1 we can check that

$$h((\nabla^{(\alpha)}_X K)(Y,Z), W) = -\frac{1}{2}(\nabla^{(\alpha)}_X C)(Y,Z,W) + 2\alpha h(K_X K_Y Z, W). \tag{2.11}$$

Definition 2.2 (*cf. Remark 3.3 of [19]*)**.** For any nonnegative integer k, we can define a $(1, k+1)$-tensor field K^k on M by

$$K^k(X_1, \ldots, X_{k+1}) = K_{X_1} \cdots K_{X_k} X_{k+1}$$

for any $X_1, \ldots, X_{k+1}$. If $[K_X, K_Y] = 0$ for all X and Y, we call the difference tensor is commutable, then K^k is totally symmetric. Hence K^k vanishes identically if and only if $(K_v)^k v = 0$ for all vectors v. Denote by m the smallest number such that the symmetric tensor K^m is identically zero at the point p. Then for any tangent vector v at p, we have $(K_v)^m v = 0$ and there exists a tangent vector u at p such that $h((K_u)^{m-1}u, u) \neq 0$.

Lemma 2.1 (*cf. Lemma 3.3 of [19]*)**.** *If $[K_Y, K_Z] = 0$ for all Y and Z, then K_X is nilpotent for each X. In particular, $(K_X)^n = 0$.*

Lemma 2.2 (*cf. Proposition 3.1 of [30]*)**.** *Let M be a non-degenerate affine hypersurface with constant sectional curvature and $\widehat{\nabla} K = 0$. Then either M is a hyperquadric, i.e., $K = 0$, or $K \neq 0$ and M is an affine hypersphere with flat affine metric.*

Lemma 2.3 (*cf. Lemma 4.1 of [20], see also Lemma 3.2 of [19] for $\alpha = 1$*)**.** *If $\nabla^{(\alpha)} K = 0$ for some nonzero constant α, and $K \neq 0$, then M is an improper affine hypersphere with flat affine metric. Moreover, M is $\nabla^{(\alpha)}$-flat.*

Definition 2.3 (*cf. Lemma 4.2 of [19]*). A nonzero vector v is called a *null vector* if $h(v, v) = 0$. Define the *null space* $N \subset T_pM$ by

$$h(x, y) = 0, \quad \forall\, x, y \in N,$$

then the null space is a linear subspace of T_pM, whose maximal dimension is s. If $\dim N = s$ and a null vector v satisfies $h(v, u) = 0, \forall\, u \in N$, then $v \in N$.

For a non-degenerate affine hypersurface M, up to a sign of affine normal ξ if necessary, we always assume that the negative index s of affine metric h satisfies $n - 2s \geq 0$. Denote by $I(V)$ the negative index of a non-degenerate subspace V of T_pM, and δ the standard Kronecker delta. We follow above conventions and definitions in the rest of this paper.

3. Affine hypersurfaces with commutable difference tensor

Let M be a non-degenerate affine hypersurface with commutable difference tensor. In this section we will construct a canonical basis on the hypersurface for some special values of m, defined in Definition 2.2. First we prove the following two lemmas.

Lemma 3.1. *If* $[K_X, K_Y] = 0$ *for all* X *and* Y, *then*

$$\nabla^{(\alpha)} K = \nabla K + (1 - \alpha) K^2. \tag{3.1}$$

Proof. From $\nabla^{(\alpha)}_X Y = \nabla_X Y - (1 - \alpha) K_X Y$ and the assumption, we see that

$$\begin{aligned}(\nabla^{(\alpha)}_X K)(Y, Z) &= \nabla^{(\alpha)}_X K(Y, Z) - K(\nabla^{(\alpha)}_X Y, Z) - K(Y, \nabla^{(\alpha)}_X Z)\\ &= \nabla_X K(Y, Z) - K(\nabla_X Y, Z) - K(Y, \nabla_X Z) + (1 - \alpha)[K_Y, K_X]Z + (1 - \alpha) K_Z K_X Y\\ &= (\nabla_X K)(Y, Z) + (1 - \alpha) K^2(X, Y, Z). \quad \square\end{aligned}$$

Lemma 3.2. *If* $[K_X, K_Y] = 0$ *for all* X *and* Y, *then* $K^{2s+1} = 0$.

Proof. By Lemma 2.1, Definition 2.2 and its index, it is sufficient to prove $m \leq 2s+1$ for $2s+1 < n$. Otherwise, we suppose that $2s + 1 < m \leq n$. Then there exists a tangent vector u such that $h((K_u)^{m-1}u, u) \neq 0$, and define

$$\begin{aligned}&x_1 = (K_u)^{m-1}u, \qquad x_2 = (K_u)^{m-2}u, \ldots,\\ &x_s = (K_u)^{m-s}u, \qquad x_{s+1} = (K_u)^{m-s-1}u.\end{aligned}$$

Then $(K_u)^\ell x_\ell = 0$ but $(K_u)^i x_\ell \neq 0$ for $i < \ell \leq s + 1$. It follows that $x_1, \ldots, x_{s+1}$ are linearly independent vectors. By $0 \leq m - 2\ell < m - \ell - i$ we further obtain

$$\begin{aligned}&h(x_\ell, x_\ell) = h((K_u)^{m-\ell}u, (K_u)^{m-\ell}u) = h((K_u)^m u, (K_u)^{m-2\ell}u) = 0,\\ &h(x_i, x_\ell) = h((K_u)^{m-i}u, (K_u)^{m-\ell}u) = h((K_u)^m u, (K_u)^{m-\ell-i}u) = 0.\end{aligned}$$

Hence, span$\{x_1, \ldots, x_{s+1}\}$ is an $(s + 1)$-dimensional null space. This contradicts that the maximal dimension of null space is s. $\square$

It follows from Lemmas 2.1 and 3.2 that the integer m satisfies $1 \leq m \leq \min\{2s + 1, n\}$. Next, we construct m linearly independent vectors with a canonical represent of (h, K), namely

Lemma 3.3. *There exist a subspace* $V \subset T_pM$ *spanned by* m *linearly independent vectors* $\{y_1, \ldots, y_m\}$ *such that*

$$K_{y_i} y_j = \begin{cases} y_{i+j}, & i + j \leq m,\\ 0, & \text{otherwise},\end{cases} \qquad h(y_i, y_j) = \begin{cases} \epsilon, & i + j = m + 1,\\ 0, & \text{otherwise},\end{cases} \tag{3.2}$$

where $\epsilon = 1$ *if* $h(y_1, y_m) > 0$ *or* m *is even; otherwise* $\epsilon = -1$.

Proof. By Definition 2.2 there exists a vector u such that $h((K_u)^{m-1}u, u) \neq 0$. Define

$$y_1 = u, \qquad y_i = (K_u)^{i-1}u, \quad i \leq m,$$

then $y_m \neq 0$. Since $(K_u)^{m-i} y_i = y_m$, $(K_u)^j y_i = 0$ for $i + j \geq m + 1$, we see that $y_1, \ldots, y_m$ are linearly independent vectors, and

$$h(y_i, y_j) = \begin{cases} \alpha_{i+j-1}, & i + j \leq m + 1,\\ 0, & i + j > m + 1,\end{cases} \tag{3.3}$$

where $\alpha_i := h(y_1, y_i)$ for $i \leq m$. We have chosen u such that $\alpha_m \neq 0$.

Let β be an arbitrary real number, and $x_1 = y_1 + \beta y_2, x_j = K_{x_1}x_{j-1}$ for $j \le m$, then

$$x_2 = y_2 + 2\beta y_3 + \cdots, \qquad x_3 = y_3 + 3\beta y_4 + \cdots, \ldots,$$
$$x_{m-1} = y_{m-1} + (m-1)\beta y_m, \qquad x_m = y_m.$$

Hence $h(x_1, x_{m-1}) = h(y_1, y_{m-1}) + m\beta\alpha_m$. By choosing β appropriately, we assume that $\alpha_{m-1} = 0$.

Next we assume that there exist m vectors $y_1, \ldots, y_m$ such that $\alpha_m \ne 0$ and $\alpha_{m-1} = \cdots = \alpha_{m-i} = 0$ for some $i > 0$. Put again $x_1 = y_1 + \beta y_{2+i},\ x_j = K_{x_1}x_{j-1}$ for $j \le m$. From

$$\begin{aligned}&x_2 = y_2 + 2\beta y_{3+i} + \cdots, \qquad x_3 = y_3 + 3\beta y_{4+i} + \cdots,\ \ldots,\\&x_{m-j} = y_{m-j} + (m-j)\beta y_{m-j+i+1} + \cdots,\ \ldots,\\&x_{m-i-1} = y_{m-i-1} + (k-i-1)\beta y_m,\\&x_{m-i} = y_{m-i},\ \ldots,\ x_m = y_m,\end{aligned}$$

we see that

$$\begin{aligned}&h(x_1, x_m) = h(y_1, y_m) = \alpha_m \ne 0,\\&h(x_1, x_{m-j}) = h(y_1, y_{m-j}) = \alpha_{m-j} = 0, \quad 1 \le j \le i,\\&h(x_1, x_{m-i-1}) = h(y_1, y_{m-i-1}) + (m-i)\beta\alpha_m.\end{aligned}$$

Hence, we also assume that $\alpha_{m-i-1} = 0$ by choosing β appropriately.

Continuing this process we have the vectors $y_1, \ldots, y_m$ satisfying $\alpha_m \ne 0$ and $\alpha_1 = \cdots = \alpha_{m-1} = 0$. Then, by re-scaling the null vector y_1 if necessary, we may assume that $\alpha_m = h(y_1, (K_{y_1})^{m-1}y_1) = 1$ if $\alpha_m > 0$ or m is even; otherwise, $\alpha_m = -1$. Hence, from $K_{y_i}y_j = (K_{y_1})^{i+j-1}y_1$ and (3.3) we complete the proof. □

Remark 3.1. Lemma 3.3 is similar to Dillen and Vrancken's Lemma 6.1 in [19], where the conclusion is obtained under the conditions $\nabla K = 0$ and $K^{n-1} \ne 0$.

Note that the complement subspace $V^\perp$ of V in T_pM is non-degenerate, thus $(V^\perp, h)$ is an indefinite scalar product space, we will use the following

Proposition 3.1 (*cf. pp. 260–261 of [31]*)**.** *Let $W = \mathbb{R}^n_\nu$ be an n-dimensional vector space with indefinite scalar product $\langle\cdot,\cdot\rangle$ of negative index ν. Then a linear operator P on W is self-adjoint if and only if W can be expressed as a direct sum of irreducible subspaces W_k that are mutually orthogonal (hence non-degenerate) and P-invariant, and each $P|_{W_k}$ has a matrix representation of the form either*

$$T_1 : \begin{pmatrix} \mu & & & & \\ 1 & \mu & & 0 & \\ & \ddots & \ddots & & \\ & 0 & 1 & \mu & \\ & & & 1 & \mu \end{pmatrix}$$

relative to a basis $e_1, \ldots, e_r$ of V_k with all scalar products zero except $\langle e_i, e_j\rangle = \varepsilon = \pm 1$ if $i + j = r + 1$, or

$$T_2 : \begin{pmatrix} a & b & & & & & & & & \\ -b & a & & & & & 0 & & & \\ 1 & 0 & a & b & & & & & & \\ 0 & 1 & -b & a & & & & & & \\ & & 1 & 0 & a & b & & & & \\ & & 0 & 1 & -b & a & & & & \\ & & & & & & \ddots & & & \\ & 0 & & & & & 1 & 0 & a & b \\ & & & & & & 0 & 1 & -b & a \end{pmatrix} \quad (b \ne 0)$$

relative to a basis $u_1, v_1, \ldots, u_d, v_d$ of W_k with all scalar products zero except $\langle u_i, u_j\rangle = -\langle v_i, v_j\rangle = 1$ if $i + j = d + 1$. Here r, ε and d depend on W_k.

Remark 3.2. Applying Proposition 3.1 and its notations for the symmetric operator K_{y_1} on $(V^\perp, h)$, we see that K_{y_1} on $(V^\perp, h)$ has only the direct sum representation of type T_1 with $\mu = 0$. In fact, by $(K_{y_1})^n = 0$ and

$$K_{y_1}e_r = \mu e_r,$$
$$K_{y_1}(u_d + \sqrt{-1}v_d) = (a + \sqrt{-1}b)(u_d + \sqrt{-1}v_d),$$

we see that $\mu = 0$ in T_1, and $a = b = 0$ in T_2. In the latter we get a contradiction.

Remark 3.3. For the direct sum $T_pM = V \oplus V^{\perp}$ relative to h, if $K(v, w) = 0$ holds for all $v \in V$, $w \in V^{\perp}$, the subspace $(V^{\perp}, h, K)$ satisfies (2.8) and the apolarity condition. Let m_1 be the smallest positive integer such that $K^{m_1}|_{V^{\perp}} \equiv 0$ at p. On $(V^{\perp}, h, K)$, by Lemmas 2.1 and 3.2 there hold

$$1 \le m_1 \le \min\{m, n-m, 2I(V^{\perp}) + 1\}.$$

Using Lemma 3.3 for $V^{\perp}$, we see that there exist $m + m_1$ linearly independent vectors $y_1, \dots, y_m, z_1, \dots, z_{m_1}$ on T_pM such that $h(y_i, z_k) = 0$, $K(y_i, z_k) = 0$ for all i, k, moreover,

$$K_{y_i}y_j = \begin{cases} y_{i+j}, & i+j \le m, \\ 0, & \text{otherwise}, \end{cases} \qquad h(y_i, y_j) = \begin{cases} \epsilon, & i+j = m+1, \\ 0, & \text{otherwise}, \end{cases}$$

where $\epsilon = 1$ if $h(y_1, y_m) > 0$ or m is even, otherwise $\epsilon = -1$; and

$$K_{z_k}z_\ell = \begin{cases} z_{k+\ell}, & k+\ell \le m_1, \\ 0, & \text{otherwise}, \end{cases} \qquad h(z_k, z_\ell) = \begin{cases} \epsilon_1, & k+\ell = m_1+1, \\ 0, & \text{otherwise}, \end{cases}$$

where $\epsilon_1 = 1$ if $h(z_1, z_m) > 0$ or m_1 is even, otherwise $\epsilon_1 = -1$.

By Lemma 3.3 the m-dimensional subspace V satisfies $\lfloor \frac{m}{2} \rfloor \le I(V) \le \lceil \frac{m}{2} \rceil$. According to that $(V^{\perp}, h)$ is positive definite or Lorentzian we will choose the canonical basis of T_pM.

3.1. $(V^{\perp}, h)$ is positive definite

By Lemma 3.3, $I(V^{\perp}) = 0$, or equivalently $I(V) = s$ holds if and only if m is one of the three cases:

$$m = \min\{2s+1, n\}, \quad m = 2s \quad \text{with } \epsilon = 1; \qquad m = 2s-1 \quad \text{with } \epsilon = -1. \tag{3.4}$$

Then we can choose a canonical basis of T_pM as follows.

Lemma 3.4. *If $(V^{\perp}, h)$ is positive definite, there exists a basis $\{y_1, \dots, y_n\}$ of T_pM such that*

$$K_{y_i}y_j = \begin{cases} y_{i+j}, & i+j \le m, \\ 0, & \text{otherwise}, \end{cases} \qquad h(y_i, y_j) = \begin{cases} \epsilon, & i+j = m+1, \\ \delta_i^j, & i, j \ge m+1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.5}$$

where $\epsilon = \pm 1$ is determined by (3.4).

Proof. The case $m = n$ has been proved in Lemma 4.2 of [20] and Lemma 3.2 of [30]. Hence, we are enough to prove the conclusion for $m < n$.

Let $y_1, \dots, y_m$ be the basis of V as stated in Lemma 3.3, thus

$$K_{y_i}y_j = \begin{cases} y_{i+j}, & i+j \le m, \\ 0, & \text{otherwise}, \end{cases} \qquad h(y_i, y_j) = \begin{cases} \epsilon, & i+j = m+1, \\ 0, & \text{otherwise}. \end{cases} \tag{3.6}$$

We claim $\epsilon = 1$ for $m = 2s+1$ or $m = 2s$, and $\epsilon = -1$ for $m = 2s-1$. In fact, for $m = 2s+1$, if $h(y_{s+1}, y_{s+1}) < 0$, then (3.2) shows that $I(V) = s+1$, which gives a contradiction, thus $\epsilon = 1$ for $m = 2s+1$. Note that $I(V) = s$ and $\dim V = m$, by Lemma 3.3 we easily see that $\epsilon = 1$ for $m = 2s$, and $\epsilon = -1$ for $m = 2s-1$.

Since $(V^{\perp}, h)$ is positive definite, we choose an orthonormal basis $\{y_{m+1}, \dots, y_n\}$ of $V^{\perp}$ such that the second equation of (3.5) holds. Further, Eq. (3.6) shows that V is a K_{y_1}-invariant subspace, so is $V^{\perp}$. Note that K_{y_1} is nilpotent, we get $K(y_1, w) = 0$ for $w \in V^{\perp}$. Therefore,

$$K(y_i, w) = K((K_{y_1})^{i-1}y_1, w) = (K_{y_1})^{i-1}K_{y_1}w = 0, \quad \forall\, i \le m,$$

which shows that $K(v, w) = 0$ for $w \in V^{\perp}$, $v \in V$.

On the other hand, for any fixed vector $w \in V^{\perp}$, the operator K_w satisfies $K_w v = 0$ for all $v \in V$, hence V is its invariant subspace, and so is $V^{\perp}$. Since K_w is nilpotent and $(V^{\perp}, h)$ is positive definite, this implies that $K_w w' = 0$ for $w' \in V^{\perp}$. We can summarize from the above that $K_{y_i}y_j = 0$ when either i or $j \ge m+1$. Together with (3.6) we easily get the first equation of (3.5). Lemma 3.4 has been proved. $\square$

3.2. $(V^{\perp}, h)$ *is Lorentzian*

By Lemma 3.3, $I(V^{\perp}) = 1$, or equivalently $I(V) = s - 1$ holds if and only if m is one of the three cases:

$$m = 2s-1, \qquad m = 2s-2 \quad \text{with } \epsilon = 1; \qquad m = 2s-3 \quad \text{with } \epsilon = -1. \tag{3.7}$$

By Proposition 3.1 and Remark 3.2 the symmetric operator K_{y_1} on $(V^{\perp}, h)$ has only three representations:

$$\mathrm{I}: O, \qquad \mathrm{II}: \begin{pmatrix} 0 & 0 & & 0 \\ 1 & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}, \qquad \mathrm{III}: \begin{pmatrix} 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix},$$

where O is the $n - m$ order matrix, and three canonical bases of $(V^{\perp}, h)$ have been chosen according to Proposition 3.1. Next we will choose a canonical basis of $(V^{\perp}, h)$ for the three cases, respectively.

Case I: $K(y_1, w) = 0$ for all $w \in V^{\perp}$. We easily obtain that $K(v, w) = 0$ for all $v \in V$, $w \in V^{\perp}$. By Remark 3.3 and Lemma 3.4 we can choose a canonical basis of $(V^{\perp}, h)$ as follows.

Lemma 3.5. *If $(V^{\perp}, h)$ is Lorentzian, for Case I there exist an integer m_1 satisfying $K^{m_1}|_{V^{\perp}} = 0$, $K^{m_1-1}|_{V^{\perp}} \neq 0$ with $1 \le m_1 \le \min\{m, n-m, 3\}$ and a basis $\{z_1, \ldots, z_{n-m}\}$ of $(V^{\perp}, h)$ such that $K(z_k, y_i) = 0$, $h(z_k, y_i) = 0$ and*

$$K_{z_k} z_\ell = \begin{cases} z_{k+\ell}, & k + \ell \le m_1, \\ 0, & \text{otherwise}, \end{cases} \qquad h(z_k, z_\ell) = \begin{cases} \varepsilon, & k + \ell = m_1 + 1, \\ \delta_k^\ell, & k, \ell \ge m_1 + 1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.8}$$

where $\varepsilon = -1$ for $m_1 = 1$, otherwise $\varepsilon = 1$.

Case II: There exists a basis $\{z_1, \ldots, z_{n-m}\}$ of $(V^{\perp}, h)$ such that

$$K_{y_i} z_k = \begin{cases} z_2, & i = k = 1, \\ 0, & \text{otherwise}, \end{cases} \qquad h(z_k, z_\ell) = \begin{cases} \varepsilon = \pm 1, & k + \ell = 3, \\ \delta_k^\ell, & k, \ell \ge 3, \\ 0, & \text{otherwise}, \end{cases} \tag{3.9}$$

where (3.2) has been used. Then $K(z_2, z_i) = 0$ for all $i > 1$. For the positive subspace $U := \mathrm{span}\{z_3, \ldots, z_{n-m}\}$, we can choose an orthonormal basis of U, still denoted by $\{z_3, \ldots, z_{n-m}\}$, such that $h(K_{z_1} z_k, z_\ell) = \lambda_\ell \delta_k^\ell$ for all $k, \ell \ge 3$. Set $A_{ij}^k = h(K_{z_i} z_j, z_k)$, there holds

$$K_{z_1} z_1 = \epsilon \varepsilon y_m + \varepsilon A_{11}^2 z_1 + \varepsilon A_{11}^1 z_2 + \sum_{\ell=3}^{n-m} A_{11}^\ell z_\ell,$$

thus $K(z_1, z_2) = K_{y_1} K_{z_1} z_1 = \varepsilon A_{11}^2 z_2$, it follows from the nilpotency of K_{z_1} that $K(z_1, z_2) = 0$. Further, $K_{z_1} z_\ell = \varepsilon A_{11}^\ell z_2 + \lambda_\ell z_\ell$, and $(K_{z_1})^n z_\ell = \varepsilon A_{11}^\ell \lambda_\ell^{n-1} z_2 + \lambda_\ell^n z_\ell$ for $\ell > 2$. Again the nilpotency of K_{z_1} shows that $\lambda_\ell = 0$. Now, we can rechoose an orthonormal basis $\{z'_3, \ldots, z'_{n-m}\}$ of U such that $z'_3 = \beta^{-1} \sum_{\ell=3}^{n-m} A_{11}^\ell z_\ell$ if $\beta := \sqrt{\sum_{\ell=3}^{n-m} (A_{11}^\ell)^2} \neq 0$. Then summing above we see that

$$K_{z_1} z_1 = \epsilon \varepsilon y_m + \varepsilon A_{11}^1 z_2 + \beta z'_3, \qquad K_{z_2} \equiv 0,$$
$$K_{z_1} z'_4 = \cdots = K_{z_1} z'_{n-m} = 0, \qquad K_{z_1} z'_3 = \varepsilon \beta z_2.$$

By $0 = K_{z_1} K_{z_1} z'_3 = K_{z'_3} K_{z_1} z_1 = \beta K_{z'_3} z'_3$, we divide Case II into two subcases:

II-1: $\beta = 0$; II-2: $\beta \neq 0$.

For case II-1, we easily see that (U, h) is a kernel subspace of all operators K_v with $v \in U^{\perp}$. Hence the positive definite subspace (U, h) is an invariant subspace of K_u for all $u \in U$, then the nilpotency of K_u shows that $K_x y = 0$ for all $x, y \in U$.

For case II-2, we have $K_{z'_3} z'_3 = 0$ and $K_{z_1} K_{z_1} z_1 = \varepsilon \beta^2 z_2 \neq 0$, thus $m \ge 3$. Since (U, h) is positive definite and is an invariant subspace of $K_{z'_3}$, the nilpotency of $K_{z'_3}$ shows that $K_{z'_3} u = 0$ for all $u \in U$. As before, we see that (U, h) is an invariant subspace of all operators K_u with $u \in U$, then the nilpotency of K_u again shows that $K_x y = 0$ for all $x, y \in U$.

Note that $\beta = 0$ if $m = 2$ or $m = n - 2$. Summing above we have proved the following

Lemma 3.6. *If $(V^{\perp}, h)$ is Lorentzian, for Case* II *we see that $m \ge 2$, $n \ge m+2$ and there exists a basis $\{z_1, \ldots, z_{n-m}\}$ of $(V^{\perp}, h)$ such that there hold (3.9) with $h(z_k, y_i) = 0$ for all i, k, and*

$$\begin{cases} K_{z_1} z_1 = \epsilon \varepsilon y_m + \varepsilon a z_2 + \beta z_3, & K_{z_1} z_3 = \varepsilon \beta z_2, \\ K_{z_1} z_4 = \cdots = K_{z_1} z_{n-m} = 0, & K_{z_2} \equiv 0, \\ K(z_i, z_j) = 0, \quad \forall\, i, j \ge 3, \end{cases} \tag{3.10}$$

where $\varepsilon = \pm 1$, $\epsilon = \pm 1$ is determined by (3.7), and a, β are constant, $\beta = 0$ if $m = 2$ or $m = n - 2$.

Case III: There exists a basis $\{z_1, \ldots, z_{n-m}\}$ of $(V^\perp, h)$ such that

$$K_{y_i}z_k = \begin{cases} z_2, & i=k=1,\\ z_3, & i+k=3,\\ 0, & \text{otherwise}, \end{cases} \qquad h(z_k, z_\ell) = \begin{cases} 1, & k+\ell=4,\\ \delta_k^\ell, & k,\ell\geq 4,\\ 0, & \text{otherwise}, \end{cases} \tag{3.11}$$

where (3.2) has been used. Note that $m \geq 3$ by the definition of m, and $K_{z_2}z_k = K_{z_3}z_k = 0$ for all $k > 2$. As before, we can choose an orthonormal basis of the positive subspace $\bar{U}$ spanned by $\{z_4, \ldots, z_{n-m}\}$, still denoted by $\{z_3, \ldots, z_{n-m}\}$, such that $h(K_{z_1}z_k, z_\ell) = 0$ for all $k, \ell \geq 4$. Set $A_{ij}^k = h(K_{z_i}z_j, z_k)$, then

$$\begin{aligned} K_{z_1}z_1 &= \epsilon y_{m-1} + A_{11}^3 z_1 + A_{11}^2 z_2 + A_{11}^1 z_3 + \sum_{\ell=4}^{n-m} A_{11}^\ell z_\ell,\\ K_{z_1}z_2 &= K_{y_1}K_{z_1}z_1 = \epsilon y_m + A_{11}^3 z_2 + A_{11}^2 z_3,\\ K_{z_1}z_3 &= K_{z_2}z_2 = K_{y_1}K_{z_1}z_2 = A_{11}^3 z_3,\\ K_{z_1}z_\ell &= A_{11}^\ell z_3, \qquad \ell = 4, \ldots, n-m. \end{aligned}$$

Then the nilpotency of K_{z_1} shows that $A_{11}^3 = 0$. Now, we can rechoose an orthonormal basis $\{z_4', \ldots, z_{n-m}'\}$ of $\bar{U}$ such that $z_4' = \gamma^{-1}\sum_{\ell=4}^{n-m} A_{11}^\ell z_\ell$ if $\gamma := \sqrt{\sum_{\ell=4}^{n-m}(A_{11}^\ell)^2} \neq 0$. Hence,

$$\begin{aligned} &K_{z_1}z_1 = \epsilon y_{m-1} + A_{11}^2 z_2 + A_{11}^1 z_3 + \gamma z_4', \qquad K_{z_3} \equiv 0, \qquad K_{z_2}z_2 = 0,\\ &K_{z_1}z_2 = \epsilon y_m + A_{11}^2 z_3, \qquad K_{z_1}z_j' = 0, \quad j \geq 5,\\ &K_{z_1}z_4' = \gamma z_3, \qquad K_{z_2}z_k' = 0, \quad k \geq 4. \end{aligned}$$

By $0 = K_{z_1}K_{z_1}z_4' = K_{z_4'}K_{z_1}z_1 = \gamma K_{z_4'}z_4'$, we divide Case III into two subcases:

III-1: $\gamma = 0$; III-2: $\gamma \neq 0$.

For case III-1, we easily see that $(\bar{U}, h)$ is a kernel subspace of all operators K_v with $v \in \bar{U}^\perp$. Since $(\bar{U}, h)$ is positive and is an invariant subspace of K_u for all $u \in \bar{U}$, then the nilpotency of K_u shows that $K_x y = 0$ for all $x, y \in \bar{U}$.

For case III-2, we have $K_{z_4'}z_4' = 0$. Since $(\bar{U}, h)$ is positive definite and is an invariant subspace of $K_{z_4'}$, the nilpotency of $K_{z_4'}$ shows that $K_{z_4'}u = 0$ for all $u \in \bar{U}$. Similarly, $(\bar{U}, h)$ is an invariant subspace of all operators K_u with $u \in \bar{U}$, then the nilpotency of K_u again shows that $K_x y = 0$ for all $x, y \in \bar{U}$. Summing above we have proved the following

Lemma 3.7. *If $(V^\perp, h)$ is Lorentzian, for Case* III *we see that $m \geq 3$, $n \geq m+3$ and there exists a basis $\{z_1, \ldots, z_{n-m}\}$ of $(V^\perp, h)$ such that there hold (3.11) with $h(z_k, y_i) = 0$, and*

$$\begin{cases} K_{z_1}z_1 = \epsilon y_{m-1} + bz_2 + az_3 + \gamma z_4, & K_{z_3} \equiv 0,\\ K_{z_1}z_2 = \epsilon y_m + bz_3, & K_{z_1}z_i = 0, i \geq 5,\\ K_{z_1}z_4 = \gamma z_3, & K_{z_2}z_j = 0, j \geq 2,\\ K(z_k, z_\ell) = 0, \quad k, \ell = 4, \ldots, n-m, & \end{cases} \tag{3.12}$$

where a, b, γ are constant, $\epsilon = \pm 1$ is determined by (3.7), and $\gamma = 0$ if $n = m+3$.

4. Affine hypersurfaces with $\nabla^{(\alpha)}K = 0$

In this section, we pay our attention to the hypersurfaces with $\nabla^{(\alpha)}K = 0$. Assume that $F : M \hookrightarrow \mathbb{R}^{n+1}$ is a non-degenerate affine hypersurface satisfying $\nabla^{(\alpha)}K = 0$. If $\alpha = 0$ we further assume that $J = 0$ and affine metric is of constant sectional curvature. It follows from (2.6), Lemmas 2.2 and 2.3 that either M is a hyperquadric, i.e., $K = 0$, or $K \neq 0$ and M is an improper affine hypersphere with flat indefinite affine metric and $J = 0$. Moreover, for the latter there hold (2.8) and $R^{(\alpha)} = 0$. From above we can prove

Lemma 4.1. *Let M be the hypersurface above. Then for any positive integer k,*

$$\nabla^k h(X_1, \ldots, X_{k+2}) = (-1)^k \prod_{i=0}^{k-1}[(1-\alpha)i + 2]h(K_{X_1}\cdots K_{X_k}X_{k+1}, X_{k+2}),$$

where $X_1, \ldots, X_{k+2}$ are tangent vector fields.

Proof. From (2.4) we see that the lemma is true for $k = 1$ or $K = 0$. Assume $K \neq 0$. By induction we suppose that the lemma is satisfied for all values for $1 \leq k \leq r-1$. Since M is $\nabla^{(\alpha)}$-flat, we can obtain a local frame field $\{X_1, \ldots, X_n\}$ such that $\nabla^{(\alpha)}_{X_i} X_j = 0$. Together with $\nabla^{(\alpha)} K = 0$ and Definition 2.1, we easily get

$$\nabla^{(\alpha)}_{X_1}(K_{X_2} \cdots K_{X_r} X_{r+1}) = 0, \qquad \nabla_{X_i} Y_j = (1-\alpha) K_{X_i} Y_j.$$

Combining this with (2.10) and the assumption we see that

$$\begin{aligned}
\nabla^r h(X_1, \ldots, X_{r+2}) &= X_1(\nabla^{r-1}h)(X_2, \ldots, X_{r+2}) - (\nabla^{r-1}h)(\nabla_{X_1}X_2, \ldots, X_{r+2}) \\
&\quad - \cdots - (\nabla^{r-1}h)(X_2, \ldots, \nabla_{X_1}X_{r+2}) \\
&= (-1)^{r-1}\prod_{i=0}^{r-2}[(1-\alpha)i+2] X_1 h(K_{X_2}\cdots K_{X_r}X_{r+1}, X_{r+2}) \\
&\quad + (-1)^r \prod_{i=0}^{r-2}[(1-\alpha)i+2](1-\alpha)(r+1)h(K_{X_1}\cdots K_{X_r}X_{r+1}, X_{r+2}) \\
&= \left[(\nabla^{(\alpha)}_{X_1}h)(K_{X_2}\cdots K_{X_r}X_{r+1}, X_{r+2}) - (1-\alpha)(r+1)h(K_{X_1}\cdots K_{X_r}X_{r+1}, X_{r+2})\right] \\
&\quad \times (-1)^{r-1}\prod_{i=0}^{r-2}[(1-\alpha)i+2] \\
&= (-1)^r \prod_{i=0}^{r-2}[(1-\alpha)i+2][(1-\alpha)(r-1)+2]h(K_{X_1}\cdots K_{X_r}X_{r+1}, X_{r+2}) \\
&= (-1)^r \prod_{i=0}^{r-1}[(1-\alpha)i+2]h(K_{X_1}\cdots K_{X_r}X_{r+1}, X_{r+2}).
\end{aligned}$$

Hence the lemma is true for $k = r$. Lemma 4.1 has been proved. □

Remark 4.1. Note that Lemma 4.1 reduces to Lemma 3.4 of [19] when $\alpha = 1$.

From now on, we consider the case $K \neq 0$. Then M has the commutable difference tensor, thus we have all the conclusions of Section 3. By Definition 2.2 let m be the smallest integer such that the symmetric tensor K^m is identically zero at a point p, then $2 \leq m \leq \min\{2s+1, n\}$.

Proof of Main Theorem. As shown above, M is an improper affine hypersphere with flat affine metric and $J = 0$. Lemma 4.1 further shows that $\nabla^m h = 0$ but $\nabla^{m-1} h \neq 0$. By Proposition 2 of [1] we see that M is locally affine equivalent to the graph immersion of a polynomial function of degree $m+1$ with constant Hessian determinant.

Since M is ∇-flat, we can extend a given basis $\{e_1, \ldots, e_n\}$ to ∇-parallel coordinates $(x_1, \ldots, x_n)$. Then the position vector F satisfies

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\xi. \tag{4.1}$$

After an affine transformation, we may assume that $\{F_*(\frac{\partial}{\partial x_1})(0), \ldots, F_*(\frac{\partial}{\partial x_n})(0), \xi\}$ is the standard basis of $\mathbb{R}^{n+1}$ and $F(0) = 0$. With these initial conditions, Eq. (4.1) can be solved, and one obtains that M is locally given in the standard coordinates by $x_{n+1} = f(x_1, \ldots, x_n)$, where f is the function determined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \tag{4.2}$$

with the initial conditions $\frac{\partial f}{\partial x_i}(0) = 0$ and $f(0) = 0$. By $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$ and Lemma 4.1, for any integer $k \geq 2$ (4.2) further gives that

$$\begin{aligned}
\frac{\partial^k f}{\partial x_{i_1} \ldots \partial x_{i_k}} &= (\nabla^{k-2}h)\left(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_k}}\right) \\
&= (-1)^k \prod_{i=0}^{k-3}[(1-\alpha)i+2]h\left(K^{k-2}\left(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_{k-1}}}\right), \frac{\partial}{\partial x_{i_k}}\right).
\end{aligned} \tag{4.3}$$

Finally, using the Taylor expansion, $\nabla^m h = 0$ and $\frac{\partial}{\partial x_i}(0) = e_i$, we can obtain

$$\begin{aligned} f &= \sum_{k=2}^{m+1} \frac{1}{k!} \sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \ldots \partial x_{i_k}}(0) x_{i_1} \cdots x_{i_k} \\ &= \sum_{k=2}^{m+1} (-1)^k \frac{\prod_{i=0}^{k-3}[(1-\alpha)i+2]}{k!} \sum_{i_1,\ldots,i_k=1}^{n} h(K^{k-2}(e_{i_1},\ldots,e_{i_{k-1}}), e_{i_k}) x_{i_1} \cdots x_{i_k}. \end{aligned} \tag{4.4}$$

Note that the function $\Phi(x_1,\ldots,x_n;\alpha)$ of (1.2) can be rewritten as

$$\Phi(x_1,\ldots,x_n;\alpha) = \sum_{k=2}^{n+1} (-1)^k \prod_{i=0}^{k-3}[(1-\alpha)i+2] \sum_{\substack{i_1+2i_2+\cdots+ni_n=n+1 \\ i_1+\cdots+i_n=k}} \frac{x_1^{i_1}\cdots x_n^{i_n}}{i_1!i_2!\cdots i_n!}. \tag{4.5}$$

According to the canonical basis on the hypersurfaces of Section 3, by (4.4) we next show the partial classification of the hypersurfaces considered. As their corollary, Main Theorem easily follows.

First, for $m = 2$ we get the following

Proposition 4.1. *Let M be the affine hypersurface of Main Theorem with $m = 2$. Then there exists a positive integer r with $2r \le n$ such that M is locally affine equivalent to the graph immersion of the polynomial function*

$$x_{n+1} = \sum_{i=1}^{r} y_i w_i + \frac{1}{2}\sum_{k=1}^{n-2r} \epsilon_k u_k^2 - \frac{1}{3}\sum_{i,j,\ell=1}^{r} A_{ij}^{\ell} w_i w_j w_\ell,$$

where A_{ij}^{ℓ} ($i, j, \ell \in \{1,\ldots,r\}$) are totally symmetric constants, $\epsilon_k = \pm 1$ and $(y_1,\ldots,y_r, w_1,\ldots,w_r, u_1,\ldots,u_{n-2r}, x_{n+1})$ are the standard coordinates of $\mathbb{R}^{n+1}$. The converse is also true.

Proof. From $m = 2$ and (3.1) it follows that $\nabla^{(\alpha)}K = \nabla K = 0$. Now, Proposition 3.1 immediately follows from the classification of affine hypersurfaces with $\nabla K = 0$ and $K^2 = 0$ (cf. Theorem 4.3 of [19]). □

Remark 4.2 (*cf. Remark 4.2 of [19]*). In Proposition 4.1, r is the dimension of null subspace span$\{K_x y | x,\ y \in T_pM\}$, thus $0 < r \le s$. In particular, if $n = 2$ then $r = 1$ and M is the Cayley surface described by

$$x_3 = v_1 w_1 - \tfrac{1}{3} w_1^3.$$

If $n = 3$, then $r = 1$ and we have the hypersurface

$$x_4 = v_1 w_1 + \tfrac{1}{2}\epsilon_1 u_1^2 - \tfrac{1}{3} w_1^3.$$

Finally, if $n = 4$ then either $r = 1$ or $r = 2$. By the classification of homogeneous polynomials of degree 3 in 2 variables, we get the following hypersurfaces:

$$\begin{aligned} x_5 &= v_1 w_1 + \tfrac{1}{2}(\epsilon_1 u_1^2 + \epsilon_2 u_2^2) - \tfrac{1}{3} w_1^3, \\ x_5 &= v_1 w_1 + v_2 w_2 - \tfrac{1}{3}(w_1^3 + w_2^3), \\ x_5 &= v_1 w_1 + v_2 w_2 - \tfrac{1}{3} w_1 w_2^2, \\ x_5 &= v_1 w_1 + v_2 w_2 - \tfrac{1}{3} w_1 w_2 (w_1 + w_2). \end{aligned}$$

Note that the m-dimensional non-degenerate subspace $V \subset T_pM$, defined in Lemma 3.3, satisfies $\lfloor \frac{m}{2} \rfloor \le I(V) \le \lceil \frac{m}{2} \rceil$. If $(V^\perp, h)$ is positive definite, i.e., $m = \min\{2s+1, n\}$, $m = 2s$, or $m = 2s-1$ with $I(V) = s$, we have the following

Proposition 4.2. *Let M be the affine hypersurface of Main Theorem with $m \ge 2$. If $m = \min\{2s+1, n\}$, $m = 2s$, or $m = 2s-1$ with $I(V) = s$, then M is locally affine equivalent to the graph immersion of the function*

$$x_{n+1} = \epsilon\Phi(x_1,\ldots,x_m;\alpha) + \frac{1}{2}\sum_{j=m+1}^{n} x_j^2,$$

where $\Phi(x_1,\ldots,x_m;\alpha)$ is given by (1.2), $(x_1,\ldots,x_{n+1})$ are the standard coordinates of $\mathbb{R}^{n+1}$, and $\epsilon = 1$ for $m = \min\{2s+1, n\}$ or $m = 2s$, $\epsilon = -1$ for $m = 2s-1$.

Proof. By Lemma 3.4 there exists a basis $\{e_1, \dots, e_n\}$ of T_pM such that

$$K_{e_i}e_j = \begin{cases} e_{i+j}, & i+j \le m, \\ 0, & \text{otherwise}, \end{cases} \qquad h(e_i, e_j) = \begin{cases} \epsilon, & i+j = m+1, \\ \delta_i^j, & i, j \ge m+1, \\ 0, & \text{otherwise}, \end{cases} \tag{4.6}$$

where $\epsilon = \pm 1$ is determined by (3.4). Since M is ∇-flat, we can extend the above basis to ∇-parallel coordinates $(x_1, \dots, x_n)$. Using the method as above we see that M is locally given by a polynomial function $x_{n+1} = f(x_1, \dots, x_n)$ of degree $m+1$, where f is determined by (4.4), where $e_i = \frac{\partial}{\partial x_i}(0)$. By the initial conditions (4.6) it follows from (4.4) that

$$f = \epsilon \Phi(x_1, \dots, x_m; \alpha) + \frac{1}{2} \sum_{j=m+1}^{n} x_j^2,$$

where $\Phi(x_1, \dots, x_m; \alpha)$ is given by (1.2) or (4.5). □

Remark 4.3. When $m = n$, Proposition 4.2 (resp. $\alpha = 0$, $\alpha = 1$) reduces to the main result of [20] (resp. Main Theorem of [30], Theorem 6.2 of [19]), while the method here based on (4.4) is more direct. The method implies that affine hypersurfaces with $\nabla^{(\alpha)}K = 0$ are completely determined by the properties of (h, K) at a point, thus we conjecture that such hypersurfaces are locally homogeneous.

Next, if $(V^{\perp}, h)$ is Lorentzian, i.e., $m = 2s-1$, $m = 2s-2$ or $m = 2s-3$ with $I(V) = s-1$, there exist three distinct cases as in Section 3.2.

For Case I, by Lemmas 3.3 and 3.5 there exist an integer m_1 satisfying $1 \le m_1 \le \min\{m, n-m, 3\}$ and a basis $\{y_1, \dots, y_m, z_1, \dots, z_{n-m}\}$ of T_pM such that there hold (3.2) and (3.8) with $K(z_k, y_i) = 0$, $h(z_k, y_i) = 0$ for all k, i. Similar to the proof of Proposition 4.2, by (4.4) and the above basis corresponding to $m_1 = 1, 2, 3$ we can obtain the following

Proposition 4.3. *Let M be the affine hypersurface of Main Theorem with $m \ge 2$. If $m = 2s-1$, $m = 2s-2$, or $m = 2s-3$ with $I(V) = s-1$, for Case* I *M is locally affine equivalent to one of the graph immersions of the three functions*

$$x_{n+1} = \epsilon \Phi(x_1, \dots, x_m; \alpha) - \tfrac{1}{2} x_{m+1}^2 + \tfrac{1}{2} \sum_{i=m+2}^{n} x_i^2,$$

$$x_{n+1} = \epsilon \Phi(x_1, \dots, x_m; \alpha) + x_{m+1}x_{m+2} - \tfrac{1}{3} x_{m+1}^3 + \tfrac{1}{2} \sum_{i=m+3}^{n} x_i^2, \quad 4 \le m+2 \le n,$$

$$x_{n+1} = \epsilon \Phi(x_1, \dots, x_m; \alpha) + \Phi(x_{m+1}, x_{m+2}, x_{m+3}; \alpha) + \tfrac{1}{2} \sum_{i=m+4}^{n} x_i^2, \quad 6 \le m+3 \le n,$$

where $(x_1, \dots, x_{n+1})$ are the standard coordinates of $\mathbb{R}^{n+1}$, and $\epsilon = 1$ for $m = 2s-1$ or $m = 2s-2$, $\epsilon = -1$ for $m = 2s-3$.

For Case II, by Lemmas 3.3 and 3.6 there exists a basis $\{y_1, \dots, y_m, z_1, \dots, z_{n-m}\}$ of T_pM such that there hold (3.2), (3.9) and (3.10) with $h(z_k, y_i) = 0$ for all i, k. Similar to Proposition 4.2, we can extend the above basis to local ∇-parallel coordinates $\{x_1, \dots, x_n\}$, and p has coordinates 0. As before M is locally given by a polynomial function $x_{n+1} = f(x_1, \dots, x_n)$, where f is determined by (4.4) and the initial conditions $\{y_1, \dots, y_m, z_1, \dots, z_{n-m}\}$. Then we easily obtain

$$\begin{aligned} f &= \epsilon \Phi(x_1, \dots, x_m; \alpha) + \varepsilon x_{m+1}x_{m+2} - \varepsilon x_1 x_{m+1}^2 - \tfrac{a}{3} x_{m+1}^3 \\ &\quad - \beta x_{m+1}^2 x_{m+3} + \tfrac{3-\alpha}{12} \beta^2 x_{m+1}^4 + \tfrac{1}{2} \sum_{i=m+3}^{n} x_i^2, \end{aligned} \tag{4.7}$$

where $\varepsilon = \pm 1$, and $\epsilon = \pm 1$ is determined by (3.7). Now, we have proved for Case II the following

Proposition 4.4. *Let M be the affine hypersurface of Main Theorem with $m \ge 2$. If $m = 2s-1$, $m = 2s-2$, or $m = 2s-3$ with $I(V) = s-1$, for Case* II *we see that $n \ge m+2$ and M is locally affine equivalent to the graph immersion of the function* (4.7), *where a, β are constant, and $\beta = 0$ if $m = 2$ or $m = n-2$.*

For Case III, by Lemmas 3.3 and 3.7 there exists a basis $\{y_1, \dots, y_m, z_1, \dots, z_{n-m}\}$ of T_pM such that there hold (3.2), (3.11) and (3.12) with $h(z_k, y_i) = 0$ for all i, k. Similar to Proposition 4.2, we can extend the above basis to local ∇-parallel coordinates $\{x_1, \dots, x_n\}$, and p has coordinates 0. By the previous method we see that M is locally given by a polynomial function $x_{n+1} = f(x_1, \dots, x_n)$, where f is determined by (4.4) and the initial conditions $\{y_1, \dots, y_m, z_1, \dots, z_{n-m}\}$. Then we easily obtain

$$f = \epsilon \Phi(x_1, \dots, x_m; \alpha) + x_{m+1}x_{m+3} + \tfrac{1}{2} x_{m+2}^2 - 2x_1 x_{m+1} x_{m+2} - x_2 x_{m+1}^2$$

$$- bx_{m+1}^2 x_{m+2} - \gamma x_{m+1}^2 x_{m+4} - \tfrac{a}{3} x_{m+1}^3 + \tfrac{3-\alpha}{2} x_1^2 x_{m+1}^2 + \tfrac{(3-\alpha)b}{3} x_1 x_{m+1}^3$$
$$+ \tfrac{(3-\alpha)(b^2+\gamma^2)}{12} x_{m+1}^4 + \tfrac{1}{2} \sum_{i=m+4}^{n} x_i^2, \tag{4.8}$$

where $\epsilon = \pm 1$ is determined by (3.7). Now, we have proved for Case III the following

Proposition 4.5. *Let M be the affine hypersurface of Main Theorem with $m \geq 2$. If $m = 2s - 1$, $m = 2s - 2$, or $m = 2s - 3$ with $I(V) = s - 1$, for Case* III *we see that $m \geq 3$, $n \geq m + 3$ and M is locally affine equivalent to the graph immersion of the function (4.8), where a, b, γ are constant, and $\gamma = 0$ if $n = m + 3$.*

If M is locally strongly convex, i.e., $s = 0$, then the vanishing Pick invariant shows that M is a locally strongly convex hyperquadric. Taking $s = 1, 2$ in Propositions 4.2–4.5 respectively, we see that the remaining part of Main Theorem immediately follows. □

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