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Bach flow

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ABSTRACT

In this paper, we study the Bach flow which is defined as

$$\frac{\partial}{\partial t}g_{ij} = -B_{ij}$$

where B_{ij} is the Bach tensor. Among other things, we study the solitons to the Bach flow. We also study the Bach flow on a four-dimensional Lie group, in which we study the convergence of the Bach flow.

1. Introduction

In an n-dimensional Riemannian manifold (M^n, g), $n \geq 4$, the Bach tensor, introduced by Bach in [1], is defined as

$$B_{ij} = \frac{1}{n-3}\nabla_k\nabla_l W_{ikjl} + \frac{1}{n-2}R_{kl}W_{ikjl}. \tag{1.1}$$

Hereafter, we use the Einstein summation convention: we sum over the repeat indices. Hence, (1.1) should be read as

$$B_{ij} = \frac{1}{n-3}\sum_{k,j=1}^{n}\nabla_k\nabla_l W_{ikjl} + \frac{1}{n-2}\sum_{k,j=1}^{n}R_{kl}W_{ikjl}.$$

Here

$$\begin{aligned} W_{ikjl} = R_{ikjl} &- \frac{1}{n-2}(R_{ij}g_{kl} + R_{kl}g_{ij} - R_{il}g_{kj} - R_{kj}g_{il}) \\ &+ \frac{R}{(n-1)(n-2)}(g_{ij}g_{kl} - g_{il}g_{kj}) \end{aligned} \tag{1.2}$$

is the Weyl tensor, R_{ikjl} is the Riemann curvature tensor, R_{ij} is the Ricci curvature tensor, and R is the scalar curvature of the metric g. It is easy to see that if (M^n, g) is either locally conformally flat (i.e. $W_{ikjl} = 0$) or Einstein, then (M^n, g) is *Bach-flat*: $B_{ij} = 0$. See Proposition 2.2 when g is Einstein.

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The case when $n = 4$ is the most interesting, as it is well known that (see [2]) on any compact 4-manifold (M^4, g), Bach-flat metrics are precisely the critical point of the conformally invariant functional on the space of metrics,

$$\mathcal{W}(g) = \int_M |W|_g^2 dV_g$$

where W denotes the Weyl tensor of g. Moreover, if (M^4, g) is either half conformally flat (i.e. self-dual or anti-self-dual) or locally conformal to an Einstein manifold, then its Bach tensor vanishes.

A geometric flow which was defined using the Bach tensor has been introduced in [3]: the Bach flow is defined as follows:

$$\frac{\partial}{\partial t} g_{ij} = -B_{ij}. \tag{1.3}$$

In [3], Das and Kar studied the Bach flow on some product manifolds equipped with the product metric.

In this paper, we study the Bach flow more generally. After collecting some properties of the Bach flow in Section 2, we study in Section 3 the solitons to the Bach flow. In particular, we prove that any compact soliton to the Bach flow must be Bach-flat. See Theorems 3.1 and 3.2. Finally, in Section 4, we study the Bach flow on a four-dimensional Lie groups, in which we consider the convergence of the Bach flow.

2. Some properties of Bach flow

In this section, we collect some properties of Bach flow. Along the Bach flow (1.3), we have

$$\frac{\partial}{\partial t} dV_g = \frac{1}{2} g^{ij} B_{ij} dV_g = 0, \tag{2.1}$$

since the Bach tensor is trace-free. Therefore, we have the following:

Proposition 2.1. *The Bach flow (1.3) preserves the volume of M.*

Note that the Bach flow does not preserve the conformal structure in general. Indeed, the Bach flow preserves the conformal structure only if the initial metric is Bach flat. To see this, note that if $\tilde{g} = e^{2u} g$, then

$$\widetilde{B}_{ij} = e^{-2u} B_{ij} \tag{2.2}$$

where $\widetilde{B}_{ij}$ and B_{ij} are the Bach tensors of $\tilde{g}$ and g respectively. Therefore, if $\tilde{g} = e^{2u} g$ is the solution of the Bach flow, we have

$$2e^{2u} \frac{\partial u}{\partial t} g_{ij} = e^{-2u} B_{ij} \tag{2.3}$$

by (1.3) and (2.2). Taking trace of both sides in (2.3), we get

$$\frac{\partial u}{\partial t} = 0 \tag{2.4}$$

since the Bach tensor is trace-free by the definition in (1.1). It follows from (2.4) that $\frac{\partial}{\partial t} \tilde{g}_{ij} = 0$, which implies that $\widetilde{B}_{ij} = 0$ for all $t \geq 0$. In particular, the initial metric is Bach flat.

The following proposition is well-known.

Proposition 2.2. *If g is Einstein, then g must be Bach flat.*

To be self-contained, we give the proof here.

Proof of Proposition 2.2. If g is Einstein, then

$$R_{ij} = \frac{R}{n} g_{ij}. \tag{2.5}$$

In particular, the scalar curvature R must be constant since $n \geq 4$. It follows from (2.5) that Weyl tensor can be rewritten as

$$\begin{aligned} W_{ikjl} &= R_{ikjl} - \frac{1}{n-2}(R_{ij} g_{kl} + R_{kl} g_{ij} - R_{il} g_{kj} - R_{kj} g_{il}) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{ij} g_{kl} - g_{il} g_{kj}) \\ &= R_{ikjl} - \frac{R}{n(n-1)}(g_{ij} g_{kl} - g_{il} g_{kj}). \end{aligned} \tag{2.6}$$

It follows from (1.2) and (2.5) that

$$R_{kl} W_{ikjl} = \frac{R}{n} g_{kl} W_{ikjl} = 0. \tag{2.7}$$

Therefore, it follows from (2.6) and (2.7) that the Bach tensor can be rewritten as

$$\begin{aligned} B_{ij} &= \frac{1}{n-3}\nabla_k\nabla_l W_{ikjl} + \frac{1}{n-2}R_{kl}W_{ikjl} \\ &= \frac{1}{n-3}\nabla_k\nabla_l\left(R_{ikjl} - \frac{R}{n(n-1)}(g_{ij}g_{kl} - g_{il}g_{kj})\right) \\ &= \frac{1}{n-3}\nabla_k\nabla_l R_{ikjl}. \end{aligned} \tag{2.8}$$

It follows from Bianchi identity that

$$\begin{aligned} \nabla_l R_{ikjl} &= -\nabla_i R_{kljl} - \nabla_k R_{lijl} \\ &= -\nabla_i R_{kljl} + \nabla_k R_{iljl} \\ &= -\nabla_i R_{kj} + \nabla_k R_{ij} \\ &= -\nabla_i\left(\frac{R}{n}g_{kj}\right) + \nabla_k\left(\frac{R}{n}g_{ij}\right) = 0 \end{aligned} \tag{2.9}$$

where the second last equality follows from (2.5). Now the result follows from (2.8) and (2.9). □

Therefore, the Bach flow does not preserve the Einstein condition in general. Indeed, the Bach flow preserves the Einstein condition only if the initial metric is Bach flat. To see this, we note that, by Proposition 2.2, if g is the solution of the Bach flow (1.3) and is Einstein for all $t \geq 0$, then g must be Bach flat for all $t \geq 0$.

Next we derive the evolution equation of the Ricci curvature tensor and the scalar curvature along the Bach flow.

Lemma 2.3. *Along the Bach flow (1.3), the Ricci curvature tensor of g satisfies the following evolution equation:*

$$\frac{\partial}{\partial t}R_{ij} = \frac{1}{2}\left(\Delta_g B_{ij} - \nabla_k\nabla_i B_{jk} - \nabla_k\nabla_j B_{ik}\right).$$

Proof. Note that (see the proof of Lemma 2.3 in [4])

$$\begin{aligned} \frac{\partial}{\partial t}R_{ij} &= \nabla_k\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right) - \nabla_i\left(\frac{\partial}{\partial t}\Gamma_{kj}^k\right) \\ &= \frac{1}{2}\left[-\Delta_g\left(\frac{\partial}{\partial t}g_{ij}\right) - \nabla_i\nabla_j\left(g^{kl}\frac{\partial}{\partial t}g_{kl}\right) + \nabla_k\nabla_i\left(\frac{\partial}{\partial t}g_{jk}\right) + \nabla_k\nabla_j\left(\frac{\partial}{\partial t}g_{ik}\right)\right]. \end{aligned}$$

Combining this with (1.3) and using the fact that the Bach tensor is trace-free, we get the result. □

Lemma 2.4. *Along the Bach flow (1.3), the scalar curvature of g satisfies the following evolution equation:*

$$\frac{\partial}{\partial t}R = R_{ij}B_{ij} - \nabla_j\nabla_i B_{ij}.$$

Proof. Since $R = g^{ij}R_{ij}$ where (g^{ij}) is the inverse of (g_{ij}), the result follows from Lemma 2.3 and

$$\frac{\partial}{\partial t}g^{ij} = B_{ij}$$

by (1.3). □

By Lemma 5.1 in [5], we have

$$\nabla_j B_{ij} = \frac{n-4}{(n-2)^2}C_{ijk}R_{jk} \tag{2.10}$$

where C_{ijk} is the Cotton tensor defined by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R).$$

Especially, in dimension four, the Bach tensor is divergence-free, i.e.

$$\nabla_i B_{ij} = 0.$$

Combining this with Lemmas 2.3 and 2.4, we have

Corollary 2.5. *In dimension four, the scalar curvature of g satisfies the following evolution equations along the Bach flow* (1.3)*:*

$$\frac{\partial}{\partial t}R = R_{ij}B_{ij}.$$

To conclude this section, we note that the short time existence was known for the flow

$$\frac{\partial}{\partial t}g_{ij} = B_{ij} + \frac{1}{12}\Delta_g R\, g_{ij}$$

in dimension 4 (see [6]). It would be interesting to see if one can prove the short time existence of the Bach flow (1.3) by following the argument in [6]. See also [7] and [8].

3. Soliton to the Bach flow

In this section, we study the soliton to the Bach flow. Soliton is a self-similar solution to geometric flow. More precisely, given a Riemannian manifold (M, g_0), $g(t)$ is the soliton of the geometric flow

$$\frac{\partial}{\partial t}g(t) = F(g(t)) \quad \text{with } g(0) = g_0, \tag{3.1}$$

if

$$g(t) = \sigma(t)\varphi_t^*(g_0) \tag{3.2}$$

is a solution to (3.1), where σ is a function depending only on t with $\sigma(0) = 1$ and φ_t is a family of diffeomorphisms of M such that $\varphi_0 = id_M$. For the Ricci flow

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij},$$

the Ricci soliton takes the form:

$$R_{ij} = cg_{ij} + (\mathcal{L}_X g)_{ij}.$$

To derive this, one can take the derivative of (3.2) with respect to t and evaluate it at $t = 0$. Ricci soliton has been studied extensively for the last few decades. See [9–12] and references therein for results related to Ricci soliton.

Similar to the Ricci soliton, soliton to the Bach flow is given by

$$B_{ij} = cg_{ij} + (\mathcal{L}_X g)_{ij}. \tag{3.3}$$

Here $\mathcal{L}$ denotes the Lie derivative. If the vector field X is gradient, i.e. $X = \nabla_g f$ for some smooth function f on M, then (3.3) becomes the gradient soliton:

$$B_{ij} = cg_{ij} + \nabla_i\nabla_j f. \tag{3.4}$$

We have the following:

Theorem 3.1. *Any compact gradient soliton* (3.4) *to the Bach flow* (1.3) *must be Bach-flat.*

Proof. By the definition (1.1), the Bach tensor is trace-free. Hence, taking trace of both sides of (3.4), we get

$$0 = cn + \Delta f. \tag{3.5}$$

Integrating (3.5) over M, we get $cn\mathrm{Vol}(M, g) = 0$, which implies that

$$c = 0. \tag{3.6}$$

It follows from (3.5) and (3.6) that f is a harmonic function. Since any harmonic function on a compact manifold M must be constant, we deduce from (3.4) and (3.6) that $B_{ij} = 0$, i.e. (M, g_{ij}) is Bach-flat. This proves the assertion. □

More generally, we have the following:

Theorem 3.2. *Let (M, g_{ij}, X) be a 4-dimensional compact soliton* (3.3) *to the Bach flow* (1.3)*. Then (M, g_j) must be Bach-flat and X is a Killing vector field.*

Proof. Our proof is inspired by the proof in [13]. Note that if ϕ is a symmetric $(0, 2)$-tensor field, then

$$\langle \mathcal{L}_\xi g, \phi\rangle = 2\,\mathrm{div}(i_\xi\phi) - 2(\mathrm{div}\phi)(\xi) \tag{3.7}$$

for any vector field ξ on M. Here $i_\xi\phi$ is the 1-form given by $i_\xi\phi(\cdot) = \phi(\xi, \cdot)$. Taking divergence in (3.3) and by the fact that the Bach tensor (1.1) is divergence-free in dimension four, we have

$$\mathrm{div}(\mathcal{L}_X g) = -c\ \mathrm{div}(g) = 0. \tag{3.8}$$

Next, put $\phi = \mathcal{L}_X g$ and $\xi = X$ in (3.7), we obtain

$$\|\mathcal{L}_X g\|^2 = 2\ \mathrm{div}(i_X\mathcal{L}_X g) - 2\ \mathrm{div}(\mathcal{L}_X g)(X) = 2\ \mathrm{div}(i_X\mathcal{L}_X g) \tag{3.9}$$

by (3.8). Integrating (3.9) over M, we get

$$\int_M \|\mathcal{L}_X g\|^2 = 2\int_M \mathrm{div}(i_X\mathcal{L}_X g) = 0,$$

which implies that $\mathcal{L}_X g = 0$ and hence X is Killing. Furthermore, by (3.3) and the fact that the Bach tensor is trace-free, the constant c must vanish in (3.3). Hence, (M, g_{ij}) is Bach-flat. This proves the assertion. □

Following the proof of Theorem 3.1, we have the following corollary for the noncompact case.

Corollary 3.3. *Let (M, g_{ij}, f) be a noncompact gradient soliton (3.4) to the Bach flow such that M has nonnegative sectional curvature and the gradient of f satisfying*

$$\int_M |\nabla_g f|^{\frac{n}{n-1}} < \infty,$$

then (M, g_{ij}) must be Bach-flat.

Proof. Taking trace of (3.4), again we get (3.5). If $c \geq 0$, then it follows from (3.5) that $-f$ is a subharmonic function; and if $c < 0$, then it follows from (3.5) that f is a subharmonic function. In either case, we can conclude that f is a constant function by Corollary 2' in [14], which says that if M is an n-dimensional complete noncompact Riemannian of nonnegative sectional curvature, then no nonconstant C^2 subharmonic function u has its gradient $\nabla_g u$ satisfying $\int_M |\nabla_g u|^{\frac{n}{n-1}} < \infty$. Since f is constant, we have $c = 0$. Hence, it follows from (3.4) that $B_{ij} = 0$, i.e. (M, g_{ij}) is Bach-flat. This proves the assertion. □

We also have the following result for the four-dimensional gradient soliton to the Bach flow.

Theorem 3.4. *Let (M, g_{ij}, f) be a four-dimensional gradient soliton (3.4) to the Bach flow such that M has positive or negative Ricci curvature, then (M, g_{ij}) must be Bach-flat.*

Proof. Applying ∇_j to (3.4) and using (2.10), we get

$$\frac{n-4}{(n-2)^2}C_{jik}R_{ik} = \nabla_i B_{ij} = \nabla_i\nabla_i\nabla_j f = \nabla_j\Delta f + R_{jk}\nabla_k f.$$

Combining this with (3.5), we get

$$\frac{n-4}{(n-2)^2}C_{jik}R_{ik} = R_{jk}\nabla_k f.$$

Especially, if $n = 4$, we have

$$R_{jk}\nabla_k f = 0 \tag{3.10}$$

In particular, if the Ricci curvature tensor is positive definite or negative definite, then we can conclude from (3.10) that f is constant. By (3.5), we have $c = 0$. Hence, it follows from (3.4) that $B_{ij} = 0$, i.e. (M, g_{ij}) is Bach-flat. This proves the assertion. □

In the noncompact case, the following result shows the existence of non-trivial gradient solitons to the Bach flow.

Theorem 3.5. *Suppose $(\mathbb{R}^2, g^0)$ is the 2-dimensional Euclidean space equipped with the flat metric g^0, (S^2, g^1) is the 2-dimensional sphere equipped with the standard metric g^1. The product manifold $\mathbb{R}^2 \times S^2$ equipped with the product metric $g^0 \times g^1$ is a non-trivial gradient soliton to the Bach flow with $c = -\frac{1}{12}$ in (3.4), for any function $f = f(x, y)$ in $\mathbb{R}^2$ of the form*

$$f(x, y) = \frac{1}{12}(x^2 + y^2) + C \tag{3.11}$$

where C is a constant.

Proof. Suppose that (M, g^M) and (N, g^N) are two-dimensional Riemannian manifolds. We are going to use the Greek letter μ, ν, α to represent the indices of M and the English letter i, j, k to represent the indices of N. Then the Bach tensor of the product manifold $M \times N$ equipped with the product metric $g^M \times g^N$ splits as follows: (see Eqs. (10) and (11) in [3])

$$\begin{aligned} B_{\mu\nu} &= \frac{1}{3}\nabla_\mu\nabla_\nu R_{g^M} - \frac{1}{3}g^M_{\mu\nu}\left[\nabla_\alpha\nabla_\alpha R_{g^M} - \frac{1}{2}\nabla_k\nabla_k R_{g^N} + \frac{1}{4}\Big((R_{g^M})^2 - (R_{g^N})^2\Big)\right] \text{ in } M,\\ B_{ij} &= \frac{1}{3}\nabla_i\nabla_j R_{g^N} - \frac{1}{3}g^N_{ij}\left[\nabla_k\nabla_k R_{g^N} - \frac{1}{2}\nabla_\alpha\nabla_\alpha R_{g^M} + \frac{1}{4}\Big((R_{g^N})^2 - (R_{g^M})^2\Big)\right] \text{ in } N. \end{aligned} \tag{3.12}$$

Here R_{g^M} and R_{g^N} are the scalar curvatures of g^M and g^N respectively. Since the scalar curvatures of g^0 and g^1 are given by $R_{g^0} = 0$ and $R_{g^1} = 1$ respectively, it follows from (3.12) that

$$\begin{aligned} B_{\mu\nu} &= \frac{1}{12}g^0_{\mu\nu} \quad \text{in } \mathbb{R}^2,\\ B_{ij} &= -\frac{1}{12}g^1_{ij} \quad \text{in } S^2. \end{aligned} \tag{3.13}$$

Combining (3.13) with the equation of the gradient soliton of the Bach flow (3.4), we obtain

$$\begin{aligned} \frac{1}{12}g^0_{\mu\nu} &= cg^0_{\mu\nu} + \nabla_\mu\nabla_\nu f \quad \text{in } \mathbb{R}^2,\\ -\frac{1}{12}g^1_{ij} &= cg^1_{ij} + \nabla_i\nabla_j f \quad \text{in } S^2. \end{aligned} \tag{3.14}$$

Therefore, if $c = -\frac{1}{12}$ and f depends only on $\mathbb{R}^2$, then the second equation in (3.14) is satisfied. On the other hand, if $c = -\frac{1}{12}$, then the first equation in (3.14) implies that

$$\frac{1}{6}g^0_{\mu\nu} = \nabla_\mu\nabla_\nu f \quad \text{in } \mathbb{R}^2,$$

which implies that $f = f(x, y)$ is a function in $\mathbb{R}^2$ satisfying

$$f_{xx} = f_{yy} = \frac{1}{6} \text{ and } f_{xy} = 0.$$

Solving these, we conclude that

$$f(x, y) = \frac{1}{12}(x^2 + y^2) + C$$

for some constant C. This proves the assertion. □

In fact, we have the following:

Theorem 3.6. *All the nontrivial gradient solitons (3.4) to the Bach flow of the product manifold $\mathbb{R}^2 \times S^2$ equipped with the product metric $g^0 \times g^1$ must be in the form of (3.11).*

Proof. To see this, we can follow the proof of Theorem 3.5 to obtain (3.14). Taking trace of the second equation in (3.14), we get

$$\Delta_{g^1} f = -2c - \frac{1}{6} \text{ in } S^2. \tag{3.15}$$

If $c \geq -1/12$, then f is superharmonic by (3.15); and if $c \leq -1/12$, then f is subharmonic by (3.15). In either case, since S^2 is compact, f must be a constant function on S^2. This implies that f is a function depending only on $\mathbb{R}^2$ and $c = -1/12$ by (3.15). Now we can follow the remaining part of the proof of Theorem 3.5 to conclude that f must be in the form of (3.11). This proved the assertion. □

Suppose $(\mathbb{H}^2, g^{-1})$ is the 2-dimensional hyperbolic space equipped with the standard metric g^{-1}. Following the proof of Theorem 3.5, we can also prove the following:

Theorem 3.7. *The product manifold $\mathbb{R}^2 \times \mathbb{H}^2$ equipped with the product metric $g^0 \times g^{-1}$ is a non-trivial gradient soliton to the Bach flow with $c = -\frac{1}{12}$ in (3.4), for any function $f = f(x, y)$ in $\mathbb{R}^2$ of the form*

$$f(x, y) = \frac{1}{12}(x^2 + y^2) + C$$

where C is a constant.

Proof. Since the scalar curvatures of g^0 and g^{-1} are given by $R_{g^0}=0$ and $R_{g^{-1}}=-1$ respectively, it follows from (3.12) that

$$B_{\mu\nu}=\frac{1}{12}g^0_{\mu\nu}\quad\text{in }\mathbb{R}^2,$$
$$B_{ij}=-\frac{1}{12}g^{-1}_{ij}\quad\text{in }\mathbb{H}^2.$$

Combining these with the equation of the gradient soliton of the Bach flow (3.4), we obtain

$$\begin{aligned}\frac{1}{12}g^0_{\mu\nu}&=cg^0_{\mu\nu}+\nabla_\mu\nabla_\nu f\quad\text{in }\mathbb{R}^2,\\ -\frac{1}{12}g^{-1}_{ij}&=cg^{-1}_{ij}+\nabla_i\nabla_j f\quad\text{in }\mathbb{H}^2.\end{aligned}\tag{3.16}$$

Now we can follow the remaining part of the proof of Theorem 3.5 to get the result. □

In fact, we can classify the gradient solitons to the Bach flow of the product manifold $\mathbb{R}^2\times\mathbb{H}^2$ equipped with the product metric $g^0\times g^{-1}$. To this end, we can follow the proof of Theorem 3.7 to obtain (3.16). It follows from (3.16) that $f=f_1+f_2$ where f_1 and f_2 are functions depending only on $\mathbb{R}^2$ and $\mathbb{H}^2$ respectively. On the other hand, f_1 and f_2 satisfy

$$\begin{aligned}\frac{1}{12}g^0_{\mu\nu}&=cg^0_{\mu\nu}+\nabla_\mu\nabla_\nu f_1\quad\text{in }\mathbb{R}^2,\\ -\frac{1}{12}g^{-1}_{ij}&=cg^{-1}_{ij}+\nabla_i\nabla_j f_2\quad\text{in }\mathbb{H}^2.\end{aligned}\tag{3.17}$$

If we denote the coordinates of $\mathbb{R}^2$ by (x_1,x_2), then it follows from the first equation of (3.17) that

$$\frac{\partial^2 f_1}{\partial x_1^2}=\frac{\partial^2 f_1}{\partial x_2^2}=\frac{1}{12}-c\text{ and }\frac{\partial^2 f_1}{\partial x_1\partial x_2}=0.\tag{3.18}$$

It follows from $\frac{\partial^2 f_1}{\partial x_1\partial x_2}=0$ that $f_1(x_1,x_2)=h_1(x_1)+h_2(x_2)$. Combining this with the first equation in (3.18), we conclude that h_1 and h_2 are quadratic functions in x_1 and x_2 respectively. That is,

$$f_1(x_1,x_2)=\Big(\frac{1}{24}-\frac{c}{2}\Big)(x_1^2+x_2^2)+C_1x_1+C_2x_2+C$$

for some constants C_1, C_2, and C. On the other hand, if we denote the coordinates of $\mathbb{H}^2$ by (w_1,w_2), i.e. $\mathbb{H}^2=\{(w_1,w_2)\in\mathbb{R}^2|w_2>0\}$, then the metric g^{-1} is given by

$$g^{-1}_{ij}=\frac{\delta_{ij}}{(w_2)^2}.\tag{3.19}$$

Using the formula, we can compute the Christoffel symbols of g^{-1}:

$$\Gamma^1_{11}=\Gamma^2_{12}=\Gamma^2_{21}=\Gamma^1_{22}=0,\quad \Gamma^1_{12}=\Gamma^1_{21}=\Gamma^2_{22}=-\frac{1}{w_2},\quad \Gamma^2_{11}=\frac{1}{w_2}.$$

This implies that

$$\begin{aligned}\nabla_1\nabla_1 f_2&=\frac{\partial^2 f_2}{\partial w_1^2}-\Gamma^2_{11}\frac{\partial f_2}{\partial w_2}=\frac{\partial^2 f_2}{\partial w_1^2}-\frac{1}{w_2}\frac{\partial f_2}{\partial w_2},\\ \nabla_1\nabla_2 f_2&=\frac{\partial^2 f_2}{\partial w_1\partial w_2}-\Gamma^1_{12}\frac{\partial f_2}{\partial w_1}=\frac{\partial^2 f_2}{\partial w_1\partial w_2}+\frac{1}{w_2}\frac{\partial f_2}{\partial w_1},\\ \nabla_2\nabla_2 f_2&=\frac{\partial^2 f_2}{\partial w_2^2}-\Gamma^2_{22}\frac{\partial f_2}{\partial w_2}=\frac{\partial^2 f_2}{\partial w_2^2}+\frac{1}{w_2}\frac{\partial f_2}{\partial w_2}.\end{aligned}\tag{3.20}$$

Combining (3.19), (3.20), and the second equation of (3.17), we obtain

$$\begin{aligned}\frac{\partial^2 f_2}{\partial w_1^2}-\frac{1}{w_2}\frac{\partial f_2}{\partial w_2}+\Big(\frac{1}{12}+c\Big)\frac{1}{w_2^2}&=0,\\ \frac{\partial^2 f_2}{\partial w_1\partial w_2}+\frac{1}{w_2}\frac{\partial f_2}{\partial w_1}&=0,\\ \frac{\partial^2 f_2}{\partial w_2^2}+\frac{1}{w_2}\frac{\partial f_2}{\partial w_2}+\Big(\frac{1}{12}+c\Big)\frac{1}{w_2^2}&=0.\end{aligned}\tag{3.21}$$

Since

$$\frac{\partial}{\partial w_1}\left(\frac{\partial f_2}{\partial w_2}+\frac{1}{w_2}f_2\right)=\frac{\partial^2 f_2}{\partial w_1\partial w_2}+\frac{1}{w_2}\frac{\partial f_2}{\partial w_1}=0,$$

by the second equation of (3.21), the function $\frac{\partial f_2}{\partial w_2}+\frac{1}{w_2}f_2$ depends only on w_2. That is,

$$\frac{\partial f_2}{\partial w_2}+\frac{1}{w_2}f_2=h(w_2)$$

for some function h. Differentiate it with respect to w_2, we get

$$\frac{\partial^2 f_2}{\partial w_2^2}+\frac{1}{w_2}\frac{\partial f_2}{\partial w_2}-\frac{1}{w_2^2}f_2=h'(w_2).$$

Combining this with the third equation of (3.21), we obtain

$$-\left(\frac{1}{12}+c\right)\frac{1}{w_2^2}-\frac{1}{w_2^2}f_2=h'(w_2).$$

From this, we see that f_2 depends only on w_2. In particular, we have $\frac{\partial^2 f_2}{\partial w_1^2}=0$. Hence, it follows from the first equation of (3.21) that

$$-\frac{1}{w_2}\frac{\partial f_2}{\partial w_2}+\left(\frac{1}{12}+c\right)\frac{1}{w_2^2}=0.$$

Solving it, we obtain

$$f_2(w_1,w_2)=\left(\frac{1}{12}+c\right)\log w_2.$$

Combining all these, we have the following:

Theorem 3.8. *All the gradient solitons (3.4) to the Bach flow of the product manifold $\mathbb{R}^2\times\mathbb{H}^2$ equipped with the product metric $g^0\times g^{-1}$ must be in the form*

$$f(x_1,x_2,w_1,w_2)=\left(\frac{1}{24}-\frac{c}{2}\right)(x_1^2+x_2^2)+C_1x_1+C_2x_2+\left(\frac{1}{12}+c\right)\log w_2+C$$

for $(x_1,x_2)\in\mathbb{R}^2$ and $(w_1,w_2)\in\mathbb{H}^2$, where C_1, C_2, and C are constants.

4. A Lie group in four dimension

In [15], Isenberg–Jackson–Lu considered the Ricci flow on locally homogeneous closed 4-manifolds. In particular, they studied the Ricci flow on 4-dimensional unimodular Lie groups.

In this section, we consider the Bach flow on a four-dimensional Lie group, which is the class $U1[1,1,1]$ appeared in section A2. of [15].

Let θ_i be the frame of 1-forms dual to X_i. Assume that the solution to the Bach flow takes the form

$$g(t)=A(t)(\theta_1)^2+B(t)(\theta_2)^2+C(t)(\theta_3)^2+D(t)(\theta_4)^2 \tag{4.1}$$

with the initial condition

$$g(0)=\lambda_1(\theta_1)^2+\lambda_2(\theta_2)^2+\lambda_3(\theta_3)^2+\lambda_4(\theta_4)^2. \tag{4.2}$$

Then $\bar{X}_1=\frac{1}{\sqrt{A}}X_1$, $\bar{X}_2=\frac{1}{\sqrt{B}}X_2$, $\bar{X}_3=\frac{1}{\sqrt{C}}X_3$, $\bar{X}_4=\frac{1}{\sqrt{D}}X_4$ is an orthonormal frame with respect to the metric $g=g(t)$. Let $W=w_1\bar{X}_1+w_2\bar{X}_2+w_3\bar{X}_3+w_4\bar{X}_4$. Then the Ricci curvature tensor of g is given by (see P.355 in [15]):

$$Ric(W,W)=0\cdot w_1^2+0\cdot w_2^2+0\cdot w_3^2-\frac{2(k^2+k+1)}{D}w_4^2.$$

In particular, we have

$$Ric(\bar{X}_i,\bar{X}_i)=0 \text{ for } 1\le i\le 3 \text{ and } Ric(\bar{X}_4,\bar{X}_4)=-\frac{(2k^2+k+1)}{D}. \tag{4.3}$$

In four dimension, the Bach tensor can be written as (see P.304 in [16] or see P.411 in [17]):

$$B_{ij}=\nabla_p\nabla_jR_{ip}-\frac{1}{2}\nabla_p\nabla_pR_{ij}+\frac{1}{3}RR_{ij}-R_{pi}R_{pj}+\frac{1}{12}\left(3\sum_{r,s=1}^{4}(R_{rs})^2-R^2\right)\delta_{ij}. \tag{4.4}$$

It follows from (4.4) that the Bach tensor is diagonal if the Ricci tensor of the left-invariant metric is diagonal. Moreover, it follows from (4.3) and (4.4) that

$$\bar{B}_{ii} = \frac{2(k^2+k+1)^2}{3D^2} \text{ for } 1 \leq i \leq 3 \text{ and } \bar{B}_{44} = -\frac{2(k^2+k+1)^2}{D^2} \tag{4.5}$$

where $\bar{B}_{ii}$ is the Bach tensor with respect to $\bar{X}_i$. This implies that the Bach flow is equivalent to

$$\begin{aligned} \frac{dA}{dt} &= -\frac{2(k^2+k+1)^2 A}{3D^2}, \ \frac{dB}{dt} = -\frac{2(k^2+k+1)^2 B}{3D^2}, \\ \frac{dC}{dt} &= -\frac{2(k^2+k+1)^2 C}{3D^2}, \text{ and } \frac{dD}{dt} = \frac{2(k^2+k+1)^2}{D}. \end{aligned}$$

It follows that

$$D^2 = \lambda_4 + 4(k^2+k+1)^2 t. \tag{4.6}$$

Hence, we have

$$\frac{dA}{dt} = -\frac{2(k^2+k+1)^2 A}{3D^2} = -\frac{2(k^2+k+1)^2 A}{3(\lambda_4 + 4(k^2+k+1)^2 t)}$$

which gives

$$A = \lambda_1 \left(1 + \frac{4(k^2+k+1)^2}{\lambda_4} t\right)^{-\frac{1}{6}}. \tag{4.7}$$

Similarly, we get

$$B = \lambda_2 \left(1 + \frac{4(k^2+k+1)^2}{\lambda_4} t\right)^{-\frac{1}{6}} \text{ and } C = \lambda_3 \left(1 + \frac{4(k^2+k+1)^2}{\lambda_4} t\right)^{-\frac{1}{6}}. \tag{4.8}$$

Next, we compute the curvature decay of $g(t)$. The sectional curvatures of $g(t)$ are given by (see P.356 in [15])

$$\begin{aligned} K(X_1,X_2) &= -\frac{k}{D}, \quad K(X_1,X_3) = \frac{k+1}{D}, \quad K(X_2,X_3) = \frac{k(k+1)}{D}, \\ K(X_1,X_4) &= -\frac{1}{D}, \quad K(X_2,X_4) = -\frac{k^2}{D}, \quad K(X_3,X_4) = -\frac{(k+1)^2}{D}. \end{aligned} \tag{4.9}$$

These curvatures of the solution $g(t)$ decay at the rate $1/\sqrt{t}$. Pick a point $p \in M$. Combining (4.6)–(4.9), we can conclude that the solution $(M, g(t), p)$ of the Bach flow collapses to a line the pointed Gromov–Hausdorff topology.

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