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The tangential Cauchy–Fueter complex on the quaternionic Heisenberg group ☆

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ABSTRACT

The Cauchy–Fueter operator on the quaternionic space $\mathbb{H}^n$ induces the tangential Cauchy–Fueter operator on the boundary of a domain. The quaternionic Heisenberg group is a standard model of the boundaries. By using the Penrose transformation associated to a double fibration of homogeneous spaces of $\mathrm{Sp}(2N, \mathbb{C})$, we construct an exact sequence on the quaternionic Heisenberg group, the tangential k-Cauchy–Fueter complex, resolving the tangential k-Cauchy–Fueter operator $Q_0^{(k)}$. $Q_0^{(1)}$ is the tangential Cauchy–Fueter operator. The complex gives the compatible conditions under which the non-homogeneous tangential k-Cauchy–Fueter equations $Q_0^{(k)}u = f$ are solvable. The operators in this complex are left invariant differential operators on the quaternionic Heisenberg group. This is a quaternionic version of $\overline{\partial}_b$-complex on the Heisenberg group in the theory of several complex variables.

1. Introduction

For a point $q = (q_0, \ldots, q_{n-1}) \in \mathbb{H}^n$, write

$$q_l = x_{4l+1} + x_{4l+2}\mathbf{i} + x_{4l+3}\mathbf{j} + x_{4l+4}\mathbf{k}, \tag{1.1}$$

$l = 0, \ldots, n-1$. For a domain Ω in $\mathbb{H}^n$, the *Cauchy–Fueter operator* $D_0 : C^1(\Omega, \mathbb{H}) \to C(\Omega, \mathbb{H}^n)$ is defined as

$$D_0 f = \left(\overline{\partial}_{q_0} f, \ldots, \overline{\partial}_{q_{n-1}} f\right)^t, \tag{1.2}$$

for $f \in C^1(\Omega, \mathbb{H})$, where ^{t} is the transport, and

$$\overline{\partial}_{q_l} = \partial_{x_{4l+1}} + \mathbf{i}\partial_{x_{4l+2}} + \mathbf{j}\partial_{x_{4l+3}} + \mathbf{k}\partial_{x_{4l+4}}. \tag{1.3}$$

A function $f : \Omega \to \mathbb{H}$ is called *regular in* Ω if $D_0 f(q) = 0$ for any $q \in \Omega$. Similar to the Dolbeault complex in the theory of several complex variables, there exists an exact sequence, the Cauchy–Fueter complex, resolving the Cauchy–Fueter

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operator D_0 [1–7]: the sequence

$$\begin{aligned} 0 \to \quad & \mathcal{R}(\mathbb{R}^{4n}, \mathbb{C}^2) \xrightarrow{D_0} \mathcal{R}(\mathbb{R}^{4n}, \mathbb{C}^{2n}) \xrightarrow{D_1} \mathcal{R}(\mathbb{R}^{4n}, \Lambda^3\mathbb{C}^{2n}) \\ & \xrightarrow{D_2} \mathcal{R}(\mathbb{R}^{4n}, \mathbb{C}^2 \otimes \Lambda^4\mathbb{C}^{2n}) \xrightarrow{D_3} \cdots \xrightarrow{D_{2n-2}} \mathcal{R}(\mathbb{R}^{4n}, \odot^{2n-3}\mathbb{C}^2 \otimes \Lambda^{2n}\mathbb{C}^{2n}) \to 0 \end{aligned} \tag{1.4}$$

is exact except at the first spot, where D_j are differential operators of the first order except D_1, which is of the second order. Here $\mathcal{R}(\mathbb{R}^{4n}, \odot^k \mathbb{C}^2 \otimes \Lambda^j\mathbb{C}^{2n})$ is the ring of $\odot^k \mathbb{C}^2 \otimes \Lambda^j\mathbb{C}^{2n}$-valued polynomials over $\mathbb{R}^{4n}$. In [8] we solved the non-homogeneous Cauchy–Fueter equations

$$D_0 u = f, \tag{1.5}$$

on $\mathbb{H}^n$ under the compatible condition

$$D_1 f = 0, \tag{1.6}$$

and find the compactly supported solution u to the non-homogeneous equation (1.5) if f is also compactly supported. This solution allows us to prove the Hartogs' phenomenon for quaternionic regular functions in any domain (see also [9]). D_0 is the first one of a family of linear differential operators $D_0^{(k)}$, and there exists an exact sequence, the k-Cauchy–Fueter complex, resolving the k-Cauchy–Fueter operator $D_0^{(k)}$ [4,10]. The Hartogs' phenomenon for k-regular functions is also proved in [10].

As in the case of several complex variables, the Hartogs' phenomenon leads to the concept of domains of holomorphy. There already exist the concept of plurisubharmonic functions on $\mathbb{H}^n$ and the concept of a pseudoconvex domain in $\mathbb{H}^n$ [11]. To develop the theory of several quaternionic variables, we need to solve the non-homogeneous k-Cauchy–Fueter equations in a holomorphic domain or in a pseudoconvex domain and to solve the corresponding D-Neumann problem, etc. In the theory of several complex variables, one way to solve the non-homogeneous $\overline{\partial}$-equations in a smooth pseudoconvex domain is to solve the $\overline{\partial}_b$-equations on its boundary, and there is a natural $\overline{\partial}_b$-complex on the boundary. This leads us to consider the tangential Cauchy–Fueter operator and to find the counterpart of $\overline{\partial}_b$-complex.

For a domain $\Omega \subset \mathbb{H}^{n+1}$, a vector

$$\overline{Z} = \sum_{l=0}^{n} a_l \overline{\partial}_{q_l}$$

is called a *quaternionic tangential vector field* on the boundary $\partial\Omega$ if $\overline{Z}\rho \equiv 0$, where ρ is a defining function of Ω. Suppose $\partial_{q_n}\rho \neq 0$ locally. Then

$$\overline{Z}_l = \overline{\partial}_{q_l} - \overline{\partial}_{q_l}\rho(\overline{\partial}_{q_n}\rho)^{-1}\overline{\partial}_{q_n}, \tag{1.7}$$

$l = 0, \ldots, n-1$, are quaternionic tangential vector fields. The *tangential Cauchy–Fueter operator* is defined as $f \longmapsto (\overline{Z}_0 f, \ldots, \overline{Z}_{n-1} f)^t$ locally. A $\mathbb{H}$-valued distribution f on $\partial\Omega$ is said to be *Cauchy–Fueter* (or briefly, *CF*) if $\overline{Z}f = 0$ in the sense of distributions for each quaternionic tangential vector field $\overline{Z}$. It is easy to see that the restriction to $\partial\Omega$ of a function regular in a neighborhood of $\partial\Omega$ is a CF function. So CF functions are abundant. CF functions have already been applied to determine the extremals for the Sobolev inequality on the quaternionic Heisenberg group and to solve the quaternionic contact Yamabe problem (see [12] and references therein). It is interesting and important to develop a theory of CF functions.

Chang and Markina [13,14] have constructed the Szegö kernel on the unit ball in $\mathbb{H}^2$, equivalently, on the Siegel upper half space in $\mathbb{H}^2$. Consider the *boundary of the Siegel upper half space*:

$$\mathcal{S}_\kappa = \left\{ q = (q_0, \ldots, q_n) \in \mathbb{H}^{n+1}; \operatorname{Re} q_n - \sum_{l=0}^{n-1} \kappa_l |q_l|^2 = 0 \right\}, \tag{1.8}$$

where $\kappa_l = \pm 1$. We can identify $\mathcal{S}_\kappa$ with $\mathcal{H}_\kappa = \mathbb{H}^n \times \operatorname{Im}\mathbb{H}$, the *quaternionic Heisenberg group* with the multiplication given by

$$(q, t) \cdot (\widetilde{q}, \widetilde{t}) = \left(q + \widetilde{q}, t + \widetilde{t} + 2\sum_{l=0}^{n-1} \kappa_l \operatorname{Im}(\overline{q}_l \widetilde{q}_l) \right), \tag{1.9}$$

where $q, \widetilde{q} \in \mathbb{H}^n$, $t, \widetilde{t} \in \operatorname{Im}\mathbb{H}$. The neutral element is $(0, 0)$ and the inverse of (q, t) is $(-q, -t)$. In real coordinates, the multiplication is given by the following formula (cf. Remark 5.2.1):

$$(x, t) \cdot (y, s) = \left(x + y, t_\beta + s_\beta + 2\sum_{l=0}^{n-1} \sum_{i,j=1}^{4} \kappa_l b_{ij}^\beta x_{4l+i} y_{4l+j}, \right), \tag{1.10}$$

where $\beta = 1, 2, 3$, $x = (x_1, x_2, \ldots, x_{4n}) \in \mathbb{R}^{4n}$, $t = (t_1, t_2, t_3) \in \mathbb{R}^3$, y and s are defined similarly, and b^1, b^2, b^3 are antisymmetric matrices

$$b^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad b^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad b^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{1.11}$$

It is easy to see that matrices b^1, b^2, b^3 satisfy the *commutating relation of quaternions*:

$$(b^1)^2 = (b^2)^2 = (b^3)^2 = -\mathrm{id}, \qquad b^1b^2 = b^3. \tag{1.12}$$

The vector fields given by

$$X_{4l+j} = \partial_{x_{4l+j}} + 2\kappa_l \sum_{\beta=1}^{3}\sum_{i=1}^{4} b_{ij}^{\beta} x_{4l+i}\partial_{t_\beta}, \tag{1.13}$$

$l = 0, \dots, n-1, j = 1, \dots, 4$, are left invariant on $\mathscr{H}_\kappa$ and satisfy the following commutating relation

$$[X_{4l+i}, X_{4l'+j}] = 4\kappa_l\delta_{ll'} \sum_{\beta=1}^{3} b_{ij}^{\beta}\partial_{t_\beta}. \tag{1.14}$$

By the projection $\pi : \mathscr{S}_\kappa \longrightarrow \mathscr{H}_\kappa$, it is shown in Section 5.1 that the tangential Cauchy–Fueter operator of the boundary $\mathscr{S}_\kappa$ of the Siegel upper half space is mapped to the tangential Cauchy–Fueter operator $\overline{Q} : C^1(\mathscr{H}_\kappa, \mathbb{H}) \to C(\mathscr{H}_\kappa, \mathbb{H}^n)$,

$$\overline{Q}f = (\overline{Q}_0 f, \dots, \overline{Q}_{n-1}f)^t$$

with

$$\overline{Q}_l = X_{4l+1} + \mathbf{i}X_{4l+2} + \mathbf{j}X_{4l+3} + \mathbf{k}X_{4l+4}, \tag{1.15}$$

$l = 0, \dots, n-1$. The equation $\overline{Q}_l f = 0$ for $f = f_1 + f_2\mathbf{i} + f_3\mathbf{j} + f_4\mathbf{k}$ can be written as

$$\begin{pmatrix} X_{4l+1} + \mathbf{i}X_{4l+2} & -X_{4l+3} - \mathbf{i}X_{4l+4} \\ X_{4l+3} - \mathbf{i}X_{4l+4} & X_{4l+1} - \mathbf{i}X_{4l+2} \end{pmatrix} \begin{pmatrix} \phi^0 \\ \phi^1 \end{pmatrix} = 0, \tag{1.16}$$

where $\phi^0 = f_1 + \mathbf{i}f_2, \phi^1 = f_3 - \mathbf{i}f_4$. Denote

$$\begin{pmatrix} \nabla_{2l+1,1'} & \nabla_{2l+1,2'} \\ \nabla_{2l+2,1'} & \nabla_{2l+2,2'} \end{pmatrix} := \varepsilon_l \begin{pmatrix} X_{4l+1} + \mathbf{i}X_{4l+2} & -X_{4l+3} - \mathbf{i}X_{4l+4} \\ X_{4l+3} - \mathbf{i}X_{4l+4} & X_{4l+1} - \mathbf{i}X_{4l+2} \end{pmatrix}, \tag{1.17}$$

where $\varepsilon_l = 1$ if $\kappa_l = 1$ and $\varepsilon_l = \mathbf{i}$ if $\kappa_l = -1$. Then $\overline{Q}f = 0$ can be written as $\nabla_{AA'}\phi^{A'} = 0$ briefly. Here and in the following we use the Einstein convention of taking summation over repeated indices. The repeated indices A' and A are taken over 1, 2 and $1, \dots, 2n$, respectively. We will use the matrix

$$(\varepsilon_{A'B'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (\epsilon_{AB}) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{1.18}$$

to raise or lower indices A' and A, respectively. For example, $\nabla_A^{A'} = \varepsilon^{A'B'}\nabla_{AB'}$, where $(\varepsilon^{A'B'}) = -(\varepsilon_{A'B'})$ is the inverse of $(\varepsilon_{A'B'})$. Denote by $\{e_{A'_1\dots A'_s}\}$ the basis of $\odot^s\mathbb{C}^2$, the symmetric product of $\mathbb{C}^2$. Define the *tangential k-Cauchy–Fueter operator*

$$\begin{aligned} &Q_0^{(k)} : C^\infty(\mathbb{R}^{4n+3}, \odot^k\mathbb{C}^2) \longrightarrow C^\infty(\mathbb{R}^{4n+3}, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n}), \\ &fe_{A'_1\dots A'_k} \mapsto \nabla_A^{A'_s} fe_{A'_1\dots\widehat{A'_s}\dots A'_k} \otimes e_A, \end{aligned} \tag{1.19}$$

where $\widehat{A'_s}$ means that A'_s is omitted. Here and in the following we only write down the action of $Q_0^{(k)}$ on one component for simplicity, where f is a scalar function. The tangential 1-Cauchy–Fueter operator $Q_0^{(1)} = \nabla_A^{A'}f_{A'} = \nabla_{AA'}f^{A'}$ is equivalent to the tangential Cauchy–Fueter operator $\overline{Q}$.

Let $V^{(j)}$ be the irreducible representation of $\mathfrak{sp}(2n, \mathbb{C})$ with the highest weight to be the j-th fundamental weight ω_j. $\mathbb{C}^{2n} \otimes V^{(j)}$ has a unique irreducible subrepresentation of $\mathfrak{sp}(2n, \mathbb{C})$ isomorphic to $V^{(j\pm1)}$ (cf. Lemma 4.1.1). Let $\mathrm{pr}_{j\pm1}^j$ be $\mathfrak{sp}(2n, \mathbb{C})$-equivariant projection from $\mathbb{C}^{2n} \otimes V^{(j)}$ to $V^{(j\pm1)}$. For $v \in V^{(j)}$, $e_A \in \mathbb{C}^{2n}$, denote $e_A \wedge_0 v := \mathrm{pr}_{j+1}^j(e_A \otimes v)$ and $e_A.v := \mathrm{pr}_{j-1}^j(e_A \otimes v)$.

Similar to the $\overline{\partial}_b$-complex in the theory of several complex variables, which is important to investigate CR functions, we can construct an exact sequence, the *tangential k-Cauchy–Fueter complex*, resolving the $Q_0^{(k)}$ on the quaternionic Heisenberg group. Denote

$$\mathbb{T}_1 := -\mathbf{i}\partial_{x_{4n+1}}, \qquad \mathbb{T}_2 := \partial_{x_{4n+2}} - \mathbf{i}\partial_{x_{4n+3}}, \qquad \mathbb{T}_3 := \partial_{x_{4n+2}} + \mathbf{i}\partial_{x_{4n+3}}. \tag{1.20}$$

Theorem 1.0.1. *If $0 \le k \le n-2$, the sequence*

$$\begin{aligned} 0 \to\ &\mathscr{R}(\mathbb{R}^{4n+3}, \odot^k\mathbb{C}^2) \xrightarrow{Q_0^{(k)}} \mathscr{R}(\mathbb{R}^{4n+3}, \odot^{k-1}\mathbb{C}^2 \otimes V^{(1)}) \xrightarrow{Q_1^{(k)}} \cdots \longrightarrow \mathscr{R}(\mathbb{R}^{4n+3}, V^{(k)}) \\ &\xrightarrow{Q_k^{(k)}} \mathscr{R}(\mathbb{R}^{4n+3}, V^{(k+2)}) \xrightarrow{Q_{k+1}^{(k)}} \cdots \longrightarrow \mathscr{R}(\mathbb{R}^{4n+3}, \odot^{n-k-2}\mathbb{C}^2 \otimes V^{(n)}) \\ &\xrightarrow{Q_{n-1}^{(k)}} \mathscr{R}(\mathbb{R}^{4n+3}, \odot^{n-k}\mathbb{C}^2 \otimes V^{(n)}) \xrightarrow{Q_n^{(k)}} \cdots \xrightarrow{Q_{2n-1}^{(k)}} \mathscr{R}(\mathbb{R}^{4n+3}, \odot^{2n-k}\mathbb{C}^2) \to 0 \end{aligned} \tag{1.21}$$

is exact except at the first spot, where the operators $Q_j^{(k)}, j = 0, \dots, k-1$, *are given by*

$$fe_{A'_1\dots A'_{k-j}} \otimes v \mapsto \nabla_A^{A'_s} fe_{A'_1\dots\hat{A}'_s\dots A'_{k-j}} \otimes e_A \wedge_0 v, \tag{1.22}$$

for $v \in V^{(j)}$; *the operators* $Q_j^{(k)}, j = k+1, \dots, n-2$, *are given by*

$$fe_{A'_1\dots A'_{j-k-1}} \otimes v \mapsto \nabla_{AA'} fe_{A'A'_1\dots A'_{j-k-1}} \otimes e_A \wedge_0 v, \tag{1.23}$$

for $v \in V^{(j+1)}$; *the operators* $Q_j^{(k)}, j = n, \dots, 2n-1$, *are given by*

$$fe_{A'_1\dots A'_{j-k}} \otimes v \mapsto \nabla_{AA'} fe_{A'A'_1\dots A'_{j-k}} \otimes e_A . v, \tag{1.24}$$

for $v \in V^{(2n-j)}$; *the operator* $Q_{n-1}^{(k)}$ *is given by*

$$fe_{A'_1\dots A'_{n-k-2}} \otimes v \mapsto \mathbb{T}_j f \cdot e_{\sigma_j A'_1\dots A'_{n-k-2}} \otimes v, \tag{1.25}$$

for $v \in V^{(n)}$, *where* $\sigma_1 := 1'2', \sigma_2 =: 1'1', \sigma_3 := 2'2'$; $Q_k^{(k)}$ *is an operator of the second order given by*

$$f v \mapsto \nabla_{AA'} \nabla_B^{A'} fe_A \wedge_0 e_B \wedge_0 v, \tag{1.26}$$

for $v \in V^{(k)}$. *Here* $\mathcal{R}(\mathbb{R}^{4n+3}, \odot^s \mathbb{C}^2 \otimes V^{(t)})$ *is the ring of* $\odot^s \mathbb{C}^2 \otimes V^{(t)}$*-valued polynomials over* $\mathbb{R}^{4n+3}$.

The case $k > n-2$ is similar. Note that $Q_j^{(k)}$ are differential operators with variable coefficients, while differential operators in usual complexes [6] are of constant coefficients. The important application of the tangential k-Cauchy–Fueter complex is to give the compatible conditions under which the non-homogeneous tangential k-Cauchy–Fueter equations $Q_0^{(k)} u = f$ are solvable, i.e., $Q_1^{(k)} f = 0$ (cf. Remark 6.1 in [13]).

The proof of this theorem goes as follows. Let G be the complex semisimple Lie group $\mathrm{Sp}(2(n+2), \mathbb{C})$, and let P, Q and R be its parabolic subgroups, whose Lie algebra are given by (2.1), $Q = P \cap R$. Consider the double fibration:

$$\begin{tikzcd} & G/Q \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ G/R & & G/P \end{tikzcd} \tag{1.27}$$

Here G/P, G/Q and G/R are generalized flag varieties. We use the Penrose transform [15] associated with this double fibration to construct an exact sequence of sheaves on G/P. In Section 2, we give the preliminaries on Lie algebra $\mathfrak{sp}(2N, \mathbb{C})$, its parabolic subalgebras and irreducible homogeneous sheaves. In Section 3, firstly, we get the exact sequence of relative Bernstein–Gelfand–Gelfand resolution

$$0 \to \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda_k) \to \mathcal{O}_{\mathfrak{q}}(\nu_1) \xrightarrow{\tilde{\partial}_1^{(k)}} \mathcal{O}_{\mathfrak{q}}(\nu_2) \xrightarrow{\tilde{\partial}_2^{(k)}} \cdots,$$

resolving the pulling back of the irreducible homogeneous sheaf $\mathcal{O}_{\mathfrak{r}}(\lambda_k)$ on G/R, where $\mathcal{O}_{\mathfrak{q}}(\nu_j)$ are irreducible homogeneous sheaves on G/Q, and λ_k and ν_j are suitable dominant integral weights for $\mathfrak{r}$ and $\mathfrak{q}$, respectively. Then, calculate their higher direct image sheaves under τ by using the Borel–Bott–Weil theorem. We calculate the hypercohomology spectral sequence of a double complex: the Čech cochain complex $C^q(\mathcal{U}, \mathcal{O}_{\mathfrak{q}}(\nu_p))$, where $\mathcal{U}$ is a covering of $Y = \tau^{-1}X \subset G/Q$ and X is an affine open set in G/P. It converges to $H^{p+q}(Y, \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda_k))$. The later one vanishes except $p + q \le 1$. We get an exact sequence of sheaves on G/P which follows from $E_1^{q,p}$.

In Section 4, we use the representation theory to find the explicit form of the invariant operators $Q_j^{(k)}$. Recently, Colombo et al. [7] determined the invariant operators $D_j^{(1)}$ of the Cauchy–Fueter complex (1.21) also by using representation theory (see also [5]). In Section 5, by using the embedding of the quaternionic Heisenberg group into G/P, we pull back the exact sequence of sheaves on G/P to get the exact sequence in Theorem 1.0.1.

2. Preliminaries on $\mathfrak{sp}(2N, \mathbb{C})$, parabolic subalgebras and irreducible homogeneous sheaves

2.1. Parabolic subgroups and parabolic subalgebras

Let $\mathfrak{g}$ be semisimple complex Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}$ and the set of roots Δ:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right).$$

Let S be a subset of $\{1, \dots, N\}$, $\mathfrak{h}_S$ be the span of H_i with $i \in S$, and $\mathfrak{g}_S$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}_S$ and $\mathfrak{g}_{\pm\alpha_i}$ with $i \in S$. $\mathfrak{g}_S$ is a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}_S$, root system $\Delta_S := \Delta \cap \sum_{i \in S} \mathbb{Z}\alpha_i$, positive root system

$\Delta_S^+ := \Delta^+ \cap \Delta_S$ and simple roots $\{\alpha_i; i \in S\}$. We have the decomposition $\mathfrak{g}_S = \mathfrak{n}_S^+ \oplus \mathfrak{h}_S \oplus \mathfrak{n}_S^-$ with $\mathfrak{n}_S^+ = \sum_{\alpha\in\Delta_S^+} \mathfrak{g}_\alpha$, $\mathfrak{n}_S^- = \sum_{\alpha\in\Delta_S^+} \mathfrak{g}_{-\alpha}$. Set

$$\mathfrak{u}_S^+ := \sum_{\alpha\in\Delta^+\setminus\Delta_S^+} \mathfrak{g}_\alpha, \qquad \mathfrak{u}_S^- := \sum_{\alpha\in\Delta^+\setminus\Delta_S^+} \mathfrak{g}_{-\alpha}, \qquad \mathfrak{p}_S := \mathfrak{l} \oplus \mathfrak{u}_S^+, \qquad \mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{g}_S.$$

$\mathfrak{l}$ is a reductive subalgebra of $\mathfrak{g}$ with derived algebra $\mathfrak{g}_S$ and center $\mathfrak{z}(\mathfrak{l}) \subset \mathfrak{h}$. $\mathfrak{p}_S$ is a *parabolic subalgebra* of $\mathfrak{g}$. Set $\mathfrak{a}_S := \mathfrak{z}(\mathfrak{l})$. We will denote a parabolic subalgebra of $\mathfrak{g}$ by the diagram crossing through all nodes in the Dynkin diagram for $\mathfrak{g}$ which correspond to the simple roots in $\{\alpha_i; i \notin S\}$. We will consider $G = \mathrm{Sp}(2N, \mathbb{C})$, $\mathfrak{g} = \mathfrak{sp}(2N, \mathbb{C})$, where

$$N = n + 2,$$

and its parabolic subalgebras

$$\begin{aligned} \mathfrak{p} &= \bullet\!-\!\times\!-\!\bullet \cdots \Longleftarrow \bullet, \\ \mathfrak{q} &= \times\!-\!\times\!-\!\bullet \cdots \Longleftarrow \bullet, \\ \mathfrak{r} &= \times\!-\!\bullet\!-\!\bullet \cdots \Longleftarrow \bullet. \end{aligned} \tag{2.1}$$

Namely,

$$\mathfrak{p} = \mathfrak{p}_{S_2}, \qquad \mathfrak{q} = \mathfrak{p}_{S_{1,2}}, \qquad \mathfrak{r} = \mathfrak{p}_{S_1}, \tag{2.2}$$

with $S_2 = \{1, \ldots, N\} \setminus \{2\}$, $S_{1,2} = \{1, \ldots, N\} \setminus \{1, 2\}$, $S_1 = \{1, \ldots, N\} \setminus \{1\}$. $\mathfrak{p}$ and $\mathfrak{r}$ are maximal parabolic subalgebras. A weight ω_j satisfying $\langle \omega_j, \alpha_k^\vee \rangle = \delta_{jk}$ for each k is called a *fundamental weight*. Here for any root α its *coroot* is denoted by $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$. Hence $\lambda = \sum \langle \lambda, \alpha_j^\vee \rangle \omega_j$. We indicate a weight $\lambda \in \mathfrak{h}^*$ by inscribing the coefficient $\langle \lambda, \alpha^\vee \rangle$ over a node of the Dynkin diagram for $\mathfrak{g}$ corresponding to the simple root α. A weight λ is *dominant* (or *integral*) *for* $\mathfrak{p}_S$ if coefficients over uncrossed nodes are non-negative (or integral), respectively. The finite dimensional irreducible representations of $\mathfrak{p}_S$ are in a one-to-one correspondence to dominant integral weights for $\mathfrak{p}_S$. For example,

$$\lambda_k = \overset{-2-k}{\times}\!\!-\!\!-\!\!-\bullet-\!\!-\!\!-\bullet \cdots \bullet \Longleftarrow \bullet\ , \tag{2.3}$$

is a dominant integral weight for $\mathfrak{q}$ and $\mathfrak{r}$ given by (2.1), where 0 is omitted over the nodes without coefficient.

Let $P_S^+ = \{\lambda \in \mathfrak{h}^*; 0 \le \lambda(H_i) \in \mathbb{Z}, i \in S\}$. P_S^+ parametrizes the set of finite dimensional $\mathfrak{p}_S$-modules (its restriction to $\mathfrak{u}_S^+$ is trivial) which remains irreducible under $\mathfrak{g}_S$. We denote the module corresponding to λ by $F_{\mathfrak{p}_S}(\lambda)$. For $\lambda \in P_S^+$, let δ be its restriction to $\mathfrak{g}_S$. Since $\mathfrak{a}_S$ commutes with $\mathfrak{g}_S$, $\mathfrak{a}_S$ has to be scalar on the representation if $\dim \mathfrak{a}_S = 1$, i.e.,

$$\lambda(h) = \nu I, \quad h \in \mathfrak{a}_S$$

for some $\nu \in \mathbb{C}$. So the irreducible representation λ is completely determined by (δ, ν).

2.2. $\mathfrak{g} = \mathfrak{sp}(2N, \mathbb{C})$

It is the space of all $2N \times 2N$ matrices

$$X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \tag{2.4}$$

satisfying $B^t = B$ and $C^t = C$, where A, B and C are all $N \times N$ matrices. The subalgebra $\mathfrak{h}$ of matrices diagonal in this representation is a Cartan subalgebra, which is spanned by $H_j = E_{j,j} - E_{N+j,N+j}$. We fix a dual basis $\{L_j\}$ of $\mathfrak{h}^*$, i.e., $\langle L_j, H_i \rangle = \delta_{ji}$. We have root spaces:

$$\begin{aligned} \mathfrak{g}_{L_i-L_j} &= \mathbb{C}(E_{i,j} - E_{N+j,N+i}), \\ \mathfrak{g}_{L_i+L_j} &= \mathbb{C}(E_{i,N+j} + E_{j,N+i}), \\ \mathfrak{g}_{-L_i-L_j} &= \mathbb{C}(E_{N+i,j} + E_{N+j,i}), \\ \mathfrak{g}_{2L_i} &= \mathbb{C}E_{i,N+i}, \\ \mathfrak{g}_{-2L_i} &= \mathbb{C}E_{N+i,i}, \end{aligned} \tag{2.5}$$

$i, j = 1, \ldots, N$, $i \neq j$. So the set Δ of roots of $\mathfrak{sp}(2N, \mathbb{C})$ are vectors $\pm L_i \pm L_j \in \mathfrak{h}^*$. $\Delta = \Delta^+ \cup \Delta^-$ with the set of positive roots to be

$$\Delta^+ = \{L_i + L_j\}_{i\le j} \cup \{L_i - L_j\}_{i<j}.$$

Simple positive roots are $\alpha_1 = L_1 - L_2, \ldots, \alpha_{N-1} = L_{N-1} - L_N$ and $\alpha_N = 2L_N$. Up to a constant, the Killing form $\langle \cdot, \cdot \rangle$ on $\mathfrak{sp}(N, \mathbb{C})$ satisfies

$$\langle H_i, H_j \rangle = \delta_{ij}, \qquad \langle L_i, L_j \rangle = \delta_{ij},$$

from which we see that $\omega_i = L_1 + \cdots + L_i$ constitutes fundamental weights, i.e. $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. $H_{\alpha_i} = H_i - H_{i+1}$, $i = 1, \ldots, N-1$, and $H_{\alpha_N} = H_N$ (cf. Section 16.1 in [16]).

We have the following decomposition

$$\mathfrak{g} = \mathfrak{p}_{S_2} \oplus \mathfrak{u}_{S_2}^-, \qquad \mathfrak{l} = \mathfrak{g}_{S_2} \oplus \mathfrak{z}(\mathfrak{l}), \qquad \mathfrak{h} = \mathfrak{h}_{S_2} \oplus \mathfrak{z}(\mathfrak{l}), \tag{2.6}$$

with $\mathfrak{p}_{S_2} = \mathfrak{g}_{S_2} \oplus \mathfrak{u}_{S_2}^+$, and

$$\mathfrak{a}_{S_2} := \mathfrak{z}(\mathfrak{l}) = \mathbb{C}H_{\mathfrak{z}}, \qquad H_{\mathfrak{z}} = H_1 + H_2. \tag{2.7}$$

Elements of $\mathfrak{u}_{S_2}^-$ are matrices of the following form:

$$\begin{pmatrix} 0 & 0 & & & & & \\ 0 & 0 & & & & & \\ * & * & & & & & \\ \vdots & \vdots & & & & & \\ * & * & \cdots & 0 & 0 & * & \cdots \\ * & * & \cdots & 0 & 0 & * & \cdots \\ * & * & & & & & \\ \vdots & \vdots & & & & & \end{pmatrix}.$$

Denote

$$\begin{aligned} &Y_{i,1} := E_{2+i,1} - E_{N+1,N+2+i}, \qquad Y_{i,2} := E_{2+i,2} - E_{N+2,N+2+i}, \\ &Y_{n+i,1} := E_{N+2+i,1} + E_{N+1,2+i}, \qquad Y_{n+i,2} := E_{N+2+i,2} + E_{N+2,2+i}, \\ &T_1 := -\frac{1}{4}(E_{N+2,1} + E_{N+1,2}), \qquad T_2 := -\frac{1}{2}E_{N+1,1}, \qquad T_3 := -\frac{1}{2}E_{N+2,2}, \end{aligned} \tag{2.8}$$

$i = 1, \ldots, n$. Here we choose the coefficients in the definition of T_s to make the brackets satisfy (2.10). Then we have the decomposition

$$\mathfrak{g} = \mathfrak{u}_{S_2}^- \oplus \mathfrak{a}_{S_2} \oplus \mathfrak{g}_{S_2} \oplus \mathfrak{u}_{S_2}^+ = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

with $\mathfrak{g}_k := \{X \in \mathfrak{g}; [H_{\mathfrak{z}}, X] = kX\}$,

$$\begin{aligned} &\mathfrak{g}_0 = \mathfrak{a}_{S_2} \oplus \mathfrak{g}_{S_2}, \\ &\mathfrak{g}_{-1} = \{Y_{i,1}, Y_{i,2}; i = 1, \ldots, 2n\}, \\ &\mathfrak{g}_{-2} = \{T_1, T_2, T_3\}. \end{aligned} \tag{2.9}$$

It is easy to see that:

$$\begin{aligned} &[Y_{i,1}, Y_{n+i,1}] = 4T_2, \\ &[Y_{i,2}, Y_{n+i,2}] = 4T_3, \\ &[Y_{i,1}, Y_{n+i,2}] = [Y_{i,2}, Y_{n+i,1}] = 4T_1, \end{aligned} \tag{2.10}$$

and all other brackets vanish. We relabel these elements as

$$\begin{aligned} &Y_{4l+1} = Y_{l+1,1}, \qquad Y_{4l+2} = Y_{l+1,2}, \\ &Y_{4l+3} = Y_{n+l+1,1}, \qquad Y_{4l+4} = Y_{n+l+1,2}, \quad l = 0, \ldots, n-1, \\ &Y_{4n+1} := T_1, \qquad Y_{4n+2} := T_2, \qquad Y_{4n+3} := T_3. \end{aligned} \tag{2.11}$$

Then,

$$\begin{aligned} &[Y_{4l+1}, Y_{4l+3}] = 4Y_{4n+2}, \\ &[Y_{4l+2}, Y_{4l+4}] = 4Y_{4n+3}, \\ &[Y_{4l+1}, Y_{4l+4}] = [Y_{4l+2}, Y_{4l+3}] = 4Y_{4n+1}, \end{aligned} \tag{2.12}$$

and all other brackets vanish.

2.3. *Homogeneous vector bundles and irreducible homogeneous sheaves $\mathcal{O}_{\mathfrak{p}}(\lambda)$*

Given a finite dimensional representation ρ of a parabolic subgroup P on a vector space V, the *homogeneous vector space* $G \times_P V$ over G/P is the quotient space of $G \times V$ under the equivalent relation $(g, v) \sim (gp, \rho(p^{-1})v)$, $p \in P$, and the map $(g, v) \mapsto gP$ induces a projection $G \times V \to G/P$. The group G acts on $G \times_P V$ by left translation, which is denoted by τ. We have a trivialization $U_o \times V$ around $o = eP$ and $U_{go} \times V$ around $go = gP$. The sheaf of germs of holomorphic sections of $G \times_P V$ is denoted by $\mathcal{O}_{\mathfrak{p}}(V)$.

Let $\mathcal{O}_G(V)$ be the sheaf of germs of V-valued holomorphic functions on G and let $\mathcal{O}_G(V)^P$ be the sheaf of germs of V-valued holomorphic functions on G satisfying

$$f(gp) = \rho(p^{-1})f(g), \quad g \in G, p \in P.$$

A local section f of $\mathcal{O}_G(V)^P$ is in one-to-one correspondence to a local section s_f of the homogeneous sheaf $\mathcal{O}_{\mathfrak{p}}(V)$:

$$s_f(gP) = (g, f(g)).$$

For each integral weight λ for $\mathfrak{g}$ which is dominant for $\mathfrak{p}$ one obtains an irreducible representation of P and hence an *irreducible homogeneous sheaf* $\mathcal{O}_{\mathfrak{p}}(\lambda) := \mathcal{O}_{\mathfrak{p}}(F_{\mathfrak{p}}(-\lambda))$ on G/P, where $F_{\mathfrak{p}}(-\lambda)$, the dual of $F_{\mathfrak{p}}(\lambda)$, has the lowest weight vector of weight $-\lambda$. The reason we consider $\mathcal{O}_{\mathfrak{p}}(F_{\mathfrak{p}}(-\lambda))$, and not $\mathcal{O}_{\mathfrak{p}}(F_{\mathfrak{p}}(\lambda))$, is that the Borel–Bott–Weil theorem holds for such sheaves.

3. Construct an exact sequence of sheaves over G/P

3.1. The Hasse diagram and the relative Bernstein–Gelfand–Gelfand resolution

The reflection σ_α associated to a root α acts on a weight λ as $\sigma_\alpha(\lambda) = \lambda - \langle\lambda, \alpha^\vee\rangle\alpha$. The reflect associated to the simple root with coefficient b acts on a weight as follows (cf. Section 4.1 in [15]):

$$\begin{aligned}
&\overset{a}{\bullet}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet} \longmapsto \overset{a+b}{\bullet}\!-\!\overset{-b}{\bullet}\!-\!\overset{b+c}{\bullet},\\
&\overset{a}{\bullet}\!-\!\overset{b}{\bullet}\Leftarrow\overset{c}{\bullet} \longmapsto \overset{a+b}{\bullet}\!-\!\overset{-b}{\bullet}\Leftarrow\overset{b+c}{\bullet},\\
&\cdots\overset{c}{\bullet}\Leftarrow\overset{b}{\bullet}\longmapsto\cdots\overset{c+2b}{\bullet}\Leftarrow\overset{-b}{\bullet}.
\end{aligned} \tag{3.1}$$

For a weight $\lambda \in \mathfrak{h}^*$, the *affine action* of an element w of the Weyl group W is

$$w.\lambda = w(\lambda + \rho) - \rho, \qquad \rho = \sum_{j=1}^{N} \omega_j, \tag{3.2}$$

where $\omega_j, j = 1, \ldots, N$, are fundamental weights.

The Weyl group admits the structure of a direct graph as follows: for $w, w' \in W$, write $w \to w'$ if $l(w') = l(w) + 1$ and $w' = \sigma_\alpha w$ for some $\alpha \in \Delta(\mathfrak{g})$, where $l(w)$ is the length of w. Let $W_{\mathfrak{p}}$ be the Weyl group of a reductive Levi factor of $\mathfrak{p}$. Let $\rho^{\mathfrak{p}} = \sum_{j\notin S} w_j$ for $\mathfrak{p} = \mathfrak{p}_S$. It is known that $W_{\mathfrak{p}}$ is the stabilizer of $\rho^{\mathfrak{p}}$ in W. Let $W^{\mathfrak{p}}$ be the set of minimal length right coset representatives of the subgroup $W_{\mathfrak{p}}$ of $W_{\mathfrak{g}}$. There is a one-to-one correspondence between the elements of $W^{\mathfrak{p}}$ and the orbit of $\rho^{\mathfrak{p}}$ under the action of $W_{\mathfrak{g}}$. If each member of the orbit is connected to $\rho^{\mathfrak{p}}$ by one or more paths of simple reflection, the corresponding member of $W^{\mathfrak{p}}$ has reduced expressions which are obtained by taking their composition in the reverse order. So $W^{\mathfrak{p}}$ has an induced subgraph structure, and is called the *Hasse diagram*. Consider the fibration $G/Q \xrightarrow{\eta} G/R$, whose fibre has the form $R/Q = L_S/L_S \cap Q$, where L_S is the semisimple part of the reductive Levi factor of R. The Weyl group of L_S is $W_{\mathfrak{r}}$. We can define the relative Hasse diagram $W_{\mathfrak{r}}^{\mathfrak{q}}$ with $W_{\mathfrak{r}}$ playing the role $W_{\mathfrak{g}}$ and $W_{\mathfrak{q}}$ playing the role $W_{\mathfrak{p}}$ in the non-relative case (see p. 41 [15] for the algorithm of calculating Hasse diagrams). Hence to calculate $W_{\mathfrak{r}}^{\mathfrak{q}}$ for parabolic subalgebras $\mathfrak{r}$, $\mathfrak{q}$ of $\mathfrak{sp}(2N, \mathbb{C})$ given by (2.1), we first calculate the orbit of $\times\!-\!\overset{1}{\times}\!-\!\bullet\cdots \Leftarrow \bullet$ under the action of $W_{\mathfrak{r}}$. It is easy to see that

$$\begin{aligned}
&\times\!-\!\overset{1}{\times}\!-\!\bullet\cdots\bullet\Leftarrow\bullet \xrightarrow{\sigma_2} \overset{1}{\times}\!-\!\overset{-1}{\times}\!-\!\overset{1}{\bullet}\cdots\bullet\Leftarrow\bullet \xrightarrow{\sigma_3} \cdots\\
&\xrightarrow{\sigma_{N-1}} \overset{1}{\times}\!-\!\times\!-\!\bullet\cdots\overset{-1}{\bullet}\Leftarrow\overset{1}{\bullet} \xrightarrow{\sigma_N} \overset{1}{\times}\!-\!\times\!-\!\bullet\cdots\overset{1}{\bullet}\Leftarrow\overset{-1}{\bullet}\\
&\xrightarrow{\sigma_{N-1}} \overset{1}{\times}\!-\!\times\cdots\overset{1}{\bullet}\!-\!\overset{-1}{\bullet}\Leftarrow\bullet \xrightarrow{\sigma_{N-2}} \cdots \xrightarrow{\sigma_2} \overset{2}{\times}\!-\!\overset{-1}{\times}\cdots\bullet\!-\!\bullet\Leftarrow\bullet.
\end{aligned}$$

So $W_{\mathfrak{r}}^{\mathfrak{q}}$ has a single element of each length from 0 to $2n + 1$: $W_{\mathfrak{r}}^{\mathfrak{q}} = \{\Sigma_0, \ldots, \Sigma_{2n+1}\}$ with

$$\begin{aligned}
\Sigma_0 &= \mathrm{id},\\
\Sigma_1 &= \sigma_2,\\
&\vdots\\
\Sigma_{n+1} &= \sigma_2\cdots\sigma_{n+2},\\
\Sigma_{n+2} &= \sigma_2\cdots\sigma_{n+2}\sigma_{n+1},\\
&\vdots\\
\Sigma_{2n+1} &= \sigma_2\cdots\sigma_{n+2}\sigma_{n+1}\cdots\sigma_2.
\end{aligned} \tag{3.3}$$

Since λ_k is a dominant integral weight for $\mathfrak{r}$, $\eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda_k)$ has a *relative Bernstein–Gelfand–Gelfand resolution* (*relative BGG resolution*) (cf. Theorem 8.4.1, Sections 8.7 and 9.1 in [15]):

$$0 \to \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda_k) \to \Delta^0_\eta(\lambda_k) \xrightarrow{\widetilde{\partial}^{(k)}_0} \Delta^1_\eta(\lambda_k) \xrightarrow{\widetilde{\partial}^{(k)}_1} \cdots \xrightarrow{\widetilde{\partial}^{(k)}_{2n}} \Delta^{2n+1}_\eta(\lambda_k) \to 0, \tag{3.4}$$

with

$$\Delta^j_\eta(\lambda_k) := \bigoplus_{w\in W^{\mathfrak{q}}_{\mathfrak{r}};\, l(w)=j} \mathcal{O}_{\mathfrak{q}}(w.\lambda_k) = \mathcal{O}_{\mathfrak{q}}(\nu_j), \qquad \nu_j := \Sigma_j.\lambda_k, \tag{3.5}$$

where $\widetilde{\partial}^{(k)}_j$ are $\mathfrak{g}$-invariant operators. By using the rules (3.1) and the definition of affine action (3.2), it is elementary to check that $\nu_0 = \lambda_k$,

$$\nu_j := \Sigma_j.\lambda_k = \overset{-2-k+j}{\times} - \overset{-j-1}{\times} - \bullet \cdots \overset{1}{\bullet} \cdots \bullet \Leftarrow \bullet, \tag{3.6}$$

where 1 appears in the $(2+j)$-th node, $j = 1, \ldots, n$, and

$$\nu_{n+1} := \Sigma_{n+1}.\lambda_k = \overset{-k+n}{\times} - \overset{-n-3}{\times} - \bullet \cdots \bullet \cdots \bullet \Leftarrow \overset{1}{\bullet}, \tag{3.7}$$

and

$$\nu_{n+1+j} := \Sigma_{n+1+j}.\lambda_k = \overset{-k+n+j}{\times} - \overset{-j-n-3}{\times} - \bullet \cdots \overset{1}{\bullet} \cdots \bullet \Leftarrow \bullet, \tag{3.8}$$

where 1 appears in the $(n+2-j)$-th node, $j = 1, \ldots, n$.

3.2. *The spectral sequence and the exact sequence on G/P*

Cf. Sections 7.2–7.3 and Remark 9.2.9 in [15,10]. Recall the *hypercohomology spectral sequence* for a resolution

$$\mathcal{F} \to \Delta^\bullet_\eta \tag{3.9}$$

of coherent sheaves over X, where X is an open affine subset of G/P. Let $\mathcal{U}$ be a good affine cover of $Y = \tau^{-1}(X) \subset G/Q$. The differential d of the resolution (3.9) and δ of the Čech cochain complex $C^\bullet(\mathcal{U}, \Delta^\bullet_\eta)$ make

$$C^q(\mathcal{U}, \Delta^p_\eta) \tag{3.10}$$

a double complex. Deriving with respect to d gives

$$E^{0,q}_1 = C^q(\mathcal{U}, \mathcal{F}), \qquad E^{p,q}_1 = 0 \quad \text{for } p \neq 0. \tag{3.11}$$

Then deriving with respect to δ gives

$$E^{0,q}_2 = H^q(\mathcal{U}, \mathcal{F}), \qquad E^{p,q}_2 = 0 \quad \text{for } p \neq 0. \tag{3.12}$$

This spectral sequence converges to the total cohomology of $C^q(\mathcal{U}, \Delta^p_\eta)$, which is just the cohomology of $\mathcal{F}$ on Y.

Alternatively, deriving with respect to δ first gives

$$E^{p,q}_1 = H^q(Y, \Delta^p_\eta), \tag{3.13}$$

which is the E_1 term of the hypercohomology spectral sequence converging to the cohomology of $\mathcal{F}$ on Y.

Applying the hypercohomology spectral sequence to the relative BGG resolution (3.4) of $\eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda)$, we get

$$E^{p,q}_1 = H^q(Y, \Delta^p_\eta(\lambda_k)) \Rightarrow H^{p+q}(Y, \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda_k)), \tag{3.14}$$

i.e., the spectral sequence converges to the cohomology of $\eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda)$ on $Y = \tau^{-1}(X)$. Note that the Dynkin diagram of the fibres of $Y \to X$ is obtained by deleting from the Dynkin diagram for $\mathfrak{q}$ all crossed nodes (and incident edges) shared with $\mathfrak{p}$ and then deleting all connected components with no crossed nodes (Section 2.4 in [15]). Thus the fibre is $\times = \mathbb{C}P^1$, and so the cohomology $H^{p+q}(Y, \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda))$ vanishes except $p + q \leq 1$.

It follows from X being affine that the Leray spectral sequence collapses to give isomorphisms:

$$H^q(Y, \Delta^p_\eta(\lambda_k)) \cong \Gamma(X, \tau^q_*\Delta^p_\eta(\lambda_k)) = \Gamma(X, \tau^q_*\mathcal{O}_{\mathfrak{q}}(\nu_p)), \tag{3.15}$$

where $\tau^q_*\Delta^p_\eta(\lambda)$ denotes the q-th direct image.

Proposition 3.2.1. *The hypercohomology spectral sequence has the E_1 terms of the form*

$$\left|\begin{array}{cccccc} \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots \\ E^{0,1}_1 & \cdots & E^{k,1}_1 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & E^{k+2,0}_1 & \cdots \\ \hline \end{array}\right. \tag{3.16}$$

with

$$E_1^{j,1} = \Gamma(X, \mathcal{O}_{\mathfrak{p}}(\mu_j)), \qquad E_1^{j',0} = \Gamma(X, \mathcal{O}_{\mathfrak{p}}(\mu_{j'})),$$

where $j = 0, \ldots, k, j' = k+2, \ldots, 2n+1$,

$$\mu_j = \overset{k-j}{\times} - \overset{-2-k}{\times} - \bullet \cdots \overset{1}{\bullet} \cdots \bullet \Longleftarrow \bullet, \quad j = 0, \ldots, k, \tag{3.17}$$

where 1 *appears in the* $(2+j)$*-th node, and*

$$\begin{aligned} \mu_j &= \overset{-2-k+j}{\times} - \overset{-j-1}{\times} - \bullet \cdots \overset{1}{\bullet} \cdots \bullet \Leftarrow \bullet, \quad j = k+2, \ldots, n, \\ \mu_{n+1+j} &= \overset{-k+n+j}{\times} - \overset{-j-n-3}{\times} - \bullet \cdots \overset{1}{\bullet} \cdots \bullet \Leftarrow \bullet, \quad j = 0, \ldots, n, \end{aligned} \tag{3.18}$$

where 1 *appears in the* $(2+j)$*-th node and the* $(n+2-j)$*-th node, respectively.*

Proof. We can apply the Borel–Bott–Weil theorem to compute the higher direct image $\tau_*^q \mathcal{O}_{\mathfrak{q}}(\nu_p)$ (Section 5.3 in [15]). Consider the orbit $\{w.\nu;\ w \in W_{\mathfrak{p}}^{\mathfrak{q}}\}$. If none of these weights is $\mathfrak{p}$-dominant, then all higher direct images of $\mathcal{O}_{\mathfrak{q}}(\nu)$ vanish. Otherwise, there is a unique $\mathfrak{p}$-dominant weight $w.\nu$, and $\tau_*^{l(w)} \mathcal{O}_{\mathfrak{r}}(\nu) = \mathcal{O}_{\mathfrak{p}}(w.\nu)$, whereas all other higher direct images vanish.

If $j \leq k$, $\mu_j = \sigma_1.\nu_j$ is $\mathfrak{p}$-dominant with μ_j given by (3.17) by direct calculation. If $j = k+1$,

$$\nu_{k+1} = \overset{-1}{\times} - \overset{-k-2}{\times} - \bullet \cdots \overset{1}{\bullet} \cdots \bullet \Longleftarrow \bullet, \tag{3.19}$$

is singular, and so there is no non-zero direct image. If $j \geq k+2$, ν_j is regular, and so only the zeroth direct images is non-trivial: $\mu_j = \nu_j$. □

It follows that $E_*^{p,q}$ $(* = 2, 3)$ terms are

$$\begin{array}{|cccccc} \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots \\ E_*^{0,1} & \cdots & E_*^{k,1} & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & E_*^{k+2,0} & \cdots \\ \hline \end{array} \tag{3.20}$$

and $E_3^{p,q} = E_\infty^{p,q}$. Since the spectral sequence converges to $H^{p+q}(Y, \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda_k))$, which vanishes except $p+q \leq 1$ (3.14), we see that all $E_3^{p,q} = E_\infty^{p,q} = 0$ except $E_3^{0,1}$. Therefore, $E_2^{1,1} = \cdots = E_2^{k-1,1} = 0$, $E_2^{k+3,0} = \cdots = E_2^{2n+1,0} = 0$, i.e., $E_1^{0,1} \xrightarrow{d_1} \cdots \xrightarrow{d_1} E_1^{k,1}$ and $E_1^{k+2,0} \xrightarrow{d_1} E_1^{k+3,0} \xrightarrow{d_1} \cdots$ are both exact, and $0 = E_3^{k,1} = \ker d_2 : E_2^{k,1} \to E_2^{k+2,0}$, $E_2^{k+2,0} = \operatorname{im} d_2 : E_2^{k,1} \to E_2^{k+2,0}$ (by $E_3^{k+2,0} = 0$). This together with the fact that

$$E_2^{k,1} = \frac{E_1^{k,1}}{\operatorname{im} d_1 : E_1^{k-1,1} \to E_1^{k,1}} \quad \text{and} \quad E_2^{k+2,0} = \ker d_1 : E_1^{k+2,0} \to E_1^{k+3,0}, \tag{3.21}$$

implies that d_2 induces a well defined morphism $\Delta : E_1^{k,1} \to E_1^{k+2,0}$ such that $\ker \Delta = \operatorname{im} d_1 : E_1^{k-1,1} \to E_1^{k,1}$ and $\operatorname{im} \Delta = \ker d_1 : E_1^{k+2,0} \to E_1^{k+3,0}$. So we get an exact sequence

$$E_1^{0,1} \xrightarrow{d_1} \cdots \xrightarrow{d_1} E_1^{k,1} \xrightarrow{\Delta} E_1^{k+2,0} \xrightarrow{d_1} \cdots. \tag{3.22}$$

Recall that a sequence of sheaves is called *exact* if the sequence of stalks at each point is exact. Note that X is an arbitrarily chosen affine open set in G/P and the induced morphisms in (3.22) are $\mathfrak{g}$-morphisms between sheaves of $\mathfrak{g}$-modules. We obtain the following theorem.

Theorem 3.2.1. *For* $k = 0, \ldots, n$,

$$\begin{aligned} &0 \to \quad H^1(G/Q, \eta^{-1}O_{\mathfrak{r}}(\lambda_k)) \to \mathcal{O}_{\mathfrak{p}}(\mu_0) \xrightarrow{\partial_0^{(k)}} \cdots \xrightarrow{\partial_{k-1}^{(k)}} \mathcal{O}_{\mathfrak{p}}(\mu_k) \\ &\quad \xrightarrow{\partial_k^{(k)}} \mathcal{O}_{\mathfrak{p}}(\mu_{k+2}) \xrightarrow{\partial_{k+1}^{(k)}} \cdots \xrightarrow{\partial_{2n-1}^{(k)}} \mathcal{O}_{\mathfrak{p}}(\mu_{2n+1}) \to 0 \end{aligned} \tag{3.23}$$

are exact sequences of sheaves on G/P*, where operators* $\partial_j^{(k)}$ *induced from* $\widetilde{\partial}_j^{(k)}$ *are also* $\mathfrak{g}$*-invariant, and* μ_j *are* $\mathfrak{p}$*-dominant integral weights given by* (3.17) *and* (3.18)*.*

In notations in Section 2.1, the irreducible representation $F_{\mathfrak{p}}(-\mu_j)$ is completely determined by $(\delta, \nu_{F_{\mathfrak{p}}(-\mu_j)})$ with

$$\nu_{F_{\mathfrak{p}}(-\mu_j)} = -((k-j)\omega_1 - (k+2)\omega_2 + \omega_{j+2})(H_1 + H_2) = j + k + 2, \tag{3.24}$$

for $j = 0, \dots, k$, by $\omega_j = L_1 + \cdots + L_j$, L_j dual to H_j, and

$$\nu_{F_{\mathfrak{p}}(-\mu_j)} = -((-2-k+j)\omega_1 - (j+1)\omega_2 + \omega_{j+2})(H_1 + H_2) = j + k + 2, \tag{3.25}$$

for $j = k+2, \dots, n$, and

$$\nu_{F_{\mathfrak{p}}(-\mu_{n+1+j})} = -((-k+n+j)\omega_1 - (j+n+3)\omega_2 + \omega_{n+2-j})(H_1 + H_2) = j + k + n + 4, \tag{3.26}$$

for $j = 0, \dots, n$.

4. Determination of invariant operators

Let us determine the invariant operators in the exact sequence (3.23) on G/P by using the representation theory.

4.1. Some $\mathfrak{g}_{S_2}$-modules

Recall that $V^{(j)}$ is the irreducible representation of $\mathfrak{sp}(2n, \mathbb{C})$ with the highest weight to be the j-th fundamental weight ω_j, $j = 1, \dots, n$. Let $\varphi_j : \wedge^j \mathbb{C}^{2n} \to \wedge^{j-2} \mathbb{C}^{2n}$ be defined by

$$\varphi_j(v_1 \wedge \cdots \wedge v_j) = \sum_{i<j} \epsilon(v_s, v_t)(-1)^{s+t-1} v_1 \wedge \cdots \wedge \widehat{v_s} \wedge \cdots \widehat{v_t} \wedge \cdots \wedge v_j, \tag{4.1}$$

where $\widehat{v_s}$ means that v_s is omitted, and $\epsilon(\cdot, \cdot)$ is the bilinear form defined by the matrix (ϵ_{AB}) in (1.18). It is known that $V^{(j)} = \ker \varphi_j$ (cf. Theorem 17.5 in [16]). Consequently,

$$\Lambda^2\mathbb{C}^{2n} = V^{(2)} \oplus \mathbb{C}\psi, \qquad \mathrm{pr}_{V^{(2)}}(e_{A_1} \wedge e_{A_2}) = e_{A_1} \wedge e_{A_2} - \frac{1}{n}\epsilon_{A_1A_2}\psi, \tag{4.2}$$

where $\psi = \sum_{i=1}^{n} e_i \wedge e_{n+i}$ (cf. Section 17.3 in [16]). The image of φ_3 is $\mathbb{C}^{2n}$, and so

$$\mathrm{pr}_{V^{(3)}}(e_{A_1} \wedge e_{A_2} \wedge e_{A_3}) = e_{A_1} \wedge e_{A_2} \wedge e_{A_3} - \frac{1}{n-1}\psi \wedge (\epsilon_{A_2A_3}e_{A_1} - \epsilon_{A_1A_3}e_{A_2} + \epsilon_{A_1A_2}e_{A_3}). \tag{4.3}$$

From these formulae, we can write down pr_1^2 and pr_3^2.

By definition, if we embed a matrix $M \in \mathfrak{sp}(2n, \mathbb{C})$ as an element of $\mathfrak{g}_{S_2} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sp}(2n, \mathbb{C})$, we have

$$M.Y_{A,A'} := [M, Y_{A,A'}] = M_{BA}Y_{B,A'}, \tag{4.4}$$

$A' = 1, 2$, where the summation is taken over $B = 1, \dots, 2n$. Similarly, if we embed a matrix $M' \in \mathfrak{sl}(2, \mathbb{C})$ as an element $\mathfrak{g}_{S_2} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sp}(2n, \mathbb{C})$ (i.e., a matrix in form of (2.4) with $B = C = 0$ and A to be $\mathrm{diag}(M', 0_n)$), we have

$$M'.Y_{A,A'} := [M', Y_{A,A'}] = -M'_{A'B'}Y_{A,B'}, \tag{4.5}$$

where the summation is taken over $B' = 1, 2$.

Let $\{e_{1'}, e_{2'}\}$ and $\{e_1, \dots, e_{2n}\}$ be standard bases of $\mathbb{C}^2$ and $\mathbb{C}^{2n}$, respectively. If we identify $\mathfrak{g}_{-1}$ with $\mathbb{C}^2 \otimes \mathbb{C}^{2n}$ by taking $Y_{A,A'} \mapsto e_{A'} \otimes e_A$, then as an $\mathfrak{g}_{S_2}$-module $\mathfrak{g}_{-1}$ is isomorphic to $\mathbb{C}^2 \otimes \mathbb{C}^{2n}$ with the standard representation of $\mathfrak{sp}(2n, \mathbb{C})$ on $\mathbb{C}^{2n}$ and the representation (4.5) of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathbb{C}^2$ (it is the representation dual to the standard representation of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathbb{C}^2$), i.e.,

$$(M' \otimes M).(e_{A'} \otimes e_A) = -M'_{A'B'}e_{B'} \otimes e_A + e_{A'} \otimes M_{BA}e_B. \tag{4.6}$$

$\mathfrak{sp}(2n, \mathbb{C})$ acts trivially on $\mathfrak{g}_{-2}$. Let $\mathfrak{g}_{-1}^*$ be $\mathbb{C}^{2*} \otimes \mathbb{C}^{2n*}$ with the contragredient $\mathfrak{g}_{S_2}$-representation. Define a linear isomorphism

$$\begin{aligned} \vartheta : \mathfrak{g}_{-1} &\longrightarrow \mathfrak{g}_{-1}^*, \\ Y_{A,A'} &\longmapsto \mathrm{e}^{A'} \otimes e^A, \end{aligned} \tag{4.7}$$

$A' = 1, 2$, $A = 1, \dots, 2n$, where $\{e^{1'}, e^{2'}\}$ and $\{e^1, \dots, e^{2n}\}$ are standard bases of $\mathbb{C}^{2*}$ and $\mathbb{C}^{2n*}$, respectively. We can identify $\mathbb{C}^{2n*} \cong \mathbb{C}^{2n}$, $\mathbb{C}^{2*} \cong \mathbb{C}^2$ by taking

$$e^{A'} = \varepsilon^{A'B'}e_{B'}, \qquad e^A = \epsilon^{AB}e_B. \tag{4.8}$$

By direct calculation, we have

$$\begin{aligned} &[H_1 - H_2, T_1] = 0, \qquad [H_1 - H_2, T_2] = -2T_2, \qquad [H_1 - H_2, T_3] = 2T_3, \\ &[E_1, T_3] = 0, \qquad [E_1, T_2] = -2T_1, \qquad [E_1, T_1] = -T_3, \end{aligned}$$

for $E_1 := E_{12} - E_{N+2,N+1}$, and similarly,

$$\begin{aligned} &(E_{1'1'} - E_{2'2'}).e_{1'2'} = 0, \qquad (E_{1'1'} - E_{2'2'}).e_{1'1'} = -2e_{1'1'}, \qquad (E_{1'1'} - E_{2'2'}).e_{2'2'} = 2e_{2'2'}, \\ &E_{1'2'}.e_{2'2'} = 0, \qquad E_{1'2'}.e_{1'1'} = -2e_{1'2'}, \qquad E_{1'2'}.e_{1'2'} = -e_{2'2'} \end{aligned}$$

by $\mathfrak{sl}(2,\mathbb{C})$ acting on $e_{1'}, e_{2'}$ as (4.6). So we get an isomorphism $\widetilde{\theta} : \mathfrak{g}_{-2} \xrightarrow{\cong} \mathrm{span}_{\mathbb{C}}\{e_{1'2'}, e_{1'1'}, e_{2'2'}\} = \odot^2 \mathbb{C}^2$ as a $\mathfrak{sl}(2,\mathbb{C})$-module defined by $T_1 \mapsto e_{1'2'}, T_2 \mapsto e_{1'1'}, T_3 \mapsto e_{2'2'}$. Define a linear isomorphism

$$\begin{aligned}
&\theta : \mathfrak{g}_{-2} \longrightarrow \mathfrak{g}_{-2}^* \cong \mathrm{span}_{\mathbb{C}}\left\{e^{1'2'}, e^{1'1'}, e^{2'2'}\right\} \\
&T_1 \longmapsto e^{1'2'}, \qquad T_2 \mapsto e^{1'1'}, \qquad T_3 \mapsto e^{2'2'}.
\end{aligned} \tag{4.9}$$

Denote $\chi^1 := e^{1'2'}, \chi^2 := e^{1'1'}, \chi^3 := e^{2'2'}$.

Denote by Γ_λ the irreducible representation of G_{S_2} with the highest weight λ. The dual Γ_λ^* has the lowest weight $-\lambda$. Denote by $V(\chi)$ the irreducible representation of $\mathfrak{sp}(2n,\mathbb{C})$ with highest weight χ.

Lemma 4.1.1. *There is a unique irreducible $\mathfrak{sp}(2n,\mathbb{C})$-component $V^{(j\pm1)}$ in $\mathbb{C}^{2n} \otimes V^{(j)}$ ($j \neq n$), a unique irreducible $\mathfrak{sp}(2n,\mathbb{C})$-component $V^{(k+2)}$ in $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \otimes V^{(k)}$ and a unique irreducible $\mathfrak{sp}(2n,\mathbb{C})$-component $V^{(n)}$ in $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \otimes V^{(n)}$.*

Proof. Recall that the multiplicity of occurrence of the irreducible representation $V(\mu)$ in the tensor product $\mathbb{C}^{2n} \otimes V(\chi)$ is the dimension of space of the solutions to

$$X_i^{r_i+1} v = 0, \quad v \in (\mathbb{C}^{2n})_{\mu-\chi}, \ i = 1, \ldots, n, \tag{4.10}$$

where $X_i \in \mathfrak{sp}(2n,\mathbb{C})_{\alpha_i}$, α_i are simple roots, $r_i = \langle \chi, \alpha_i^\vee \rangle$ (cf. Theorem 5 in Section 131 in [17]). Recall that $V^{(j)} = V(\chi)$ with $\chi = \omega_j$. Then $r_i = \delta_{ji}$. Note that $e_A \in \mathbb{C}^{2n}$, $A = 1, \ldots, n$, has weight L_A (i.e. $H_i.e_A = L_A(H_i)e_A$), and $e_{n+A} \in \mathbb{C}^{2n}$, $A = 1, \ldots, n$, has weight $-L_A$ (i.e. $H_i.e_{n+A} = -L_A(H_i)e_{n+A}$). Recall $X_i = E_{i,i+1} - E_{n+i+1,n+i}$, $i = 1, \ldots, n-1$ and $X_n = E_{n,2n}$ by (2.5). It is direct to check that when $j < n$, e_A satisfies Eq. (4.10) only for $A = 1, j+1, n+j-1$ (since $X_{A-1}e_A = e_{A-1}, X_A e_{n+A} = -e_{n+A+1}, X_j^2 e_{j+1} = 0$, etc.); when $j = n$, e_A satisfies Eq. (4.10) only for $A = 1, 2n$. It follows that

$$\mathbb{C}^{2n} \otimes V^{(j)} \cong V(\omega_j + L_1) \oplus V(\omega_{j-1}) \oplus V(\omega_{j+1}), \qquad \mathbb{C}^{2n} \otimes V^{(n)} \cong V(\omega_n + L_1) \oplus V(\omega_{n-1}).$$

A similar argument shows that $V^{(k+2)}$ does not appear in $\mathbb{C}^{2n} \otimes [V(\omega_k + L_1) \oplus V(\omega_{k-1})]$ and has a multiplicity one in $\mathbb{C}^{2n} \otimes V(\omega_{k+1})$, and $V^{(n)}$ does not appear in $\mathbb{C}^{2n} \otimes V(\omega_n + L_1)$ and has a multiplicity one in $\mathbb{C}^{2n} \otimes V(\omega_{n-1})$. The result follows. □

4.2. The left invariant operators between irreducible homogeneous sheaves on G/P and generalized Verma modules

The left invariant differential operator of first order on $\mathcal{O}_G(V)$ are given by the element $X \in \mathfrak{g}$

$$Xf(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(g \exp tX),$$

for a local section f of $\mathcal{O}_G(V)$. This definition is extended to universal envelope algebra $\mathcal{U}(\mathfrak{g})$. More generally, the left invariant differential operators from $\mathcal{O}_G(V_1)$ to $\mathcal{O}_G(V_2)$ are given by $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{Hom}(V_1, V_2)$: $U \otimes v_1^* \otimes v_2$ maps $f \in \mathcal{O}_G(V_1)$ to $U\langle f, v_1^*\rangle v_2 \in \mathcal{O}_G(V_2)$.

Let $\mathcal{U}(\mathfrak{p})$ be the universal envelope algebra of $\mathfrak{p}$ and $Y \mapsto Y'$ be its principal antiautomorphism determined by $Y \mapsto -Y$ for $Y \in \mathfrak{p}$. A representation ρ of P on vector space V_1 induces a representation ρ_* of $\mathfrak{p}$ and of $\mathcal{U}(\mathfrak{p})$ on V_1. Let J be the linear span of elements

$$UY \otimes L - U \otimes (L \circ \rho_*(Y')),$$

where $U \in \mathcal{U}(\mathfrak{g})$, $Y \in \mathcal{U}(\mathfrak{p})$ and $L \in \mathrm{Hom}(V_1, V_2)$. Write

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathrm{Hom}(V_1, V_2) := \mathcal{U}(\mathfrak{g}) \otimes \mathrm{Hom}(V_1, V_2)/J.$$

The action of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathrm{Hom}(V_1, V_2)$ on $\mathcal{O}_G(V)^P$, and hence on $\mathcal{O}_{\mathfrak{p}}(V)$, is well defined since J act trivially on $\mathcal{O}_G(V)^P$:

$$\begin{aligned}
(UY \otimes L)f(g) &= L\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (Uf)(g \exp tY) = L\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho(\exp(-tY))(Uf)(g) \\
&= L\rho_*(Y')(Uf)(g) = (U \otimes L \circ \rho_*(Y'))f(g).
\end{aligned}$$

Suppose ρ_1, ρ_2 are two representations of P on vector spaces V_1 and V_2, respectively. P acts on $\mathrm{Hom}(V_1, V_2)$ by $p \in P$ sending any element L to $\rho_2(p) \circ L \circ \rho_1(p^{-1})$, and P acts on $\mathcal{U}$ by the adjoint representation. Hence we have an action of P on $\mathcal{U} \otimes_{\mathcal{U}(\mathfrak{p})} \mathrm{Hom}(V_1, V_2)$.

An element $u \in \mathcal{U}(\mathfrak{g})$ defines an operator as

$$f \mapsto \langle v^*, uf \rangle \tag{4.11}$$

for $v^* \in V^*$ and a germ $f \in \mathcal{O}_G(V)^P$. When $u \in \mathfrak{p}$,

$$\langle v^*, uf \rangle = \langle v^*, -\rho_*(u)f \rangle = \langle \rho^*(u)v^*, f \rangle$$

where ρ^* is the contragredient representation of ρ_*. Identifying the germ of the operator in (4.11) at the identity with an element of $\mathscr{U}(\mathfrak{g}) \otimes V^*$. Since $u \otimes v^*$ and $1 \otimes \rho^*(u)v^*$ agree on sections of V,

$$M_{\mathfrak{p}}(V^*) := (\mathscr{U}(\mathfrak{g}) \otimes V^*)/I = \mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{p})} V^* \tag{4.12}$$

is the space of differential operators from $\mathcal{O}_G(V)^P$ to $\mathcal{O}_G$, where I is the ideal generated by elements of form $u \otimes v^* - 1 \otimes \rho^*(u)v^*$ for $u \in \mathfrak{p}$. Namely,

$$M_{\mathfrak{p}}(V^*) = \varinjlim \mathrm{Hom}_{\mathbb{C}}(J^r(V)_{eP}, \mathbb{C}), \tag{4.13}$$

where $J^r(V)_{eP}$ is the r-th jet bundle at point eP. $M_{\mathfrak{p}}(V^*)$ is called a *generalized Verma module*.

Proposition 4.2.1. *A $\mathfrak{g}$-invariant homomorphism D from sheaf $\mathcal{O}_{\mathfrak{p}}(\lambda)$ to sheaf $\mathcal{O}_{\mathfrak{p}}(\mu)$ is a differential operator.*

Proof. A homomorphism D from sheaf $\mathcal{O}_{\mathfrak{p}}(\lambda)$ to sheaf $\mathcal{O}_{\mathfrak{p}}(\mu)$ maps an r_1-jet s of $J^r(F_{\mathfrak{p}}(-\lambda))_{eP}$ at the identity to some r_2-jet of $J^{r_2}(F_{\mathfrak{p}}(-\mu))_{eP}$ at the identity for some r_2 depending on s. So D induces a homomorphism

$$\varinjlim \mathrm{Hom}_{\mathbb{C}}(J^r(F_{\mathfrak{p}}(-\mu))_{eP}, \mathbb{C}) \to \varinjlim \mathrm{Hom}_{\mathbb{C}}(J^r(F_{\mathfrak{p}}(-\lambda))_{eP}, \mathbb{C}).$$

If D is $\mathfrak{g}$-invariant, the induced homomorphism is also $\mathfrak{g}$-invariant. By (4.13), D defines a $\mathfrak{g}$-homomorphism between generalized Verma modules $D : M_{\mathfrak{p}}(F_{\mathfrak{p}}(-\mu)^*) \to M_{\mathfrak{p}}(F_{\mathfrak{p}}(-\lambda)^*)$. A $\mathfrak{g}$-homomorphism between generalized Verma modules is a invariant differential operator (Proposition 11.2.1 in [15]). □

Proposition 4.2.2. *$\mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{p})} \mathrm{Hom}(V_1, V_2)$ can be identified with the space of G-invariant differential operators from local holomorphic sections of $\mathcal{O}_G(V_1)^P$ to that of $\mathcal{O}_G(V_2)$. Having chosen a subspace $\mathfrak{u}^-$ of $\mathfrak{g}$ complementary to $\mathfrak{p}$ and a basis $\{Y_j\}$, every element of $\mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{p})} \mathrm{Hom}(V_1, V_2)$ can be uniquely written in the following "normal form":*

$$\sum_\alpha Y^\alpha \otimes L_\alpha, \tag{4.14}$$

where $L_\alpha \in \mathrm{Hom}(V_1, V_2)$, $Y^\alpha = Y_1^{\alpha_1} Y_2^{\alpha_2} \cdots$ for an increasing index $\alpha = (\alpha_1, \alpha_2, \ldots)$.

This proposition is the holomorphic version of Proposition 1.1 in [18], which is stated for C^∞-sections. Its proof holds for holomorphic sections without modification.

4.3. Determination of the invariant differential operators

Proposition 4.3.1. *Let $G = \mathrm{Sp}(2N, \mathbb{C})$, and let $\{Y_j\}$ in (2.11) be a basis of $\mathfrak{u}^-_{S_2}$ complementary to $\mathfrak{p}_{S_2}$ in $\mathfrak{g} = \mathfrak{sp}(2N, \mathbb{C})$. Let $((\delta, \nu), V)$ and $((\delta', \nu'), V')$ be two irreducible finite dimensional $\mathfrak{p}_{S_2}$-modules, and let D be a $\mathfrak{g}$-invariant differential operator from $\mathcal{O}_{\mathfrak{p}}(V)$ to that of $\mathcal{O}_{\mathfrak{p}}(V')$. Then, if $\nu' - \nu = 1$, D can be uniquely written as*

$$D = \sum_{j=1}^{4n} Y_j \otimes F_j \tag{4.15}$$

for some $F_j \in \mathrm{Hom}(V, V')$; if $\nu' - \nu = 2$, D can be uniquely written as

$$D = \sum_{j,k=1}^{4n} Y_j Y_k \otimes F_j + \sum_{s=1}^{3} Y_{4n+s} \otimes F_s, \tag{4.16}$$

for some $F_j, F_s \in \mathrm{Hom}(V, V')$.

Proof. Let D be written in normal form (4.14) with $Y^\alpha = Y_1^{\alpha_1} \cdots Y_{4n+3}^{\alpha_{4n+3}}$. Note that

$$H_3 \cdot D = \sum_\alpha \left(\nu' - \nu - \sum_{j=1}^{4n} \alpha_j - 2 \sum_{j=s}^{3} \alpha_{4n+s} \right) Y^\alpha \otimes L_\alpha$$

by (2.9). The operator $0 = D - H_3 \cdot D$ is in normal form. When $\nu' - \nu = 1$, we must have $|\alpha| = 1$ and $\alpha_{4n+1} = \alpha_{4n+2} = \alpha_{4n+3} = 0$. So D has the form (4.15). When $\nu' - \nu = 2$, there are two cases: (1) $|\alpha| = 2$ and $\alpha_{4n+1} = \alpha_{4n+2} = \alpha_{4n+3} = 0$, (2) $|\alpha| = 1$ and $\alpha_1 = \cdots = \alpha_{4n} = 0$. Although the normal form in (4.14) takes over increasing multiindices, D can be uniquely written as the form of (4.16) since $[Y_i, Y_j] = 4Y_{4n+s}$ for some s or equals 0. So D has the form (4.16). □

Proposition 4.3.2. *Let G and $\{Y_j\}$ be as in Proposition 4.3.1 and let $\{w_\alpha\}$ be a basis of $F_{\mathfrak{p}}(-\mu_s)$ or $F_{\mathfrak{p}}(-\mu_{s+1})$. Up to a constant, $\mathfrak{g}$-invariant differential operators $\partial_s^{(k)}$ in the complex (3.23) can be written as follows:*

(1) *if $s \neq k, n-1$,*

$$\partial_s^{(k)} = Y_{A,A'} \otimes w_\alpha^* \otimes \mathrm{pr}(\vartheta Y_{A,A'} \otimes w_\alpha),$$

where pr *is the* $\mathfrak{g}_{S_2}$*-equivariant projection from* $\mathfrak{g}^*_{-1} \otimes F_{\mathfrak{p}}(-\mu_s)$ *to* $F_{\mathfrak{p}}(-\mu_{s+1})$ *for* $s < k$ *and is the* $\mathfrak{g}_{S_2}$*-equivariant projection from* $\mathfrak{g}^*_{-1} \otimes F_{\mathfrak{p}}(-\mu_{s+1})$ *to* $F_{\mathfrak{p}}(-\mu_{s+2})$ *for* $s \geq k+1$.

(2)

$$\partial^{(k)}_{n-1} = Y_{4n+j} \otimes w^*_\alpha \otimes \mathrm{pr}(\theta Y_{4n+j} \otimes w_\alpha),$$

where pr *is the* $\mathfrak{g}_{S_2}$*-equivariant projection from* $\mathfrak{g}^*_{-2} \otimes F_{\mathfrak{p}}(-\mu_n)$ *to* $F_{\mathfrak{p}}(-\mu_{n+1})$.

(3)

$$\partial^{(k)}_{k} = Y_{A,A'} Y_{B,B'} \otimes w^*_\alpha \otimes \mathrm{pr}(\vartheta Y_{A,A'} \otimes \vartheta Y_{B,B'} \otimes w_\alpha),$$

where pr *is the projection from* $\mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes F_{\mathfrak{p}}(-\mu_k)$ *to* $F_{\mathfrak{p}}(-\mu_{k+2})$.

Proof. Cf. Theorem 2.2 in [18] for determining the G-invariant differential operators of the first order on boundaries of symmetric spaces. We prove (2), (3). The proof of (1) is similar.

By (3.24)–(3.26), we have

$$\nu_{F_{\mathfrak{p}}(-\mu_{j+1})} - \nu_{F_{\mathfrak{p}}(-\mu_j)} = 1, \quad j \neq k, k+1, n$$
$$\nu_{F_{\mathfrak{p}}(-\mu_{k+2})} - \nu_{F_{\mathfrak{p}}(-\mu_k)} = \nu_{F_{\mathfrak{p}}(-\mu_{n+1})} - \nu_{F_{\mathfrak{p}}(-\mu_n)} = 2.$$

So by Proposition 4.3.1, we can write

$$\partial^{(k)}_k = Y_{A_1,A_1'} Y_{A_2,A_2'} \otimes w^*_\alpha \otimes f_{(A_1,A_1')(A_2,A_2')\alpha} + Y_{4n+j} \otimes w^*_\alpha \otimes f_{j\alpha},$$

with $f_{(A_1,A_1')(A_2,A_2')\alpha}, f_{j\alpha} \in V_2 := F_{\mathfrak{p}}(-\mu_{k+2})$, where $\{w_\alpha\}$ is a basis of $V_1 := F_{\mathfrak{p}}(-\mu_k)$. This element induces mappings $U_1 \in \mathrm{Hom}(\mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes F_{\mathfrak{p}}(-\mu_k), F_{\mathfrak{p}}(-\mu_{k+2}))$ and $U_2 \in \mathrm{Hom}(\mathfrak{g}^*_{-2} \otimes F_{\mathfrak{p}}(-\mu_k), F_{\mathfrak{p}}(-\mu_{k+2}))$ defined by

$$U_1 : \vartheta Y_{A_1,A_1'} \otimes \vartheta Y_{A_2,A_2'} \otimes w_\alpha \longmapsto f_{(A,A_1')(A_2,A_2')\alpha},$$
$$U_2 : \theta Y_{4n+j} \otimes w_\alpha \longmapsto f_{j\alpha},$$

and $\partial^{(k)}_k$ can be written as $U_1 \circ \nabla_1 + U_2 \circ \nabla_2$ with

$$\nabla_1 := Y_{A_1,A_1'} Y_{A_2,A_2'} \otimes w^*_\alpha \otimes \vartheta Y_{A_1,A_1'} \otimes \vartheta Y_{A_2,A_2'} \otimes w_\alpha,$$
$$\nabla_2 := Y_{4n+j} \otimes w^*_\alpha \otimes \theta Y_{4n+j} \otimes w_\alpha. \tag{4.17}$$

Denote by π_{V_j} the representation of G_{S_2} on V_j, and

$$m.U_1 := \pi_{V_2}(m) \circ U_1 \circ \pi_{\mathfrak{g}^*_{-1}\otimes\mathfrak{g}^*_{-1}\otimes V_1}(m^{-1}),$$
$$m.U_2 := \pi_{V_2}(m) \circ U_2 \circ \pi_{\mathfrak{g}^*_{-2}\otimes V_1}(m^{-1}). \tag{4.18}$$

If $m.w_\alpha = m_{\beta\alpha} w_\beta$ for $m \in G_{S_2}$, then $m.w^*_\alpha = (m^{-1})_{\alpha\gamma} w^*_\gamma$ by definition of the contragredient representation. It follows that $m.(w^*_\alpha \otimes w_\alpha) = w^*_\alpha \otimes w_\alpha$. Similarly, for $M \in \mathrm{Sp}(2n, \mathbb{C})$, $M.(Y_{A_1,A_1'} \otimes \vartheta Y_{A_1,A_1'}) = Y_{A_1,A_1'} \otimes \vartheta Y_{A_1,A_1'}$ by $\vartheta Y_{A_1,A_1'}$ dual to $Y_{A_1,A_1'}$, and $N.(Y_{A_1,A_1'} \otimes \vartheta Y_{A_1,A_1'}) = Y_{A_1,A_1'} \otimes \vartheta Y_{A_1,A_1'}$ for $N \in \mathrm{SL}(2, \mathbb{C})$. Consequently, $m.\nabla_i = \nabla_i$ by the formulae (4.17) of ∇_i.

Now the invariance of $\partial^{(k)}_k$ and ∇_i under G_{S_2} implies that

$$m.\partial^{(k)}_k = Y_{A_1,A_1'} Y_{A_2,A_2'} \otimes w^*_\alpha \otimes \pi_{V_2}(m) \circ U_1 \circ \pi_{\mathfrak{g}^*_{-1}\otimes\mathfrak{g}^*_{-1}\otimes V_1}(m^{-1})(\vartheta Y_{A_1,A_1'} \otimes \vartheta Y_{A_2,A_2'} \otimes w_\alpha)$$
$$+ Y_{4n+j} \otimes w^*_\alpha \otimes \otimes \pi_{V_2}(m) \circ U_2 \circ \pi_{\mathfrak{g}^*_{-2}\otimes V_1}(m^{-1})(\theta Y_{4n+j} \otimes w_\alpha).$$

It follows from $0 = \partial^{(k)}_k - m.\partial^{(k)}_k$ in normal form that

$$\pi_{V_2}(m) \circ U_1 \circ \pi_{\mathfrak{g}^*_{-1}\otimes\mathfrak{g}^*_{-1}\otimes V_1}(m^{-1})(\vartheta Y_{A_1,A_1'} \otimes \vartheta Y_{A_2,A_2'} \otimes w_\alpha) = U_1(\vartheta Y_{A_1,A_1'} \otimes \vartheta Y_{A_2,A_2'} \otimes w_\alpha). \tag{4.19}$$

Namely, U_1 is G_{S_2}-equivariant. Similarly, U_2 is also G_{S_2}-equivariant. By Lemma 4.1.1 together with

$$\mathbb{C}^2 \otimes \odot^i \mathbb{C}^2 \cong \odot^{i-1} \mathbb{C}^2 \otimes \odot^{i+1} \mathbb{C}^2$$

as $\mathfrak{sl}(2, \mathbb{C})$-modules, there is a unique irreducible component $\Gamma^*_{\mu_{k+2}}$ in $\mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \Gamma^*_{\mu_k}$ and obviously no irreducible component $\Gamma^*_{\mu_{k+2}}$ in $\mathfrak{g}^*_{-2} \otimes \Gamma^*_{\mu_k}$. By Schur's lemma, U_i is scalar on each irreducible component. So U_2 is zero. The result for $\partial^{(k)}_k$ follows.

The same argument is applied to $\partial^{(k)}_{n-1}$. By Lemma 4.1.1 again, there is a unique irreducible component $\Gamma^*_{\mu_{n+1}}$ in $\mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \Gamma^*_{\mu_n}$ and a unique irreducible component $\Gamma^*_{\mu_{n+1}}$ in $\mathfrak{g}^*_{-2} \otimes \Gamma^*_{\mu_n}$, since $\mathfrak{g}_{-2} \cong \odot^2 \mathbb{C}^2$ as $\mathfrak{g}_{S_2}$-modules. By Schur's lemma,

$$\partial^{(k)}_{n-1} = C_1 Y_{4n+j} \otimes w^*_\alpha \otimes \mathrm{pr}_1(\theta Y_{4n+j} \otimes w_\alpha) + C_2 Y_{A,A'} Y_{B,B'} \otimes w^*_\alpha \otimes \mathrm{pr}_2(\vartheta Y_{A,A'} \otimes \vartheta Y_{B,B'} \otimes w_\alpha) \tag{4.20}$$

for some constant C_1, C_2, where pr_i, $i = 1, 2$, are the projection from $\mathfrak{g}^*_{-2} \otimes F_{\mathfrak{p}}(-\mu_n)$ and $\mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes F_{\mathfrak{p}}(-\mu_n)$ to $F_{\mathfrak{p}}(-\mu_{n+1})$, respectively. pr_2 is the $\mathfrak{g}_{S_2}$-equivariant projection from $(\mathbb{C}^{2*} \otimes \mathbb{C}^{2n*}) \otimes (\mathbb{C}^{2*} \otimes \mathbb{C}^{2n*}) \otimes \odot^{n-k-2} \mathbb{C}^{2*} \otimes V^{(n)*}$ to $\odot^{n-k} \mathbb{C}^{2*} \otimes \otimes V^{(n)*}$. Let $\{e^{A'_1 \ldots A'_l}\}$ be a basis of $\odot^l \mathbb{C}^{2*}$, $\{v^\alpha\}$ is a basis of $V^{(n)*}$. Note that up to a constant,

$$\mathrm{pr}_2(e^{A'} \otimes e^A \otimes e^{B'} \otimes e^B \otimes e^{A'_1 \ldots A'_{n-k-2}} \otimes v^\alpha) = \epsilon_{AB} e^{A'B'A'_1 \ldots \ldots A'_{n-k-2}} \otimes v^\alpha,$$

which is $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sp}(2n, \mathbb{C})$-invariant. So the second sum of (4.20) is

$$C_2 \epsilon_{AB} Y_{A,A'} Y_{B,B'} \otimes e^{A'_1 \ldots A'_{n-k-2}*} \otimes v^{\alpha *} \otimes e^{A'B'A'_1 \ldots \ldots A'_{n-k-2}} \otimes v^\alpha.$$

Note that $\epsilon_{AB} Y_{A,A'} Y_{B,B'} \in \mathfrak{g}_{-2}$. So $\partial^{(k)}_{n-1}$ has the form $\sum_{s=1}^3 Y_{4n+s} \otimes F_s$ and we can assume $\partial^{(k)}_{n-1} = U_2 \circ \nabla_2$. The result follows as above. □

By Proposition 4.3.2, $\partial^{(k)}_j : \mathcal{O}_{\mathfrak{p}}(\odot^{k-j} \mathbb{C}^{2*} \otimes V^{(j)*}) \to \mathcal{O}_{\mathfrak{p}}(\odot^{k-j-1} \mathbb{C}^{2*} \otimes V^{(j+1)*})$, $j = 0, \ldots, k-1$, is written as

$$\partial^{(k)}_j = Y_{AA'} \otimes e^{A'_1 \ldots A'_{k-j}*} \otimes v^{\alpha *} \otimes \mathrm{pr}(e^A \otimes e^{A'} \otimes e^{A'_1 \ldots A'_{k-j}} \otimes v^\alpha)$$

where pr is the $\mathfrak{g}_{S_2}$-equivariant projection from $\mathbb{C}^{2*} \otimes \mathbb{C}^{2n*} \otimes \odot^{k-j} \mathbb{C}^{2*} \otimes V^{(j)*}$ to $\odot^{k-j-1} \mathbb{C}^{2*} \otimes \otimes V^{(j+1)*}$. Note that up to a constant,

$$\mathrm{pr}(e^{A'} \otimes e^{A'_1 \ldots A'_{k-j}}) = \epsilon^{A'_s A'} e^{A'_1 \ldots \hat{A}_{s'} \ldots A'_{k-j}},$$

which is $\mathfrak{sl}(2, \mathbb{C})$-invariant. Thus

$$\mathrm{pr}(e^A \otimes e^{A'} \otimes e^{A'_1 \ldots A'_{k-j}} \otimes v^\alpha) = \epsilon^{A'_s A'} e^{A'_1 \ldots \hat{A}_{s'} \ldots A'_{k-j}} \otimes e^A \wedge_0 v^\alpha,$$

and so

$$\partial^{(k)}_j (f e^{A'_1 \ldots A'_{k-j}} \otimes v^\alpha) = Y_A^{A'_s} f \, e^{A'_1 \ldots \hat{A}_{s'} \ldots A'_{k-j}} \otimes e^A \wedge_0 v^\alpha.$$

We can find the formulae of $\partial^{(k)}_j$ for the other j similarly.

Proposition 4.3.3. *The G-invariant operators $\partial^{(k)}_j$ in the exact sequences (3.23) of sheaves on G/P are as follows: for $j = 0, \ldots, k-1$,*

$$f e^{A'_1 \ldots A'_{k-j}} \otimes v \mapsto Y_A^{A'_s} f e^{A'_1 \ldots \widehat{A}_{s'} \ldots A'_{k-j}} \otimes e^A \wedge_0 v,$$

for $v \in V^{(j)*}$; for $j = k+1, \ldots, n-1$,

$$f e^{A'_1 \ldots A'_{j-k-1}} \otimes v \mapsto Y_{AA'} f e^{A'A'_1 \ldots A'_{j-k-1}} \otimes e^A \wedge_0 v,$$

for $v \in V^{(j+1)*}$; for $j = n, \ldots, 2n-1$,

$$f e^{A'_1 \ldots A'_{j-k}} \otimes v \mapsto Y_{AA'} f e^{A'A'_1 \ldots A'_{j-k}} \otimes e^A . v,$$

for $v \in V^{(2n-j)*}$; for $j = n-1$,

$$f e^{A'_1 \ldots A'_{n-k-2}} \otimes v \mapsto Y_{4n+s} f \cdot \chi^s e^{A'_1 \ldots A'_{n-k-2}} \otimes v,$$

for $v \in V^{(n)*}$; for $j = k$,

$$f v \mapsto Y_{AA'} Y_B^{A'} f e^A \wedge_0 e^B \wedge_0 v,$$

for $v \in V^{(k)*}$, which is an operator of the second order.

5. Pulling back the complex to the quaternionic Heisenberg group

5.1. Tangential Cauchy–Fueter operators on the boundary of the Siegel upper half space

$\rho = \mathrm{Re} q_n - \sum_{l=0}^{n-1} \kappa_l |q_l|^2$ is a defining function of $\mathcal{S}_\kappa$. Define the projection

$$\begin{aligned} \pi : \mathcal{S}_\kappa &\longrightarrow \mathbb{H}^n \oplus \mathrm{Im}\,\mathbb{H}, \\ \left(q_0, \ldots, q_{n-1}, \sum_{l=0}^{n-1} \kappa_l |q_l|^2 + q'_n\right) &\longmapsto (q_0, \ldots, q_{n-1}, q'_n), \end{aligned} \tag{5.1}$$

where $q'_n = t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}$. By definition,

$$\pi_*^{-1}\partial_{t_j} = \partial_{t_j}, \qquad \pi_*^{-1}\partial_{x_{4l+s}} = \partial_{x_{4l+s}} + 2\kappa_l x_{4l+s}\partial_{x_{4n+1}}, \tag{5.2}$$

for $j = 1, 2, 3$, $s = 1, \ldots, 4$, $l = 0, \ldots, n-1$. Thus,

$$\pi_*^{-1}\left(\overline{\partial}_{q_l} + 2\kappa_l q_l \overline{\partial}_{\mathrm{Im}q_n}\right) = \overline{\partial}_{q_l} + 2\kappa_l q_l \overline{\partial}_{q_n}, \tag{5.3}$$

where $\overline{\partial}_{\mathrm{Im}q_n} = \mathbf{i}\partial_{t_1} + \mathbf{j}\partial_{t_2} + \mathbf{k}\partial_{t_3}$. So the tangential Cauchy–Fueter operator of $\mathcal{S}_\kappa$ is mapped by π_* to $\overline{Q} = (\overline{Q}_0 f, \ldots, \overline{Q}_{n-1} f)^t$ with

$$\overline{Q}_l = \overline{\partial}_{q_l} + 2\kappa_l q_l \overline{\partial}_{\mathrm{Im}q_n}. \tag{5.4}$$

To find the explicit forms $\overline{Q}_l$, denote $\mathbf{i}_0 = 1$, $\mathbf{i}_1 = \mathbf{i}$, $\mathbf{i}_2 = \mathbf{j}$, $\mathbf{i}_3 = \mathbf{k}$. Note that

$$\begin{aligned}
(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k})\mathbf{i} &= -x_2 + x_1\mathbf{i} + x_4\mathbf{j} - x_3\mathbf{k}, \\
(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k})\mathbf{j} &= -x_3 - x_4\mathbf{i} + x_1\mathbf{j} + x_2\mathbf{k}, \\
(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k})\mathbf{k} &= -x_4 + x_3\mathbf{i} - x_2\mathbf{j} + x_1\mathbf{k}.
\end{aligned} \tag{5.5}$$

So the right multiplying $\mathbf{i}_j$ is (4×4)-matrix $-b^j$, i.e.

$$(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k})\mathbf{i}_\beta = -(b^\beta x)_1 - (b^\beta x)_2\mathbf{i} - (b^\beta x)_3\mathbf{j} - (b^\beta x)_4\mathbf{k} \tag{5.6}$$

where $b^\beta, \beta = 1, 2, 3$, are anti-symmetric matrices defined in (1.11), and $(b^\beta x)_j$ is the j-th element of $b^\beta x$ for $x = (x_1, x_2, x_3, x_4)^t$. Therefore,

$$(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k})\partial_{t_\beta}\mathbf{i}_\beta = \sum_{i,j=1}^{4} b_{ij}^\beta x_i \mathbf{i}_{j-1}\partial_{t_\beta}, \tag{5.7}$$

by $b_{ij}^\beta = -b_{ji}^\beta$, and so

$$(x_{4l+1} + \mathbf{i}x_{4l+2} + \mathbf{j}x_{4l+3} + \mathbf{k}x_{4l+4})(\mathbf{i}\partial_{t_1} + \mathbf{j}\partial_{t_2} + \mathbf{k}\partial_{t_3}) = \sum_{\beta=1}^{3}\sum_{i,j=1}^{4} b_{ij}^\beta x_{4l+i}\partial_{t_\beta}\mathbf{i}_{j-1}. \tag{5.8}$$

Let X_j, $j = 1, \ldots, 4n$, be real vector fields defined by

$$\overline{Q}_l = X_{4l+1} + \mathbf{i}X_{4l+2} + \mathbf{j}X_{4l+3} + \mathbf{k}X_{4l+4}. \tag{5.9}$$

Then by (5.4) and (5.8), we get

$$X_{4l+j} = \partial_{x_{4l+j}} + 2\kappa_l \sum_{\beta=1}^{3}\sum_{i=1}^{4} b_{ij}^\beta x_{4l+i}\partial_{t_\beta}, \tag{5.10}$$

which are exactly left invariant vector fields of $\mathcal{H}_\kappa$ and satisfy the commutating relation (1.14).

5.2. *Embedding the quaternionic Heisenberg group into* $\mathrm{Sp}(2N, \mathbb{C})/P$

Define the local embedding

$$\begin{aligned}
\iota_1 : \mathbb{C}^{4n+3} &\longrightarrow G/P, \\
(w_1, \ldots, w_{4n+3}) &\longmapsto \exp(wY)P,
\end{aligned} \tag{5.11}$$

where $wY := w_1 Y_1 + \cdots w_{4n+3}Y_{4n+3}$. Set $\widetilde{f} = f \circ \iota_1 : \mathbb{C}^{4n+3} \to V$ for a local section f of $\mathcal{O}_G(V)^P$ near eP. Note that

$$\begin{aligned}
(\iota_{1*}^{-1}Y_{4l+j})\widetilde{f}(w) &= Y_{4l+j}f(\exp(wY)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\exp(wY)\exp(tY_{4l+j})) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(\exp\left(wY + tY_{4l+j} + \frac{1}{2}[wY, tY_{4l+j}]\right)\right),
\end{aligned}$$

by using the Baker–Campell–Hausdorff formula

$$\exp(wY)\exp(\widetilde{w}Y) = \exp\left(wY + \widetilde{w}Y + \frac{1}{2}[wY, \widetilde{w}Y]\right),$$

for a nilpotent Lie group of step 2. By brackets in (2.12), we get

$$wY + \widetilde{w}Y + \frac{1}{2}[wY, \widetilde{w}Y] = (w + \widetilde{w})Y + \sum_{s=1}^{3} \alpha_s(w, \widetilde{w})Y_{4n+s},$$

where α_s are bilinear functions given by

$$\begin{aligned} \alpha_1 &= 2\sum_{l=0}^{n-1} w_{4l+1}\widetilde{w}_{4l+4} - w_{4l+4}\widetilde{w}_{4l+1} + w_{4l+2}\widetilde{w}_{4j+3} - w_{4l+3}\widetilde{w}_{4l+2}, \\ \alpha_2 &= 2\sum_{l=0}^{n-1} w_{4l+1}\widetilde{w}_{4l+3} - w_{4l+3}\widetilde{w}_{4l+1}, \\ \alpha_3 &= 2\sum_{l=0}^{n-1} w_{4l+2}\widetilde{w}_{4l+4} - w_{4l+4}\widetilde{w}_{4l+2}. \end{aligned} \tag{5.12}$$

Then

$$(w', w'') \circ (\widetilde{w}', \widetilde{w}'') = (w' + \widetilde{w}', w_{4n+s} + \widetilde{w}_{4n+s} + \alpha_s(w, \widetilde{w})), \tag{5.13}$$

$s = 1, 2, 3$, defines the multiplication of a nilpotent group $\mathscr{N}$ on $\mathbb{C}^{4n+3}$ such that $(\iota_1^{-1})_* Y_{4l+j}$ is a left invariant vector field on $\mathscr{N}$:

$$(\iota_1^{-1})_* Y_{4l+j} = \partial_{w_{4l+j}} + \sum_{s=1}^{3} \partial_{\widetilde{w}_{4l+j}}\alpha_s(w, \widetilde{w})\partial_{w_{4n+s}},$$

for $l = 0, \ldots, n-1$, and $(\iota_1^{-1})_* Y_{4n+j} = \partial_{w_{4n+j}}$. Here $\partial_{\widetilde{w}_{4l+j}}\alpha_s(w, \widetilde{w})$ is independent of $\widetilde{w}$.

It is well known that the quaternionic algebra $\mathbb{H}$ can be represented by 2×2 matrices with complex entries,

$$x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mapsto \begin{pmatrix} x_0 + \mathbf{i}x_1 & -x_2 - \mathbf{i}x_3 \\ x_2 - \mathbf{i}x_3 & x_0 - \mathbf{i}x_1 \end{pmatrix}.$$

Consider the complexified conjugate embedding $\iota_2 : \mathbb{C}^{4n+3} \to \mathbb{C}^{4n+3}$ given by $z \mapsto w$ with

$$\begin{pmatrix} w_{4l+1} & w_{4l+2} \\ w_{4l+3} & w_{4l+4} \end{pmatrix} = \frac{1}{2\varepsilon_l}\begin{pmatrix} z_{4l+1} - \mathbf{i}z_{4l+2} & -z_{4l+3} + \mathbf{i}x_{4l+4} \\ z_{4l+3} + \mathbf{i}z_{4l+4} & z_{4l+1} + \mathbf{i}z_{4l+2} \end{pmatrix}, \tag{5.14}$$

where ε_l is given below (1.17) and

$$w_{4n+1} = \frac{\mathbf{i}}{2}z_{4n+1}, \qquad w_{4n+2} = \frac{1}{4}(z_{4n+2} + \mathbf{i}z_{4n+3}), \qquad w_{4n+3} = \frac{1}{4}(z_{4n+2} - \mathbf{i}z_{4n+3}). \tag{5.15}$$

Under this embedding,

$$(z', z'') \circ (\widetilde{z}', \widetilde{z}'') = (z' + \widetilde{z}', z_{4n+s} + \widetilde{z}_{4n+s} + \beta_s(z, \widetilde{z})), \tag{5.16}$$

defines the multiplication of a nilpotent group $\mathscr{N}_\kappa$ on $\mathbb{C}^{4n+3}$ such that ι_2 is an isomorphism from $\mathscr{N}_\kappa$ to $\mathscr{N}$. It is easy to see that $\beta_s(z, \widetilde{z})$ are bilinear functions as follows:

$$\begin{aligned} \beta_1 &= 2\sum_{l=0}^{n-1} \kappa_l(z_{4l+1}\widetilde{z}_{4l+2} - \widetilde{z}_{4l+1}z_{4l+2} - z_{4l+3}\widetilde{z}_{4l+4} + \widetilde{z}_{4l+3}z_{4l+4}), \\ \beta_2 &= 2\sum_{l=0}^{n-1} \kappa_l(z_{4l+1}\widetilde{z}_{4l+3} - \widetilde{z}_{4l+1}z_{4l+3} + z_{4l+2}\widetilde{z}_{4l+4} - \widetilde{z}_{4l+2}z_{4l+4}), \\ \beta_3 &= 2\sum_{l=0}^{n-1} \kappa_l(z_{4l+1}\widetilde{z}_{4l+4} - \widetilde{z}_{4l+1}z_{4l+4} - z_{4l+2}\widetilde{z}_{4l+3} + \widetilde{z}_{4l+2}z_{4l+3}). \end{aligned} \tag{5.17}$$

In terms of antisymmetric matrices b^β defined in (1.11), the multiplication of the nilpotent group $\mathscr{N}_\kappa$ can be written as follows:

$$(z', z'') \cdot (\widetilde{z}', \widetilde{z}'') = \left(z' + \widetilde{z}', z_{4n+s} + \widetilde{z}_{4n+s} + 2\sum_{l=0}^{n-1}\sum_{i,j=1}^{4} \kappa_l b_{ij}^s z_{4l+i}\widetilde{z}_{4l+j},\right), \tag{5.18}$$

where $s = 1, 2, 3$, $z', \widetilde{z}' \in \mathbb{C}^{4n}$, $z'', \widetilde{z}'' \in \mathbb{C}^3$.

At last, $\iota_3 :: \mathbb{R}^{4n+3} \to \mathbb{C}^{4n+3}$ is the trivial embedding given by $x \mapsto x + i0$. ι_3 is an isomorphism from the quaternionic Heisenberg group $\mathscr{H}_\kappa$ to a subgroup of $\mathscr{N}_\kappa$. $\iota = \iota_1\iota_2\iota_3$ is an embedding of $\mathscr{H}_\kappa$ to G/P.

Remark 5.2.1. (1) The multiplications of the quaternionic Heisenberg group in (1.9) and (5.18) are the same by direct calculation

$$\begin{aligned}\Im\{(x_1-x_2\mathbf{i}-x_3\mathbf{j}-x_4\mathbf{k})(\widetilde{x}_1+\widetilde{x}_2\mathbf{i}+\widetilde{x}_3\mathbf{j}+\widetilde{x}_4\mathbf{k})\} &= (x_1\widetilde{x}_2-x_2\widetilde{x}_1-x_3\widetilde{x}_4+x_4\widetilde{x}_3)\mathbf{i}+(x_1\widetilde{x}_3-x_3\widetilde{x}_1+x_2\widetilde{x}_4-x_4\widetilde{x}_2)\mathbf{j}\\ &\quad+(x_1\widetilde{x}_4-x_4\widetilde{x}_1-x_2\widetilde{x}_3+x_3\widetilde{x}_2)\mathbf{k}\\ &= \sum_{i,j=1}^{4} b_{ij}^1 x_i\widetilde{x}_j\mathbf{i}+\sum_{i,j=1}^{4} b_{ij}^2 x_i\widetilde{x}_j\mathbf{j}+\sum_{i,j=1}^{4} b_{ij}^3 x_i\widetilde{x}_j\mathbf{k}.\end{aligned} \tag{5.19}$$

(2) The quaternionic contact manifolds can be approximated by another quaternionic Heisenberg group, whose multiplication is given by

$$(q,t)\cdot(q',t')=\left(q+q',t+t'+\mathrm{Im}(q\overline{q}')\right), \tag{5.20}$$

where $q,q'\in\mathbb{H}^n$, $t,t'\in\mathrm{Im}\mathbb{H}$ (cf. [19]). It corresponds to the right Cauchy–Fueter operator.

5.3. Pulling back the left invariant vector fields

See [10] for the pulling back operators in the case of the Cauchy–Fueter complex. Note that $z_{4l+1}=\varepsilon_l(w_{4l+1}+w_{4l+4})$, $z_{4l+2}=\mathbf{i}\varepsilon_l(w_{4l+1}-w_{4l+4})$, $z_{4l+3}=-\varepsilon_l(w_{4l+2}-w_{4l+3})$ and $z_{4l+4}=-\mathbf{i}\varepsilon_l(w_{4l+2}+w_{4l+3})$ by (5.14). It follows that

$$\begin{pmatrix}(\iota_2^{-1})_*\partial_{w_{4l+1}} & (\iota_2^{-1})_*\partial_{w_{4l+2}}\\ (\iota_2^{-1})_*\partial_{w_{4l+3}} & (\iota_2^{-1})_*\partial_{w_{4l+4}}\end{pmatrix}=\varepsilon_l\begin{pmatrix}\partial_{z_{4l+1}}+\mathbf{i}\partial_{z_{4l+2}} & -\partial_{z_{4l+3}}-\mathbf{i}\partial_{z_{4l+4}}\\ \partial_{z_{4l+3}}-\mathbf{i}\partial_{z_{4l+4}} & \partial_{z_{4l+1}}-\mathbf{i}\partial_{z_{4l+2}}\end{pmatrix}.$$

This together with the fact that ι_{2*}^{-1} maps left invariant vector fields on $\mathcal{N}$ to that on $\mathcal{N}_\kappa$ implies that

$$(\iota_2^{-1})_*(\iota_1^{-1})_*\begin{pmatrix}Y_{4l+1} & Y_{4l+2}\\ Y_{4l+3} & Y_{4l+4}\end{pmatrix}=\varepsilon_l\begin{pmatrix}\widetilde{X}_{4l+1}+\mathbf{i}\widetilde{X}_{4l+2} & -\widetilde{X}_{4l+3}-\mathbf{i}\widetilde{X}_{4l+4}\\ \widetilde{X}_{4l+3}-\mathbf{i}\widetilde{X}_{4l+4} & \widetilde{X}_{4l+1}-\mathbf{i}\widetilde{X}_{4l+2}\end{pmatrix} \tag{5.21}$$

with

$$\widetilde{X}_{4l+j}=\partial_{z_{4l+j}}+2\kappa_l\sum_{\beta=1}^{3}\sum_{i=1}^{4} b_{ij}^{\beta} z_{4l+i}\partial_{z_{4n+\beta}}, \tag{5.22}$$

and

$$\begin{aligned}&(\iota_2^{-1})_*(\iota_1^{-1})_*Y_{4n+1}=-2\mathbf{i}\partial_{z_{4n+j}}, \qquad (\iota_2^{-1})_*(\iota_1^{-1})_*Y_{4n+2}=2(\partial_{z_{4n+2}}-\mathbf{i}\partial_{z_{4n+3}}),\\ &(\iota_2^{-1})_*(\iota_1^{-1})_*Y_{4n+3}=2(\partial_{z_{4n+2}}+\mathbf{i}\partial_{z_{4n+3}}).\end{aligned} \tag{5.23}$$

For a left invariant vector field Y on G/P, we define the notation $\iota^*Y:=\iota_{2*}^{-1}\iota_{1*}^{-1}Y|_{\mathbb{R}^{4n+3}+0\mathbf{i}}$. The following proposition follows from (5.21)–(5.23).

Proposition 5.3.1. *We have*

$$\begin{aligned}&\iota^*Y_{4l+1}=\nabla_{2l+1,1'}, \qquad \iota^*Y_{4l+2}=\nabla_{2l+1,2'}, \qquad \iota^*Y_{4l+3}=\nabla_{2l+2,1'}, \qquad \iota^*Y_{4l+4}=\nabla_{2l+2,2'},\\ &\iota^*Y_{4n+1}=2\mathbb{T}_1, \qquad \iota^*Y_{4n+2}=2\mathbb{T}_2, \qquad \iota^*Y_{4n+3}=2\mathbb{T}_3\end{aligned} \tag{5.24}$$

where vectors ∇_* are given by (1.17).

Proof of Theorem 1.0.1. The pulling back of the exact sequence (3.23) of sheaves on G/P by $\iota_1\iota_2$ is an exact sequence of sheaves on $\mathbb{C}^{4n+3}$. Note that a linear differential operator maps a polynomial to a polynomial, and if a germ of holomorphic function is mapped to a polynomial by a homogeneous differential operator, there obviously exists a polynomial mapped to this polynomial. Hence the exact sequence (3.23) of sheaves on G/P implies that the sequence on $\mathbb{C}^{4n+3}$

$$\begin{aligned}0\longrightarrow\ &\mathcal{R}(\mathbb{C}^{4n+3},\odot^k\mathbb{C}^{2*})\xrightarrow{(\iota_1\iota_2)^*\partial_0^{(k)}}\mathcal{R}(\mathbb{C}^{4n+3},\odot^{k-1}\mathbb{C}^{2*}\otimes V^{(1)*})\xrightarrow{(\iota_1\iota_2)^*\partial_1^{(k)}}\cdots\\ &\xrightarrow{\iota^*\partial_{2n-1}^{(k)}}\mathcal{R}(\mathbb{C}^{4n+3},\odot^{2n-k}\mathbb{C}^{2*})\to 0\end{aligned} \tag{5.25}$$

is exact except at the first spot, where $\mathcal{R}(\mathbb{C}^{4n},\odot^s\mathbb{C}^{2*}\otimes V^{(t)*})$ is the ring of $\odot^s\mathbb{C}^{2*}\otimes V^{(t)*}$-valued polynomials over $\mathbb{C}^{4n+3}$. The restriction of the sequence (5.25) to the real subspace $\mathbb{R}^{4n+3}+i0\subset\mathbb{C}^{4n+3}$ is also exact. For $j=0,\ldots,k-1$, it follows from Proposition 4.3.3 that

$$\begin{aligned}Q_j^{(k)}:=\iota^*\partial_j^{(k)}:\mathcal{R}(\mathbb{R}^{4n+3},\odot^{k-j}\mathbb{C}^{2*}\otimes V^{(j)*})&\longrightarrow\mathcal{R}(\mathbb{C}^{4n+3},\odot^{k-j-1}\mathbb{C}^{2*}\otimes V^{(j+1)*})\\ fe^{A_1'\ldots A_{k-j}'}\otimes v&\longmapsto\nabla_A^{A_s'}fe^{A_1'\ldots\widehat{A_s'}\ldots A_{k-j}'}\otimes e^A\wedge_0 v.\end{aligned} \tag{5.26}$$

We get $Q_j^{(k)}$ in Theorem 1.0.1 by identifying $\odot^s\mathbb{C}^{2*}\otimes V^{(t)*}$ with $\odot^s\mathbb{C}^2\otimes V^{(t)}$. Similarly, we get $Q_j^{(k)}$ for other j. □

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