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Real spectral triples over noncommutative Bieberbach manifolds

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ABSTRACT

We classify and construct all real spectral triples over noncommutative Bieberbach manifolds, which are restrictions of irreducible, real, equivariant spectral triples over the noncommutative three-torus. We show that, in the classical case, the constructed geometries correspond exactly to spin structures over Bieberbach manifolds and the Dirac operators constructed for a flat metric.

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1. Introduction

Bieberbach manifolds are compact manifolds, which are quotients of the Euclidean space by a free, properly discontinuous and isometric action of a discrete group. The torus is the canonical example of a Bieberbach manifold, however, the first nontrivial example appears in dimension 2 and is a Klein bottle. The case $d = 3$ has already been described in the seminal works of Bieberbach [1,2]. In this paper we work with the dual picture, looking at the suitable algebra of functions on the Bieberbach manifold (and their noncommutative counterparts) in terms of a fixed point subalgebra of the relevant dense subalgebra of the C^* algebra of continuous functions on the three-torus and its corresponding noncommutative deformation $\mathcal{A}(\mathbb{T}^3_\theta)$.

1.1. Noncommutative Bieberbach manifolds

In this section we shall briefly recall the description of three-dimensional noncommutative Bieberbach manifolds as quotients of the three-dimensional noncommutative tori by the action of a finite discrete group. For details we refer to [3], here we present the notation and the results. Out of 10 different Bieberbach three-dimensional manifolds (six orientable, including the three-torus itself and four nonorientable ones) only six have noncommutative counterparts.

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Table 1
The action of finite cyclic groups on $A(T^3_\theta)$.

Name	$\mathbb{Z}_N$	Generator	Action of $\mathbb{Z}_N$ on U, V, W
$B2_\theta$	$\mathbb{Z}_2$	$h, h^2 = \mathrm{id}$	$h \triangleright U = -U, h \triangleright V = V^*, h \triangleright W = W^*,$
$B3_\theta$	$\mathbb{Z}_3$	$h, h^3 = \mathrm{id}$	$h \triangleright U = e^{\frac{2}{3}\pi i}U, h \triangleright V = e^{-\pi i\theta}V^*W, h \triangleright W = V^*,$
$B4_\theta$	$\mathbb{Z}_4$	$h, h^4 = \mathrm{id}$	$h \triangleright U = iU, h \triangleright V = W, h \triangleright W = V^*,$
$B6_\theta$	$\mathbb{Z}_6$	$h, h^6 = \mathrm{id}$	$h \triangleright U = e^{\frac{1}{3}\pi i}U, h \triangleright V = W, h \triangleright W = e^{-\pi i\theta}V^*W,$
$N1_\theta$	$\mathbb{Z}_2$	$h, h^2 = \mathrm{id}$	$h \triangleright U = U^*, h \triangleright V = -V, h \triangleright W = W,$
$N2_\theta$	$\mathbb{Z}_2$	$h, h^2 = \mathrm{id}$	$h \triangleright U = U^*, h \triangleright V = -V, h \triangleright W = WU^*,$

Definition 1.1 (*See [3]*)**.** Let $\mathcal{A}(\mathbb{T}^3_\theta)$ be an algebra of smooth elements on a three-dimensional noncommutative torus, which contains the polynomial algebra generated by three unitaries U, V, W satisfying relations,

$$UV = VU, \qquad UW = WU, \qquad WV = e^{2\pi i\theta}VW,$$

where $0 < \theta < 1$ is irrational. We define the algebras of noncommutative Bieberbach manifolds as the fixed point algebras of the following actions of finite groups G on $\mathcal{A}(\mathbb{T}^3_\theta)$, which are combined in Table 1.

We have shown in [3] that the actions of the cyclic groups $\mathbb{Z}_N, N = 2, 3, 4, 6$ on the noncommutative three-torus, as given in Table 1is free. The aim of this paper is to study and classify flat (i.e. restricted from flat geometries of the torus $\mathcal{A}(\mathbb{T}^3_\theta)$) real spectral geometries over the orientable noncommutative Bieberbachs.

2. Spectral triples over Bieberbachs

Since each noncommutative Bieberbach algebra is a subalgebra of the noncommutative torus, a restriction of the spectral triple over the latter to the subalgebra, gives a generic spectral triple over a noncommutative Bieberbach manifold, which might be, however, reducible. By restriction of a spectral triple $(\mathcal{A}, \pi, \mathcal{H}, D, J)$ to a subalgebra, $\mathcal{B} \subset \mathcal{A}$, we understand the triple $(\mathcal{B}, \pi, \mathcal{H}', D', J')$ where π' is the restriction of π to $\mathcal{B}$, $\mathcal{H}' \subset \mathcal{H}$ is a subspace invariant under the action of $\mathcal{B}$, D and J, so that D', J' are their restrictions to $\mathcal{H}'$ (note that in the even case this must apply also to γ).

In what follows we shall show that, in fact, each spectral triple over Bieberbach is a restriction of a spectral triple over the torus, first showing that each spectral triple over Bieberbach can be lifted to a noncommutative torus.

2.1. The lift and the restriction of spectral triples

Lemma 2.1. *Let* $(\mathrm{BN}_\Theta, \mathcal{H}, J, D)$ *be a real spectral triple over a noncommutative Bieberbach manifold* BN_θ*. Then, there exists a spectral triple over a three-torus, such that this triple is its restriction.*

Proof. In [3] we showed that the crossed product algebra $\mathcal{A}(\mathbb{T}^3_\theta) \rtimes \mathbb{Z}_N$ is isomorphic to the matrix algebra of the noncommutative Bieberbach manifold algebra:

$$\mathcal{A}(\mathbb{T}^3_\theta) \rtimes \mathbb{Z}_N \sim \mathrm{BN}_\theta \otimes M_N(\mathbb{C}).$$

First, let us recall that any spectral triple $\mathcal{A}, \mathcal{H}, D, J$ could be lifted to a spectral triple over $\mathcal{A} \otimes M_n(\mathbb{C})$. Indeed, if we take $\mathcal{H}' = \mathcal{H} \otimes M_n(\mathbb{C})$ with the natural representation $\pi'(a \otimes m)(h \otimes M) = \pi(a)h \otimes mM$, the diagonal Dirac operator and $J'(h \otimes M) = Jh \otimes UM^\dagger U^\dagger$, for an arbitrary unitary $U \in M_n(\mathbb{C})$ it is easy to see that we obtain again a real spectral triple. Applying this to the case of BN_θ, and identifying $\mathrm{BN}_\theta \otimes M_N(\mathbb{C})$ using the above isomorphism we obtain a spectral triple over $\mathcal{A}(\mathbb{T}^3_\theta) \rtimes \mathbb{Z}_N$. As $\mathcal{A}(\mathbb{T}^3_\theta)$ is a subalgebra of $\mathcal{A}(\mathbb{T}^3_\theta) \rtimes \mathbb{Z}_N$ by restriction we obtain, in turn, a spectral triple over a three-dimensional noncommutative torus. In fact, it is easy to see that we obtain a spectral triple, which is equivariant with respect to the action of a $\mathbb{Z}_N$ group. Clearly, the fact that we have a representation of the crossed product algebra is just a rephrasing of the fact that we have a $\mathbb{Z}_N$-equivariant representation of $\mathcal{A}(\mathbb{T}^3_\theta)$. By definition, the Dirac operator lifted from the spectral triple over BN_θ commutes with the group elements which are identified as matrices in $M_N(\mathbb{C})$. A little care is required to show that the lift of J would properly commute with the generator of $\mathbb{Z}_N$. However, since the lift of J involves a matrix U, it is sufficient to use a matrix, which provides a unitary equivalence between the generator h and its inverse h^{-1} of $\mathbb{Z}_N$ in $M_N(\mathbb{C})$. Simple computation shows that the following $U, U_{00} = 1, U_{kl} = \delta_{k,N-l}$ for $k, l = 1, \ldots, N-1$ is the one providing the equivalence.

Hence, by this construction, we obtain a real, $\mathbb{Z}_N$-equivariant spectral triple over $\mathcal{A}(\mathbb{T}^3_\theta)$. It is easy to see that the original spectral triple over BN_θ is a restriction of the constructed spectral triple by taking the invariant subalgebra of $\mathcal{A}(\mathbb{T}^3_\theta)$, the $\mathbb{Z}_N$-invariant subspace of $\mathcal{H}$ and the restriction of D and J. □

Remark 2.2. The procedure described above does not necessarily provide the *canonical* (equivariant) Dirac operator over $\mathcal{A}(\mathbb{T}^3_\theta)$. Indeed, even a simple example of $\mathcal{A}(\mathbb{T}^1)$ shows that the lifted Dirac operator differs from the fully equivariant one by a bounded term. There may exist, however, a fully equivariant triple, so that its restriction is the same triple we started with.

Definition 2.3. We call the geometry (spectral triple) over the noncommutative Bieberbach manifold BN_θ *flat* if it is a restriction of a *flat* spectral triple over the noncommutative three-torus, that is, the latter being equivariant with respect to the full action of $U(1) \times U(1) \times U(1)$.

2.2. Equivariant real spectral triples over $C(T^3_\theta)$

Let us take one of the eight canonical equivariant spectral triples over the noncommutative torus $\mathcal{A}(\mathbb{T}^3_\theta)$ (for a definition of equivariance see [4], for classification of equivariant real spectral triples over a noncommutative two-torus see [5], for a generalization to higher dimensions see [6]).

Let us recall, that the Hilbert space is spanned by two copies of $l^2(\mathbb{Z}^3)$, each of them with basis e_μ, where μ is a three-index and each μ_1, μ_2, μ_3 is either integer or half-integer depending on the choice of the spin structure. We parameterize spin structures by ϵ_i $i = 1, 2, 3$, which can take values 0 or $\frac{1}{2}$, so that $\mu_i - \epsilon_i$ is always integer [6].

For $\mathbf{k} = [k_1, k_2, k_3] \in \mathbb{Z}^3$ let us define the generic homogeneous element of the algebra of polynomials over the noncommutative torus:

$$x^{\mathbf{k}} = e^{\pi i \theta k_2 k_3} U^{k_1} V^{k_2} W^{k_3}.$$

We fix the representation of the algebra of the noncommutative torus (relevant for the construction of noncommutative Bieberbach manifolds) on $l^2(\mathbb{Z}^3)$ to be: as follows:

$$\pi(x^{\mathbf{k}}) e_\mu = e^{\pi i \theta (k_3 \mu_2 - \mu_3 k_2)} e_{\mu + k}, \tag{1}$$

and on the Hilbert space $\mathcal{H}$ we take it diagonal:

$$\pi(x) = \begin{pmatrix} \pi(x) & 0 \\ 0 & \pi(x) \end{pmatrix}.$$

The real structure J is of the form:

$$J = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix},$$

where

$$J_0 e_\mu = e_{-\mu}, \quad \forall \mu \in \mathbb{Z}^3 + \epsilon.$$

The most general equivariant and real Dirac operator (up to rescaling) is of the form:

$$D = \begin{pmatrix} R\delta_1 & \delta_2 + \tau\delta_3 \\ \delta_2 + \tau^* \delta_3 & -R\delta_1 \end{pmatrix}, \tag{2}$$

where R is a real parameter and τ a complex parameter (parameterizing the conformal structure of the underlying noncommutative two-torus).

The derivations δ_i, $i = 1, 2, 3$ act diagonally on the $l^2(\mathbb{Z}^3)$:

$$\delta_i e_\mu = \mu_i e_\mu.$$

Theorem 2.4 (*See [4–6]*)**.** *The spectral triple, given by* $(\mathcal{A}_\theta, \pi, J, D, \mathcal{H})$ *is an* $U(1)^3$*-equivariant, irreducible, real spectral triple.*

Note that the choice of J and the Dirac operator (2) has still some unnecessary freedom. Indeed, changing J to $-J$ does not influence any of the axioms. Combining that with a conjugation by Pauli matrices σ^1 or σ^2 we see that we might restrict ourselves to the case $R > 0$.

We shall discuss later the difference between the choice of τ or τ^*.

2.3. The equivariant action of $\mathbb{Z}_N$

Our aim in this part will be to construct all equivariant representations of the finite groups $G = \mathbb{Z}_N$, $N = 2, 3, 4, 6$ on the Hilbert space of the spectral triple for the noncommutative torus $\mathcal{A}(\mathbb{T}^3_\theta)$, which implement the group action on the algebra. The conditions for the representation ρ of G are:

- action:

$$\rho(g)\pi(a) = \pi(g \triangleright a)\rho(g), \quad \forall g \in G, a \in \mathcal{A}(\mathbb{T}^3_\theta),$$

- D-equivariance:

$$D\rho(g) = \rho(g)D, \quad \forall g \in G,$$

- J-equivariance:

$$J\rho(g) = \rho(g)J, \quad \forall g \in G.$$

We begin with two auxiliary lemmas.

Lemma 2.5. *Any equivariant representation of $\mathbb{Z}_N$ on the Hilbert space of the spectral triple must be diagonal:*

$$\rho(g) = \begin{pmatrix} \rho_+(g) & 0 \\ 0 & \rho_-(g) \end{pmatrix}$$

and commute with $\delta_1 \otimes 1$.

Proof. First of all, observe that from the definition of equivariance the element $\sigma^3 = \frac{1}{R}U^*[D, U]$ commutes with $\rho(g)$ for any $g \in G$:

$$\rho(g)\left(U^*[D, U]\right) = (g \triangleright U^*)[D, (g \triangleright U)]\rho(g) = U^*[D, U]\rho(g).$$

This happens since the action of g on U is by multiplication by a root of 1. Therefore the action of $\mathbb{Z}_N$ is diagonal and we can treat the two copies independently.

Moreover, $\sigma^3 D + D\sigma^3$ also commutes with ρ, and, since the latter is proportional to $\delta_1 \otimes 1$, we obtain the second statement. □

Further, we have:

Lemma 2.6. *Let ρ_1, ρ_2 be a representation of a finite group G on the Hilbert space $\mathcal{H}$ such that both implement its action on $\mathcal{A}(\mathbb{T}^3_\theta)$, are J-equivariant and D-equivariant. Then $\rho_1(g) = T(g)\rho_2(g)$, for every $g \in G$, where $T(g) = \zeta \cdot 1 \in M_2(\mathbb{C})$.*

Proof. From the fact that both representations implement the action we see that $T(g) = \rho_1(g)\rho_2(g^{-1})$ commutes with the representation of the algebra $\mathcal{A}(\mathbb{T}^3_\theta)$. Since if $[D, T(g)] = 0$, it commutes with $V^*[D, V]$ and $W^*[D, W]$ and therefore with any matrix in $M_2(\mathbb{C})$. Using similar arguments as in Lemma 2.5 we see that it must therefore commute with δ_2 and δ_3. Hence, $T(g)$ is an $u(1)^3$ invariant element, which commutes with D, J and the representation of the algebra. On the other hand, we know that the spectral triple from Theorem 2.4 is equivariant-irreducible, so $T(g)$ must be proportional to the identity operator. □

The strategy of finding all equivariant representations will be based on finding one, which is J-equivariant and then, with the help of the above lemma, finding all possible representations, which are J-equivariant and commute with D at the same time.

2.4. Spin structures and equivariant actions of $\mathbb{Z}_N$

In this section we shall determine which of the spin structures over the noncommutative torus may admit an equivariant action of $\mathbb{Z}_N$ for the chosen values of N. This will be a first step only, as later we shall proceed with further restrictions. Since the arguments and computations are basically identical in each case, we combine them into a single section.

We denote by h the generator of the group $\mathbb{Z}_N$, $h^N = \mathrm{id}$. As it follows from Lemma 2.5 the representation ρ of $\mathbb{Z}_N$ is diagonal:

$$\rho(h)e_{\mu_1,\mu_2,\mu_3,\pm} = \rho_\pm(h)e_{\mu_1,\mu_2,\mu_3},$$

with $\rho_\pm$ being operators on $l^2(\mathbb{Z}^3)$.

Additionally, due to J-equivariance, they satisfy:

$$J_0\rho_+(h) = \rho_-(h)J_0. \tag{3}$$

We shall introduce convenient notation for the action of the group $\mathbb{Z}_N$ on the generators (and the basis) of the algebra $\mathcal{A}(\mathbb{T}^3_\theta)$, which uses the shorthand presentation (1):

$$h \triangleright x^{\mathbf{k}} = e^{2\pi i \frac{k_1}{N}} x^{(k_1, A(k_2,k_3))}, \tag{4}$$

where $k \in \mathbb{Z}^3$ and $A \in M_2(\mathbb{Z})$ is the following matrix for each of $N = 2, 3, 4, 6$, respectively:

2	3	4	6
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

Theorem 2.7. *The action of $\mathbb{Z}_N$ on the Hilbert space of the equivariant spectral triple over $\mathcal{A}(\mathbb{T}^3_\theta)$, which implements the action (4) on the algebra and which commutes with J and D is possible only for some of the spin structures for the torus ϵ_2, ϵ_3,*

- *For $N = 2$: no restrictions on ϵ_2, ϵ_3,*
- *For $N = 3, 6 : \epsilon_2 = \epsilon_3 = 0$,*
- *For $N = 4 : \epsilon_2 = \epsilon_3$.*

Remark 2.8. Before we begin with the proof, observe that the above theorem lists *some* of the necessary restrictions but not necessarily *all* of them. We shall see that in some cases ($N = 2$, in particular) there are some additional constraints on ϵ_1 (which are not stated in the above theorem).

Before we start with the proof, we shall need lemmas, which restrict the parameters of the Dirac and the action of the discrete group.

Lemma 2.9. *The only D-equivariant and J-equivariant action of $\mathbb{Z}_N$ on $\mathcal{H}$, which implements the action of $\mathbb{Z}_N$ on the algebra (Table 1) is possible if:*

N	3	4	6
τ	$e^{\pm\frac{1}{3}\pi i}$	$e^{\pm\frac{1}{2}\pi i}$	$e^{\pm\frac{1}{3}\pi i}$
ϵ_2	0	ϵ	0
ϵ_3	0	ϵ	0

with no restrictions on τ in the case $N = 2$.
For every such action, there exists $\zeta \in \mathbb{C}$ such that $\rho_-(h) = \zeta\rho_+(h)$:

N	2	3	4	6
τ	–	$e^{\pm\frac{1}{3}\pi i}$	$e^{\pm\frac{1}{2}\pi i}$	$e^{\pm\frac{1}{3}\pi i}$
ζ	-1	$-\tau$	τ^*	τ^*

Proof. Consider $\omega_V = \pi(V^*)[D, \pi(V)]$ and $\omega_W = \pi(W^*)[D, \pi(W)]$:

$$\omega_V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \omega_W = \begin{pmatrix} 0 & \tau \\ \tau^* & 0 \end{pmatrix}.$$

Using the fact that ρ implements the action of $\mathbb{Z}_N$ on the algebra we have:

$$\begin{bmatrix} \rho(h) & 0 \\ 0 & \rho(h) \end{bmatrix} \begin{bmatrix} \omega_V \\ \omega_W \end{bmatrix} = A^T \begin{bmatrix} \omega_V \\ \omega_W \end{bmatrix} \begin{bmatrix} \rho(h) & 0 \\ 0 & \rho(h) \end{bmatrix}. \tag{5}$$

Rewriting the above identities we get, in each of the cases:

$$\begin{aligned} &N = 2 : \ \rho_+(h) = -\rho_-(h), \qquad \tau\rho_+(h) = -\rho_-(h)\tau, \\ &N = 3 : \ \rho_+(h) = (\tau - 1)\rho_-(h), \qquad \tau\rho_+(h) = -\rho_-(h), \\ &N = 4 : \ \rho_+(h) = \tau\rho_-(h), \qquad \tau\rho_+(h) = -\rho_-(h), \\ &N = 6 : \ \rho_+(h) = \tau\rho_-(h), \qquad \tau\rho_+(h) = (\tau - 1)\rho_-(h). \end{aligned} \tag{6}$$

which are self-consistent only if

$$\begin{aligned} &N = 3, 6 : \ \tau = e^{\pm\frac{1}{3}\pi i}, \\ &N = 4 : \ \tau = e^{\pm\frac{1}{2}\pi i}. \end{aligned} \tag{7}$$

There are no restrictions for τ in the case $N = 2$. □

Next, consider the vector $e_{\boldsymbol{\epsilon}} = e_{\epsilon_1,\epsilon_2,\epsilon_3}$, which is a generating and separating vector for $l^2(\mathbb{Z}^3)$ for a given spin structure determined by $\boldsymbol{\epsilon}$. In order to determine $\rho_\pm(h)$ it is sufficient to see its action on $e_{\boldsymbol{\epsilon}}$.

Lemma 2.10. *There exist elements $\sigma_+, \sigma_- \in \mathcal{A}(\mathbb{T}^3_\theta)$, which are polynomials in the generators, such that:*

$$\rho_+(h)e_{\boldsymbol{\epsilon}} = \sigma_+ e_{\boldsymbol{\epsilon}}, \qquad \rho_-(h)e_{\boldsymbol{\epsilon}} = \sigma_- e_{\boldsymbol{\epsilon}}.$$

Proof. Clearly, since $\rho_\pm(h)$ are proportional to each other it is sufficient to show it for $\rho_+(h)$. As the action of $\mathbb{Z}_N$ commutes with D, it commutes with D^2, which is diagonal, hence $\rho_+(h)$ commutes with the restriction of D^2 to $l^2(\mathbb{Z}^3)$. The vector $e_{\boldsymbol{\epsilon}}$ is an eigenvector of D^2 and the eigenspace of the same eigenvalue is finite dimensional and spanned by the vectors e_{μ_1,μ_2,μ_3}. Since each of such vectors could be obtained by acting with a homogeneous polynomial in the generators of the noncommutative three-torus on $e_{\boldsymbol{\epsilon}}$ any vector in this subspace is of the form $\sigma_+ e_{\boldsymbol{\epsilon}}$, where $\sigma_+ \in \mathcal{A}(\mathbb{T}^3_\theta)$ is a polynomial in the generators. □

Proof of Theorem 2.7. We know that the representation of the discrete group $\mathbb{Z}_N$ on the generating vectors e_ϵ is by the action of algebra elements $\sigma_\pm$, proportional to each other. We shall consider now the action of the commutator $[D, \rho(h)]$ on $e_{\epsilon,\pm}$. Since we have already established that $\rho(h)$ commutes with δ_1, we have:

$$(\zeta(\delta_2 + \tau\delta_3)\sigma_+ - \sigma_+(\delta_2 + \tau\delta_3))\, e_\epsilon = 0,$$
$$\left((\delta_2 + \tau^*\delta_3)\sigma_+ - \zeta\sigma_+(\delta_2 + \tau^*\delta_3)\right) e_\epsilon = 0,$$

where we have already taken into account that $\sigma_- = \zeta\sigma_+$. If we call $\partial = \delta_2 + \tau\delta_3$, these equations could be rewritten as:

$$(\zeta\partial(\sigma_+) + (\zeta - 1)\sigma_+\partial)\, e_\epsilon = 0,$$
$$\left(\partial^*(\sigma_+) + (1 - \zeta)\sigma_+(\partial^*)\right) e_\epsilon = 0,$$

and further, taking into account that $\partial e_\epsilon = (\epsilon_2 + \tau\epsilon_3)e_\epsilon$:

$$(\zeta\partial(\sigma_+) + (\zeta - 1)\sigma_+(\epsilon_2 + \tau\epsilon_3))\, e_\epsilon = 0,$$
$$\left(\partial^*(\sigma_+) + (1 - \zeta)\sigma_+(\epsilon_2 + \tau^*\epsilon_3)\right) e_\epsilon = 0.$$

The elements acting on e_ϵ are polynomials from $\mathcal{A}(\mathbb{T}^3_\theta)$, so, since e_ϵ is a separating vector they must vanish identically, therefore we obtain the following equations:

$$\zeta\,\partial(\sigma_+) = (1 - \zeta)(\epsilon_2 + \tau\epsilon_3)\sigma_+,$$
$$\partial^*(\sigma_+) = (\zeta - 1)(\epsilon_2 + \tau^*\epsilon_3)\sigma_+,$$

and we need to look for possible solutions in the polynomial algebra generated by U, V, W depending on the values of ϵ_2, ϵ_3 and ζ, with the latter fixed in Lemma 2.9.

First of all, observe that in the case $N = 2$, $1 - \zeta = 2$ and there are no restrictions on ϵ_2 and ϵ_3 and σ_+ proportional to $V^{-2\epsilon_2}W^{-2\epsilon_3}$ is the solution.

In any other case $N = 3, 4, 6$, $\epsilon_2 = \epsilon_3 = 0$ admits a solution $\sigma_+ = 1$. If $\epsilon_2 = 0$ and $\epsilon_3 = \frac{1}{2}$ we would need $\frac{1}{2}\zeta^{-1}(1 - \zeta)\tau$ to be in $\mathbb{Z} + \tau\mathbb{Z}$, as these are the eigenvalues of ∂. Taking the values of ζ and τ as established in Lemma 2.9 we see that this is not possible. Similarly we find that no solutions are possible in the case $\epsilon_2 = \frac{1}{2}$ and $\epsilon_3 = 0$. If $\epsilon_2 = \epsilon_3 = \frac{1}{2}$ we need to compute a formula for the eigenvalue of ∂ to σ_+ as an eigenvector:

$$\zeta^{-1}(1 - \zeta)(\epsilon_2 + \tau\epsilon_3) = \frac{1}{2}(1 - \zeta)(1 + \tau)\zeta^{-1},$$

which, for $N = 3, 4, 6$, is:

$$N = 3: \ \frac{1}{2}(1 - \zeta)(1 + \tau)\zeta^{-1} = 1 + \frac{1}{2},$$
$$N = 4: \ \frac{1}{2}(1 - \zeta)(1 + \tau)\zeta^{-1} = -1,$$
$$N = 6: \ \frac{1}{2}(1 - \zeta)(1 + \tau)\zeta^{-1} = \left(-\frac{3}{2} + i\frac{\sqrt{3}}{4}\right)$$

and it is easy to check that only in the case $N = 4$ can this be an eigenvalue of ∂, as the latter must be in $\mathbb{Z} + \tau\mathbb{Z}$. For $N = 3$ we have $\frac{3}{2}$, which is not in $\mathbb{Z}$; for $N = 6$, the imaginary part is $\frac{\sqrt{3}}{4}$, which is not an integer multiple of the imaginary part of τ either.

Therefore, apart from the case $N = 2$, where there are solutions for any ϵ_2, ϵ_3, and the case $\epsilon_2 = \epsilon_3 = 0$ for $N = 3, 4, 6$, the only other nontrivial solution is for $N = 4$ and $\epsilon_2 = \epsilon_3 = \frac{1}{2}$ and it is $\sigma_+ = V^*$. □

Let us summarize the results of this section: we have established that the desired actions of $\mathbb{Z}_N$ exist only for some of the spin structures over the noncommutative three-torus and, if they exist, they are unique up to multiplication by a scalar. This will allow us to classify, case by case, all possible actions and hence all possible restrictions of the spectral triple over a noncommutative torus to the fixed point subalgebra of the noncommutative Bieberbachs.

2.5. All equivariant actions of $\mathbb{Z}_N$

Here we restrict ourselves to the actions only for the spin structures determined in the previous section and we proceed case by case. However, we begin with a general lemma:

Lemma 2.11. *For a given spin structure on $\mathcal{A}(\mathbb{T}^3_\theta)$ the following is a most general J-equivariant and diagonal action of $\mathbb{Z}_N$ on $l^2(\mathbb{Z}^3)$, which implements the action on the algebra from Table 1:*

$$\rho_\pm(h)e_{\mu_1,\mu_2,\mu_3,\pm} = \beta_\pm e^{\frac{2\pi i}{N}(\mu_1 \pm \epsilon_1)} e_{\mu_1, A(\mu_2,\mu_3),\pm}. \tag{8}$$

where A is the matrix defined in Section 2.4, and $\beta_\pm$ satisfy:

$$(\beta_\pm)^N = 1, \qquad \beta_+ = (\beta_-)^*.$$

Proof. First we demonstrate that it implements the action, using (1) and (4), using the fact that the bilinear functional $\phi(\mathbf{x}, \mathbf{y}) = x_3y_2 - y_3x_2$ is invariant under A, that is $\phi(\mathbf{x}, \mathbf{y}) = \phi(A\mathbf{x}, A\mathbf{y})$, where $x, y \in \mathbb{R}$. We have:

$$\begin{aligned}&\left(\rho_\pm(h)\pi(x^{\mathbf{k}}) - \pi(h \triangleright x^{\mathbf{k}})\rho_\pm(h)\right) e_{\mu_1,\mu_2,\mu_3,\pm}\\ &= \beta_\pm \left(e^{\frac{2\pi i}{N}(\mu_1 \pm \epsilon_1 + k_1)} - e^{2\pi i \frac{k_1}{N}} e^{\frac{2\pi i}{N}(\mu_1 \pm \epsilon_1)}\right) e^{\pi i \theta(k_3\mu_2 - \mu_3 k_2)} e_{\mu_1+k_1, A(\mu_2+k_2, \mu_3+k_3),\pm}\\ &= 0.\end{aligned}$$

Furthermore, for the J-equivariance we have:

$$\begin{aligned}\rho(h)Je_{\mu_1,\mu_2,\mu_3,\pm} &= \pm\beta_\mp e^{\frac{-2\pi i}{N}(\mu_1 \pm \epsilon_1)} e_{-\mu_1, -A(\mu_2,\mu_3),\mp},\\ J\rho(h)e_{\mu_1,\mu_2,\mu_3,\pm} &= \pm\left(\beta_\pm e^{\frac{2\pi i}{N}(\mu_1 \pm \epsilon_1)}\right)^* e_{-\mu_1, A(-\mu_2,-\mu_3),\mp}\end{aligned}$$

and that enforces the relation: $\beta_+ = (\beta_-)^*$. □

Next, let us check the invariance of D with respect to the above action of $\mathbb{Z}_N$.

Lemma 2.12. *In addition to the constraints 2.9, the action (8) commutes with the Dirac operator only if the following conditions are satisfied:*

- $N = 2$: $\epsilon_1 = \frac{1}{2}$, $\beta_+ = \sigma$,
- $N = 3$: $\tau = -e^{\frac{2}{3}\pi i\sigma}$, $\beta_+ = -e^{\frac{4}{3}\pi i\epsilon_1}\tau = e^{\frac{2}{3}\pi i(2\epsilon_1+\sigma)}$,
- $N = 4$: $\epsilon_1 = \frac{1}{2}$ *and* $\tau = e^{\frac{\sigma}{2}\pi i}$, $\beta_+ = \kappa e^{\frac{1}{4}\pi i(\sigma-1)}$,
- $N = 6$: $\epsilon_1 = \frac{1}{2}$ *and* $\tau = e^{\frac{\sigma}{3}\pi i}$, $\beta_+ = \kappa e^{\frac{1}{6}\pi i(\sigma-1)}$.

In all cases $\sigma = \pm 1$, $\kappa = \pm 1$ are parameters.

Proof. First of all, observe that since the action of h is diagonal on the Hilbert space, we can consider separately the diagonal and the off-diagonal parts of D, denoted D_d and D_o, respectively. For the diagonal part we have:

$$[\rho(h), D_d]e_{\mu_1,\mu_2,\mu_3,\pm} = e^{\frac{2\pi i}{N}(\mu_1 \pm \epsilon_1)}\left(\pm\beta_\pm R\mu_1 \mp \beta_\pm R\mu_1\right) e_{\mu_1, A(\mu_2,\mu_3),\pm} = 0$$

whereas for the off-diagonal part:

$$[\rho(h), D_o]e_{\mu_1,\mu_2,\mu_3,+} = e^{\frac{2\pi i}{N}\mu_1}\left(e^{-\frac{2\pi i}{N}\epsilon_1}\beta_-(\mu_2 + \tau^*\mu_3) - e^{\frac{2\pi i}{N}\epsilon_1}\beta_+((A\boldsymbol{\mu})_2 + \tau^*(A\boldsymbol{\mu})_3)\right) e_{\mu_1, A(\mu_2,\mu_3),-}.$$

To see the restrictions we need to require that the expression in the large parentheses vanishes for all μ_1, μ_2, μ_3. Note that the action on the other part of the Hilbert space (spanned by $e_{\mu_1,\mu_2,\mu_3,-}$) will lead to equivalent conditions. Next we proceed case by case.

$N = 2$: The condition is:

$$e^{-\pi i\epsilon_1}\beta_-(\mu_2 + \tau^*\mu_3) - e^{\pi i\epsilon_1}\beta_+((-\mu_2) + \tau^*(-\mu_3)) = 0,$$

and it is satisfied if and only if $\beta_- = -\beta_+e^{2\pi i\epsilon_1}$. Taking into account that $\beta_\pm$ must be real ($\beta_\pm^2 = 1$) and therefore equal to each other we see that this is possible only if $\epsilon_1 = \frac{1}{2}$, and there are no further restrictions on β_+.

$N = 3$: Here we have:

$$e^{-\frac{2}{3}\pi i\epsilon_1}\beta_-(\mu_2 + \tau^*\mu_3) - e^{\frac{2}{3}\pi i\epsilon_1}\beta_+((-\mu_2 - \mu_3) + \tau^*(\mu_2)) = 0.$$

Using $\beta_- = (\beta_+)^*$ this implies:

$$\begin{aligned}&e^{\frac{4}{3}\pi i\epsilon_1}\beta_+(-1 + \tau^*) = (\beta_+)^*,\\ &-e^{\frac{4}{3}\pi i\epsilon_1}\beta_+ = (\beta_+)^*\tau^*.\end{aligned}$$

The nontrivial solution, which exists due to the restriction (7) and satisfies $(\beta_+)^3 = 1$ and $\beta_+(\beta_+)^* = 1$ is:

$$\tau = e^{-\frac{1}{3}\pi i\sigma}, \qquad \beta_+ = -e^{\frac{4}{3}\pi i\epsilon_1}\tau = e^{\frac{2}{3}\pi i(2\epsilon_1+\sigma)},$$

where $\sigma = \pm 1$.

We see that the coefficients $\beta_\pm$ are in fact fixed by the choice of the spin structure ϵ_1. The potential additional freedom comes from the choice of the Dirac on $\mathcal{A}(\mathbb{T}^3_\theta)$ and can be seen in the choice of the admissible parameters τ.

$N = 4$: In this case we obtain for any μ_2, μ_3:

$$e^{\pi i\epsilon_1}\beta_+\left(-(\mu_3) + \tau^*\mu_2\right) = \beta_-(\mu_2 + \tau^*\mu_3).$$

Using $\beta_+ = (\beta_-)^*$ this implies:

$$e^{\pi i\epsilon_1}\beta_+ = -\tau^*(\beta_+)^*,$$

and then, moving β_+ to one side we obtain:

$$(\beta_+)^2 = -\tau^* e^{\pi i\epsilon_1},$$

Since $(\beta_\pm)^4 = 1$ then $(\beta_+)^2 = \pm 1$ but because $\tau = \sigma i$, $\sigma = \pm 1$, the equality is possible only if $\epsilon_1 = \frac{1}{2}$.

In the end we have:

$$\tau = e^{\frac{\sigma}{2}\pi i}, \qquad \beta_+ = \pm e^{\frac{1}{4}\pi i(\sigma-1)}.$$

$N = 6$: This can only be equal if for any μ_2, μ_3:

$$e^{\frac{2}{3}\pi i\epsilon_1}\beta_+\left(-\mu_3 + \tau^*(\mu_2 + \mu_3)\right) = \beta_-(\mu_2 + \tau^*\mu_3).$$

Using $\beta_+ = (\beta_-)^*$ this implies:

$$e^{\frac{2}{3}\pi i\epsilon_1}\beta_+(-1 + \tau^*) = (\beta_+)^*\tau^*,$$

$$e^{\frac{2}{3}\pi i\epsilon_1}\beta_+\tau^* = (\beta_+)^*.$$

This can be solved:

$$\tau = e^{\frac{\sigma}{3}\pi i}, \qquad \beta_+ = \pm e^{\frac{1}{6}\pi i(\sigma-2\epsilon_1)}, \quad \sigma = \pm 1.$$

However, using further $(\beta_\pm)^6 = 1$ we obtain:

$$\sigma - 2\epsilon_1 \in 2\mathbb{Z},$$

which enforces $\epsilon_1 = \frac{1}{2}$ and then:

$$\beta_+ = \pm e^{\frac{1}{6}\pi i(\sigma-1)}. \quad \square$$

3. Real flat spectral triples

In this section we classify all real spectral triples over Bieberbach manifolds, which arise from flat real spectral triples over the noncommutative torus. Let us recall once again that *flat* means full equivariance with respect to the action of the full isometry group of the noncommutative torus $\mathcal{A}(\mathbb{T}^3)_\theta$, which is $U(1)^3$.

To simplify the notation we shall need the notion of a generalized Dirac-type operator on the circle, $D_{\alpha,\beta}$, which is an operator with the eigenvalues:

$$\lambda_k = \alpha(k + \beta),$$

where $\alpha, \beta \in \mathbb{R}$, $-1 < \beta < 1$ and $k \in \mathbb{Z}$.

The η invariant of the operator $D_{\alpha,\beta}$ (see [7, Lemma 5.5]) is:

$$\eta(D_{\alpha,\beta}) = \mathrm{sgn}(\beta) - 2\beta.$$

If we have an operator with the same spectrum, however, with a multiplicity $M > 1$, then the η invariant is M-multiple of the computed value.

We shall denote the spectra as *multisets*, which are a pair (A, m), where A is (in our case) a discrete subset of $\mathbb{C}$ and $m : A \to \mathbb{Z}_+$ is a positive, integer-valued function. A multiset is, in fact, a convenient way to present a (discrete) spectrum of an operator with the multiplicities of the eigenvalues taken into account. All standard operations on sets extend in a natural way to multisets, allowing us to write, for instance $K(A, m)$, to be the multiset $(A, K\,m)$ for any positive integer K.

So, in particular, the spectrum of $D_{\alpha,\beta}$ shall be denoted (as a multiset) by $\mathcal{S}^1_{\alpha,\beta}$:

$$\mathcal{S}^1_{\alpha,\beta} = \left(\{\alpha(k + \beta),\ k \in \mathbb{Z}, \alpha, \beta \in \mathbb{R}\}, 1\right),$$

since every eigenvalue has multiplicity 1.

Using an analogous notation, we shall denote the spectrum of the chosen Dirac operator for a given spin structure over $\mathcal{A}(\mathbb{T}^3_\theta)$ by $\mathcal{S}^3_{\epsilon_1,\epsilon_2,\epsilon_3}(\tau)$, which corresponds to the set of eigenvalues (with multiplicities):

$$\left\{\pm\sqrt{R^2\mu_1^2 + |\mu_2 + \tau\mu_3|^2},\ \mu_i + \epsilon_i \in \mathbb{Z}, i = 1, 2, 3\right\}.$$

Observe that $2\mathcal{S}^1_{R,\epsilon_1}$ can be seen as a subset (in the multiset sense) of $\mathcal{S}^3_{\epsilon_1,0,0}(\tau)$, to see it, take $\mu_2 = \mu_3 = 0$.

Since both multisets ($2\mathcal{S}^1_{R,\epsilon_1}$ and $\mathcal{S}^3_{\epsilon_1,0,0}(\tau)$) are symmetric with respect to the change of sign (each element λ is mapped to λ) their η invariants (and hence the η invariant of their union or difference) must vanish.

Using this notation of multisets will allow us to describe the spectrum of Bieberbach in terms of the spectra of the Dirac (which depends on τ) on the noncommutative three-torus and the spectrum of the Dirac-type operator on the circle with the help of set operations. This will facilitate the analysis of the asymmetry of the spectra.

3.1. $N = 2$

Let us recall that:

$$\rho(h)e_{\mu_1,\mu_2,\mu_3,\pm} = \sigma(-1)^{\mu_1\pm\frac{1}{2}}e_{\mu_1,-\mu_2,-\mu_3,\pm}.$$

We have eight possibilities of real spectral triples. First, the choice of the spin structures over the noncommutative torus given by ϵ_2 and ϵ_3, and then the choice of the sign of $\beta_+ = \pm 1$.

We need to distinguish two cases.

3.1.1. $\epsilon_2 = \frac{1}{2}$ *or* $\epsilon_3 = \frac{1}{2}$

Fixing $\beta_+ = \sigma = \pm 1$ we have the invariant subspace of the Hilbert space of the spectral triple over the noncommutative torus spanned by the following vectors:

$$\frac{1}{\sqrt{2}}\left(e_{2k\mp\frac{1}{2}+j,\mu_2,\mu_3,\pm} + (-1)^j\sigma e_{2k\mp\frac{1}{2}+j,-\mu_2,-\mu_3,\pm}\right), \tag{9}$$

for $j = 0, 1$, with $k \in \mathbb{Z}$, $\mu_i \in \mathbb{Z} + \epsilon_i$, $i = 2, 3$. The spectrum of the Dirac operator, when restricted to that spaces consists of:

$$\lambda = \pm\sqrt{R^2\left(2k \mp \frac{1}{2} + j\right)^2 + |\mu_2 + \tau\mu_3|^2}. \tag{10}$$

As one can see, there is no asymmetry in the spectrum, hence the η invariant vanishes. In fact the spectrum of this Dirac is just the same as the spectrum of the Dirac on the noncommutative torus, with the multiplicities halved, so it is:

$$\frac{1}{2}\mathcal{S}^3_{\frac{1}{2},\epsilon_2,\epsilon_3}(\tau).$$

3.1.2. $\epsilon_2 = 0$ *and* $\epsilon_3 = 0$

Clearly, for $\mu_2 \neq 0$ or $\mu_3 \neq 0$ the vectors (9) are still the invariant vectors and the spectrum of the Dirac restricted to that subspace, is still given by (10). This part of the spectrum is, however, not the entire spectrum of the Dirac but only a part of it, namely:

$$\frac{1}{2}\left(\mathcal{S}^3_{\frac{1}{2},0,0}(\tau) \setminus 2\mathcal{S}^1_{R,\frac{1}{2}}\right),$$

which means that we are taking out the points of the spectrum with $\mu_2 = \mu_3 = 0$. Note that the factor 2 is needed to count correctly multiplicities.

However, in the case $\mu_2 = \mu_3 = 0$ there are the following invariant subspaces:

$$e_{2k\mp\frac{1}{2}\sigma,0,0,\pm}, \tag{11}$$

and the spectrum of the Dirac operator, restricted to that subspace (call it D_σ) consists of the following numbers:

$$\lambda = \pm R\left(2k \mp \frac{1}{2}\sigma\right), \tag{12}$$

for $k \in \mathbb{Z}$. It is easy to see that these spectra are:

$$\lambda_- = R\left(2k + \frac{1}{2}\right), \qquad \sigma = -1, \quad k \in \mathbb{Z},$$

with the set of all such numbers is $\mathcal{S}^1_{2R,\frac{1}{4}}$, and

$$\lambda_+ = R\left(2k - \frac{1}{2}\right), \qquad \sigma = 1, \quad k \in \mathbb{Z},$$

with the set of all of them $\mathcal{S}^1_{2R,\frac{3}{4}}$.

In each case the multiplicity of the eigenvalues for this part of the spectrum is 2 and the spectra give different η invariant for different values of σ:

$$\eta(D_\sigma) = -\sigma\,.$$

Expressed as a multiset, the spectrum is:

$$\frac{1}{2}\left(\mathcal{S}^3_{\frac{1}{2},0,0}(\tau) \setminus 2\mathcal{S}^1_{R,\frac{1}{2}}\right) \cup 2\mathcal{S}^1_{2R,\pm\frac{1}{4}},$$

3.2. $N = 3$

All possible representations are:

$$\rho(h)e_{\mu_1,\mu_2,\mu_3,\pm} = (-1)^{2\epsilon_1} e^{\frac{2}{3}\pi i(\mu_1 \pm \sigma)} e_{\mu_1,-\mu_2-\mu_3,\mu_2,\pm}.$$

Here we have $\epsilon_2 = \epsilon_3 = 0$. For μ_2, $\mu_3 \neq 0$ the invariant vectors are:

$$\frac{1}{\sqrt{3}}\left(e_{3k+3\epsilon_1\mp\sigma+j,\mu_2,\mu_3,\pm} + e^{\frac{2}{3}j\pi i} e_{3k+3\epsilon_1\mp\sigma+j,-\mu_2-\mu_3,\mu_2,\pm} + e^{\frac{4}{3}j\pi i} e_{3k+3\epsilon_1\mp\sigma+j,\mu_3,-\mu_2-\mu_3,\pm}\right) \tag{13}$$

for $j = 0, 1, 2$.

The eigenvalues of D on this subspace are:

$$\lambda = \pm\sqrt{R^2(3k + 3\epsilon_1 \mp \sigma + j)^2 + (\mu_2)^2 + (\mu_3)^2 + \mu_2\mu_3}, \tag{14}$$

and the corresponding multiset is (independently of σ)

$$\frac{1}{3}\left(\mathcal{S}^3_{\epsilon_1,0,0}\left(e^{\frac{1}{3}\pi i}\right) \setminus 2\mathcal{S}^1_{R,\epsilon_1}\right).$$

In the case with $\mu_2 = \mu_3 = 0$ we have the following invariant eigenvectors:

$$e_{3k+3\epsilon_1\mp\sigma,0,0,\pm}, \tag{15}$$

so that they are eigenvectors of D to the eigenvalues:

$$\pm R(3k + 3\epsilon_1 \mp \sigma), \quad k \in \mathbb{Z}. \tag{16}$$

and the spectrum of the Dirac operator, restricted to that subspace is the set:

$$2\mathcal{S}^1_{3R,\epsilon_1-\frac{\sigma}{3}}\,.$$

The η invariant is, in each of the four possible cases:

$$\begin{aligned}
&\sigma = +1, \qquad \epsilon_1 = \frac{1}{2} :\ \eta = \frac{4}{3},\\
&\sigma = +1, \qquad \epsilon_1 = 0 :\ \eta = -\frac{2}{3},\\
&\sigma = -1, \qquad \epsilon_1 = \frac{1}{2} :\ \eta = -\frac{4}{3},\\
&\sigma = -1, \qquad \epsilon_1 = 0 :\ \eta = \frac{2}{3}.
\end{aligned}$$

It is no surprise that some of the η invariants differ by sign as, in fact, the change $\tau \to \tau^*$ corresponds to the change $D \to -D$ on the subspace considered and gives, in fact, the same geometry.

Therefore we have in the end two distinct spin structures for each choice of the Dirac, each projected out of a different spin structure from $\mathcal{A}(\mathbb{T}^3_\theta)$, which are distinguishable by the η invariants of the Dirac operators. The case $\sigma = 1$ is the situation discussed in [7]. The spectrum, as a multiset is:

$$\frac{1}{3}\left(\mathcal{S}^3_{\epsilon_1,0,0}\left(e^{\frac{1}{3}\pi i}\right) \setminus 2\mathcal{S}^1_{R,\epsilon_1}\right) \cup 2\mathcal{S}^1_{3R,\epsilon_1-\frac{\sigma}{3}}.$$

3.3. $N = 4$

We begin with the recall of the representation:

$$\rho(h)e_{\mu_1,\mu_2,\mu_3,\pm} = \kappa e^{\frac{\pi i}{2}(\mu_1 \pm \frac{\sigma}{2})} e_{\mu_1,-\mu_3,\mu_2,\pm},$$

where $\sigma = \pm 1$ was defined earlier and $\kappa = \pm 1$.

Here we have $\epsilon_1 = \frac{1}{2}$ and a very similar situation to that of $N = 2$. We have two cases:

3.3.1. $\epsilon_2 = \epsilon_3 = \frac{1}{2}$

The invariant subspace of the Hilbert space of the spectral triple over the noncommutative torus is spanned by the following vectors:

$$\frac{1}{2}\sum_{p=0}^{3}\left(\kappa^j e^{\frac{j}{2}\pi i}\rho^p(h)\left(e_{4k\mp\frac{\sigma}{2}+j,\mu_2,\mu_3,\pm}\right)\right), \tag{17}$$

for $k \in \mathbb{Z}$ and $j = 0, 1, 2, 3$, where $\sigma = \pm 1$, $\kappa = \pm 1$.

The spectrum is symmetric, with the eigenvalues:

$$\lambda = \pm\sqrt{R^2\left(4k \mp \frac{1}{2}\sigma + j\right)^2 + |\mu_2|^2 + |\mu_3|^2}, \quad k \in \mathbb{Z}, j = 0, 1, 2, 3. \tag{18}$$

and it is clear to see that it is, in fact, the original spectrum of the Dirac on the $\mathcal{A}(\mathbb{T}^3_\theta)$, with $\frac{1}{4}$ of its multiplicities:

$$\frac{1}{4}\mathcal{S}^3_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\left(e^{\frac{1}{2}\pi i}\right).$$

3.3.2. $\epsilon_2 = \epsilon_3 = 0$

Again, similarly as in the $n = 2$ case if $\mu_2 \neq 0$ (note here this enforces $\mu_3 \neq 0$) we have the same part of the spectrum, on the subspace spanned by the same vectors (17), which are still the invariant vectors for $\mu_2 \neq 0$.

Again, this is only a part of the spectrum of the Dirac on $\mathcal{A}(\mathbb{T}^3_\theta)$, namely:

$$\frac{1}{4}\left(\mathcal{S}^3_{\frac{1}{2},0,0}\left(e^{\frac{1}{2}\pi i}\right) \setminus 2\mathcal{S}^1_{R,\frac{1}{2}}\right),$$

which means that we are not counting the spectrum for eigenvectors with $\mu_2 = \mu_3 = 0$ and the remaining part of the spectrum comes with the multiplicity 1 instead of 4.

In the latter case $\mu_2 = \mu_3 = 0$, direct computation shows that the following vectors span the invariant subspaces:

$$e_{4k\mp\frac{1}{2}\sigma-1+\kappa,0,0,\pm}, \tag{19}$$

and the spectrum of the Dirac operator, restricted to that subspace, contains all the following eigenvalues (with multiplicity 2):

$$\lambda = R\left(4k - \frac{1}{2}\sigma + 1 - \kappa\right), \quad k \in \mathbb{Z}. \tag{20}$$

We have, in all possible cases $\sigma = \pm 1$, $\kappa = \pm 1$ the spectrum (as a multiset)

$$2\mathcal{S}^1_{4R,\frac{1}{8}(2-2\kappa-\sigma)},$$

which gives the following η invariants:

$$\begin{aligned}
&\sigma = +1, \quad \kappa = +1: \ \eta = -\frac{3}{2},\\
&\sigma = +1, \quad \kappa = -1: \ \eta = \frac{1}{2},\\
&\sigma = -1, \quad \kappa = +1: \ \eta = \frac{3}{2},\\
&\sigma = -1, \quad \kappa = -1: \ \eta = -\frac{1}{2}.
\end{aligned}$$

Again, the cases $\sigma = +1$ and $\sigma = -1$ are related by the map $D \to -D$, while the case presented in [7] corresponds to $\sigma = -1$. As a multiset, the spectrum in this case is:

$$\frac{1}{4}\left(\mathcal{S}^3_{\frac{1}{2},0,0}\left(e^{\frac{1}{2}\pi i}\right) \setminus 2\mathcal{S}^1_{R,\frac{1}{2}}\right) \cup 2\mathcal{S}^1_{4R,\frac{1}{8}(2-2\kappa-\sigma)}.$$

3.3.3. $N = 6$

We recall the representation:

$$\rho(h)e_{\mu_1,\mu_2,\mu_3,\pm} = \kappa e^{\frac{\pi i}{3}(\mu_1 \pm \frac{\sigma}{2})} e_{\mu_1,-\mu_3,\mu_2+\mu_3,\pm},$$

where $\sigma = \pm 1$, $\kappa = \pm 1$.

Here, one repeats most of the arguments from the $N = 3$ case. First of all, there exists a part of the invariant subspace of the Hilbert space where the spectrum of D is symmetric and is:

$$\frac{1}{6}\left(\mathcal{S}^3_{\epsilon_1,0,0} \setminus 2\mathcal{S}^1_{R,\epsilon_1}\right).$$

We do not write an explicit expression for the vectors spanning this subspace, the formula is analogous to the ones derived earlier for $N = 3$.

Similarly, there exists additional invariant vectors:

$$e_{6k\mp\frac{1}{2}\sigma-\frac{3}{2}+\frac{3}{2}\kappa,0,0\pm}, \quad k \in \mathbb{Z},$$

where $\sigma = \pm 1$ and $\kappa = \pm 1$.

The eigenvalues are:

$$\lambda = \pm R\left(6k - \frac{1}{2}\sigma + 3\frac{1-\kappa}{2}\right), \quad k \in \mathbb{Z} \tag{21}$$

and this gives the spectrum $2\mathcal{S}^1_{6R,\,\frac{1-\kappa}{4}-\frac{\sigma}{12}}$.

We have, in all possible cases $\sigma = \pm 1$, $\kappa = \pm 1$ the spectra give the following η invariants:

$$\begin{aligned}
&\sigma = +1, \qquad \kappa = +1: \ \eta = -\frac{5}{3},\\
&\sigma = +1, \qquad \kappa = -1: \ \eta = \frac{1}{3},\\
&\sigma = -1, \qquad \kappa = +1: \ \eta = \frac{5}{3},\\
&\sigma = -1, \qquad \kappa = -1: \ \eta = -\frac{1}{3}.
\end{aligned}$$

Again, we see that the case presented in [7] is the one $\sigma = -1$, and, as a multiset the spectrum is:

$$\frac{1}{6}\left(\mathcal{S}^3_{\epsilon_1,0,0}\left(e^{\frac{1}{3}\pi i}\right) \setminus 2\mathcal{S}^1_{R,\epsilon_1}\right) \cup 2\mathcal{S}^1_{6R,\,\frac{1-\kappa}{4}-\frac{\sigma}{12}}.$$

To summarize the results of this section we collect all the information in Table 2, where the parameterization of the spin structures for each of the studied noncommutative Bieberbach manifolds is given, together with the parameterization of the spectra of flat Dirac operators. The multiplicity of the eigenvalues is there implicit in the chosen parameterization; however, using the multiset notation (which is used in this section) it can be easily compared to the multiplicity of eigenvalues in the spectra of the flat Dirac operator on the noncommutative three-torus.

3.4. Irreducibility

The notion of irreducibility of real spectral triples is a delicate one and one can distinguish at least three classical notions of irreducibility:

- *Full irreducibility*: there is no proper subspace of the Hilbert space, invariant under the algebra, D and J.
- *J irreducibility*: there is no proper subspace of the Hilbert space, invariant under the algebra and J.
- *D irreducibility*: there is no proper subspace of the Hilbert space, invariant under the algebra and D.

Observe that the canonical equivariant spectral triple for the noncommutative torus $\mathcal{A}(\mathbb{T}^3_\theta)$ is irreducible in either way.

It is easy to see that the constructed real spectral triples over Bieberbachs are irreducible in each of the above ways. Indeed, assuming the contrary (in either of the above cases) and identifying the Hilbert space of the constructed triples with a subspace of the triple for the noncommutative torus, one would be able to construct a subspace of that Hilbert space, which would be invariant by the full algebra of the torus and D and/or J, respectively. We briefly sketch here the method: if $\mathcal{H}_r \subset \mathcal{H}_B$ is such a subspace reducing the triple over Bieberbachs identified as a subspace of the Hilbert space for the torus triple, one just needs to take $\oplus_{k=0}^{N-1} U^k \mathcal{H}_r$. This will be a proper subspace of the Hilbert space of the spectral triple over a noncommutative torus, as the spaces $U^k \mathcal{H}_B$, where they are disjoint for each $0 \leq k < N$.

Let us remark, however, that the triples constructed earlier are all the only ones *flat* and irreducible (in all three ways). The assumption of flatness was necessary, as only in such a case there exists a classification of real spectral triples

Table 2
Spectra of flat Dirac operators over BN_θ.

BN_θ	Spin structure	Spectrum of D	Parameters
$B2_\theta$	$\epsilon_2 = \epsilon_3 = 0$ $\sigma = \pm 1$,	$\pm\sqrt{R^2\mu_1^2 + \lvert\mu_2 + \tau\mu_3\rvert^2}$ $\pm R(2k \mp \frac{1}{2}\sigma)$	$\mu_1 \in \mathbb{Z} + \frac{1}{2}$, $\mu_2, \mu_3 \in \mathbb{Z}$, $\mu_2 > 0$, $k \in \mathbb{Z}$
	$\epsilon_2 = \frac{1}{2}$ $\sigma = \pm 1$	$\pm\sqrt{R^2\mu_1^2 + \lvert\mu_2 + \tau\mu_3\rvert^2}$	$\mu_1 \in \mathbb{Z} + \frac{1}{2}$, $\mu_2, \mu_3 \in \mathbb{Z} + \frac{1}{2}$, $\mu_2 > 0$.
	$\epsilon_2 = 0, \epsilon_3 = \frac{1}{2}$ $\sigma = \pm 1$	$\pm\sqrt{R^2\mu_1^2 + \lvert\mu_2 + \tau\mu_3\rvert^2}$	$\mu_1 \in \mathbb{Z} + \frac{1}{2}$, $\mu_i \in \mathbb{Z} + \frac{1}{2}$, $\mu_3 > 0$.
$B3_\theta$	$\epsilon_1 = 0, \frac{1}{2}$	$\pm\sqrt{R^2\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_2\mu_3}$	$\mu_1 \in \mathbb{Z} + \epsilon_1$ $(\mu_2, \mu_3) \in (\mathbb{Z}^2 \setminus (0,0))/\mathcal{R}$
		$\pm R(3k + 3\epsilon_1 \mp 1)$	$(x, y) \overset{\mathcal{R}}{\sim} (-x - y, x)$, $k \in \mathbb{Z}$
$B4_\theta$	$\epsilon_2 = \epsilon_3 = \frac{1}{2}$ $\kappa = \pm 1$	$\pm\sqrt{R^2\mu_1^2 + \mu_2^2 + \mu_3^2}$	$\mu_1 \in \mathbb{Z} + \frac{1}{2}$ $\mu_2, \mu_3 \in \mathbb{Z} + \frac{1}{2}$, $\mu_2, \mu_3 > 0$
	$\epsilon_2 = \epsilon_3 = 0$ $\kappa = \pm 1$	$\pm\sqrt{R^2\mu_1^2 + \mu_2^2 + \mu_3^2}$ $\pm R(4k + \frac{1}{2} + 1 - \kappa)$	$\mu_1 \in \mathbb{Z} + \frac{1}{2}$ $\mu_2, \mu_3 \in \mathbb{Z}$, $\mu_2, \mu_3 > 0$, $k \in \mathbb{Z}$
$B6_\theta$	$\kappa = \pm 1$	$\pm\sqrt{R^2\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_2\mu_3}$	$\mu_1 \in \mathbb{Z} + \epsilon_1$ $(\mu_2, \mu_3) \in (\mathbb{Z}^2 \setminus (0,0))/\mathcal{R}$
		$\pm R(6k + \frac{1}{2} + 3\frac{1-\kappa}{2})$	$(x, y) \overset{\mathcal{R}}{\sim} (-y, x + y)$, $k \in \mathbb{Z}$

over a noncommutative torus. Although one can easily construct triples, which are reductions of the flat triple over a noncommutative torus (basically by creating a direct sum of restrictions to $U^k\mathcal{H}_B$ within the Hilbert space of the triple over $\mathcal{A}(\mathbb{T}^3_\theta)$) they will necessarily be the reducible ones in the sense of D-reducibility. For this reason they are not described in the paper.

4. Conclusions

The examples we have provided are interesting for geometry and noncommutative geometry for several reasons. First of all, noncommutative Bieberbach manifolds, though only mildly noncommutative, are not examples of θ-deformations, since classical Bieberbach manifolds do not admit a free action of the torus. Therefore we have a genuine new class of noncommutative examples. Furthermore, our computations of real spectral triples attempts to answer the natural question about the existence of different spin structures and their explicit construction. Our paper, building on the results from [5,6] shows that for this interesting class of noncommutative geometries, with the additional assumption (which states that we are looking for flat, irreducible geometries) we can provide a complete classification.

The classical limit of presented noncommutative geometries might have a place in the studies of models of flat compact space geometries in the context of cosmology [8,9]. The construction of different inequivalent spectral triples suggest that there might be some differences between particle models built using them, in particular, twisting the obtained geometries with the projective modules, which represent nontrivial torsion in the K-theory of these objects [3].

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