

## Accepted Manuscript

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PII: S0393-0440(17)30227-9

DOI: <https://doi.org/10.1016/j.geomphys.2017.09.008>

Reference: GEOPHY 3072

To appear in: *Journal of Geometry and Physics*

Received date: 5 April 2017

Revised date: 25 May 2017

Accepted date: 25 September 2017

Please cite this article as: F. Besnard, N. Bizi, On the definition of spacetimes in noncommutative geometry: Part I, *Journal of Geometry and Physics* (2017), <https://doi.org/10.1016/j.geomphys.2017.09.008>

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# On the definition of spacetimes in Noncommutative Geometry: Part I

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May 25, 2017

## Abstract

In this paper we consider semi-riemannian time and space oriented manifolds of even dimension, and characterize the Lorentzian and antilorentzian signatures in terms of a time-orientation 1-form and a natural Krein product on spinor fields. It turns out that all the data available in Noncommutative Geometry (the algebra of functions, the Krein space of spinor fields, the representation of the algebra on it, the Dirac operator, charge conjugation and chirality), but nothing more, play a role in this characterization. It thus yields a possible definition extending Connes' notion of even spectral triple to the Lorentzian setting.

KEYWORDS : Noncommutative geometry

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# 1 Introduction

Noncommutative geometry, as initiated by Alain Connes, is an operator algebraic framework which generalizes Riemannian manifolds in a way which harmoniously gathers continuous and discrete spaces, as well as truly noncommutative examples. This generalization happens to be just enough to allow the recasting of the Standard Model of particle physics as a noncommutative Kaluza-Klein theory. This last application has been a strong motivation for the search of a semi-Riemannian extension of Connes’ noncommutative geometry.

There have been several important attempts in that direction already, but this endeavour is plagued with difficulties. Some are analytical: among them we find the noncompactness issue and the definition of the bosonic spectral action (For important progresses on the first problem, see [C-G-R-S 14]. On the fermionic spectral action, see [Dun 15b].) A different kind of problem is the characterization of the physical Lorentzian signature among the general semi-Riemannian ones. This has been investigated in particular in [Bes 15a], [Fra 11]. The present paper exclusively deals with this last issue: we will give a characterization of the Lorentzian signature (in even dimension) using the tools available in Noncommutative Geometry. This will yield a natural notion of “noncommutative spacetime”, to be explored in a companion paper.

The paper is organized as follows: after this introduction, we make a somewhat long excursion into Clifford algebras in section 2. We first give a description of Wick rotations in terms of commuting real structures, then we recall the existence of Krein structures on spinor modules which are compatible with a given real structure, and use them to characterize the Wick rotations to Euclidean signature (*Robinson’s alternative*, proposition 8). Finally our goal is achieved in theorem 1, which characterizes the Lorentzian and antilorentzian signatures by the possibility of using a single vector to turn the Krein structure on a given spinor module into a Hilbert structure. This section also makes the connection between the real structures on the Clifford algebra and the charge conjugation operator which is used in noncommutative geometry, and also contains some extra material on Clifford algebras, e.g. the  $\sigma$ -product, which extends a given metric to the whole algebra.

In section 3 we globalize the notions introduced in section 2 to semi-riemannian manifolds: real structures on Clifford bundles, Krein structures on spinor bundle. We point out that Krein structures compatible with a given real structure exist if and only if the manifold is space or time orientable, according to the signature (theorem 2). We also give a detailed explanation of the way in which the geometric properties of the spin connection are encoded in the algebraic properties of the Dirac operator and charge conjugation. Finally we translate our local characterization of Lorentzian signature from the previous section into global terms (theorem 3), namely in terms of an orientation 1-form. In the conclusion we put forward the definition of *spectral spacetimes*, an antilorentzian extension of spectral triples inspired by theorem 3.

In the whole paper some notions are defined in the bulk of the text. When it happens, the name of the notion is always italicized.

## 2 Local constructions

### 2.1 General definitions and conventions

In this section we will be interested in the complex Clifford algebra generated by a real vector space  $V$  of even dimension  $n = 2k$  equipped with a non-degenerate quadratic form  $Q$  of signature  $(p, q)$ . The bilinear form associated with  $Q$  will be denoted by  $B$ . Our notations and conventions are summarized below:

- As is traditional in physics, what we call the signature of a real bilinear symmetric (or complex sesquilinear hermitian) form is the triple  $(n_+, n_-, n_0)$ , where  $n_0$  is the nullity,  $n_-$  the negative and  $n_+$  the positive indices of inertia. When the form is non-degenerate, we leave out  $n_0 = 0$  and write  $(n_+, n_-)$  instead.
- We give names to the following cases : *Lorentz signature* is  $(n-1, 1)$ , *anti-Lorentz signature* is  $(1, n-1)$ , *Euclidean signature* is  $(n, 0)$  and *neutral signature* is  $(n, n)$ .
- The real Clifford algebra  $Cl(V, Q)$  is defined using the following convention<sup>1</sup>

$$v^2 = +Q(v)$$

for all  $v \in V$ . In later sections, in the context of manifolds, we will put the emphasis on the bilinear form and write  $Cl(V, B)$  instead of  $Cl(V, Q)$ . We let  $\mathbb{C}l(V) = Cl(V, Q) \otimes \mathbb{C}$  be its complexification, and  $V^{\mathbb{C}}$  be the complexification of  $V$ . We consider that  $V \subset Cl(V, Q) \subset \mathbb{C}l(V)$  in the natural way, i.e. we do not write down explicitly the embedding  $i : V \rightarrow \mathbb{C}l(V, Q)$ . We still denote by  $Q$  the natural extension of  $Q$  such that  $v^2 = Q(v)$  holds for all  $v \in V^{\mathbb{C}}$ .

---

<sup>1</sup>We suggest the name “anti-Clifford” for the algebra defined using the convention  $v^2 = -Q(v)$ . We think that making the sign conventions explicit by using the words anti-Clifford and antilorentzian would be greatly beneficial to the mathematical physics community.

- We let  $c : \mathbb{C}l(V) \rightarrow \mathbb{C}l(V)$  be the real structure defined by  $c(a \otimes \lambda) = a \otimes \bar{\lambda}$ .
- We let  $T : \mathbb{C}l(V) \rightarrow \mathbb{C}l(V)$  be the unique linear antiautomorphism which restricts to the identity on  $V$ . It is called *the principal anti-involution*. We often write  $a^T$  instead of  $T(a)$ .
- We note that  $c \circ T = T \circ c$  is the unique antilinear antiautomorphism of  $\mathbb{C}l(V)$  which extends  $\text{Id}_V$ . We will write  $a^\times = c(a^T)$ .
- We let  $\gamma$  be the *principal involution*, that is, the unique automorphism of  $\mathbb{C}l(V)$  which extends  $-\text{Id}_V$ . This a grading operator which decomposes the Clifford algebra into the sum  $\mathbb{C}l(V) = \mathbb{C}l^0(V) \oplus \mathbb{C}l^1(V)$  of its even and odd parts.
- For any  $a$  and invertible  $g$  we write  $L_g(a) = ga$ ,  $R_g(a) = ag$ ,  $Ad_g(a) = gag^{-1}$ . The (complex) Clifford group is defined by  $\Gamma_{\mathbb{C}} = \{g \in \mathbb{C}l(V) \mid Ad_g(V^{\mathbb{C}}) \subset V^{\mathbb{C}}\}$ , the Pin group is  $Pin(Q) = \{g \in \Gamma_{\mathbb{C}} \mid c(g) = g \text{ and } gg^T = \pm 1\}$ , and the Spin group is  $Spin(Q) = Pin(Q) \cap \mathbb{C}l(V, Q)^0$ . Remember that the Clifford group is generated by non-isotropic vectors, and that its elements satisfy  $gg^\times \in \mathbb{R}$ .
- Let  $e_1, \dots, e_n$  be a pseudo-orthonormal basis of  $V$ . Then we denote by  $\omega = e_1 \dots e_n$  the volume element. It depends on the pseudo-orthonormal basis chosen only up to a sign, which we can fix by choosing an orientation of  $V$ . The volume element anticommutes with every odd element of  $\mathbb{C}l(V)$ . It has the following properties :

$$\omega^T = (-1)^{\frac{n}{2}} \omega, \omega^2 = (-1)^{\frac{n}{2}+q}$$

- If  $\rho$  is a spinor representation, we set  $\chi := (-i)^{\frac{n}{2}+q} \rho(\omega)$ . It is called the chirality operator and always satisfies  $\chi^2 = 1$ . When the representation is  $c$ -admissible (to be defined later), it will also satisfy  $\chi^\times = (-1)^q \chi$ .

Our general reference on Clifford algebras is [Cru 90]. For future use we note the following fact : let  $\phi : \mathbb{C}l(V) \rightarrow \mathbb{C}l(V)$  be an automorphism or antiautomorphism which stabilizes  $V^{\mathbb{C}}$ . Then  $\phi \circ T$  and  $T \circ \phi$  coincide on  $V^{\mathbb{C}}$ , hence on the whole algebra. Thus  $T$  and  $\phi$  commute.

## 2.2 Real structures and local Wick rotations

What we wish to do in this section is to understand Wick rotations algebraically. We are given at the start the vector space  $V$  equipped with a quadratic form  $Q$ . The piece of data  $(V, Q)$  is equivalent to  $\mathbb{C}l(V, Q)$ . We stress that the Clifford algebra is not to be seen only as a real algebra, but as a real algebra equipped with a particular set of generators, namely  $V$ . Since we want to change the signature of the quadratic form, it is natural to embed  $\mathbb{C}l(V, Q)$  in its complexification  $\mathbb{C}l(V)$  which will remain constant when  $Q$  is varied. Recovering  $Q$  from  $\mathbb{C}l(V)$  amounts to fix a particular real form for the complex algebra

$\mathbb{C}l(V)$ , by way of a *real structure*, i.e. an involutive antilinear automorphism which stabilizes  $V^{\mathbb{C}}$ . The real structure which has  $\mathbb{C}l(V, Q)$  as its set of fixed points will be denoted by  $c$  throughout the text. Clearly, the data  $(V, Q)$  and  $(\mathbb{C}l(V), c)$  are equivalent. Given a general real structure  $\sigma$  we define:

- The real subspace of  $\sigma$ -real vectors  $V_{\sigma} := \{v \in V^{\mathbb{C}} | \sigma(v) = v\}$ .
- The map  $u_{\sigma} : v \mapsto \frac{v + \sigma(v)}{2} + i \frac{v - \sigma(v)}{2}$ , which is easily seen to be an isomorphism of real vector spaces from  $V$  onto  $V_{\sigma}$ .
- The bilinear form  $B_{\sigma}(v, v') := \frac{1}{2}(B(\sigma(v), v') + B(\sigma(v'), v))$  on  $V$ , and its associated quadratic form  $Q_{\sigma}(v) := B(\sigma(v), v)$ .

Since  $\sigma(v)$  does not in general belong to  $V$  it is not obvious at first sight that  $B_{\sigma}(v, v')$  is real. However we can observe that  $B_{\sigma}(v, v') = \frac{1}{2}(\sigma(v)v' + v'\sigma(v) + \sigma(v')v + v\sigma(v'))$  and on this form it is clear that  $\sigma(B_{\sigma}(v, v')) = B_{\sigma}(v, v')$ , hence this complex number is in fact real.

We also note that the restriction of the quadratic form  $Q$  to  $V_{\sigma}$  is real, since  $\sigma(w^2) = \sigma(w)^2 = w^2$  for every  $w \in V_{\sigma}$ . Furthermore, since  $V_{\sigma} + iV_{\sigma} = V^{\mathbb{C}}$ , the complex algebra generated by  $V_{\sigma}$  is  $\mathbb{C}l(V)$ , which we can then identify with  $\mathbb{C}l(V_{\sigma}, Q|_{V_{\sigma}}) \otimes \mathbb{C}$ . In particular we note that  $Q|_{V_{\sigma}}$  is non-degenerate.

Finally the calculation  $Q(u_{\sigma}(v)) = u_{\sigma}(v)^2 = \frac{1}{2}(v\sigma(v) + \sigma(v)v) = Q_{\sigma}(v)$  shows that  $u_{\sigma}$  is an isometry from  $(V, Q_{\sigma})$  to  $(V_{\sigma}, Q)$ .

Hence we have defined from  $\sigma$  a non-degenerate quadratic form  $Q_{\sigma}$  on  $V$  and a real subspace  $V_{\sigma}$  of  $V^{\mathbb{C}}$  such that  $V_{\sigma} \oplus iV_{\sigma} = V^{\mathbb{C}}$  and  $Q|_{V_{\sigma}}$  is real.

This construction can be inverted. Consider a  $n$ -dimensional real vector space  $W$  of  $V^{\mathbb{C}}$  such that  $Q$  is real on  $W$ . Then  $W \oplus iW = V^{\mathbb{C}}$  (indeed, if there exists a nonzero  $w \in W$  such that  $iw \in W$  then  $Q(w + iw)$  is not real). Define  $\sigma : V^{\mathbb{C}} \rightarrow V^{\mathbb{C}}$  to be antilinear and satisfy  $\sigma|_W = \text{Id}_W$ . It is easy to check that  $\sigma(v_1 v_2 + v_2 v_1) = \sigma(v_1)\sigma(v_2) + \sigma(v_2)\sigma(v_1)$ , hence  $\sigma$  extends as an antilinear algebra automorphism.

Hence we have the following one-to-one correspondence:

$$\{\text{real structures on } \mathbb{C}l(V)\} \simeq \{n\text{-dimensional real subspaces } W \subset V^{\mathbb{C}} \text{ such that } Q|_W \text{ is real}\}$$

Since  $c$  is a fixed “background” structure, it is natural to be particularly interested in real structures which commute with it, and we will call them *admissible real structures*. They turn out to correspond to Wick rotations of the quadratic form  $Q$ .

**Lemma 1** *The following are equivalent.*

1. *The real structure  $\sigma$  is admissible.*
2. *The subspace  $V_{\sigma}$  is stable by  $c$ .*

3. The subspace  $V$  is stable by  $\sigma$ .
4. The real structure  $\sigma$  restricts to a  $Q$ -orthogonal symmetry of  $V$ .
5. The subspaces  $V_+ := V \cap V_\sigma$  and  $V_- := V \cap iV_\sigma$  form a  $Q$ -orthogonal decomposition of  $V$ .

**Proof:** It is immediate that (1) $\Rightarrow$ (3), and for the converse it suffices to observe that  $\sigma \circ c$  and  $c \circ \sigma$  are two algebra automorphisms of  $\mathbb{C}l(V)$  which coincide on  $V$ . The equivalence between (1) and (2) is obtained by symmetry.

Of course (4) $\Rightarrow$ (3) is trivial, and to see that (3) $\Rightarrow$ (4) we observe that  $Q(\sigma(v)) = \sigma(v)^2 = \sigma(v^2) = \sigma(Q(v)) = Q(v)$ , hence  $\sigma$  restricts to a  $Q$ -orthogonal transformation of  $V$  which is moreover involutive.

Finally we see that  $V_+$  is the  $+1$ -eigenspace and  $V_-$  is the  $-1$ -eigenspace of the  $\mathbb{R}$ -linear operator  $\sigma|_V$ , hence (5) and (4) are equivalent.  $\blacksquare$

Let us consider an admissible real structure  $\sigma$ . Using the decomposition  $V = V_+ \oplus V_-$  given by point 5 above, we see that if  $v = v_+ + v_-$ , with  $v_\pm \in V_\pm$ , then the isometry

$$u_\sigma : (V, Q_\sigma) \longrightarrow (V_\sigma, Q|_{V_\sigma})$$

takes the simple form

$$v \longmapsto v_+ + iv_-$$

Hence  $u_\sigma$  “puts an  $i$ ” in front of the elements of  $V_-$ : this is what is called a *Wick rotation*. Clearly  $Q_\sigma$  is positive definite iff  $Q$  is positive definite on  $V_+$  and negative definite on  $V_-$ . In this latter case we say that  $u$  is a rotation “to Euclidean signature”.

As the proposition below shows, admissible real structures can be expressed in terms of particular elements of the Clifford group.

**Proposition 1** 1. The real structures are of the form  $\sigma = Ad_b \circ c$  with  $b \in \Gamma_{\mathbb{C}}$  such that  $bc(b) = \lambda \in \mathbb{R}$ . We also have  $b^T = \alpha b$ ,  $\alpha = \pm 1$ ,  $b^\times = \alpha c(b)$ .

2. The real structure  $\sigma = Ad_b \circ c$  with  $b \in \Gamma_{\mathbb{C}}$  is admissible iff  $c(b) = e^{i\theta}b$ , with  $\theta \in \mathbb{R}$ . In this case we can choose  $b$  to satisfy  $c(b) = b$  and  $b^2 = \lambda = \pm 1$ , in which case we say that it is real and normalized. Then  $b^\times = \alpha b$ ,  $b$  belongs to  $Pin(Q)$ , and is unique up to a sign.

**Proof:** It is obvious that  $\sigma$  is an antilinear automorphism which preserves  $V^{\mathbb{C}}$  when it is of the form  $Ad_b \circ c$  with  $b$  in the Clifford group. If  $bc(b) = \lambda \in \mathbb{R}$  it is moreover an involution since  $\sigma^2(a) = bc(b)ac(b)^{-1}b^{-1} = a$  for all  $a \in \mathbb{C}l(V)$ .

Conversely, if  $\sigma$  is a real structure, then  $\sigma \circ c$  is an automorphism of the Clifford algebra which preserves  $V^{\mathbb{C}}$ , and it is then of the form  $Ad_b$  with  $b \in \Gamma_{\mathbb{C}}$ . Hence  $\sigma = Ad_b \circ c$ . Since  $\sigma$  is an involution we have  $bc(b) = \lambda \in \mathbb{C}$  by the calculation above. Since  $c(b)$  is then equal to  $b^{-1}$  up to a constant, it commutes with  $b$ , from which we obtain that  $\lambda$  is real.

The other properties follow from the ones just proved: since  $b \in \Gamma_{\mathbb{C}}$ ,  $bb^{\times}$  is a constant, hence  $c(b)b^T$  is a constant, and from  $c(b)b = \lambda \in \mathbb{R}$  we get that  $b$  is proportional to  $b^T$ . The involutory property of  $T$  forces the proportionality constant to be a sign  $\alpha$ . Then  $b^{\times} = c(b^T) = \alpha c(b)$ .

Now it is easy to check that  $\sigma$  commutes with  $c$  iff  $c(b) = e^{i\theta}b$  for some  $\theta \in \mathbb{R}$ . Then  $b' = \frac{e^{i\theta/2}}{\sqrt{|\lambda|}}b$  is normalized and real and one has  $Ad_b = Ad_{b'}$ . If  $b''$  is another normalized and real element such that  $Ad_{b''} = Ad_b$  then  $b'' = \mu b'$  with  $\mu \in \mathbb{C}$ . From reality one has  $\mu \in \mathbb{R}$ , and from normalization one has  $\mu^2 = 1$ .

Since  $b'$  satisfies  $b'(b')^{\times} = \pm 1$  and is real it is in the Pin group.  $\blacksquare$

At this point one might like to have an example of a non-admissible real structure. For this, consider  $\mathbb{R}^2$  with an Euclidean metric and  $(e_1, e_2)$  an orthonormal basis. Then for  $t \in \mathbb{R}$ , let  $b_t$  be the Clifford group element  $b_t = \cosh t + i(\sinh t)e_1e_2$ . Then  $c(b_t) = b_{-t} = b_t^{-1}$ . Hence  $b_t$  satisfies the hypotheses of the first part of proposition 1, but not the second (except if  $t = 0$ ). If we denote by  $V$  the vector of components of  $v \in V^{\mathbb{C}}$  in the chosen basis, then the real structure  $\sigma_t = Ad_{b_t} \circ c$  is given matricially by  $V \mapsto O_t \bar{V}$  where  $O_t$  is the matrix

$$O_t = \begin{pmatrix} \cosh 2t & i \sinh 2t \\ -i \sinh 2t & \cosh 2t \end{pmatrix}$$

One then easily shows that the metric  $B_{\sigma_t}$  associated to the real structure  $\sigma_t = Ad_{b_t} \circ c$  is  $\cosh 2t$  times the original metric. More generally, one can show in (even) dimension  $n$  and in the Euclidean case that there always exists an orthonormal basis of  $V$  in which a general real structure is given by a matrix which is a direct sum of  $I_p$ ,  $-I_q$ , and  $2 \times 2$  blocks  $\pm O_{t_k}$  with  $O_{t_k}$  as above.

### 2.3 Hermitian forms on the complex Clifford algebra

Wick rotations to Euclidean signature are very special. This can be best seen by extending the quadratic form  $Q_{\sigma}$  to the whole Clifford algebra. The automorphism  $\sigma \circ T$  will then appear naturally as an adjunction, and its exceptional character when  $Q_{\sigma}$  is positive definite will be made manifest.

Let  $(e_i)_{1 \leq i \leq n}$  be a pseudo-orthonormal basis of  $V^{\mathbb{C}}$ . For any subset  $I \subset \{1, \dots, n\}$  we write  $e_I = e_{i_1} \dots e_{i_k}$  where  $i_1 < \dots < i_k$  are the elements of  $I$ . We know that  $(e_I)_{I \subset \{1, \dots, n\}}$  is a basis of  $\mathcal{Cl}(V)$ . Hence there is a projection map  $\tau_n : \mathcal{Cl}(V) \rightarrow \mathbb{C}$  which sends an element of  $\mathcal{Cl}(V)$  to its coordinate on  $1 = e_{\emptyset}$ . The map  $\tau_n$  is called the *normalized trace*, a name justified by the following proposition (we refer to [Gar 11] p. 100 for the proof).

**Proposition 2** *The projection  $\tau_n$  does not depend on the chosen pseudo-orthonormal basis. It is the unique linear form  $\tau_n : \mathcal{Cl}(V) \rightarrow \mathbb{R}$  such that  $\tau_n(ab) = \tau_n(ba)$  for all  $a, b \in \mathcal{Cl}(V)$  and  $\tau_n(1) = 1$ . It also satisfies  $\tau_n(a^T) = \tau_n(a)$ .*

It is also easy to see that if  $a \in \mathcal{Cl}(V)$ , then  $\tau_n(\gamma(a)) = \tau_n(a)$ , and if  $\sigma$  is a real structure, then  $\tau_n(\sigma(a)) = \overline{\tau_n(a)}$ .

**Remark:** We can see the above proposition as the reason behind the fact that a product of distinct gamma-matrices is always traceless.



Given the normalized trace and a real structure, it is very natural to define

$$(a, b)_\sigma := \tau_n(\sigma(a^T)b) \quad (1)$$

for all  $a, b \in \mathbb{Cl}(V)$ . We call it the  $\sigma$ -product. It has remarkable properties.

**Proposition 3** *The  $\sigma$ -product is a non-degenerate hermitian form on  $\mathbb{Cl}(V)$ . The associated quadratic form restricts to  $Q_\sigma$  on  $V$  and to  $Q$  on  $V_\sigma$ . It satisfies*

$$(w_1 \dots w_k, w_1 \dots w_k)_\sigma = Q(w_1) \dots Q(w_k) \quad (2)$$

for any vectors  $w_1, \dots, w_k \in V_\sigma$ . Moreover if  $(e_i)_{1 \leq i \leq n}$  is a pseudo-orthonormal basis of  $V_\sigma$  for  $Q$  then  $(e_I)_{I \subset \{1, \dots, n\}}$  is a pseudo-orthonormal basis of  $\mathbb{Cl}(V)$  for the  $\sigma$ -product.

**Proof:** It is obvious that  $(\cdot, \cdot)_\sigma$  is sesquilinear. Moreover we have  $(b, a)_\sigma = \tau_n(\sigma(b^T)a) = \tau_n(\sigma(b)^T a) = \tau_n(a^T \sigma(b)) = \overline{\tau_n(\sigma(a^T)b)} = \overline{(a, b)_\sigma}$ .

If  $v \in V$ , then  $(v, v)_\sigma = \tau_n(\sigma(v)v) = \frac{1}{2}\tau_n(\sigma(v)v + v\sigma(v)) = Q_\sigma(v)$ .

Let us prove (2). We have :

$$\begin{aligned} (w_1 \dots w_k, w_1 \dots w_k)_\sigma &= \tau_n(\sigma(w_k^T) \dots \sigma(w_1^T) w_1 \dots w_k) \\ &= \tau_n(w_k \dots w_1 w_1 \dots w_k) \\ &= w_1^2 \dots w_k^2 \\ &= Q(w_1) \dots Q(w_k) \end{aligned}$$

Let  $(e_i)_{1 \leq i \leq n}$  be a pseudo-orthonormal basis of  $V_\sigma$ . Then  $(e_I, e_I)_\sigma = \pm 1$  by property (2). Moreover if  $I \neq J$ ,  $(e_I, e_J)_\sigma = \tau_n(e_I^T e_J) = \pm \tau_n(e_{I \Delta J}) = 0$ . Hence  $(e_I)_{I \subset \{1, \dots, n\}}$  is a pseudo-orthonormal basis of  $\mathbb{Cl}(V)$ , which shows that  $(\cdot, \cdot)_\sigma$  is not degenerate.  $\blacksquare$

Note that the multiplicative property (2) of the  $\sigma$ -product does not generalize to elements of  $V$ . The correct symmetrical statement which apply to elements of  $V$  uses the  $c$ -product instead of the  $\sigma$ -product.

**Remark:** When  $g$  is an element of the Clifford group,  $(g, g)_c = \tau_n(g^\times g) = g^\times g$  because  $g^\times g$  is real. It is then customary to write  $N(g) = g^\times g$  and call it the *spinor norm*. Note also that the  $\sigma$ -product is related with the Killing form of the Lie algebra of the spin group<sup>2</sup>. More precisely, one can show that the Killing form  $K$  of  $so(Q)$ , the Lie algebra of  $\text{Spin}(Q)$  satisfies  $K(x, y) = 8(n-2)(x, y)_c$ , for all elements  $x, y \in so(p, q)$  (see appendix A for the proof).

Here is an elegant (and useful) property of the  $\sigma$ -product.

**Lemma 2** *Let  $\phi : (V, B) \rightarrow (V', B')$  be an isometry between two real vector spaces equipped with nondegenerate bilinear forms. Let  $c, c'$  be the canonical real structures on  $\mathbb{Cl}(V, B)$  and  $\mathbb{Cl}(V', B')$  respectively. Then the isomorphism  $\tilde{\phi} : \mathbb{Cl}(V, B) \rightarrow \mathbb{Cl}(V', B')$  which canonically extends  $\phi$  transforms  $(\cdot, \cdot)_c$  into  $(\cdot, \cdot)_{c'}$ .*

<sup>2</sup>We thank Christian Brouder for suggesting this link.

**Proof:** For any  $a, b \in Cl(V, B)$  we have

$$\begin{aligned} (\tilde{\phi}(a), \tilde{\phi}(b))_{c'} &= \tau'_n(c'(\tilde{\phi}(a)^T)\tilde{\phi}(b)) \\ &= \tau'_n(c'(\tilde{\phi}(a^T))\tilde{\phi}(b)), \text{ since } \phi(V) = V' \\ &= \tau'_n \circ \phi(c(a^T)b) \\ &= \tau_n(c(a^T)b) = (a, b)_c \end{aligned}$$

where the last step follows from the uniqueness of the normalized trace.  $\blacksquare$

**Remark:** Care must be taken in applying this lemma. For instance if we use it on  $u_\sigma$  we obtain  $(\tilde{u}_\sigma(a), \tilde{u}_\sigma(b))_\sigma = \tau_n(c(a^T) *_\sigma b)$ , where  $*_\sigma$  is the Clifford product corresponding to the quadratic form  $Q_\sigma$ . In particular if  $a = b = v$  we obtain  $(u_\sigma(v), u_\sigma(v))_\sigma = \tau_n(v *_\sigma v) = Q_\sigma(v) = Q(u_\sigma(v))$  which is correct.

Taking a  $Q$ -pseudo-orthonormal basis  $(e_i)_{1 \leq i \leq n}$  in  $V$ , then the decomposition

$$\bigoplus_{k=0}^n V^k = Cl(V, Q) \quad (3)$$

where  $V^k = \text{Vect}\{e_I | |I| = k\}$  is immediately seen to be orthogonal for the  $c$ -product. The following proposition shows in particular that this decomposition is independent of the chosen basis.

**Proposition 4** *Let  $\phi : (V, B) \rightarrow (V, B)$  be an isometry. Then  $\tilde{\phi}(V^k) = V^k$  for all  $k$ .*

**Proof:** Clearly  $\tilde{\phi}(V^k) = V^k$  for  $k = 0, 1$ . Let us suppose that  $\tilde{\phi}(V^j) = V^j$  for  $j \leq k$ . The sum  $V^0 \oplus \dots \oplus V^k \oplus V^{k+1}$  is orthogonal for the  $c$ -product, hence by the lemma the sum  $\tilde{\phi}(V^0) \oplus \dots \oplus \tilde{\phi}(V^k) \oplus \tilde{\phi}(V^{k+1}) = V^0 \oplus \dots \oplus V^k \oplus \tilde{\phi}(V^{k+1})$  also is. Since we obviously have  $\tilde{\phi}(V^{k+1}) \subset V^0 \oplus \dots \oplus V^{k+1}$  we obtain that  $\tilde{\phi}(V^{k+1}) = V^{k+1}$ .  $\blacksquare$

Here is another way to understand why the decomposition (3) is independent of the chosen basis. There is a well-known vector space isomorphism  $\Theta$  from the exterior algebra  $\Lambda V$  to the Clifford algebra which is defined by

$$\begin{aligned} \Lambda V &\longrightarrow Cl(V, Q) \\ v_1 \wedge \dots \wedge v_r &\longmapsto \frac{1}{r!} \sum_{\sigma} \epsilon(\sigma) v_{\sigma(1)} \dots v_{\sigma(r)} \end{aligned} \quad (4)$$

Since  $\Theta(e_{i_1} \wedge \dots \wedge e_{i_k}) = e_{i_1} \dots e_{i_k}$  for distinct elements  $e_{i_1}, \dots, e_{i_k}$  of the pseudo-orthonormal basis, we see that  $V^k = \Theta(\Lambda^k V)$ . Moreover, there is a well-known way to extend the bilinear form  $B$  to the exterior algebra, which is to decree that  $\Lambda^j V$  and  $\Lambda^k V$  are orthogonal for  $j \neq k$  and to define

$$(u_1 \wedge \dots \wedge u_k, v_1 \wedge \dots \wedge v_k)_B := \det((B(u_i, v_j))_{1 \leq i, j \leq k})$$

It is left to reader to check that  $\Theta$  is an isometry from  $(\Lambda V, (.,.)_B)$  to  $(Cl(V, Q), (.,.)_c)$ . Of course what we have described using  $c$  can be extended to a general real structure  $\sigma$ . However, when passing to the complexification, the reference to the real structure, or quadratic form, vanishes, so that the decomposition

$$\bigoplus_{k=0}^n V^k \otimes \mathbb{C} = Cl(V) \quad (5)$$

is orthogonal for all  $\sigma$ -products. The vector space  $V^k \otimes \mathbb{C}$  is described in a basis-independent way as  $V^k \otimes \mathbb{C} = \Theta(\Lambda^k V^{\mathbb{C}})$ , where we still write  $\Theta$  the natural extension of the isomorphism (4) to complex scalars.

If  $U : Cl(V) \rightarrow Cl(V)$  is a linear map, we will write  $U^{\times\sigma}$  for its adjoint relative to the  $\sigma$ -product. We will also write

$$a^{\times\sigma} := \sigma(a^T)$$

for an element  $a \in Cl(V)$ . The two notations are consistent thanks to the property

$$L_a^{\times\sigma} = L_{a^{\times\sigma}} \quad (6)$$

which is easily checked. We will now see a simple yet important result which tells us how the signature of  $(.,.)_\sigma$  depends on that of  $Q_\sigma$ . We call it *Garling's alternative*, since the only place where we could locate it is [Gar 11] (p 101), where a direct combinatorial proof is given. We give below a slightly different proof based on the following lemma.

**Lemma 3** *Let  $(K, (.,.))$  be a finite dimensional space equipped with a non-degenerate hermitian form. If there exists  $U \in End(K)$  such that  $U^\times U = -Id_K$ , then  $(.,.)$  is neutral.*

**Proof:** We have  $(U\psi, U\eta) = (\psi, U^\times U\eta) = -(\psi, \eta)$  for all  $\psi, \eta \in K$ . Thus if  $K = K_+ \oplus K_-$  is an orthogonal decomposition of  $K$  into subspaces where  $(.,.)$  is positive definite and negative definite respectively, we see that  $K = UK_+ \oplus UK_-$  is an orthogonal decomposition where the signs are swapped. Since  $\dim(UK_\pm) = \dim K_\pm$  we conclude by Sylvester's law of inertia.  $\blacksquare$

**Proposition 5** (*Garling's alternative*) *The  $\sigma$ -product on  $Cl(V)$  is positive definite whenever  $Q_\sigma$  is. It is neutral in every other case.*

**Proof:** Let  $(e_i)_{1 \leq i \leq n}$  be a pseudo-orthonormal basis of  $V_\sigma$ . Using property (2) of proposition 1 on the basis  $(e_I)$  we see that  $(.,.)_\sigma$  is positive definite if  $Q|_{V_\sigma}$  is, and we know that  $Q|_{V_\sigma}$  and  $Q_\sigma$  have the same signature.

If  $Q_\sigma$  (or equivalently  $Q|_{V_\sigma}$ ) is not positive definite, then there exists  $i$  such that  $e_i^2 = -1$ . Let  $U = L_{e_i}$ . Since  $L_{e_i}^{\times\sigma} = L_{e_i^{\times\sigma}} = L_{e_i}$ , hence  $UU^\times = -Id_{Cl(V)}$  and we conclude by lemma 3.  $\blacksquare$

We can now characterize Wick rotations to Euclidean signature in terms of the corresponding real structure. The notations are the same as in lemma 1.

**Proposition 6** *Let  $\sigma$  be an admissible real structure. The following properties are equivalent.*

1. *The couple  $(V_+, V_-)$  of  $Q$ -orthogonal supplementary subspaces of  $V$  is such that  $Q$  is positive definite on  $V_+$  and negative definite on  $V_-$ .*
2. *The quadratic form  $Q_\sigma$  on  $V$  is positive definite.*
3. *The restriction of the quadratic form  $Q$  to  $V_\sigma$  is positive definite.*
4. *The  $\sigma$ -product  $(\cdot, \cdot)_\sigma$  is positive definite.*

When these properties are satisfied we say that  $\sigma$  is an *Euclidean real structure*. Notice that when  $\sigma = Ad_b \circ c$  is an Euclidean real structure, and  $b$  is normalized, the signs such that  $b^\times = \pm b$  and  $b^2 = \pm 1$  are no longer independent. Indeed,  $b^{\times\sigma} = b^{-1}b^\times b = b^\times$ , hence  $bb^\times$  is a positive operator and can only be equal to 1.

**Remark:** Another specific feature of Euclidean real structures is that the involution  $a \mapsto a^{\times\sigma}$  and the norm  $\|a\|_{\infty, \sigma} = \sup_{\|b\|_\sigma=1} \|ab\|_\sigma$  turn  $\mathbb{C}l(V)$  into a  $C^*$ -algebra.

## 2.4 Krein products on spinor spaces

In the rest of this section we let  $\rho : \mathbb{C}l(V) \rightarrow \text{End}(K)$  be a representation of  $\mathbb{C}l(V)$  on a finite dimensional complex vector space  $K$  equipped with a non-degenerate hermitian form  $(\cdot, \cdot)^K$ . We call such a form a *Krein product*. Remember that a *Krein space* (see [Bog 74]) is a vector space equipped with a non-degenerate hermitian form  $(\cdot, \cdot)$  and at least one operator  $\beta$ , called a *fundamental symmetry* such that  $\beta^2 = 1$  and  $(\cdot, \beta \cdot)$  is positive definite and induces a complete topology. In finite dimension the existence of  $\beta$  is obvious, thus  $K$  equipped with a Krein product is a Krein space.

We will say that  $\rho$  is  $\sigma$ -compatible iff

$$\forall a \in \mathbb{C}l(V), \forall \psi, \phi \in K, (\rho(a)\psi, \phi) = (\psi, \rho(a^{\times\sigma})\phi)$$

This is clearly equivalent to ask  $\rho(v)$  to be Krein-self adjoint for all  $v \in V$ . When the Krein product is  $\sigma$ -compatible we will denote by  $A^{\times\sigma}$  the Krein adjoint of  $A \in \text{End}(K)$ .

We can always build a  $\sigma$ -compatible Krein product on  $K$ . First we can suppose without loss of generality that  $\rho$  is irreducible. Then we can use  $\rho$  to transfer  $\times_\sigma$  from  $\mathbb{C}l(V)$  to an antilinear antiautomorphism  $A \mapsto A^{\times\sigma}$  of  $\text{End}(K)$ . Such an antiautomorphism is necessarily the adjoint operation for some non-degenerate hermitian form  $(\cdot, \cdot)_\sigma^K$ . To see this, use the simplicity of  $\text{End}(K)$  to pick a  $\beta$  such that  $A^{\times\sigma} = \beta A^\dagger \beta^{-1}$  where  $A^\dagger$  is the adjoint for some scalar product  $\langle \cdot, \cdot \rangle$ . Then it is a simple matter to verify that  $(\cdot, \cdot)_\sigma^K := \langle \cdot, \beta^{-1} \cdot \rangle$  has the required property.

Since this procedure is essentially in [Rob 88], we will call it *Robinson's transfert principle*.

**Proposition 7** *There exists a non-degenerate hermitian form  $(\cdot, \cdot)_\sigma^K$  on  $K$  such that  $\rho$  is  $\sigma$ -compatible, and when  $\rho$  is irreducible it is unique up to multiplication by a non-zero real number.*

We haven't proven the uniqueness part yet. It is a simple consequence of Riesz' representation theorem, but we state it here as a separate lemma for future reference. The proof is left to the reader.

**Lemma 4** *Let  $(K, (\cdot, \cdot))$  be a space equipped with a non-degenerate hermitian form, and let  $(\cdot, \cdot)'$  be a hermitian form such that for all  $\psi, \phi \in K$ , and for all  $A \in B(K)$ , one has  $(A\psi, \phi)' = (\psi, A^\times \phi)'$ , where  $A^\times$  is the adjoint of  $A$  for  $(\cdot, \cdot)$ . Then there exists  $\lambda \in \mathbb{R}$  such that  $(\cdot, \cdot)' = \lambda(\cdot, \cdot)$ .*

Every minimal left ideal of  $\mathcal{Cl}(V)$  is an irreducible module for the representation of  $\mathcal{Cl}(V)$  by left multiplication. Such a module is of the form  $S_e := \mathcal{Cl}(V)e$  where  $e$  is a primitive idempotent of  $\mathcal{Cl}(V)$ , that is an idempotent which cannot be decomposed as a sum of two others. The module  $S_e$  is sometimes called an *algebraic spinor module*, and its elements are *algebraic spinors*. It is then natural to restrict  $(\cdot, \cdot)_\sigma$  to  $S_e$  and ask if it defines a  $\sigma$ -compatible Krein product. In fact  $\sigma$ -compatibility is immediate, but the restriction of  $(\cdot, \cdot)_\sigma$  can be degenerate. Of course when  $Q_\sigma$  is positive definite everything works fine :  $(\cdot, \cdot)_\sigma$  is positive definite and its restriction to every algebraic spinor module is a  $\sigma$ -compatible scalar product. We will use this fact in the following proposition, which is implicit in [Rob 88].

**Remark:** The positive definite case is all we really need, but we can wonder what happens in general. It turns out that we just have to compute  $ee^{\times\sigma}$ : either it is zero, in which case  $(\cdot, \cdot)_\sigma$  vanishes on  $S_e$ , or it is not, in which case  $(\cdot, \cdot)_\sigma$  restricts to a  $\sigma$ -compatible Krein product on  $S_e$ . Moreover, in this latter case, there exists a unique primitive idempotent  $f \in \mathcal{Cl}(V)$  such that  $f^{\times\sigma} = f$  and  $S_e = S_f$ .

**Proposition 8** (*Robinson's alternative*) *Let  $K$  be an irreducible spinor module and  $(\cdot, \cdot)^K$  be a  $\sigma$ -compatible Krein spinor product.*

*If  $Q_\sigma$  is positive definite, then  $(\cdot, \cdot)^K$  is definite, if  $Q_\sigma$  is not positive definite, then  $(\cdot, \cdot)^K$  is neutral.*

**Proof:** Let us call  $\rho : \mathcal{Cl}(V) \rightarrow \text{End}(K)$  the representation homomorphism.

If  $Q_\sigma$  is not positive definite, let  $e_i \in V_\sigma$  be such that  $e_i^2 = -1$ , set  $U = \rho(e_i)$  and use lemma 3.

If  $Q_\sigma$  is positive definite, then let  $U : K \rightarrow S_e$ , where  $S_e$  is a minimal left ideal in  $\mathcal{Cl}(V)$ , be a Clifford-module isomorphism, which we know exists by irreducibility of  $K$ . By Garling's alternative,  $(\cdot, \cdot)_\sigma$  is positive definite, hence its restriction to  $S_e$  also is. We now transport  $(\cdot, \cdot)_\sigma$  to  $K$  thanks to  $U$  by the formula  $(\psi, \eta)_U := (U\psi, U\eta)_\sigma$ , for all  $\psi, \eta \in K$ . Let us check that  $(\cdot, \cdot)_U$  is  $\sigma$ -compatible:

$$\begin{aligned}
 \forall a \in \mathbb{C}l(V), (\rho(a)\psi, \eta)_U &= (U\rho(a)\psi, U\eta)_\sigma \\
 &= (L_a U\psi, U\eta)_\sigma \text{ by the intertwining property of } U \\
 &= (U\psi, L_{a^\times \sigma} U\eta)_\sigma \\
 &= (U\psi, U\rho(a^{\times \sigma})\eta)_\sigma \\
 &= (\psi, \rho(a^{\times \sigma})\eta)_U
 \end{aligned}$$

Now  $(\psi, \psi)_U = 0 \Rightarrow (U\psi, U\psi)_\sigma = 0 \Rightarrow U\psi = 0 \Rightarrow \psi = 0$ , hence  $(\cdot, \cdot)_U$  is definite, therefore  $(\cdot, \cdot)^K$  is by lemma 4.  $\P$

Of course multiplication by  $-1$  turns  $\sigma$ -compatible Krein products into  $\sigma$ -compatible Krein products. Putting together Robinson's alternative and proposition 7, we can therefore characterize the Euclidean signature of  $Q_\sigma$  in the following way.

**Corollary 1** *If the form  $Q_\sigma$  is positive definite then there exists a  $\sigma$ -compatible scalar product on every irreducible spinor module. If there exists at least one  $\sigma$ -compatible scalar product on at least one irreducible spinor module then  $Q_\sigma$  is positive definite.*

We now turn our attention to the case where  $\sigma$  is admissible.

**Lemma 5** *Let  $\rho$  be a  $c$ -compatible representation of  $\mathbb{C}l(V)$  on a Krein space  $(K, (\cdot, \cdot))$ . Let  $\sigma = \text{Ad}_b \circ c$  be an admissible real structure. Let  $x \in \mathbb{C}l(V)$  and let  $(\cdot, \cdot)_x = (\cdot, \rho(x) \cdot)$ . Then  $(\cdot, \cdot)_x$  is a  $\sigma$ -compatible Krein product on  $K$  iff  $x = x^\times$  and  $x$  is proportional to  $b^{-1}$ .*

**Proof:** The two properties are easily seen to be sufficient. Let us prove that they are necessary. First we note that since  $\mathbb{C}l(V)$  is simple and  $\rho \neq 0$ , then  $\rho$  is injective. For  $(\cdot, \cdot)_x$  to be sesquilinear  $x$  must satisfy  $\rho(x) = \rho(x)^\times$  which is equivalent to  $x = x^\times$  since  $\rho$  is injective and  $c$ -compatible. Now we note that  $a^{\times \sigma} = \sigma(a^T) = \sigma \circ c \circ c(a^T) = \text{Ad}_b(a^\times) = ba^\times b^{-1}$ .

We must have  $\rho(a^{\times \sigma}) = \rho(a)^{\times \sigma} = \rho(x)^{-1} \rho(a)^\times \rho(x)$  and we obtain that  $xb$  is a scalar by the injectivity of  $\rho$ .  $\P$

Now if  $b$  is real then  $b^\times = \pm b$  according to proposition 1, and we can take  $x = b^{-1}$  or  $x = ib^{-1}$  in the above lemma. Let us store the result we obtain as a proposition.

**Proposition 9** *Let  $\rho$  be a  $c$ -compatible representation of  $\mathbb{C}l(V)$  on a Krein space  $(K, (\cdot, \cdot))$ . Let  $\sigma = \text{Ad}_b \circ c$  be an admissible real structure with  $b$  real.*

1. *If  $b^\times = b$ ,  $(\cdot, \cdot)_b := (\cdot, \rho(b)^{-1} \cdot)$  is a  $\sigma$ -compatible Krein product on  $K$ .*
2. *If  $b^\times = -b$ ,  $(\cdot, \cdot)_b := (\cdot, i\rho(b)^{-1} \cdot)$  is a  $\sigma$ -compatible Krein product on  $K$ .*

## 2.5 Characterization of the Lorentz and anti-Lorentz signatures

First we fix some terminology. If  $Q$  has the Lorentz signature, the open light cone  $C$  of  $V$  is the non-convex cone of those  $v$  such that  $Q(v) < 0$ . It consists of two connected components which we arbitrarily call  $C_+$  and  $C_-$ . If  $Q$  has the anti-Lorentz signature, the open light cone  $C$  is defined by  $Q(v) > 0$  and  $C_+/C_-$  are defined accordingly.

Let  $L$  be a subspace in  $V$  and  $s_L$  be the orthogonal symmetry with respect to  $L$ . That is, if  $V = L \oplus W$  is an orthogonal decomposition, then  $s_L$  is the identity on  $L$  and minus the identity on  $W$ .

Our starting point will be the following evident observation:

- $Q$  has the Lorentz signature iff there exists a line  $L$  such that  $B(-s_L(\cdot), \cdot)$  has Euclidean signature.
- $Q$  has the anti-Lorentz signature iff there exists a line  $L$  such that  $B(s_L(\cdot), \cdot)$  has Euclidean signature.

Note that in both cases the line will belong to the open light cone.

Now all we have to do is to translate this observation in terms of admissible real structures, and use the characterization of Euclidean signature we arrived at in the previous subsection. In order to do this we first review some basic facts about the implementation of orthogonal symmetries with the adjoint action of vectors in Clifford algebras. Special care is needed about the signs.

Let  $v$  be a non-zero vector in  $V$ . Then  $v$  is invertible and  $v^{-1} = \frac{1}{Q(v)}v$ . The adjoint action of  $v$  on a vector  $u \in V$  satisfies  $Ad_v(v) = vvv^{-1} = v$  and if  $w \perp v$ ,  $Ad_v(w) = vvw^{-1} = -wv v^{-1} = -w$ . Thus  $Ad_v = s_L$  where  $L$  is the line  $\mathbb{R}v$ .

Now notice that for any  $u \in V$ ,  $Ad_\omega(u) = \omega u \omega^{-1} = -u \omega \omega^{-1} = -u$  since  $u$  is odd. Thus  $Ad_{\omega v} = Ad_\omega \circ Ad_v = -s_L$ .

Moreover, since  $c(\omega) = \omega$  and  $c(v) = v$ , the real structures  $\sigma_v := Ad_v \circ c$  and  $\sigma_{\omega v} := Ad_{\omega v} \circ c$  are admissible. We have thus obtained the following:

**Lemma 6** *Let  $L$  be a line in  $V$ . Then*

- *The unique admissible real structure which restricts to  $s_L$  is  $\sigma_v := Ad_v \circ c$ , where  $v \in L$  is a non-zero vector.*
- *The unique admissible real structure which restricts to  $-s_L$  is  $\sigma_{\omega v} := Ad_{\omega v} \circ c$ , where  $v \in L$  is a non-zero vector.*

Putting together what we have learned so far, we obtain:

**Theorem 1** *Let  $V$  be a finite dimensional real vector space of even dimension with non-degenerate quadratic form  $Q$ . Let  $\rho$  be a  $c$ -compatible irreducible representation on the Krein space  $(K, (\cdot, \cdot))$ . Then:*

1. *The signature of  $Q$  is antilorentzian iff there exists  $v \in V$  such that  $(\cdot, \cdot)_v := (\cdot, \rho(v)^{-1} \cdot)$  is definite.*

2. If  $n = 2$  or  $n = 6$  modulo 8, then the signature of  $Q$  is Lorentzian iff there exists  $v \in V$  such that  $(.,.)_v := (., \rho(\omega v)^{-1}.)$  is definite.
3. If  $n = 0$  or  $n = 4$  modulo 8, then the signature is Lorentzian iff there exists  $v \in V$  such that  $(.,.)_v := (., i\rho(\omega v)^{-1}.)$  is definite.

In every case,  $(.,.)_v$  is definite  $\Leftrightarrow v \in C$ , and there exists  $\lambda = \pm 1$  such that  $\lambda(.,.)_v$  positive definite  $\Leftrightarrow v \in C_+$  and  $\lambda(.,.)_v$  negative definite  $\Leftrightarrow v \in C_-$ .

**Proof:** 1. We know that  $Q$  has anti-Lorentz signature iff there exists a line  $L$  such that  $B(s_L(.,.))$  is positive definite. Observe that  $B(s_L(.,.)) = Q_\sigma$ , where  $\sigma = Ad_v \circ c$ , for any non-zero  $v \in L$ . We also know that  $(.,.)_v$  is a  $\sigma$ -compatible Krein product on  $K$  by proposition 9, and we conclude by Robinson's alternative.

2. Case 2 and 3 are similar to case 1 with the extra complication that  $b = \omega v$  satisfies  $b^\times = b$  for  $n = 2, 6 \bmod 8$  and  $b^\times = -b$  for  $n = 0, 4 \bmod 8$ .

It is obvious that a line  $L$  is such that  $B(s_L(.,.))$  is positive definite iff  $L \setminus \{0\}$  lies inside the light cone. Moreover multiplication by  $-1$  exchanges  $C_+$  with  $C_-$  and  $(.,.)_v$  with  $-(.,.)_v$ , so that it only remains to show that  $(\psi, \psi)_v$  keeps a constant sign for a fixed non-zero  $\psi$  and  $v$  varying continuously in one component of  $C$ , which is immediate by continuity of  $\rho$ .  $\blacksquare$

The presentation of this result can be cleaned up a little bit. First we could write  $\rho(v)$  instead of  $\rho(v)^{-1}$  and so on, since they differ only by a non-zero constant. However, it could prove misleading in the noncommutative situation, hence we prefer to refrain from doing so. More importantly, cases 2 and 3 can be put together by saying that the signature is Lorentzian iff there exists  $v \in V$  such that  $(.,.)_v := (., \rho(v)^{-1}\chi.)$  is definite, where  $\chi$  is chirality operator of noncommutative geometry. Now we can notice that the hermitian form  $(.,.)$  is  $\sigma$ -compatible iff  $(.,.)' := i(., \chi.)$  is  $\gamma \circ \sigma$ -compatible. A Krein product compatible with the graded real structure  $\gamma \circ c$  is characterized by the fact that the Clifford action of real vectors is anti-self-adjoint. The theorem shows that this kind of convention is well-adapted for the Lorentzian signature, whereas the  $c$ -compatible Krein product we have been using suits the antilorentzian signature better.

We will not delve much into such matters, but the above characterization of the Lorentzian/antilorentzian signature can be generalized easily. If  $Q$  is of signature  $(p, q)$  and  $p$  is odd, then we can find  $p$  vectors  $v_1, \dots, v_p$  such that  $\sigma = Ad_g \circ c$ , with  $g = v_1 \dots v_p$ , is an Euclidean real structure. The converse is also true, and these vectors will then automatically satisfy  $v_i^2 > 0$ . If  $p$  is even we have to take  $g = v_1 \dots v_p$  and we will have  $v_i^2 < 0$ . The problem now is to characterize the elements of the Clifford group which are of this form. For this we can use the canonical isomorphism  $\Theta : \Lambda V \rightarrow Cl(V)$ . Thus we need to consider two different classes of quadratic forms: even and odd ones, meaning that  $p$  and  $q$  are both even or both odd, according to the case. We can then say that:



- If  $Q$  is odd, then  $p$  is the only integer such that  $\exists g \in \Theta(\Lambda^p V)$ ,  $Ad_g \circ c$  is an Euclidean real structure.
- If  $Q$  is even then  $q$  is the only integer such that  $\exists g \in \Theta(\Lambda^q V)$ ,  $Ad_g \circ c$  is an Euclidean real structure.

We could then translate these properties in terms of the Krein structures on spinor modules.

## 2.6 Real structure and charge conjugation. KO-dimension tables

In noncommutative geometry, the real structure is defined as an antilinear operator on the Krein space which is the abstract substitute of the space of sections of the spinor bundle. When it is viewed in this way, it is sometimes called the *charge conjugation*, and we will adopt this terminology in order to distinguish it from its Clifford counterpart. To begin this section let us see how the two notions relate locally.

**Proposition 10** *Let  $\rho : \mathbb{C}l(V) \rightarrow \text{End}(K)$  be an irreducible  $c$ -compatible representation. Then there exists an antilinear operator  $\mathcal{C} : K \rightarrow K$  implementing  $c$ , i.e. such that*

$$\rho(c(a)) = \mathcal{C}\rho(a)\mathcal{C}^{-1}$$

*for all  $a \in \mathbb{C}l(V)$ . This operator can be chosen to satisfy  $\mathcal{C}^2 = \tilde{\epsilon}$  and  $\mathcal{C}^\times \mathcal{C} = \tilde{\kappa}$ , with  $\tilde{\epsilon}$  and  $\tilde{\kappa}$  some signs, and in that case is unique up to multiplication by  $e^{i\theta}$ ,  $\theta \in \mathbb{R}$ .*

*Let  $\sigma = Ad_b \circ c$  be an admissible real structure, with  $b$  in the Clifford group such that  $b^2 = \lambda = \pm 1$ ,  $b^\times = \lambda' b$ ,  $\lambda' = \pm 1$ , and  $c(b) = b$ . Let  $B = \rho(b)$ . Then  $\sigma$  is implemented on  $K$  by  $\mathcal{C}_\sigma = BC = CB$ .*

*We have  $\mathcal{C}_\sigma^2 = \lambda \tilde{\epsilon}$ ,  $\mathcal{C}_\sigma^\times \mathcal{C}_\sigma = \lambda \lambda' \tilde{\kappa}$ , and if  $(\cdot, \cdot)_\sigma$  is a  $\sigma$ -compatible Krein product on  $K$ , then  $(\mathcal{C}_\sigma)^\times = \lambda \mathcal{C}_\sigma^\times$  and  $(\mathcal{C}_\sigma)^\times \mathcal{C}_\sigma = \lambda' \mathcal{C}^\times \mathcal{C}$ .*

**Proof:** Fix any basis of  $K$  and denote by  $c.c. : \psi \mapsto \bar{\psi}$  the complex conjugation of coordinates with respect to this basis. We also denote by  $c.c. : A \mapsto \bar{A}$  the antilinear involution on operators induced by complex conjugation of coordinates, that is :  $\bar{A}\psi = \overline{A\bar{\psi}}$ . Let us write  $\tilde{c} = \rho \circ c \circ \rho^{-1}$ . Since  $\tilde{c}$  is an antilinear involution of  $\text{End}(K)$ ,  $\tilde{c} \circ c.c.$  is a linear (involutive) automorphism and is thus of the form  $Ad_C$  for some  $C \in \text{End}(K)$ . We let  $\mathcal{C} = C \circ c.c.$  and we obtain that  $\rho(c(a)) = \mathcal{C}\rho(a)\mathcal{C}^{-1}$ .

The fact that  $c$  is an involution translates as  $\mathcal{C}^2 = \tilde{\epsilon}$ , with  $\tilde{\epsilon}$  a constant. Let  $\psi \in K$  be such that  $\mathcal{C}\psi \neq 0$ . Then  $\mathcal{C}^3\psi = \tilde{\epsilon}\mathcal{C}\psi = \mathcal{C}(\tilde{\epsilon}\psi)$ , which shows that  $\tilde{\epsilon}$  is real. From the fact that  $\rho$  is  $c$ -compatible we obtain that  $\rho(c(a))^\times = \rho(c(a)^\times) = \rho(c(a^\times))$  and it boils down to  $\mathcal{C}\mathcal{C}^\times = \tilde{\kappa}$ , with  $\tilde{\kappa}$  a constant, which must be real. Calculating  $\mathcal{C}^2(\mathcal{C}^\times)^2$  we find that  $\tilde{\kappa}^2 = \tilde{\epsilon}^2$ . Dividing  $\mathcal{C}$  by  $\sqrt{|\tilde{\kappa}|}$  we can suppose that  $\tilde{\kappa} = \pm 1$  and  $\tilde{\epsilon} = \pm 1$ . It is now easy to prove the uniqueness up to a phase.

The second part is an obvious consequence of proposition 1. We have  $\mathcal{C}_\sigma^\times \mathcal{C}_\sigma = \mathcal{C}^\times B^\times B \mathcal{C} = \lambda \lambda' \mathcal{C}^\times \mathcal{C} = \lambda \lambda' \tilde{\kappa}$ . If  $\lambda = 1$  then  $(\mathcal{C}_\sigma)^\times = B^{-1} \mathcal{C}_\sigma^\times B = \mathcal{C}_\sigma^\times$  (see proposition 9), and if  $\lambda = -1$ , then  $(\mathcal{C}_\sigma)^\times = (iB)^{-1} \mathcal{C}_\sigma^\times (iB) = -\mathcal{C}_\sigma^\times$ .

Hence we have  $(\mathcal{C}_\sigma)^\times \mathcal{C}_\sigma = \lambda \mathcal{C}_\sigma^\times \mathcal{C}_\sigma = \lambda \mathcal{C}^\times B^\times B \mathcal{C} = \lambda \lambda' \mathcal{C}^\times B^2 \mathcal{C} = \lambda' \mathcal{C}^\times \mathcal{C}$ .  $\blacksquare$

If the representation is not irreducible, we can break it up into irreducible parts and use the proposition on each one of them. The signs  $\tilde{\epsilon}$  and  $\tilde{\kappa}$  can still be defined globally since they do not depend on the representation, but the phase  $\theta$  can vary from block to block.

Let us use proposition 10 to see how the KO table of signs change when we perform a Wick rotation.

If we start from an antilorentzian quadratic form  $Q$ , a Wick rotation to Euclidean signature corresponds to  $b = v$  with  $v^2 = 1$ ,  $v^\times = v$ , hence  $\lambda' = \lambda = 1$ .

In the Lorentzian case  $b = \omega v$  with  $v^2 = -1$ , hence  $b^2 = -\omega^2 v^2 = (-1)^{\frac{n}{2}+1}$ , and  $b^\times = v^\times \omega^\times = (-1)^{\frac{n}{2}} v \omega = (-1)^{\frac{n}{2}+1} b$ . Hence one has  $\lambda = \lambda' = (-1)^{\frac{n}{2}+1}$  in this case.

We introduce the hopefully obvious notations  $\mathcal{C}_L$ ,  $\chi_L$ ,  $\mathcal{C}_E$ ,  $\chi_E$ ,  $\mathcal{C}_{AL}$ ,  $\chi_{AL}$ . Then  $\mathcal{C}_E = \mathcal{C}_{AL} \rho(v) = \mathcal{C}_L \rho(\omega v)$ . According to the above proposition we then have:

$$\mathcal{C}_E^2 = \mathcal{C}_{AL}^2; \mathcal{C}_E^* \mathcal{C}_E = \mathcal{C}_{AL}^\times \mathcal{C}_{AL}$$

when Wick rotating from antilorentzian to Euclidean signature, but

$$\mathcal{C}_E^2 = (-1)^{\frac{n}{2}+1} \mathcal{C}_L^2; \mathcal{C}_E^* \mathcal{C}_E = (-1)^{\frac{n}{2}+1} \mathcal{C}_L^\times \mathcal{C}_L$$

when Wick rotating from Lorentzian to Euclidean signature. Here we have used the notation  $*$  instead of  $\times_\sigma$  since we are dealing with a Hilbert adjoint.

When going from antilorentzian/Lorentzian to Euclidean signature, the volume element changes in the following way :  $\omega_{AL/L} = e_1 \dots e_n \mapsto \omega_E := u_\sigma(\omega_{AL/L}) = i^q \omega_{AL/L}$ . But the Euclidean chirality element is  $\chi_E = (-i)^{\frac{n}{2}} \rho(\omega_E)$  whereas  $\chi_{AL/L} = (-i)^{\frac{n}{2}+q} \rho(\omega_{AL/L})$ . Hence the two expressions only differ by a minus sign:

$$\chi_E = -\chi_L, \chi_E = -\chi_{AL}$$

Finally we note that in both cases  $\mathcal{C}_E$  has an additional  $-1$  factor in its commutation relation with  $\chi_E$  with respect to the one of  $\mathcal{C}_{AL/L}$ .

Summarizing we have:

- Antilorentzian  $\rightarrow$  Euclidean :  $\tilde{\epsilon} \mapsto \tilde{\epsilon}$ ,  $\tilde{\kappa} \mapsto \tilde{\kappa}$ ,  $\tilde{\epsilon}'' \mapsto -\tilde{\epsilon}''$
- Lorentzian  $\rightarrow$  Euclidean :  $\tilde{\epsilon} \mapsto (-1)^{\frac{n}{2}+1} \tilde{\epsilon}$ ,  $\tilde{\kappa} \mapsto (-1)^{\frac{n}{2}+1} \tilde{\kappa}$ ,  $\tilde{\epsilon}'' \mapsto -\tilde{\epsilon}''$

These rules permit to fill the Lorentzian and antilorentzian KO table of signs from the Euclidean one, using the fact that the metric dimension modulo 8 is preserved by a Wick rotation (the KO dimension, of course, is not).

KO dim = $n = p - q$ [8]	0	2	4	6
$J^2 = \epsilon$	1	-1	-1	1
$J\chi = \epsilon''\chi J, \mathcal{C}\chi = \epsilon''\chi\mathcal{C}$	1	-1	1	-1
$\mathcal{C}^2 = \tilde{\epsilon}$	1	1	-1	-1
$J^\times J = \kappa$	1	1	1	1
$\mathcal{C}^\times \mathcal{C} = \tilde{\kappa}$	1	1	1	1

Table 1: Euclidean table of KO signs

The real Clifford algebra  $Cl(V, Q)$  can be directly recovered as the algebra of all  $a \in Cl(V)$  such that  $[\mathcal{C}, \rho(a)] = 0$ . In noncommutative geometry, one generally uses the operator  $J$  which selects  $Cl(V, -Q)$  (as originally in [Connes 95]). To pass from one to the other is easy:

$$J = \chi\mathcal{C}$$

We call  $\mathcal{C}$  the *ungraded charge conjugation* and  $J$  the *graded charge conjugation* operator. In this work it is generally more convenient to use  $\mathcal{C}$ , but we stick to the traditional  $J$  in definitions and statements of theorem.

The signs  $\epsilon, \epsilon', \epsilon''$  which are given in the definition of spectral triples correspond to the commutation rules of  $J$  with  $D$  and  $\chi$  and to the square of  $J$ . In the Lorentzian/antilorentzian context there is also a sign arising from  $J^\times J$ .

When we pass from the  $\mathcal{C}$  convention (signs with tildes) to the  $J$  convention (signs without tildes) and vice versa we must use the following rules:

- $\tilde{\epsilon}'' = \epsilon''$  ( $J$  and  $\mathcal{C}$  have the same commutation rules with  $\chi$ ).
- $\tilde{\epsilon}' = -\epsilon'$  ( $J$  and  $D$  have a commutation sign opposite to the one of  $\mathcal{C}$  and  $D$ ). In even dimension  $\epsilon' = 1$  and  $\tilde{\epsilon}' = -1$  always. Hence we do not let these signs appear in the table.
- $\tilde{\epsilon} = \epsilon''\epsilon$  (because  $J^2 = \chi\mathcal{C}\chi\mathcal{C} = \epsilon''\mathcal{C}^2$ ).
- In the Lorentzian/antilorentzian case :  $\tilde{\kappa} = -\kappa$  (since  $J^\times J = (\chi\mathcal{C})^\times \chi\mathcal{C} = -\mathcal{C}^\times \mathcal{C}$ ), in the Euclidean case,  $\tilde{\kappa} = \kappa$ .

## 2.7 Examples

In this section, which is optional, we derive the compatible Krein products on spinors in some usual representations. We illustrate how the light cone appears in each of the three different cases of theorem 1 in concrete computations with gamma matrices. Let us fix the notations. In each case we take  $V = \mathbb{R}^n$ , with a pseudo-orthonormal basis  $(e^\mu)_{\mu=0,\dots,n-1}$ . We have raised the indices to be consistent with physicists conventions. We will have  $B(e^0, e^0) = -1$  in the Lorentz case and  $B(e^0, e^0) = 1$  in the anti-Lorentz case, and  $e^0$  will belong to  $C^+$  in both cases.

$n$ [8]	0	2	4	6
KO dim = $p - q$ [8]	2	0	6	4
$\epsilon$	-1	1	1	-1
$\epsilon'' = \tilde{\epsilon}''$	-1	1	-1	1
$\tilde{\epsilon} = \epsilon''\epsilon$	1	1	-1	-1
$\kappa$	-1	-1	-1	-1
$\tilde{\kappa}$	1	1	1	1

Table 2: Antilorentzian table of KO signs.

$n$ [8]	0	2	4	6
KO dim = $p - q$ [8]	6	0	2	4
$\epsilon$	1	1	-1	-1
$\epsilon'' = \tilde{\epsilon}''$	-1	1	-1	1
$\tilde{\epsilon} = \epsilon''\epsilon$	-1	1	1	-1
$\kappa$	1	-1	1	-1
$\tilde{\kappa}$	-1	1	-1	1

Table 3: Lorentzian table of KO signs.

We will use a representation (for instance Dirac or Majorana<sup>3</sup>) in which the gamma matrices  $\gamma^\mu := \rho(e^\mu)$  satisfy  $(\gamma^\mu)^\dagger = \pm \gamma^\mu$  where the sign is the same as in  $(\gamma^\mu)^2 = \pm 1$ , and  $\dagger$  is the adjoint relatively to the canonical scalar product on  $K = \mathbb{C}^N$ ,  $N = 2^{n/2}$ .

We will consider a  $c$ -compatible Krein product on  $K$  denoted by  $(.,.)$ . Its adjunction operation is denoted  $A \mapsto A^\times$ . We denote by  $\beta$  the matrix of the form  $(.,.)$  in the canonical basis, that is,  $(.,.) = \langle ., \beta. \rangle$  where  $\langle ., . \rangle$  is the canonical scalar product. We know that  $(.,.)$  is determined up to a constant, which we reduce to a sign by requiring that  $\beta^2 = 1$ .

The  $c$ -compatibility of  $\rho$  is equivalent to the requirement that  $(\gamma^\mu)^\times = \gamma^\mu$  for all  $\mu$ , which is in turn equivalent to  $\beta(\gamma^\mu)^\dagger \beta^{-1} = \gamma^\mu$ . We see then that :

- In the anti-Lorentz case,  $\beta$  is determined up to a sign by :

$$\beta \gamma^0 \beta^{-1} = \gamma^0, \quad \beta \gamma^k \beta^{-1} = -\gamma^k, k = 1, \dots, n-1, \quad \beta^\dagger = \beta, \beta^2 = 1 \quad (7)$$

- In the Lorentz case,  $\beta$  is determined up to a sign by :

$$\beta \gamma^0 \beta^{-1} = -\gamma^0, \quad \beta \gamma^k \beta^{-1} = \gamma^k, k = 1, \dots, n-1, \quad \beta^\dagger = \beta, \beta^2 = 1 \quad (8)$$

The obvious solution to (7) is  $\beta = \pm \gamma^0$ . The more or less obvious solution to (8) is  $\beta = \pm \gamma^0 \rho(\omega)$  if  $n = 2, 6$  [8] and  $\beta = \pm i \gamma^0 \rho(\omega)$  if  $n = 0, 4$  [8]. In

<sup>3</sup>In fact the Dirac representation is more natural, since the Dirac basis is at the same time orthonormal for the scalar product  $\langle ., . \rangle_{e^0}$  and pseudo-orthonormal for the Krein-product.

other words in either case  $\beta = \pm \gamma^0 \chi$  where  $\chi$  is the chirality operator (the “ $\gamma^5$  matrix” in this context).

Now the part about the light cone in theorem 1 tells us in the anti-Lorentz case that  $v \in C$  if and only if  $\beta \rho(v)$  is a definite hermitian matrix. In the Lorentz case it is  $\beta \chi \rho(v)$  which must be definite, but since the two  $\chi$  cancel, we obtain in both cases that  $v \in C \Leftrightarrow \gamma^0 \rho(v)$  is a definite hermitian matrix.

### 3 Global constructions

Let  $(M, g)$  be a connected orientable semi-Riemannian manifold of dimension  $n$  and signature  $(p, q)$ . In this section the notion of spacelike (resp. timelike) will be associated with the positive (resp. negative) index of inertia. Note that this clashes with the relativistic convention in the antilorentzian case ! We see no way of avoiding this problem. We will warn the reader when a confusion could arise. We will revert to the usual convention in the following sections when we are exclusively concerned with the antilorentzian/Lorentzian cases.

Given the Clifford bundle  $Cl(M, g)$  and its complexification  $\mathbb{C}l(M)$  we now seek to make the constructions of the preceding section global. We start with real structures.

#### 3.1 Global real structures

A (global) real structure  $\sigma$  on  $\mathbb{C}l(M)$  is an involutive bundle map which is antilinear and respect products over each fibre. Since  $\mathbb{C}l(M)$  is given as the complexification of  $Cl(M, g)$ , we start with a given real structure  $c$ . We will be interested in real structures  $\sigma$  commuting with  $c$  and such that the metric  $g(\sigma(\cdot), \cdot)$  is Euclidean, i.e. *Euclidean real structures*. In view of proposition 6 such a real structure  $\sigma$  defines an orthogonal splitting  $TM = E_s \oplus E_t$  of the tangent bundle of  $M$  into spacelike and timelike subbundles, and the restriction of  $\sigma$  to  $TM$  is the orthogonal symmetry with respect to  $E_s$ . Conversely, given such a splitting (which always exists, see [F-N 12], section 3), we can define the spatial orthogonal symmetry globally, and extend it by the universal property of the Clifford bundle. If we compose the result with  $c$  we obtain an Euclidean real structure. Hence there is no obstruction to the existence of such objects.

However,  $\sigma$  is given at each point  $x \in M$  by  $\sigma_x = Ad_{b_x} \circ c_x$  where  $b_x$  is an element of the Clifford group at  $x$ . Hence globally we will have  $\sigma = Ad_b \circ c$  where  $b$  is a section of the Clifford group such that  $Ad_b$  is smooth. But it is clear that there will generally be an obstruction to the existence of a smooth  $b$ . To get a feel of what this obstruction might be let us consider the antilorentzian case. We know that  $\sigma$  defines an orthogonal splitting  $TM = E_s \oplus E_t$  of the tangent bundle, and in view of lemma 6,  $b$  itself is a section of  $E_s$ . Asking this section to be smooth is asking for the existence of a *time-orientation*<sup>4</sup>, that is a nonvanishing timelike (in the sense of relativity) vector field ([Beem 81], p 5).

<sup>4</sup>Reader beware ! What we have called spacelike in this semi-Riemannian context is what is called timelike in Relativity when the mostly plus convention is adopted.

To deal with the general case we need a formal definition of space and time orientations in the semi-Riemannian case<sup>5</sup>. Recall that the kernel of a  $p$ -form  $\omega_x$  is the subspace of  $T_x M$  consisting of those vectors  $v$  such that  $\omega(v, \dots, \cdot)$  is the null  $p-1$ -form.

**Lemma 7** *The following claims are equivalent.*

1. *There exists a  $p$ -form  $\omega$  such that  $\text{Ker}(\omega_x)$  is a timelike  $q$ -dimensional subspace of  $T_x M$  for all  $x \in M$ .*
2. *There exists a  $p$ -form  $\omega$  such that for every linearly independent family  $(v_1, \dots, v_p)$  of spacelike tangent vectors at  $x$ ,  $\omega_x(v_1, \dots, v_p) \neq 0$ .*
3. *Given any decomposition  $TM = E_s \oplus E_t$  of the tangent bundle into the sum of a spacelike and timelike subbundle, there exists a non-vanishing top form  $\omega^s$  of  $E_s$ .*

**Proof:** To see that (1) entails (2) consider any family  $\mathcal{F} = (v_1, \dots, v_p)$  of spacelike tangent vectors at  $x$  and let  $S_x \subset T_x M$  be the linear subspace they span. Since  $\text{Ker}\omega_x$  is timelike and  $q$ -dimensional, one has  $\text{Ker}\omega_x \oplus S_x = T_x M$ . Call  $\pi_x$  the projection onto  $S_x$  defined by this decomposition. It is immediate that  $\omega_x(u_1, \dots, u_p) = \lambda \det_{\mathcal{F}}(\pi_x(u_1), \dots, \pi_x(u_p))$  for any vectors  $u_1, \dots, u_p$ , where  $\det_{\mathcal{F}}$  is the determinant in the basis  $\mathcal{F}$  of  $S_x$  and  $\lambda \in \mathbb{R}$ . Clearly  $\lambda \neq 0$  since  $\omega_x$  does not vanish identically. Hence  $\omega_x(v_1, \dots, v_p) \neq 0$ .

The proof that (2) entails (3) is immediate. To obtain (1) from (3), consider the projection  $\pi$  on  $E_s$  defined by  $TM = E_s \oplus E_t$ , and extend  $\omega^s$  by the formula

$$\omega_x(u_1, \dots, u_p) := \omega_x^s(\pi(u_1), \dots, \pi(u_p))$$

Then  $\omega$  is a  $p$ -form on  $TM$  and its kernel at  $x$  is clearly  $(E_t)_x$ . ¶

A similar lemma clearly holds with timelike/spacelike reversed. When the properties stated in the lemma hold, we will say  $(M, g)$  is *space (respectively) time orientable*. A form on  $TM$  with the two first properties will be called a space (resp. time) orientation.

If  $\omega'$  is a space orientation  $p$ -form on  $M$ , a linearly independent family  $(v_1, \dots, v_p)$  of spacelike vectors at  $x$  will be said to be positively oriented iff  $\omega'_x(v_1, \dots, v_p) > 0$ . A similar definition can be given with time-orientations. Note that space/time orientation forms actually provide more information than the notion of orientation on families of spacelike/timelike vectors just defined. For example, suppose a Lorentz manifold is time-orientable and consider a 1-form  $\omega''$  with spacelike kernel. Then a timelike vector  $v$  at  $x$  will be said to be positively oriented, or *future-directed* iff  $\omega''_x(v) > 0$ . Using the musical isomorphism  $\sharp$  provided by  $g$  one can define a vector field  $\xi = (\omega'')^\sharp$ , which is timelike, since it is orthogonal to the distribution of spacelike subspaces  $\text{Ker}(\omega'')$ , and non-vanishing. This is indeed the usual definition of time-orientation for

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<sup>5</sup>Surprisingly we could not locate any in the literature.

Lorentz manifolds, even if the notion of future/past directed timelike vectors would just require the continuous choice of a half light-cone at each point of the manifold.

In the same vein we say that  $M$  is (totally) orientable if there exists a  $n$ -form  $\omega$  such that  $\omega_x$  has zero kernel (that is,  $\omega$  is non-vanishing) for all  $x$ .

Now the canonical isomorphism  $\Theta_x : \Lambda T_x M \simeq Cl(T_x M, g_x)$  gives rise to a bundle isomorphism (see for instance [L-M 89], prop. 3.5, chap. 2). We can compose it with the musical isomorphism and obtain a canonical isomorphism of vector bundles  $\Theta : \Lambda T^* M \simeq Cl(M, g)$ . If  $\omega$  is a space orientation form, then at each  $x$  we can decompose  $T_x M$  into an orthogonal sum  $\text{Ker}(\omega_x) \oplus S_x$ , where  $S_x$  is spacelike. If  $\mathcal{B} = (v_1, \dots, v_p)$  is an orthonormal basis of  $S_x$ , we obtain immediately that  $\omega_x(u_1, \dots, u_p) = \lambda_x \det_{\mathcal{B}}(\pi(u_1), \dots, \pi(u_p))$  with  $\pi$  the orthogonal projection on  $S_x$ . Hence  $b_x := \Theta_x(\omega_x) = \lambda_x v_1 \dots v_p$  is seen to be an element of the Clifford group at  $x$ . Thus  $b = \theta(\omega)$  is a smooth section of the Clifford group bundle. Conversely if  $b$  is smooth section of the Clifford group bundle which is locally of the form  $v_1 \dots v_p$ , with  $v_i$  spacelike, we easily see that  $\theta^{-1}(b)$  is a spatial orientation. Similar consideration apply to time orientations.

We can use these observations to globalize the discussion at the end of subsection 2.5. Moreover since we have assumed  $M$  to be orientable, space orientations can be converted into time orientations, saving us the trouble distinguishing the even and odd cases, and of translating the words timelike/spacelike when we revert to the traditional convention. We can thus answer our question about the obstruction to the definition of real structure through smooth sections of the Clifford group bundle in the following way:

- An orientable semi-Riemannian manifold admits Euclidean real structures defined through smooth sections of the Clifford group bundle iff it is time and space orientable.

Now recall that a time-oriented Lorentzian/antilorentzian manifold is called a *spacetime*. Hence an oriented Lorentzian/antilorentzian manifold admits Euclidean real structures defined through smooth sections of the Clifford bundle iff it is a *spacetime*.

### 3.2 Hermitian forms on the spinor bundle

In this subsection we consider a spin-c manifold  $M$  with metric  $g$  of signature  $(p, q)$  and spinor bundle  $S$ . We call  $\rho$  the representation  $\rho : \mathbb{C}l(M) \rightarrow \text{End}(S)$ . We will often write  $a.\Psi$  instead of  $\rho(a)\Psi$  in order to simplify notations. Let  $H : x \mapsto H_x$  be a smooth field of nowhere degenerate hermitian forms on  $S$ . We call this object a *spinor metric*. For any  $A \in \text{End}(S)$  we denote by  $A^\times$  the map  $x \mapsto A(x)^\times$ , where  $A(x)^\times$  is the adjoint relatively to  $H(x)$ . As in the local case, we say that  $H$  is  $c$ -compatible if  $\rho(a)^\times = \rho(a^\times)$  for all sections  $a$  of  $\mathbb{C}l(M)$ . This is equivalent to ask that  $\rho(\xi)$  be self-adjoint for all vector fields  $\xi \in \Gamma(TM)$ . What we want now is to globalize Robinson's transfert principle. The following theorem extends results in [Baum 80].

**Theorem 2** *There exists a  $c$ -compatible spinor metric iff  $(M, g)$  is time orientable when  $p, q$  are even, and iff  $(M, g)$  is space orientable when  $p, q$  are odd.*

**Proof:** We deal with the odd case only, the even case being completely similar.

Let us suppose that there exists a  $c$ -compatible spinor metric  $H$ . Consider a covering of  $M$  by open sets  $(U_\alpha)_{\alpha \in A}$  such that  $TM$  and  $S$  are trivial over  $U_\alpha$ , and a subordinate partition of unity  $(f_\alpha)_{\alpha \in A}$ . For each  $x \in M$  we let  $I_x$  be the finite set of indices  $\alpha$  such that  $f_\alpha(x) \neq 0$ . On each  $U_\alpha$  let us choose a section  $\psi_\alpha$  of the spinor bundle which is constant in some trivialization  $S|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^{2^{n/2}}$  and non-vanishing.

Let us define the  $p$ -form

$$\omega(X_1, \dots, X_p) = \sum_{\alpha} f_{\alpha}(x) H_x(\psi_{\alpha}, i^r \Theta(X_1 \wedge \dots \wedge X_p) \psi_{\alpha}) \quad (9)$$

The integer  $r$  is  $\lfloor \frac{p}{2} \rfloor$ . It ensures that  $\omega$  is a real  $p$ -form. Indeed:

$$\begin{aligned} \Theta(X_1 \wedge \dots \wedge X_p)^{\times} &= \frac{1}{p!} \sum_{\sigma} \epsilon(\sigma) X_{\sigma(p)} \dots X_{\sigma(1)} \\ &= \frac{(-1)^r}{p!} \sum_{\tau} \epsilon(\tau) X_{\tau(1)} \dots X_{\tau(p)} \\ &= (-1)^r \Theta(X_1 \wedge \dots \wedge X_p) \end{aligned}$$

where  $(-1)^r$  is the signature of the reversal permutation  $(1, \dots, p) \mapsto (p, \dots, 1)$ . Let  $(e_1, \dots, e_p, e_{p+1}, \dots, e_n)$  be a pseudo-orthonormal basis of  $T_x M$  such that  $e_1, \dots, e_p$  are spacelike. Then we know that  $H_x(\cdot, i^r(e_1 \dots e_p)^{-1} \cdot)$  is definite. Since  $(e_1 \dots e_p)^{-1} = e_p \dots e_1 = (-1)^r e_1 \dots e_p$ , we have either:

$$\forall \alpha \in I_x, H_x(\psi_{\alpha}, i^r e_1 \dots e_p \psi_{\alpha}) > 0$$

or

$$\forall \alpha \in I_x, H_x(\psi_{\alpha}, i^r e_1 \dots e_p \psi_{\alpha}) < 0$$

In both cases we have  $\omega_x(e_1, \dots, e_p) \neq 0$ . Now, since  $\omega$  is multilinear alternate, we have for any vectors  $u_1, \dots, u_p \in \text{Span}(e_1, \dots, e_p)$ :

$$\omega_x(u_1, \dots, u_p) = \det(u_i^j)_{1 \leq i, j \leq p} \omega_x(e_1, \dots, e_p) \quad (10)$$

where  $u_i^j$  is the  $j$ -th component of  $u_i$  in the basis  $(e_1, \dots, e_p)$ . Since  $(e_1, \dots, e_p)$  is any orthonormal family of spatial vectors, this shows that  $\omega_x(u_1, \dots, u_p) \neq 0$  for any linearly independent family of spatial vectors. Thus  $\omega$  is a space orientation.

Conversely, since  $M$  is space-orientable, there exists a Euclidean real structure  $\sigma = Ad_b \circ c$  where  $b$  is a smooth section of the Clifford group bundle. On the Riemannian manifold  $(M, g_{\sigma})$  it is well-known that there exists a  $\sigma$ -compatible spinor metric. Let us call  $\tilde{H}$  such a spinor metric. Then  $H(\cdot, \cdot) = \tilde{H}(\cdot, i^r b \cdot)$  defines a  $c$ -compatible spinor metric for  $M$ .  $\blacksquare$

Of course a  $c$ -compatible spinor metric is unique only up to multiplication by a nonvanishing real function.



### 3.3 The Dirac operator and $c$ -compatibility

In the case of a manifold we have seen that we can express neatly the  $c$ -compatibility of a spinor metric by saying that vector fields are self-adjoint. Unfortunately this cannot be directly generalized in the noncommutative setting. It is therefore important to have an alternative formulation which is better suited to this generalization.

We will need to consider a spin manifold. Our definition of a spin structure on a space and time oriented spin- $c$  manifold  $(M, g, S)$  is the algebraic one, which is directly applicable in noncommutative geometry: it is antilinear bundle map  $J : S \rightarrow S$  which satisfies  $J\rho(a)J^{-1} = -\rho(c(a))$  and  $J^2 = \pm \text{Id}_S$ . If  $S$  is equipped with a  $c$ -compatible spinor metric  $H$ , we will also have  $JJ^\times = \pm \text{Id}_S$  (see proposition 10). We refer to [Sch 00] for the equivalence of this definition and the traditional one involving the possibility of lifting the frame bundle to a spin bundle.

By progressively enriching the structure we can define various kinds of connections on  $S$ . We will use the notation

$$(\Psi, \Psi')_H = \int_M H_x(\Psi(x), \Psi'(x)) \sqrt{|\det(g)|} dx \quad (11)$$

We call this the *Krein product* on spinor fields, a name which will be justified later on.

**Definition 1** *Let  $(M, g, S)$  be a spin- $c$  manifold and  $H$  be a spinor metric on  $S$ . Let  $\nabla$  be a connection on  $S$ .*

1. *If  $\nabla_X(a \cdot \Psi) = (\nabla_X^{LC} a) \cdot \Psi + a \cdot \nabla_X \Psi$ , for all sections  $a$  of the Clifford bundle, vector fields  $X$  and spinor fields  $\Psi, \Psi'$  with compact support, then  $\nabla$  is said to be a Clifford connection.*
2. *If  $X \cdot (\Psi, \Psi')_H = (\nabla_X \Psi, \Psi')_H + (\Psi, \nabla_X \Psi')_H$ , for all  $X, \Psi, \Psi'$  as above,  $\nabla$  is said to be metric.*
3. *If in addition  $M$  is a spin manifold and  $J$  is a spin structure on it, then  $\nabla$  is said to preserve spin if  $\nabla_X J = J \nabla_X$  for all vector field  $X$ .*
4. *If  $M$  is a spin manifold and is space and time oriented,  $J$  is a spin structure on it, and  $H$  is a  $c$ -compatible metric on  $S$ , then  $\nabla$  is said to be a spin connection if it is a Clifford connection which is metric and preserves spin.*

**Remark:** Let  $x, y \in M$ , let  $\lambda$  be a curve joining these points. Let  $h_\lambda : S_x \rightarrow S_y$  be the parallel transport operator of  $\nabla$  along  $\lambda$ . Similarly, we denote by  $h_\lambda^{LC}$  the parallel transport of the Levi-Civita connection along  $\lambda$ . Since this is an isometry from  $(T_x M, g_x)$  onto  $(T_y M, g_y)$  we can consider its canonical extension  $\tilde{h}_\lambda^{LC}$  which is an isomorphism between  $\text{Cl}(T_x M)$  and  $\text{Cl}(T_y M)$ . The above infinitesimal definitions can be given integrated forms. More precisely, with the same hypotheses as in definition 1, we can say that

1.  $\nabla$  is a Clifford connection iff for all  $x, y, \lambda$ ,  $h_\lambda$  intertwines the action of the Clifford algebras at  $x$  and at  $y$  on the spinors. More precisely, this means that for all  $a \in \text{Cl}(T_x M)$  the following diagram commutes

$$\begin{array}{ccc} S_x & \xrightarrow{a} & S_x \\ h_\lambda \downarrow & & \downarrow h_\lambda \\ S_y & \xrightarrow{\tilde{h}_\lambda^{LC}(a)} & S_y \end{array}$$

2.  $\nabla$  is metric iff the parallel transport  $h_\lambda$  is an isometry from  $(S_x, H_x)$  onto  $(S_y, H_y)$ .
3.  $\nabla$  preserves spin iff the following diagram commutes for all  $x, y, \lambda$ :

$$\begin{array}{ccc} S_x & \xrightarrow{J_x} & S_x \\ h_\lambda \downarrow & & \downarrow h_\lambda \\ S_y & \xrightarrow{J_y} & S_y \end{array}$$

The interest of the integrated versions is that they continue to make sense in a discrete context. Going from the integrated to the infinitesimal properties is not difficult using the formula

$$(\nabla_X \Psi)(x) = \lim_{t \rightarrow 0} \frac{h_\lambda^{-1}(\Psi(\lambda(t)) - \Psi(x))}{t}$$

where  $\lambda : [0; 1] \rightarrow M$  is a curve such that  $\lambda(0) = x$  and  $(\frac{d}{dt}\lambda)(0) = X$ , and the similar formula for  $\nabla^{LC}$ . Conversely we use the uniqueness of the solution to the parallel transport equation with a given initial condition.

Two connections on  $S$  differ by an  $\text{End}(S)$ -valued 1-form, and it is easy to see that this 1-form must be scalar-valued in the case of Clifford connections. If the connections are also metric, the 1-form has values in  $i\mathbb{R}$ , and if they commute with  $J$  it has values in  $\mathbb{R}$ , so we obtain the uniqueness of the spin connection. Moreover the spin connection always exists (see theorem 9.8 in [GB-V-F 01] for the Riemannian case, [Bizi XX] for the general case).

Every Clifford connection gives rise to a Dirac operator which will have the local form  $D = -i \sum_\mu \gamma(dx^\mu) \nabla_{\partial_\mu}$ , where  $\gamma : T^*M \rightarrow \text{End}(S)$  is the composition of the musical isomorphism defined by  $g$ , and the representation  $\rho$  (hence  $\gamma(dx^\mu)\psi = \rho((dx^\mu)^\sharp)\psi$ ).

In noncommutative geometry we will be left with (an abstract version of) the sections of the spinor bundle, Krein product, Dirac operator and spin structure, with no direct hold on the connection, Clifford algebra and vector fields. The following facts are then crucial:

1. The Dirac operator commutes with the spin structure iff the latter commutes with the Clifford connection and anti-commutes with vector fields.
2. The Dirac operator is symmetric (i.e. formally self-adjoint) with respect to the Krein product  $(\cdot, \cdot)_H$  iff the spinor metric  $H$  is  $c$ -compatible and the Clifford connection is metric with respect to it.

Let  $(M, g)$  be a semi-riemannian spin manifold, with spinor bundle  $S$  and spin structure  $J$ . Let  $\nabla$  be a Clifford connection and  $D$  be the Dirac operator associated to it. We write  $\nabla_\mu := \nabla_{\frac{\partial}{\partial x^\mu}}$ .

**Proposition 11** *The Dirac operator and spin structure commute if and only if  $\{J, X\} = [J, \nabla_X] = 0$  for all  $X \in \Gamma(TM)$ .*

**Proof:** The fact that  $\{J, X\} = [J, \nabla_X] = 0$  are sufficient conditions is a straightforward calculation.

Let us prove that they are necessary. Let  $f$  be a smooth real-valued function. Since  $[J, D] = [J, f] = 0$ , we deduce that  $J$  commutes with  $[D, f] = -i\gamma(df)$ . We infer easily that  $J$  anti-commutes with any real vector field or differential form. Next we consider the differential operator  $\nabla' = J^{-1}\nabla J$ . This operator is a Clifford connection. Indeed, for any  $X, Y \in \Gamma(TM)$ ,  $\psi \in \Gamma(S)$ :

$$\begin{aligned} \nabla'_X(Y \cdot \psi) &= J^{-1}\nabla_X J(Y \cdot \psi) \\ &= -J^{-1}\nabla_X(Y \cdot J\psi) \\ &= -J^{-1}[(\nabla_X^{LC} Y) \cdot J\psi + Y \cdot \nabla_X(J\psi)] \\ &= (\nabla_X^{LC} Y) \cdot \psi + Y \cdot \nabla'_X \psi \end{aligned} \quad (12)$$

There thus exists a complex-valued one-form  $\omega$  such that:  $\nabla'_X - \nabla_X = \omega(X)$ . This can also be written:  $\omega(X) = J^{-1}[\nabla_X, J]$ . We have:

$$\begin{aligned} 0 &= [J, D] \\ &= [J, -i \sum_\mu \gamma(dx^\mu) \nabla_\mu] \\ &= -i \sum_\mu \gamma(dx^\mu) [J, \nabla_\mu] \\ &= -iJ \sum_\mu \gamma(dx^\mu) \omega_\mu \end{aligned}$$

This implies that  $\omega = 0$ , and thus that  $[\nabla_X, J] = 0$ . ¶

Let  $H$  be a spinor metric and  $(\cdot, \cdot)_H$  be the associated product on spinor fields defined by (11).

**Proposition 12** *The Dirac operator is symmetric if and only if  $X^\times = X$  for all  $X \in \Gamma(TM)$ , and the Clifford connection  $\nabla$  is metric for  $H$ .*

**Remark:** Our spinor bundle  $S$  thus has to be a Dirac bundle (see [L-M 89], p 114).

**Proof:** Let us assume that the Dirac operator is symmetric. All the scalar functions, vector and spinor fields appearing in this proof are assumed to have compact support. Let  $f$  be a real-valued smooth function. Then  $f$  is self-adjoint. This implies that  $[D, f] = -i\gamma(df)$  is anti-self-adjoint. We easily conclude from this that all real vector fields and all real-valued 1-forms are self-adjoint. In

particular, the  $\gamma(dx^\mu)$  are self-adjoint. Let  $\psi, \phi$  be spinor fields with compact support. We have:

$$\begin{aligned} 0 &= (\psi, D\phi)_H - (D\psi, \phi)_H \\ &= \int \sqrt{|g|} \sum_{\mu} [H(-i\gamma(dx^\mu)\nabla_{\mu}\psi, \phi) - H(\psi, -i\gamma(dx^\mu)\nabla_{\mu}\phi)] dx \\ 0 &= i \int \sqrt{|g|} \sum_{\mu} [H(\nabla_{\mu}\psi, \gamma(dx^\mu)\phi) + H(\psi, \gamma(dx^\mu)\nabla_{\mu}\phi)] dx \end{aligned}$$

We also have that:

$$\begin{aligned} [\nabla_{\mu}, \gamma(dx^\mu)] &= \gamma(\nabla_{\mu}^{LC}(dx^\mu)), \\ &= -\sum_{\alpha} \Gamma_{\mu\alpha}^{\mu} \gamma(dx^\alpha) \end{aligned}$$

Hence

$$\begin{aligned} \sum_{\mu} [\nabla_{\mu}, \gamma(dx^\mu)] &= -\sum_{\mu, \alpha} \Gamma_{\mu\alpha}^{\mu} \gamma(dx^\alpha) \\ &= -\sum_{\alpha} \frac{\partial_{\alpha} \sqrt{|g|}}{\sqrt{|g|}} \gamma(dx^\alpha) \end{aligned}$$

from which we infer that  $\sum_{\mu} \gamma(dx^\mu) \nabla_{\mu} = \sum_{\mu} \nabla_{\mu} \gamma(dx^\mu) + \sum_{\mu} \gamma(dx^\mu) (\partial_{\mu} \sqrt{|g|}) / \sqrt{|g|}$ . Substituting in the integral above gives us:

$$\int dx \sum_{\mu} [\sqrt{|g|} H(\nabla_{\mu}\psi, \gamma(dx^\mu)\phi) + \sqrt{|g|} H(\psi, \nabla_{\mu}(\gamma(dx^\mu)\phi)) + (\partial_{\mu} \sqrt{|g|}) H(\psi, \gamma(dx^\mu)\phi)] = 0$$

Finally, an integration by part yields:

$$\int dx \sqrt{|g|} \sum_{\mu} [H(\nabla_{\mu}\psi, \gamma(dx^\mu)\phi) + H(\psi, \nabla_{\mu}(\gamma(dx^\mu)\phi)) - \partial_{\mu} H(\psi, \gamma(dx^\mu)\phi)] dx = 0$$

for all  $\psi, \phi \in \Gamma_c(S)$ . Now, the expression between brackets can be proven to be  $\mathcal{C}^\infty(M, \mathbb{C})$ -linear in  $\phi$  (and anti-linear in  $\psi$  as well). Indeed,  $H(\nabla_{\mu}\psi, \gamma(dx^\mu)\phi)$  is clearly linear in  $\phi$ . let  $f \in \mathcal{C}^\infty(M, \mathbb{C})$ . We replace  $\phi$  by  $f\phi$  in the two remaining terms:

$$\begin{aligned} H(\psi, \nabla_{\mu}(\gamma(dx^\mu)f\phi)) - \partial_{\mu} H(\psi, \gamma(dx^\mu)f\phi) &= H(\psi, \nabla_{\mu}[f(\gamma(dx^\mu)\phi)]) - \partial_{\mu}[fH(\psi, \gamma(dx^\mu)\phi)] \\ &= H(\psi, (\partial_{\mu}f)\gamma(dx^\mu)\phi) + f\nabla_{\mu}(\gamma(dx^\mu)\phi) \\ &\quad - (\partial_{\mu}f)H(\psi, \gamma(dx^\mu)\phi) - f\partial_{\mu}H(\psi, \gamma(dx^\mu)\phi) \\ &= f[H(\psi, \nabla_{\mu}(\gamma(dx^\mu)\phi)) - \partial_{\mu}H(\psi, \gamma(dx^\mu)\phi)] \end{aligned}$$

Thus, for all  $f \in \mathcal{C}^\infty(M, \mathbb{C})$  and  $\psi, \phi \in \Gamma_c(S)$ :

$$\int dx \sqrt{|g|} f \sum_{\mu} [H(\nabla_{\mu}\psi, \gamma(dx^\mu)\phi) + H(\psi, \nabla_{\mu}(\gamma(dx^\mu)\phi)) - \partial_{\mu} H(\psi, \gamma(dx^\mu)\phi)] = 0$$

which implies that:

$$\sum_{\mu} [H(\nabla_{\mu}\psi, \gamma(dx^{\mu})\phi) + H(\psi, \nabla_{\mu}(\gamma(dx^{\mu})\phi)) - \partial_{\mu}H(\psi, \gamma(dx^{\mu})\phi)] = 0 \quad (13)$$

at every  $x$ . Now suppose that  $\nabla^0$  is a local spin connection defined around  $x$ . We know that  $\nabla = \nabla^0 + A$  where  $A$  is a scalar 1-form, and that  $\nabla$  is metric if and only if  $A$  has pure imaginary values. Using (13) we find that

$$\sum_{\mu} (A_{\mu} + \bar{A}_{\mu}) H(\psi, \gamma(dx^{\mu})\phi) = 0 \quad (14)$$

for all spinor fields  $\psi, \phi$  with small enough compact support containing  $x$ . Since  $H$  is non-degenerate the orthogonal of  $\{\gamma(dx^2)\phi, \dots, \gamma(dx^n)\phi\}$  for  $H$  is a subbundle of dimension  $2^{\frac{n}{2}} - n + 1 \geq 1$  of the spinor bundle. Since the orthogonal of  $\{\gamma(dx^1)\phi, \dots, \gamma(dx^n)\phi\}$  has codimension 1 in it, we can, at least locally, consider a spinor field  $\psi$  such that  $H(\psi, \gamma(dx^{\mu})\phi) = 0$  for  $\mu = 2, \dots, n$  and  $H(\psi, \gamma(dx^1)\phi) \neq 0$ . Hence  $A_1 + \bar{A}_1 = 0$ , and of course the same can be done for the other indices. The Clifford connection  $\nabla$  is thus metric.

The converse can be proven easily following the same steps. It is in fact a standard result of spin geometry. See for example [L-M 89].  $\blacksquare$

### 3.4 The canonical spectral “triple” of a semi-Riemannian manifold

In this section we consider a space and time orientable spin manifold  $M$ , with a given spinor bundle  $S$  equipped with  $c$ -compatible spinor metric  $H$  and representation  $\rho$ , charge conjugation  $J$ , and the canonical Dirac operator  $D$  corresponding to the spin connection  $\nabla$ .

#### 3.4.1 The Krein space of spinor fields

We let  $\Gamma_c^{\infty}(S)$  denotes the space of smooth sections of  $S$  with compact support. On this space we have already defined the hermitian form  $(\cdot, \cdot)_H$ . Since  $(M, g)$  is space orientable we can consider a space-orientation<sup>6</sup> form  $\beta$ , and the corresponding smooth field of Clifford group elements  $b = \Theta(\beta)$ . In fact we can suppose without loss of generality that  $b_x$  lies in the Pin group for all  $x$ , since here  $c(b_x) = b_x$  and we have  $b_x^2 \in \mathbb{R}$ . Hence we can replace  $b$  with  $b/\sqrt{|b^2|}$  (see proposition 1 for details). We set  $B = \rho \circ b$ , and we recall that we can define a definite spinor metric  $(\cdot, \cdot)_b$  by premultiplying spinors by  $B^{-1}$  or  $iB^{-1}$  according to the negativity index of the metric  $g$  (see proposition 9). For instance in case  $B = B^{\times}$

$$(\Psi, \Phi)_b = \int_M H_x(\Psi(x), B_x^{-1}\Phi(x)) \sqrt{-g} dx \quad (15)$$

<sup>6</sup>Once again, in the antilorentzian convention this is a time-orientation !

is a definite hermitian form on  $\Gamma_c^\infty(S)$ . Since  $M$  is connected we can suppose that it is positive definite up to an overall change of sign when choosing  $\beta$ . We call  $K$  the Hilbert space obtained by completing  $\Gamma_c^\infty(S)$  with respect to this scalar product. It turns out that this completion does not depend on the choice of  $\beta$  (see [Baum 80] section 3.3.1 or [Bizi XX]). Moreover, as we have already noticed at the end of section 2.3, since  $Ad_b \circ c$  is an Euclidean real structure, we have  $b_x^2 = 1$  if  $b_x^\times = b_x$  and  $b_x^2 = -1$  if  $b_x^\times = -b_x$ . In either case  $B^2 = 1$ . We then obtain canonically a Krein space  $K$  with fundamental symmetry  $B$ .

The canonical spectral “triple” of a Riemannian spin manifold of even dimension is the following bunch of objects : the pre- $C^*$ -algebra  $\mathcal{A} = \mathcal{C}^\infty(M)$ , Hilbert space of  $L^2$ -spinor fields  $\mathcal{H}$ , representation  $\pi$  of  $\mathcal{A}$  on  $\mathcal{H}$  by pointwise multiplication, canonical Dirac operator  $D$ , charge conjugation operator  $J$  and chirality operator  $\chi$ . We see that in the semi-Riemannian space and time orientable context we must replace  $\mathcal{H}$  with the Krein space  $\mathcal{K}$  described above, but without fixing a particular fundamental symmetry: doing so would be equivalent to fix a particular space orientation form.

### 3.4.2 The antilorentzian and Lorentzian cases

Given what we have done just above, characterizing the Lorentzian/antilorentzian signature of a manifold metric purely in terms of the data available in non-commutative geometry is easy. Beware that we now, and for the rest of the paper, revert to the usual convention of calling “timelike” the vectors such that  $g(v, v) > 0$  in the antilorentzian case. Recall that if  $\omega$  is a 1-form we denote by  $\gamma(\omega)$  the Clifford multiplication by  $\omega$ , that is  $\gamma(\omega) = \rho(\gamma^\sharp)$ .

**Theorem 3** *Let  $(M, g)$  be a semi-Riemannian space and time orientable spin manifold of even dimension, with given  $c$ -compatible spinor metric  $H$ , spin structure  $J$  and chirality operator  $\chi$ . Let  $K$  be the Krein space of spinor fields equipped with the Krein product  $(\cdot, \cdot)_H$ . Then  $(M, g)$  is*

1. *antilorentzian iff there exists a never vanishing 1-form  $\beta$  such that  $J\beta J^{-1} = -\beta$  and  $(\cdot, \gamma(\beta)^{-1} \cdot)_H$  is positive definite.*
2. *Lorentzian iff there exists a never vanishing 1-form  $\beta$  such that  $J\beta J^{-1} = -\beta$  and  $(\cdot, \gamma(\beta)^{-1} \chi \cdot)_H$  is positive definite.*

## 4 Conclusion

In view of theorem 3, and setting aside the analytical questions (for this, see [D-P-R 13]) we are led to the following definition, which is meant to be the antilorentzian counterpart of spectral triples.

**Definition 2** *An even antilorentzian spectral spacetime of KO-dimension  $k \bmod 8$  is given by a multiplet  $S = (\mathcal{A}, K, (\cdot, \cdot), \pi, D, J, \chi)$  where:*

1. *the couple  $(K, (\cdot, \cdot))$  is a Krein space, with adjunction denoted by  $\times$ ,*

2.  $\mathcal{A}$  is an algebra and  $\pi$  is a faithful representation of it on  $K$ ,
3. the “Dirac operator”  $D$  on  $K$  is such that  $D^\times = D$ ,
4. the “chirality operator”  $\chi \in B(K)$  is such that  $\chi^2 = 1$ ,  $[\pi(a), \chi] = 0$  for all  $a \in A$ ,  $\chi D = -D\chi$  and  $\chi^\times = -\chi$ ,
5. the “charge conjugation”  $J$  is an antilinear operator on  $K$  which is required to satisfy:

$$J^2 = \epsilon, \quad JD = DJ, \quad J\chi = \epsilon''\chi J, \quad J^\times J = \kappa$$

with  $\epsilon, \epsilon'', \kappa = \pm 1$ , according to table 2.

and such that

6. there exists an invertible noncommutative 1-form  $\beta$ , called a time-orientation, which satisfies the following properties:
  - (a) it is Krein self-adjoint,
  - (b) it is imaginary ( $J\beta J^{-1} = -\beta$ ),
  - (c) the hermitian form  $\langle \cdot, \cdot \rangle_\beta := (\cdot, \beta^{-1} \cdot)$  is positive definite,
  - (d)  $\pi(\mathcal{A})^{*\beta} = \pi(\mathcal{A})$  (where  $*_\beta$  is the adjoint for  $\langle \cdot, \cdot \rangle_\beta$ ).

Condition 6d, which we call the *reconstructibility condition*, is where  $C^*$ -structures enter into the game. Since we want to consider all possible time-orientations, may have in principle several distinct Hilbert adjoints, hence several  $C^*$ -structures (easily shown to be isomorphic) induced on  $\mathcal{A}$  by  $\pi$ . This is the reason why we cannot fix a  $C^*$ -structure on  $\mathcal{A}$  right from the start.

We will explore the consequences of this definition, and give discrete commutative and noncommutative examples, in a companion paper.

## A Relation between the $\sigma$ -product and the Killing form

Let  $\sigma$  be a real structure. Using the isomorphism  $Cl(V, g_\sigma) \simeq Cl(V_\sigma, g)$ , we see that the Lie algebra  $\mathfrak{g}_\sigma$  of the spin group  $\text{Spin}(V, g_\sigma)$  is identified with  $\Theta(\Lambda^2 V_\sigma)$ . In particular  $\mathfrak{g}_c \simeq so(p, q) \simeq \Theta\Lambda^2 V$ .

Now let  $(e_i)_{1 \leq i \leq n}$  be a pseudo-orthonormal basis of  $V_\sigma$ . Then  $(e_i e_j)_{1 \leq i, j \leq n}$  is a pseudo-orthonormal basis of  $\Theta(\Lambda^2 V_\sigma)$  for  $(\cdot, \cdot)_\sigma$ , with  $(e_i e_j, e_i e_j)_\sigma = e_i^2 e_j^2 = \kappa_i \kappa_j$  with obvious notations. Hence for the Killing form of  $\mathfrak{g}_\sigma$  we have

$$K(x, y) := \text{Tr}(ad_x \circ ad_y) = \sum_{1 \leq a < b \leq n} \kappa_a \kappa_b (e_a e_b, [x, [y, e_a e_b]])_\sigma$$

In particular we have

$$\begin{aligned} K(e_i e_j, e_k e_l) &= \sum_{1 \leq a < b \leq n} \kappa_a \kappa_b (e_a e_b, [e_i e_j, [e_k e_l, e_a e_b]])_\sigma \\ &:= \sum_{1 \leq a < b \leq n} \kappa_a \kappa_b T_{ab} \end{aligned} \quad (16)$$

with

$$\begin{aligned} T_{ab} &= \tau_n(e_b e_a e_i e_j e_k e_l e_a e_b) - \tau_n(e_b e_a e_i e_j e_a e_b e_k e_l) - \tau_n(e_b e_a e_k e_l e_a e_b e_i e_j) \\ &\quad + \tau_n(e_b e_a e_a e_b e_k e_l e_i e_j) \end{aligned}$$

We note that if  $e_i e_j$  commutes with  $e_a e_b$  then the first and second terms on the one hand, and third and fourth terms on the other, cancel each other. Hence  $T_{ab}$  vanishes unless  $\{a; b\}$  and  $\{i; j\}$  have exactly one element in common, in which case  $e_a e_b$  anticommutes with  $e_i e_j$  and we have

$$T_{ab} = 4\kappa_a \kappa_b \tau_n(e_i e_j e_k e_l) = -4\kappa_a \kappa_b (e_i e_j, e_k e_l)_\sigma$$

Replacing in (16) we obtain  $K(e_i e_j, e_k e_l) = -4 \times 2(n-2)(e_i e_j, e_k e_l)_\sigma$  since there are  $2(n-2)$  couples having exactly one element in common with  $\{a; b\}$ . Hence we have

$$K(x, y) = 8(n-2)(x, y)_\sigma$$

for all  $x, y \in \mathfrak{g}_\sigma$ .

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