



Elliptic solutions to integrable nonlinear equations and many-body systems

A. Zabrodin*

National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russian Federation
 Steklov Mathematical Institute of Russian Academy of Sciences, Gubkina str. 8, Moscow, 119991, Russian Federation



ARTICLE INFO

Article history:

Received 3 June 2019

Accepted 27 August 2019

Available online 29 August 2019

Keywords:

Integrable equations

Elliptic solutions

Dynamics of poles

Integrable many-body systems

ABSTRACT

We review elliptic solutions to integrable nonlinear partial differential and difference equations (KP, mKP, BKP, Toda) and derive equations of motion for poles of the solutions. The pole dynamics is given by an integrable many-body system (Calogero–Moser, Ruijsenaars–Schneider). The basic tool is the auxiliary linear problems for the wave function which yield equations of motion together with their Lax representation. We also discuss integrals of motion and properties of the spectral curves.

© 2019 Elsevier B.V. All rights reserved.

1. Introduction

The study of singular solutions to integrable nonlinear partial differential equations and dynamics of their poles was initiated by the seminal paper [2], where elliptic and rational solutions to the Korteweg–de Vries and Boussinesq equations were investigated. It was discovered there that the poles move as particles of the integrable many-body Calogero–Moser system [4,5,20] with some additional restrictions in the phase space. Later in [6,12] it was shown that in the case of the more general Kadomtsev–Petviashvili (KP) equation the connection with many-body systems becomes most natural: the dynamics of poles of rational solutions to the KP equation is isomorphic to the Calogero–Moser system of particles with rational pairwise interaction potential $1/(x_i - x_j)^2$.

Elliptic (double periodic in the complex plane) solutions to the KP equation were studied by Krichever in [13], where it was shown that poles x_i of the elliptic solutions move according to the equations of motion

$$\ddot{x}_i = 4 \sum_{k \neq i} \wp'(x_i - x_k) \quad (1.1)$$

of the Calogero–Moser system of particles with the elliptic interaction potential $\wp(x_i - x_j)$ ($\wp$ is the Weierstrass $\wp$ -function). Here dot means derivative with respect to the time t_2 . The method suggested by Krichever consists in substituting the elliptic solution not in the KP equation itself but in the auxiliary linear problems for it, using a suitable pole ansatz for the wave function depending on a spectral parameter. The wave function is defined by its poles and residues at the poles. This method allows one to obtain the equations of motion together with the Lax representation for them:

$$\dot{L} = [M, L], \quad (1.2)$$

where matrices L , M depend on x_i and $\dot{x}_i$ and on the spectral parameter.

* Correspondence to: National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russian Federation.
 E-mail address: zabrodin@itep.ru.

The further development is Shiota's work [27], where it was shown that the correspondence between rational solutions to the KP equation and the Calogero–Moser system with rational potential (when $\wp(x)$ degenerates to $1/x^2$) can be extended to the level of *hierarchies*. There it was proved that the evolution of poles with respect to the higher times t_k of the infinite KP hierarchy is governed by higher Hamiltonians H_k of the integrable Calogero–Moser system.

Another way to derive the equations of motion for poles of singular solutions to the KP equation was suggested in [1]. It consists in parametrizing the wave function through its poles x_i and zeros y_i , which is basically equivalent to substitution of the pole ansatz to the modified KP (mKP) equation. This does not allow one to derive the Lax representation but instead leads to the so-called self-dual form of equations of motion [1].

Elliptic solutions to the matrix KP equation were studied in [15]; they give rise to the spin generalization of the Calogero–Moser system. Spin degrees of freedom are related to the matrix residues at the poles of the solutions (which are fixed in the scalar case). In this paper we will restrict ourselves by the scalar (one-component) case.

Solutions of the B -version of the KP equation (BKP), which are elliptic functions of the time variable $x = t_1$, were recently discussed in [24]. It was shown that the poles move according to the following equations of motion:

$$\ddot{x}_i + 6 \sum_{j \neq i} (\dot{x}_i + \dot{x}_j) \wp'(x_i - x_j) - 72 \sum_{j \neq k \neq i} \wp(x_i - x_j) \wp'(x_i - x_k) = 0. \quad (1.3)$$

Here dot means derivative with respect to the time t_3 . Instead of the Lax representation, these equations of motion admit a matrix representation of the form

$$\dot{L} = [M, L] + P(L - \Lambda I), \quad (1.4)$$

where P is a traceless matrix, I is the unity matrix and Λ is the eigenvalue of the Lax matrix L depending on the spectral parameter (Manakov's triple representation [19]). Matrix elements of the matrices L, M depend on $x_i, \dot{x}_i$ and on the spectral parameter.

Poles x_i of solutions to the 2D Toda lattice (2DTL), which are elliptic functions of the discrete time variable $x = t_0$, satisfy the elliptic Ruijsenaars–Schneider many-body system [26] (a relativistic version of the Calogero–Moser system):

$$\ddot{x}_i = \sum_{j \neq i} \dot{x}_i \dot{x}_j \frac{\wp'(x_i - x_j)}{\wp(\eta) - \wp(x_i - x_j)} \quad (1.5)$$

(see [17]). Here dot means derivative with respect to the time t_1 . In the limit $\eta \rightarrow 0$ one recovers the Calogero–Moser system. The Ruijsenaars–Schneider system is known to be integrable. Similarly to the Calogero–Moser model, it admits a Lax representation which can be obtained by substituting the pole ansatz for the wave function into the semi-difference auxiliary linear problem. Another approach to the connection between special solutions to the KP and Toda equations and integrable many-body systems was developed in [25]. A self-dual form of the equations of motion also exists (see [32]). It is closely connected with the integrable discrete time version of the Ruijsenaars–Schneider system [22], with equations of motion having the form of the nested Bethe ansatz equations.

This paper is a review of the results mentioned above and related topics. We give a detailed derivation of the equations of motion for poles of elliptic solutions to the KP, mKP, BKP and 2DTL equations together with their Lax (or Manakov's triple) representation and construction of the spectral curve. Most of the material is contained in the existing literature in one or another form. However, some points are new. Among them is the derivation of the t_3 -dynamics of poles of elliptic solutions to the KP equation and equations of motion for poles of elliptic solutions to the Novikov–Veselov equation (which is a member of the 2-component BKP hierarchy).

2. Elliptic solutions to the KP equation

2.1. The KP equation

The KP equation is the first member of an infinite KP hierarchy of partial differential equations with independent variables ("times") $t_1 = x, t_2, t_3, t_4, \dots$ [9,11]. The KP equation is equivalent to the Zakharov–Shabat ("zero curvature") condition $\partial_{t_3} A_2 - \partial_{t_2} A_3 + [A_2, A_3] = 0$ for the differential operators

$$A_2 = \partial_x^2 + 2u, \quad A_3 = \partial_x^3 + 3u\partial_x + w. \quad (2.1)$$

It has the form of a system for two dependent variables u, w :

$$\begin{cases} 2w_x = 3u_{t_2} + 3u_{xx} \\ 2u_{t_3} - w_{t_2} = 6uu_x + 2u_{xxx} - w_{xx}. \end{cases} \quad (2.2)$$

Excluding w from this system, one obtains the KP equation for the function u :

$$3u_{t_2 t_2} = \left(4u_{t_3} - 12uu_x - u_{xxx} \right)_x. \quad (2.3)$$

The Zakharov–Shabat equation (and, therefore, the KP equation) is the compatibility condition for the auxiliary linear problems

$$\partial_{t_2} \psi = A_2 \psi, \quad \partial_{t_3} \psi = A_3 \psi \quad (2.4)$$

for the wave function ψ which depends on a spectral parameter z .

The general solution to the whole KP hierarchy is given in terms of the tau-function $\tau = \tau(t_1, t_2, t_3, \dots)$. The change of the dependent variables from u, w to the tau-function

$$u = \partial_x^2 \log \tau, \quad w = \frac{3}{2} (\partial_x^3 \log \tau + \partial_{t_2} \partial_x \log \tau) \quad (2.5)$$

brings equations (2.2) into the bilinear form [11]

$$(D_1^4 - 4D_1 D_3 + 3D_2^2) \tau \cdot \tau = 0, \quad (2.6)$$

where D_i are the Hirota operators. Their action is defined by the rule

$$P(D_1, D_2, D_3, \dots) f(t_i) \cdot g(t_i) = P(\partial_{y_1}, \partial_{y_2}, \partial_{y_3}, \dots) f(t_i + y_i) g(t_i - y_i) \Big|_{y_i=0}$$

for any polynomial $P(D_1, D_2, D_3, \dots)$. The solution to the linear equations (2.4) can be expressed through the tau-function according to the formula [9]

$$\psi = A(z) \exp \left(\sum_{k \geq 1} t_k z^k \right) \frac{\tau(t_1 - z^{-1}, t_2 - \frac{1}{2} z^{-2}, t_3 - \frac{1}{3} z^{-3}, \dots)}{\tau(t_1, t_2, t_3, \dots)}. \quad (2.7)$$

Here z is the spectral parameter and $A(z)$ is a normalization factor.

2.2. The t_2 -dynamics of poles of elliptic solutions

Our aim is to study double-periodic (elliptic) in the variable x solutions of the KP equation. For such solutions the tau-function is an “elliptic quasi-polynomial” in the variable x :

$$\tau = C e^{Q(x, t_2, t_3, \dots)} \prod_{i=1}^N \sigma(x - x_i) \quad (2.8)$$

where $Q(x, t_2, t_3, \dots)$ is a quadratic form in the times t_i and

$$\sigma(x) = \sigma(x | \omega, \omega') = x \prod_{s \neq 0} \left(1 - \frac{x}{s} \right) e^{\frac{x}{s} + \frac{x^2}{2s^2}}, \quad s = 2\omega m + 2\omega' m' \quad \text{with integer } m, m',$$

is the Weierstrass σ -function with quasi-periods $2\omega, 2\omega'$ such that $\text{Im}(\omega'/\omega) > 0$. It is connected with the Weierstrass ζ - and $\wp$ -functions by the formulas $\zeta(x) = \sigma'(x)/\sigma(x)$, $\wp(x) = -\zeta'(x) = -\partial_x^2 \log \sigma(x)$. We set $Q = cx^2 + bt_2 x + \dots$ with some constants b, c . The roots x_i are assumed to be all distinct. Correspondingly, the function $u = \partial_x^2 \log \tau$ is an elliptic function with double poles at the points x_i :

$$u = - \sum_{i=1}^N \wp(x - x_i) + 2c. \quad (2.9)$$

The poles depend on the times t_2, t_3 .

According to Krichever’s method [13], the basic tool for studying t_2 -dynamics of poles is the auxiliary linear problem $\partial_{t_2} \psi = A_2 \psi$ for the function ψ , i.e.,

$$\partial_{t_2} \psi = \partial_x^2 \psi + 2u \psi. \quad (2.10)$$

Since the coefficient function u is double-periodic, one can find double-Bloch solutions $\psi(x)$, i.e., solutions such that $\psi(x + 2\omega) = b\psi(x)$, $\psi(x + 2\omega') = b'\psi(x)$ with some Bloch multipliers b, b' . Eqs. (2.7), (2.8) tell us that the wave function ψ has simple poles at the points x_i . The pole ansatz for the ψ -function is

$$\psi = e^{xz + t_2 z^2 + t_3 z^3} \sum_{i=1}^N c_i \Phi(x - x_i, \lambda), \quad (2.11)$$

where the coefficients c_i do not depend on x . Here we use the function

$$\Phi(x, \lambda) = \frac{\sigma(x + \lambda)}{\sigma(\lambda)\sigma(x)} e^{-\zeta(\lambda)x} \quad (2.12)$$

which has a simple pole at $x = 0$ (ζ is the Weierstrass ζ -function). The expansion of Φ as $x \rightarrow 0$ is

$$\Phi(x, \lambda) = \frac{1}{x} + \alpha_1 x + \alpha_2 x^2 + \dots, \quad x \rightarrow 0,$$

where $\alpha_1 = -\frac{1}{2} \wp(\lambda)$, $\alpha_2 = -\frac{1}{6} \wp'(\lambda)$. The parameters z and λ are spectral parameters. They are going to be connected by equation of the spectral curve (see below). Using the quasiperiodicity properties of the function Φ ,

$$\Phi(x + 2\omega, \lambda) = e^{2(\zeta(\omega)\lambda - \zeta(\lambda)\omega)} \Phi(x, \lambda),$$

$$\Phi(x + 2\omega', \lambda) = e^{2(\zeta(\omega')\lambda - \zeta(\lambda)\omega')} \Phi(x, \lambda),$$

one concludes that ψ given by (2.11) is indeed a double-Bloch function with Bloch multipliers

$$b = e^{2(\omega z + \zeta(\omega)\lambda - \zeta(\lambda)\omega)}, \quad b' = e^{2(\omega' z + \zeta(\omega')\lambda - \zeta(\lambda)\omega')}.$$

We will often suppress the second argument of Φ writing simply $\Phi(x) = \Phi(x, \lambda)$. We will also need the x -derivatives $\Phi'(x, \lambda) = \partial_x \Phi(x, \lambda)$, $\Phi''(x, \lambda) = \partial_x^2 \Phi(x, \lambda)$, etc.

The function $-\partial_{t_2} \psi + \partial_x^2 \psi + 2u\psi$ is also a double-Bloch function with the same Bloch multipliers. If we manage to prove that this function is free of poles, then the only possibility for it is the exponential function Ce^{ax} , which, however, has a pair of Bloch multipliers that is not equivalent to b, b' . Therefore, $C = 0$ and the function vanishes identically. Substituting (2.11) into (2.10) with u given by (2.9), we get:

$$\begin{aligned} & - \sum_i \dot{c}_i \Phi(x - x_i) + \sum_i c_i \dot{x}_i \Phi'(x - x_i) + 2z \sum_i c_i \Phi'(x - x_i) + \sum_i c_i \Phi''(x - x_i) \\ & - 2 \left(\sum_i \wp(x - x_i) \right) \left(\sum_k c_k \Phi(x - x_k) \right) + 4c \sum_i c_i \Phi(x - x_i) = 0, \end{aligned}$$

where dot means the t_2 -derivative. Different terms of this expression have poles at $x = x_i$. The highest poles are of third order but it is easy to see that they cancel identically. It is a matter of direct calculation to see that the conditions of cancellation of second and first order poles have the form

$$c_i \dot{x}_i = -2zc_i - 2 \sum_{j \neq i} c_j \Phi(x_i - x_j), \quad (2.13)$$

$$\dot{c}_i = (4c - 2\alpha_1)c_i - 2 \sum_{j \neq i} c_j \Phi'(x_i - x_j) - 2c_i \sum_{j \neq i} \wp(x_i - x_j). \quad (2.14)$$

They have to be valid for all $i = 1, \dots, N$. Introducing $N \times N$ matrices

$$L_{ij} = -\delta_{ij} \dot{x}_i - 2(1 - \delta_{ij}) \Phi(x_i - x_j), \quad (2.15)$$

$$M_{ij} = \delta_{ij}(\wp(\lambda) + 4c) - 2\delta_{ij} \sum_{k \neq i} \wp(x_i - x_k) - 2(1 - \delta_{ij}) \Phi'(x_i - x_j), \quad (2.16)$$

we can write the above conditions as a system of linear equations for the vector $\mathbf{c} = (c_1, \dots, c_N)^T$:

$$\begin{cases} L\mathbf{c} = 2z\mathbf{c} \\ \dot{\mathbf{c}} = M\mathbf{c}. \end{cases} \quad (2.17)$$

It is convenient to introduce diagonal matrices I, X, D given by $I_{ik} = \delta_{ik}$, $X_{ik} = \delta_{ik} x_i$,

$$D_{ik} = \delta_{ik} \sum_{j \neq i} \wp(x_i - x_j) \quad (2.18)$$

and off-diagonal matrices A, B given by

$$A_{ik} = (1 - \delta_{ik}) \Phi(x_i - x_k), \quad B_{ik} = (1 - \delta_{ik}) \Phi'(x_i - x_k), \quad (2.19)$$

then the matrices L, M are

$$L = -\dot{X} - 2A, \quad M = (\wp(\lambda) + 4c)I - 2B - 2D.$$

Differentiating the first equation in (2.17) with respect to t_2 , we arrive at the compatibility condition of the linear problems (2.17):

$$(\dot{I} + [L, M])\mathbf{c} = 0. \quad (2.20)$$

We have

$$\dot{L} + [L, M] = -\ddot{X} - 2\dot{A} + 2[\dot{X}, B] + 4[A, B] + 4[A, D].$$

It is straightforward to see that $\dot{A} = [\dot{X}, B]$. Next, in the [Appendix](#) it is proved that $[A, B] + [A, D] = D'$, where D' is the diagonal matrix $D'_{ik} = \delta_{ik} \sum_{j \neq i} \wp'(x_i - x_j)$. Therefore, we have the identity

$$\dot{L} + [L, M] = -\ddot{X} + 4D',$$

and so the compatibility condition states that $(-\ddot{X} + 4D')_{ii} = 0$ for all $i = 1, \dots, N$. This implies the Calogero–Moser equations of motion (1.1) together with their Lax representation (1.2).

It follows from the Lax representation that the time evolution is an isospectral transformation of the Lax matrix, so all traces $\text{tr} L^k$ and the characteristic polynomial $\det(L - 2zI)$ are integrals of motion.

The Calogero–Moser system is Hamiltonian with the Hamiltonian

$$H_2 = \sum_i p_i^2 - 2 \sum_{i < j} \wp(x_i - x_j) \quad (2.21)$$

and the Poisson brackets $\{x_i, p_k\} = \delta_{ik}$. Since $\dot{x}_i = \partial H_2 / \partial p_i = 2p_i$, the Lax matrix expressed through the momenta reads $L_{ij} = -2(\delta_{ij}p_i + (1 - \delta_{ij})\Phi(x_i - x_j))$. The Hamiltonian is given by

$$H_2 = \frac{1}{4} \text{tr} L^2 - N(N-1)\wp(\lambda). \quad (2.22)$$

The higher Hamiltonians can be defined by the formulas

$$H_k = 2^{-k} \text{tr} L^k + \sum_{m=0}^{k-2} a_m(\lambda) \text{tr} L^m, \quad (2.23)$$

where $a_m(\lambda)$ are some elliptic functions of λ which are determined by the requirement that the Hamiltonians should be λ -independent. In particular,

$$\begin{aligned} H_1 &= \frac{1}{2} \text{tr} L = -\sum_i p_i, \\ H_3 &= \frac{1}{8} \text{tr} L^3 - \frac{3}{2} (N-1)\wp(\lambda) \text{tr} L - \frac{1}{2} N(N-1)(N-2)\wp'(\lambda) \\ &= -\sum_i p_i^3 + 3 \sum_{i \neq j} p_i \wp(x_i - x_j). \end{aligned} \quad (2.24)$$

2.3. The t_3 -dynamics of poles of elliptic solutions

Similarly to the t_2 -case, we should substitute ψ given by (2.11), into the linear problem

$$\partial_{t_3} \psi = \partial_x^3 \psi + 3u \partial_x \psi + w \psi.$$

According to (2.5), (2.8), we have for elliptic solutions:

$$w = -\frac{3}{2} \sum_i \wp'(x - x_i) + \frac{3}{2} \sum_i \dot{x}_i \wp(x - x_i) + \frac{3}{2} b. \quad (2.25)$$

The substitution leads to the following equation:

$$\begin{aligned} &\sum_i \partial_{t_3} c_i \Phi(x - x_i) - \sum_i c_i \partial_{t_3} x_i \Phi'(x - x_i) \\ &= 3z^2 \sum_i c_i \Phi'(x - x_i) + 3z \sum_i c_i \Phi''(x - x_i) + \sum_i c_i \Phi'''(x - x_i) \\ &\quad - 3z \left(\sum_i \wp(x - x_i) \right) \left(\sum_j c_j \Phi(x - x_j) \right) - 3 \left(\sum_i \wp(x - x_i) \right) \left(\sum_j c_j \Phi'(x - x_j) \right) \\ &\quad - \frac{3}{2} \left(\sum_i \wp'(x - x_i) \right) \left(\sum_j c_j \Phi(x - x_j) \right) + \frac{3}{2} \left(\sum_i \dot{x}_i \wp(x - x_i) \right) \left(\sum_j c_j \Phi(x - x_j) \right) \\ &\quad + 6cz \sum_i c_i \Phi(x - x_i) + 6c \sum_i c_i \Phi'(x - x_i) + \frac{3}{2} b \sum_i c_i \Phi(x - x_i). \end{aligned}$$

The fourth order poles at $x = x_i$ cancel identically. The cancellation of the third order poles leads to the same condition (2.13) which is the eigenvalue equation $Lc = 2zc$ for the Lax matrix. The cancellation of the second order poles leads to the equations

$$\partial_{t_3} x_i c_i = -3z^2 c_i - 6cc_i - 3z \sum_{j \neq i} c_j \Phi(x_i - x_j) + 3c_i \sum_{j \neq i} \wp(x_i - x_j) + \frac{3}{2} \dot{x}_i \sum_{j \neq i} c_j \Phi(x_i - x_j). \quad (2.26)$$

Taking into account (2.13), one can rewrite these equations as

$$\partial_{t_3} x_i = -6c - \frac{3}{4} \dot{x}_i^2 + 3 \sum_{j \neq i} \wp(x_i - x_j), \quad (2.27)$$

or, in the matrix form,

$$\partial_{t_3} X = -6cI - \frac{3}{4} \dot{X}^2 + 3D. \quad (2.28)$$

Finally, the cancellation of the simple poles at $x = x_i$, with the equation $\dot{c} = Mc$ being taken into account, leads to the conditions

$$\begin{aligned} \partial_{t_3} c_i = & \frac{3}{2} z \dot{c}_i - \frac{3}{2} \sum_{j \neq i} c_j \Phi''(x_i - x_j) + \frac{3}{2} c_i \sum_{j \neq i} \wp'(x_i - x_j) + \frac{1}{2} \wp'(\lambda) c_i + \frac{3}{2} b c_i \\ & + \frac{3}{2} \dot{x}_i \sum_{j \neq i} c_j \Phi'(x_i - x_j) + \frac{3}{2} c_i \sum_{j \neq i} \dot{x}_j \wp(x_i - x_j) - \frac{3}{4} \wp(\lambda) \dot{x}_i c_i. \end{aligned}$$

Introducing the diagonal matrix $\tilde{D}_{ik} = \delta_{ik} \sum_{j \neq i} \dot{x}_j \wp(x_i - x_j)$ and the off-diagonal matrix $C_{ik} = (1 - \delta_{ik}) \Phi''(x_i - x_k)$, we can rewrite these conditions in the matrix form as

$$\partial_{t_3} c = Tc, \quad (2.29)$$

where T is the matrix

$$T = \frac{3}{4} ML - \frac{3}{2} C + \frac{3}{2} \dot{X}B + \frac{3}{2} D' + \frac{3}{2} \tilde{D} - \frac{3}{4} \wp(\lambda) \dot{X} + \frac{1}{2} (\wp'(\lambda) + 3b)I. \quad (2.30)$$

The compatibility condition of the linear system (2.29) and the equation $Lc = 2zc$ is $(\partial_{t_3} L + [L, T])c = 0$. Let us calculate $\partial_{t_3} L + [L, T]$ using the Lax equation $\dot{L} = [M, L]$. We have, after some algebra:

$$\partial_{t_3} L + [L, T] = -\partial_{t_3} \dot{X} - 3D' \dot{X} + 3([A, C] + 2[B, D] - AD' - D'A) + 3Y + \frac{3}{2} [\dot{X}, C - \wp(\lambda)A],$$

where $Y = B\dot{X}A - A\dot{X}B - [A, \tilde{D}]$. In the Appendix we prove the matrix identity $[A, C] + 2[B, D] - AD' - D'A = 0$. Moreover, it holds

$$Y_{ik} + \frac{1}{2} [\dot{X}, C - \wp(\lambda)A]_{ik} = 0 \quad \text{for } i \neq k$$

and

$$Y_{ii} = - \sum_{j \neq i} \dot{x}_j \wp'(x_i - x_j)$$

(see (A.11) in the Appendix). Therefore, we obtain that the matrix $\partial_{t_3} L + [L, T]$ is diagonal and

$$\left(\partial_{t_3} L + [L, T] \right)_{ii} = -\partial_{t_3} \dot{x}_i - 3 \sum_{j \neq i} (\dot{x}_i + \dot{x}_j) \wp'(x_i - x_j). \quad (2.31)$$

This means that the compatibility condition leads to the equations of motion

$$\partial_{t_3} \dot{x}_i + 3 \sum_{j \neq i} (\dot{x}_i + \dot{x}_j) \wp'(x_i - x_j) = 0. \quad (2.32)$$

As one can easily see, these equations coincide with the t_2 -derivatives of Eqs. (2.27). Moreover, Eqs. (2.27) and (2.32), when rewritten in terms of the moments p_i , are Hamiltonian equations for the $\tilde{H}_3$ Hamiltonian flow (here $\tilde{H}_3 = H_3 + 6cH_1$):

$$\begin{cases} \partial_{t_3} x_i = \frac{\partial \tilde{H}_3}{\partial p_i} = -6c - 3p_i^2 + 3 \sum_{j \neq i} \wp(x_i - x_j) \\ \partial_{t_3} p_i = -\frac{\partial \tilde{H}_3}{\partial x_i} = -3 \sum_{j \neq i} (p_i + p_j) \wp'(x_i - x_j) \end{cases} \quad (2.33)$$

(H_3 is given in (2.24)).

It is easy to see that the result of [2] for the KdV equation (u is independent of t_2) follows from (2.33): the equations of motion are $\partial_{t_3} x_i = -6c + 3 \sum_{j \neq i} \wp(x_i - x_j)$ on the locus defined by the equations $\sum_{j \neq i} \wp'(x_i - x_j) = 0$.

2.4. Reviving the coupling constant

The Calogero–Moser system that is obtained from the dynamics of poles comes with a fixed coupling constant. A natural question is whether other values of the coupling constant are possible. It appears that in order to revive the coupling constant one should consider the so-called $\hbar$ -version of the KP hierarchy ($\hbar$ -KP) instead of the standard KP one. The $\hbar$ -KP hierarchy is obtained by the formal change of times $t_k \rightarrow t_k/\hbar$, so that the $\hbar$ -KP equation acquires the form

$$3u_{t_2 t_2} = \left(4u_{t_3} - 12uu_x - \hbar^2 u_{xxx}\right)_x \quad (2.34)$$

and the second linear problem is

$$\hbar \partial_{t_2} \psi = \hbar^2 \partial_x^2 \psi + 2\hbar^2 \partial_x^2 \log \tau \psi. \quad (2.35)$$

Substituting the wave function in the form

$$\psi = e^{\frac{1}{\hbar} xz + \frac{1}{\hbar} t_2 z^2 + \frac{1}{\hbar} t_3 z^3} \sum_i c_i \Phi(x - x_i) \quad (2.36)$$

with the same expression for τ (2.8), we get, after writing down conditions of cancellation of the poles, the same system (2.17) with the matrices

$$L_{ij} = -\delta_{ij} \dot{x}_i - 2\hbar(1 - \delta_{ij})\Phi(x_i - x_j), \quad (2.37)$$

$$M_{ij} = (\hbar \wp(\lambda) + 4c\hbar^{-1})\delta_{ij} - 2\hbar \delta_{ij} \sum_{k \neq i} \wp(x_i - x_k) - 2\hbar(1 - \delta_{ij})\Phi'(x_i - x_j). \quad (2.38)$$

Then, repeating the calculation above, one obtains

$$\dot{L} + [L, M] = -\ddot{X} + 4\hbar^2 D'$$

and the equations of motion acquire the form

$$\ddot{x}_i = 4\hbar^2 \sum_{j \neq i} \wp'(x_i - x_j) \quad (2.39)$$

with the coupling constant $4\hbar^2$.

2.5. The spectral curve

From now on we put $\hbar = 1$, as before. The equation of the spectral curve is

$$R(z, \lambda) = \det(2zI - L(z, \lambda)) = 0. \quad (2.40)$$

As it was already mentioned, the equation of the spectral curve (the characteristic equation of the Lax matrix) is an integral of motion by virtue of the Lax equation for both t_2 and t_3 flows: $\frac{d}{dt_2} \det(2zI - L(z, \lambda)) = \frac{d}{dt_3} \det(2zI - L(z, \lambda)) = 0$.

The matrix $L = L(z, \lambda)$, which has essential singularity at $\lambda = 0$, can be represented in the form $L = \tilde{L}G^{-1}$, where matrix elements of $\tilde{L}$ do not have essential singularities and G is the diagonal matrix $G_{ij} = \delta_{ij} e^{-\zeta(\lambda)x_i}$. Therefore,

$$R(z, \lambda) = \sum_{k=0}^N R_k(\lambda) z^k,$$

where the coefficients $R_k(\lambda)$ are elliptic functions of λ with poles at $\lambda = 0$. The functions $R_k(\lambda)$ can be represented as linear combinations of $\wp$ -function and its derivatives. Coefficients of this expansion are integrals of motion. Fixing values of these integrals, we obtain via the equation $R(z, \lambda) = 0$ the algebraic curve Γ which is a N -sheet covering of the initial elliptic curve $\mathcal{E}$ realized as a factor of the complex plane with respect to the lattice generated by $2\omega, 2\omega'$.

Example ($N = 2$).

$$\det_{2 \times 2} (2zI - L(z, \lambda)) = 4z^2 + 2z(\dot{x}_1 + \dot{x}_2) + \dot{x}_1 \dot{x}_2 + 4\wp(x_1 - x_2) - 4\wp(\lambda).$$

Example ($N = 3$).

$$\begin{aligned} \det_{3 \times 3} (2zI - L(z, \lambda)) &= 8z^3 + 4z^2(\dot{x}_1 + \dot{x}_2 + \dot{x}_3) \\ &+ 2z(\dot{x}_1\dot{x}_2 + \dot{x}_1\dot{x}_3 + \dot{x}_2\dot{x}_3 + 4\wp(x_{12}) + 4\wp(x_{13}) + 4\wp(x_{23}) - 12\wp(\lambda)) \\ &+ \dot{x}_1\dot{x}_2\dot{x}_3 + 4\dot{x}_1\wp(x_{23}) + 4\dot{x}_2\wp(x_{13}) + 4\dot{x}_3\wp(x_{12}) - 4\wp(\lambda)(\dot{x}_1 + \dot{x}_2 + \dot{x}_3) - 8\wp'(\lambda), \end{aligned}$$

where $x_{ik} = x_i - x_k$.

In a neighborhood of $\lambda = 0$ the matrix $\tilde{L}$ can be written as

$$\tilde{L} = -2\lambda^{-1}(E - I) + O(1),$$

where E is the rank 1 matrix with matrix elements $E_{ij} = 1$ for all $i, j = 1, \dots, N$. The matrix E has eigenvalue 0 with multiplicity $N - 1$ and another eigenvalue equal to N . Therefore, we can write $R(z, \lambda)$ in the form

$$\begin{aligned} R(z, \lambda) &= \det(2z + 2\lambda^{-1}(E - I) + O(1)) \\ &= 2^N \left(z + (N - 1)\lambda^{-1} - f_N(\lambda) \right) \prod_{i=1}^{N-1} (z - \lambda^{-1} - f_i(\lambda)), \end{aligned} \quad (2.41)$$

where f_i are regular functions of λ at $\lambda = 0$: $f_i(\lambda) = O(1)$ as $\lambda \rightarrow 0$. This means that the function z has simple poles on all sheets at the points P_j ($j = 1, \dots, N$) of the curve Γ located above $\lambda = 0$. Its expansion in the local parameter λ on the sheets near these points is given by the multipliers in the right hand side of (2.41). So we have the following expansions of the function z near the “points at infinity” P_j :

$$\begin{aligned} z &= \lambda^{-1} + f_j(\lambda) \quad \text{near } P_j, \quad j = 1, \dots, N - 1, \\ z &= -(N - 1)\lambda^{-1} + f_N(\lambda) \quad \text{near } P_N. \end{aligned} \quad (2.42)$$

The N th sheet is distinguished, as it can be seen from (2.41), (2.42). As in [13], we call it the upper sheet.

The genus g of the spectral curve Γ can be found using the following argument. Let us apply the Riemann–Hurwitz formula to the covering $\Gamma \rightarrow \mathcal{E}$. We have $2g - 2 = \nu$, where ν is the number of ramification points of the covering, which are zeros on Γ of the function $\partial R / \partial z$. Differentiating equation (2.41) with respect to z , we can conclude that the function $\partial R / \partial z$ has simple poles at the points $P_1, \dots, P_{N-1}$ on all sheets except the upper one, where it has a pole of order $N - 1$. The number of poles of any meromorphic function is equal to the number of zeros. Therefore, $\nu = 2(N - 1)$ and so $g = N$.

The spectral curve Γ is not smooth because in general position the genus of the curve which is a N -sheet covering of an elliptic curve is $g = \frac{1}{2}N(N - 1) + 1$.

2.6. The ψ -function as the Baker–Akhiezer function on the spectral curve

Let P be a point of the curve Γ , i.e. $P = (z, \lambda)$, where z and λ are connected by the equation $R(z, \lambda) = 0$. The coefficients c_i in the pole ansatz for the function ψ , after normalization, are functions on the curve Γ : $c_i = c_i(t_2, P)$. Let us normalize them by the condition $c_i(0, P) = 1$. The non-normalized components $c_i(0, P)$ are equal to $\Delta_i(0, P)$, where $\Delta_i(0, P)$ are suitable minors of the matrix $2zI - L(0)$. They are holomorphic functions on Γ outside the points above $\lambda = 0$. After normalizing the first component, all other components $c_i(0, P)$ become meromorphic functions on Γ outside the points P_j located above $\lambda = 0$. Their poles are zeros on Γ of the first minor of the matrix $2zI - L(0)$, i.e., they are given by common solutions of Eq. (2.40) and the equation $\det(2z\delta_{ij} - L_{ij}(0)) = 0$, $i, j = 2, \dots, N$. The location of these poles depends on the initial data.

The number of the poles can be found by the following argument [13,15]. Let us consider the function F of the complex variable $\lambda \in \mathcal{E}$ defined by

$$F(\lambda) = \left(\det c_i(0, P_j(\lambda)) \right)^2,$$

where $P_j(\lambda)$ are N pre-images of the point λ under the projection $\Gamma \rightarrow \mathcal{E}$. This function is well-defined as a function of λ (since it does not depend on the order of the sheets) and has double poles at the images of the poles of c_i 's. Clearly, F only vanishes at the ramification points, where at least two columns of the matrix $c_i(0, P_j(\lambda))$ coincide. Indeed, let $P_j = (z_j, \lambda)$ be the N points above λ . Then $c_i(P_j)$ are eigenvectors of $L(\lambda)$ with the eigenvalues $2z_j$. They are linearly independent if all the z_j 's are different. Therefore, F does not vanish at such a point. Let us now assume that λ is a ramification point which is generically of order 2. It is easy to see that at such a point F has a simple zero. Indeed, let ξ be a local parameter on the curve around the ramification point, then $\lambda = \lambda_0 + \lambda_1 \xi^2 + O(\xi^3)$ in a small neighborhood of this point. The determinant is $O(\xi)$ hence $F = O(\xi^2)$ but this is precisely proportional to $\lambda - \lambda_0$. If M is the number of poles of the vector $\mathbf{c}$ on Γ , then $2M = \nu = 2(N - 1)$, hence $M = N - 1$.

Finding explicitly eigenvectors of the matrix $E - I$, one can see that in a neighborhood of the “points at infinity” P_j the functions $c_i(0, P)$ have the form

$$c_i(0, P) = \left(c_i^{(0(j))} + O(\lambda) \right) e^{-\zeta(\lambda)(x_i(0) - x_1(0))}, \quad 2 \leq i \leq N, \quad j \neq N, \quad (2.43)$$

where $\sum_{i=2}^N c_i^{(0(j))} = -1$ and

$$c_i(0, P) = \left(1 + O(\lambda) \right) e^{-\zeta(\lambda)(x_i(0) - x_1(0))}, \quad 2 \leq i \leq N, \quad j = N \quad (2.44)$$

(on the upper sheet).

The fundamental matrix $S(t_2)$ of solutions to the equation $\partial_{t_2} S = MS$, $S(0) = I$, is a regular function of λ for $\lambda \neq 0$. From the Lax equation it follows that $\mathbf{c}(t_2) = S(t_2)\mathbf{c}(0)$ is the common solution of the equations $\dot{\mathbf{c}} = M\mathbf{c}$ and $L\mathbf{c} = 2z\mathbf{c}$. Thus the vector $\mathbf{c}(t_2, P)$ has the same t_2 -independent poles as the vector $\mathbf{c}(0, P)$.

In order to find $c_i(t_2, P)$ near the pre-images of the point $\lambda = 0$ it is convenient to pass to the gauge equivalent pair $\tilde{L}$, $\tilde{M}$, where

$$\tilde{L} = G^{-1}LG, \quad \tilde{M} = -G^{-1}\partial_{t_2}G + G^{-1}MG$$

with the same diagonal matrix G as before. Let $\tilde{\mathbf{c}} = G^{-1}\mathbf{c}$ be the gauge-transformed vector $\mathbf{c} = (c_1, \dots, c_N)^T$, then our linear system is

$$\tilde{L}\tilde{\mathbf{c}} = 2z\tilde{\mathbf{c}}, \quad \partial_t \tilde{\mathbf{c}} = \tilde{M}\tilde{\mathbf{c}}.$$

By a straightforward calculation one can check that the following relation holds:

$$\tilde{M} = \lambda^{-2}I - \lambda^{-1}\tilde{L} + O(1). \quad (2.45)$$

Applying the both sides to the eigenvector $\tilde{\mathbf{c}}$ of $\tilde{L}$ with the eigenvalue $2z$, we get

$$\partial_{t_2} \tilde{\mathbf{c}} = (\lambda^{-2} - 2z\lambda^{-1})\tilde{\mathbf{c}} + O(1). \quad (2.46)$$

Therefore, since $z = \lambda^{-1} + O(1)$ on all sheets except the upper one, we have

$$\partial_{t_2} \tilde{\mathbf{c}}^{(j)} = -(z^2 + O(1))\tilde{\mathbf{c}}^{(j)}, \quad j = 1, \dots, N-1, \quad (2.47)$$

so

$$\tilde{\mathbf{c}}^{(j)}(t_2, P) = (\mathbf{c}^{(0(j))} + O(\lambda))e^{-z^2 t_2}, \quad j = 1, \dots, N-1.$$

For the vector $\tilde{\mathbf{c}}^{(N)}$ on the upper sheet we have from (2.46), recalling (2.42),

$$\partial_{t_2} \tilde{\mathbf{c}}^{(N)} = (-z^2 + k^2(\lambda) + O(1))\tilde{\mathbf{c}}^{(N)}, \quad (2.48)$$

where

$$k(\lambda) = -N\lambda^{-1} + f_N,$$

so

$$\tilde{\mathbf{c}}^{(N)}(t_2, P) = (\mathbf{e} + O(\lambda))e^{(-z^2 + k^2(\lambda))t_2}$$

(here $\mathbf{e} = (1, 1, \dots, 1)^T$). Coming back to the vector $\mathbf{c}(t_2, P)$, we obtain after normalization

$$c_i^{(j)}(t_2, P) = c_{ij}(\lambda)e^{-\zeta(\lambda)(x_i(t_2) - x_1(0)) + v_j(\lambda)t_2}, \quad (2.49)$$

where $v_j = -z^2$ for $j = 1, \dots, N-1$, $v_N = -z^2 + k^2(\lambda)$ and $c_{ij}(\lambda)$ are regular functions in a neighborhood of $\lambda = 0$. Their values at $\lambda = 0$ are

$$c_{1j}(0) = 1, \quad j = 1, \dots, N, \quad c_{ij}(0) = c_i^{(0(j))}, \quad i \geq 2, \quad j \neq N, \quad c_{iN}(0) = 1, \quad (2.50)$$

with $\sum_{i=2}^N c_i^{(0(j))} = -1$.

After investigating the analytic properties of the vector $\mathbf{c}(t_2, P)$ let us turn to the function ψ :

$$\psi(x, t_2, P) = \sum_{i=1}^N c_i(t_2, P)\Phi(x - x_i, \lambda)e^{2x + z^2 t_2}.$$

The function $\Phi(x - x_i, \lambda)$ has essential singularities at all points P_j located above $\lambda = 0$. It follows from (2.49) that in the function ψ these essential singularities cancel on all sheets except the upper one, where $\psi \propto e^{k(\lambda)x + k^2(\lambda)t_2} e^{\zeta(\lambda)x_1(0)}$. From (2.50) it follows that ψ has a simple pole at the point P_N and no poles at the points P_j for $j = 1, \dots, N-1$. As we have seen before, the function ψ also has $N-1$ poles in the finite part of the curve Γ , which do not depend on x, t_2 . These analytic properties allow one to identify the function $\psi(x, t_2, P)$ with the Baker-Akhiezer function on the spectral curve Γ with the marked point at infinity P_N .

2.7. Self-dual form of the equations of motion

Another way to derive the equations of motion is to parametrize the wave function through its poles and zeros and substitute into the linear problem (2.10). Namely, represent the wave function as $\psi = \tilde{\tau}/\tau$, then the linear problem (2.10) acquires the form

$$\partial_{t_2} \log \frac{\tilde{\tau}}{\tau} = \partial_x^2 \log(\tau \tilde{\tau}) + \left(\partial_x \log \frac{\tilde{\tau}}{\tau} \right)^2, \quad (2.51)$$

or

$$(D_2 + D_1^2)\tau \cdot \tilde{\tau} = 0, \quad (2.52)$$

which is the mKP equation in the bilinear Hirota form. Let us put $t_3 = 0$ in this subsection for simplicity. Then $\tilde{\tau} = e^{zx+z^2t_2}\hat{\tau}$, where $\hat{\tau}$ is the tau-function with shifted times $\mathbf{t} = (t_1, t_2, t_3, \dots)$: $\hat{\tau}(\mathbf{t}) = \tau(\mathbf{t} - [z^{-1}])$, where we have used the standard notation

$$\mathbf{t} \pm [z^{-1}] = \left(t_1 \pm \frac{1}{z}, t_2 \pm \frac{1}{2z^2}, t_3 \pm \frac{1}{3z^3}, \dots \right).$$

Eq. (2.51) becomes

$$\partial_{t_2} \log \frac{\hat{\tau}}{\tau} = \partial_x^2 \log(\tau \hat{\tau}) + \left(\partial_x \log \frac{\hat{\tau}}{\tau} \right)^2 + 2z \partial_x \log \frac{\hat{\tau}}{\tau}. \quad (2.53)$$

Let y_i ($i = 1, \dots, N$) be zeros of the function $\hat{\tau}$. Then we can write

$$\frac{\hat{\tau}}{\tau} = A e^{\alpha x + \beta t_2} \prod_i \frac{\sigma(x - y_i)}{\sigma(x - x_i)},$$

where A, α, β are some constants. Substituting expressions for $\tau, \hat{\tau}$ through σ -functions into (2.53), we obtain:

$$\begin{aligned} \sum_i \left(\dot{x}_i \zeta(x - x_i) - \dot{y}_i \zeta(x - y_i) \right) &= - \sum_i \left(\wp(x - x_i) + \wp(x - y_i) \right) \\ &\quad + \left(\sum_i \left(\zeta(x - x_i) - \zeta(x - y_i) \right) \right)^2 + \mu \sum_i \left(\zeta(x - x_i) - \zeta(x - y_i) \right) + \text{const}, \end{aligned}$$

where μ is a constant. Identifying residues at the poles at $x = x_i$ and $x = y_i$, we obtain the following system of first order differential equations:

$$\begin{cases} \dot{x}_i = 2 \sum_{j \neq i} \zeta(x_i - x_j) - 2 \sum_j \zeta(x_i - y_j) + \mu \\ \dot{y}_i = -2 \sum_{j \neq i} \zeta(y_i - y_j) + 2 \sum_j \zeta(y_i - x_j) + \mu. \end{cases} \quad (2.54)$$

This is the so-called self-dual form of equations of motion of the elliptic Calogero–Moser system [1,3]. For the first time it appeared in [31] as a Bäcklund transformation of the Calogero–Moser system.

It can be shown that Eqs. (2.54) are equivalent to (1.1). Indeed, let us differentiate the first equation in (2.54) with respect to t_2 :

$$\begin{aligned} \ddot{x}_i &= -2 \sum_{j \neq i} (\dot{x}_i - \dot{x}_j) \wp(x_i - x_j) + 2 \sum_j (\dot{x}_i - \dot{y}_j) \wp(x_i - y_j) \\ &= -4 \sum_{j \neq i} \left(\sum_{k \neq i} \zeta(x_i - x_k) - \sum_k \zeta(x_i - y_k) - \sum_{k \neq j} \zeta(x_j - x_k) + \sum_k \zeta(x_j - y_k) \right) \wp(x_i - x_j) \\ &\quad + 4 \sum_j \left(\sum_{k \neq i} \zeta(x_i - x_k) - \sum_k \zeta(x_i - y_k) + \sum_{k \neq j} \zeta(y_j - y_k) - \sum_k \zeta(y_j - x_k) \right) \wp(x_i - y_j). \end{aligned} \quad (2.55)$$

It can be proved [3] (see the Appendix) that the right hand side is in fact equal to $4 \sum_{j \neq i} \wp'(x_i - x_j)$. By symmetry, the same Calogero–Moser equations of motion are satisfied by y_i 's.

2.8. Calogero–moser system in discrete time

The self-dual form of equations of motion is directly connected with the integrable time discretization of the Calogero–Moser system. To see this, let us consider dynamics of poles of elliptic solutions to the semi-discrete KP equation and note that the discrete time flow n in the KP hierarchy is introduced according to the rule [10]

$$\tau^n(\mathbf{t}) = \tau(\mathbf{t} - n[z^{-1}]), \quad (2.56)$$

so that τ and $\hat{\tau}$ are tau-functions taken at two subsequent values of the discrete time. Accordingly, we can denote $x_i = x_i^n$, $y_i = x_i^{n+1}$ and rewrite Eqs. (2.54) as

$$\begin{cases} \dot{x}_i^n = 2 \sum_{j \neq i} \zeta(x_i^n - x_j^n) - 2 \sum_j \zeta(x_i^n - x_j^{n+1}) + \mu \\ \dot{x}_i^{n+1} = -2 \sum_{j \neq i} \zeta(x_i^{n+1} - x_j^{n+1}) + 2 \sum_j \zeta(x_i^{n+1} - x_j^n) + \mu. \end{cases} \quad (2.57)$$

Shifting $n \rightarrow n-1$ in the second group of equations and subtracting the second line from the first one, we get equations of motion for the Calogero–Moser system in discrete time [21,22]:

$$\sum_j \zeta(x_i^n - x_j^{n+1}) + \sum_j \zeta(x_i^n - x_j^{n-1}) - 2 \sum_{j \neq i} \zeta(x_i^n - x_j^n) = 0. \quad (2.58)$$

Remarkably, these equations coincide with the nested Bethe ansatz equations for the elliptic Gaudin model associated with the root system A_m , with the discrete time n taking values $0, 1, \dots, m+1$.

3. Elliptic solutions to the BKP equation

3.1. The BKP equation

The BKP equation is the first member of an infinite BKP hierarchy with independent variables (“times”) $t_1 = x$, $t_3, t_5, t_7, \dots$ [8,9], see also [7,18,29]. It is the following system of nonlinear partial differential equations for two dependent variables u, w :

$$\begin{cases} 3w' = 10u_{t_3} + 20u''' + 120uu' \\ w_{t_3} - 6u_{t_5} = w''' - 6u^V - 60uu''' - 60u'u'' + 6uw' - 6wu', \end{cases} \quad (3.1)$$

where prime means differentiation w.r.t. x . (Some notation in this section such as w and L, M below is the same as in the previous one but their meaning is different; we hope that this will not lead to a misunderstanding.) It is easy to see that the variable w can be excluded and the equation can be written in terms a single dependent variable $U = \int^x u dx$. Eqs. (1.5) are equivalent to the Zakharov–Shabat equation $\partial_{t_5} B_3 - \partial_{t_3} B_5 + [B_3, B_5] = 0$ for the differential operators

$$B_3 = \partial_x^3 + 6u\partial_x, \quad B_5 = \partial_x^5 + 10u\partial_x^3 + 10u'\partial_x^2 + w\partial_x. \quad (3.2)$$

Similarly to the case discussed in the previous section, the Zakharov–Shabat equation is the compatibility condition for the auxiliary linear problems

$$\partial_{t_3} \psi = B_3 \psi, \quad \partial_{t_5} \psi = B_5 \psi$$

for the wave function ψ which depends on a spectral parameter z .

The tau-function $\tau = \tau(t_1, t_3, t_5, \dots)$ of the BKP hierarchy is related to the variables u, w by the formulas

$$u = \partial_x^2 \log \tau, \quad w = \frac{10}{3} \partial_{t_3} \partial_x \log \tau + \frac{20}{3} \partial_x^4 \log \tau + 20(\partial_x^2 \log \tau)^2 \quad (3.3)$$

In terms of the tau-function, Eqs. (3.1) acquire the bilinear form [8]

$$\left(D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5 \right) \tau \cdot \tau = 0. \quad (3.4)$$

The wave function ψ can be expressed through the tau-function according to the formula [8]

$$\psi = A(z) \exp \left(\sum_{k \geq 1, k \text{ odd}} t_k z^k \right) \frac{\tau \left(t_1 - 2z^{-1}, t_3 - \frac{2}{3} z^{-3}, t_5 - \frac{2}{5} z^{-5}, \dots \right)}{\tau(t_1, t_3, t_5, \dots)}, \quad (3.5)$$

where $A(z)$ is a normalization factor.

3.2. Dynamics of poles

We are going to study dynamics of poles of elliptic in the variable $t_1 = x$ solutions of the BKP equation as functions of $t_3 = t$. For such solutions the tau-function has the same form (2.8):

$$\tau = A e^{Q(x, t, \dots)} \prod_{i=1}^N \sigma(x - x_i) \quad (3.6)$$

with a quadratic form $Q = cx^2 + \dots$ and u is given by (2.9). The basic tool for studying t -dynamics of poles is the linear problem $\partial_t \psi = B_3 \psi$ for the function ψ , i.e.,

$$\partial_t \psi = \partial_x^3 \psi + 6u \partial_x \psi. \quad (3.7)$$

As in the KP case, one can find double-Bloch solutions $\psi(x)$

$$\psi = e^{xz + tz^3} \sum_{i=1}^N c_i \Phi(x - x_i, \lambda) \quad (3.8)$$

with simple poles at $x = x_i$ and x -independent coefficients c_i (the function Φ is the same as in the previous section).

It is evident from (3.7) that the constant c in the expression (2.9) for the function u can be eliminated by the transformation $x \rightarrow x - 12ct$, $t \rightarrow t$ (or $\partial_x \rightarrow \partial_x$, $\partial_t \rightarrow \partial_t + 12c\partial_x$ for the vector fields). Because of this we put $c = 0$ from now on for simplicity.

Substituting (3.8) into (3.7) with $u = -\sum_i \wp(x - x_i)$, we get:

$$\begin{aligned} \sum_i \dot{c}_i \Phi(x - x_i) - \sum_i c_i \dot{x}_i \Phi'(x - x_i) &= 3z^2 \sum_i c_i \Phi'(x - x_i) + 3z \sum_i c_i \Phi''(x - x_i) + \sum_i c_i \Phi'''(x - x_i) \\ &- 6z \left(\sum_k \wp(x - x_k) \right) \left(\sum_i c_i \Phi(x - x_i) \right) - 6 \left(\sum_k \wp(x - x_k) \right) \left(\sum_i c_i \Phi'(x - x_i) \right). \end{aligned}$$

This expression has poles at $x = x_i$ (up to fourth order). Poles of the fourth and third order cancel identically. As it can be seen by a direct calculation, the conditions of cancellation of second and first order poles have the form

$$c_i \dot{x}_i = -(3z^2 - 3\wp(\lambda))c_i - 6z \sum_{k \neq i} c_k \Phi(x_i - x_k) - 6 \sum_{k \neq i} c_k \Phi'(x_i - x_k) + 6c_i \sum_{k \neq i} \wp(x_i - x_k), \quad (3.9)$$

$$\begin{aligned} \dot{c}_i &= 3z\wp(\lambda)c_i + 2\wp'(\lambda)c_i - 6z \sum_{k \neq i} c_k \Phi'(x_i - x_k) - 6zc_i \sum_{k \neq i} \wp(x_i - x_k) \\ &- 6 \sum_{k \neq i} c_k \Phi''(x_i - x_k) + 6c_i \sum_{k \neq i} \wp'(x_i - x_k) \end{aligned} \quad (3.10)$$

which have to be valid for all $i = 1, \dots, N$. In the matrix form, these conditions look like linear problems for a vector $\mathbf{c} = (c_1, \dots, c_N)^T$:

$$\begin{cases} L\mathbf{c} = 3(z^2 - \wp(\lambda))\mathbf{c} \\ \dot{\mathbf{c}} = M\mathbf{c}, \end{cases} \quad (3.11)$$

where

$$L = -\dot{X} - 6zA - 6B + 6D, \quad (3.12)$$

$$M = (3z\wp(\lambda) + 2\wp'(\lambda))I - 6zB - 6zD - 6C + 6D' \quad (3.13)$$

with the same matrices X, A, B, C, D, D' as in the previous section. Note that in the present case the matrices L, M depend not only on λ but also on z . The compatibility condition of the linear problems (3.11) is

$$(\dot{L} + [L, M])\mathbf{c} = 0. \quad (3.14)$$

In the Appendix we prove the following matrix identity:

$$\dot{L} + [L, M] = -12D'(L - 3(z^2 - \wp(\lambda))I) - \ddot{X} + 12D'(6D - \dot{X}) + 6\dot{D} - 6D''', \quad (3.15)$$

where $D''_{ik} = \delta_{ik} \sum_{j \neq i} \wp'''(x_i - x_j)$. Using this identity, it is straightforward to see that the compatibility condition (3.14) is equivalent to vanishing of all elements of the diagonal matrix $-\ddot{X} + 12D'(6D - \dot{X}) + 6\dot{D} - 6D'''$. Writing the diagonal elements explicitly, we get equations of motion for the poles x_i :

$$\ddot{x}_i + 6 \sum_{j \neq i} (\dot{x}_i + \dot{x}_j) \wp'(x_i - x_j) - 72 \sum_{j \neq i} \sum_{k \neq i} \wp(x_i - x_j) \wp'(x_i - x_k) + 6 \sum_{j \neq i} \wp'''(x_i - x_j) = 0.$$

Using the identity $\wp'''(x) = 12\wp(x)\wp'(x)$, we get the equations of motion (1.3). They were obtained in [24]. Their rational limit (when both periods tend to infinity and $\wp(x) \rightarrow 1/x^2$) is

$$\ddot{x}_i - 12 \sum_{j \neq i} \frac{\dot{x}_i + \dot{x}_j}{(x_i - x_j)^3} + 144 \sum_{j \neq k \neq i} \frac{1}{(x_i - x_j)^2 (x_i - x_k)^3} = 0. \quad (3.16)$$

We see that in contrast to the equations of motion for poles of elliptic solutions to the KP equation, where interaction between “particles” (poles) is pairwise, in the BKP case there is a three-body interaction.

3.3. Integrals of motion

Eqs. (1.3) do not admit a representation of Lax type. Instead, the matrix relation

$$\dot{L} + [L, M] = -12D'(L - \Lambda I), \quad (3.17)$$

where $\Lambda = 3(z^2 - \wp(\lambda))$, is equivalent to the equations of motion (1.3). This is a sort of Manakov's triple representation. In contrast to the KP case, the evolution $L \rightarrow L(t)$ of our "Lax matrix" is not isospectral. Nevertheless, the characteristic polynomial, $\det(L - \Lambda I)$, is an integral of motion. Indeed,

$$\begin{aligned} \frac{d}{dt} \log \det(L - \Lambda I) &= \frac{d}{dt} \operatorname{tr} \log(L - \Lambda I) \\ &= \operatorname{tr} \left(\dot{L}(L - \Lambda I)^{-1} \right) = -12 \operatorname{tr} D' = 0, \end{aligned}$$

where we have used (3.17) and the fact that the matrix D' is traceless (because $\wp'$ is an odd function and so $\sum_{i \neq j} \wp'(x_i - x_j) = 0$). The expression

$$R(z, \lambda) = \det(3(z^2 - \wp(\lambda))I - L)$$

is a polynomial in z of degree $2N$. Its coefficients are integrals of motion (some of them may be trivial).

Applying the same similarity transformation $L = \tilde{G}L\tilde{G}^{-1}$ as in the previous section with the matrix $G_{ij} = \delta_{ij}e^{-\zeta(\lambda)x_i}$, we conclude that the coefficients $R_k(\lambda)$ of the polynomial

$$R(z, \lambda) = \sum_{k=0}^{2N} R_k(\lambda) z^k \quad (3.18)$$

are elliptic functions of λ with poles at $\lambda = 0$.

Example ($N = 2$).

$$\begin{aligned} \det_{2 \times 2} (3(z^2 - \wp(\lambda))I - L) &= 9z^4 + 3z^2(\dot{x}_1 + \dot{x}_2 - 18\wp(\lambda)) - 36z\wp'(\lambda) - 3\wp(\lambda)(\dot{x}_1 + \dot{x}_2) \\ &\quad + \dot{x}_1\dot{x}_2 - 6(\dot{x}_1 + \dot{x}_2)\wp(x_1 - x_2) - 27\wp^2(\lambda) + 9g_2, \end{aligned}$$

where g_2 is the coefficient in the expansion of the $\wp$ -function near $x = 0$: $\wp(x) = x^{-2} + \frac{1}{20}g_2x^2 + \frac{1}{28}g_3x^4 + O(x^6)$. Therefore, for $N = 2$ there are two integrals of motion: $I_1 = \dot{x}_1 + \dot{x}_2$, $I_2 = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + 6(\dot{x}_1 + \dot{x}_2)\wp(x_1 - x_2)$.

Example ($N = 3$).

$$\begin{aligned} \det_{3 \times 3} (3(z^2 - \wp(\lambda))I - L) &= 27z^6 + 9(I_1 - 45\wp(\lambda))z^4 - 540\wp'(\lambda)z^3 \\ &\quad + \left[\frac{3}{2}I_1^2 - 3I_2 - 54\wp(\lambda)I_1 - 1215\wp^2(\lambda) + 243g_2 \right] z^2 - 36\wp'(\lambda)(I_1 + 9\wp(\lambda))z \\ &\quad + I_3 - I_1I_2 + \frac{1}{6}I_1^3 + 3\wp(\lambda)(I_2 - \frac{1}{2}I_1^2) - 27\wp^2(\lambda)I_1 + 9g_2I_1 - 135\wp^3(\lambda) - 27g_2\wp(\lambda) + 216g_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \dot{x}_1 + \dot{x}_2 + \dot{x}_3, \\ I_2 &= \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) + 6\dot{x}_1(\wp(x_{12}) + \wp(x_{13})) + 6\dot{x}_2(\wp(x_{21}) + \wp(x_{23})) \\ &\quad + 6\dot{x}_3(\wp(x_{31}) + \wp(x_{32})) - 36(\wp(x_{12})\wp(x_{13}) + \wp(x_{12})\wp(x_{23}) + \wp(x_{13})\wp(x_{23})), \\ I_3 &= \frac{1}{3}(\dot{x}_1^3 + \dot{x}_2^3 + \dot{x}_3^3) + 6\dot{x}_1^2(\wp(x_{12}) + \wp(x_{13})) + 6\dot{x}_2^2(\wp(x_{21}) + \wp(x_{23})) \\ &\quad + 6\dot{x}_3^2(\wp(x_{31}) + \wp(x_{32})) + 12\dot{x}_1\dot{x}_2\wp(x_{12}) + 12\dot{x}_1\dot{x}_3\wp(x_{13}) + 12\dot{x}_2\dot{x}_3\wp(x_{23}) \\ &\quad - 864\wp(x_{12})\wp(x_{13})\wp(x_{23}) \end{aligned} \quad (3.19)$$

are integrals of motion (here $x_{ik} \equiv x_i - x_k$).

For arbitrary N , one can prove that the quantities

$$\begin{aligned} I_1 &= \sum_i \dot{x}_i, \\ I_2 &= \frac{1}{2} \sum_i \dot{x}_i^2 + 6 \sum_{i \neq j} \dot{x}_i \wp(x_{ij}) - 18 \sum_{i \neq j \neq k} \wp(x_{ij}) \wp(x_{ik}) \end{aligned} \quad (3.20)$$

are integrals of motion. In the expression for I_2 the last sum is taken over all triples of distinct numbers i, j, k from 1 to N . The proof can be found in [24]. It is based on the following identities for the $\wp$ -function:

$$\sum_{i=1}^n \partial_{x_i} \prod_{k=1, k \neq i}^n \wp(x_i - x_k) = 0, \quad n = 2, 3, \dots \quad (3.21)$$

(for the proof we need them at $n = 3$ and $n = 4$). The proof of (3.21) is standard. The left hand side is an elliptic function of x_1 . Expanding it near possible poles at $x_1 = x_k$, $k = 2, \dots, n$, one can see that it is regular, so it is a constant independent of x_1 . By symmetry, this constant does not depend also on all the x_i 's. To see that this constant is actually zero, one can put $x_k = kx$.

Another integral of motion for arbitrary N is

$$J = \lim_{\lambda \rightarrow 0} R(\lambda^{-1}, \lambda) = \det_{1 \leq i, j \leq N} \left[\delta_{ij} \dot{x}_i - 6\delta_{ij} \sum_{k \neq i} \wp(x_{ik}) - 6(1 - \delta_{ij})\wp(x_{ij}) \right]. \quad (3.22)$$

Indeed, using the obvious formula $\Phi'(x, \lambda) = \Phi(x, \lambda) \left(\zeta(x + \lambda) - \zeta(x) - \zeta(\lambda) \right)$ and the expansion

$$\tilde{\Phi}(x, \lambda) = e^{\zeta(\lambda)x} \Phi(x, \lambda) = \lambda^{-1} + \zeta(x) + \frac{1}{2} \frac{\sigma''(x)}{\sigma(x)} \lambda + O(\lambda^2),$$

we have

$$\tilde{L}(z, \lambda) = (z - \lambda^{-1})Y(z, \lambda) + \dot{X} - 6D - 6Q + O(\lambda),$$

where Q is the off-diagonal matrix with elements $Q_{ij} = (1 - \delta_{ij})\wp(x_{ij})$ and $Y(z, \lambda)$ is a matrix which is regular at $z = \lambda^{-1}$. Therefore, $R(\lambda^{-1}, \lambda) = \det(\dot{X} - 6D - 6Q) + O(\lambda)$.

3.4. The spectral curve

The spectral curve is given by the equation

$$R(z, \lambda) = \det(3(z^2 - \wp(\lambda))I - L(z, \lambda)) = 0. \quad (3.23)$$

It is easy to see that $L(-z, -\lambda) = L^T(z, \lambda)$, so the spectral curve admits the involution $\iota : (z, \lambda) \rightarrow (-z, -\lambda)$.

As it was already mentioned, the coefficients $R_k(\lambda)$ in (3.18) are elliptic functions of λ . The functions $R_k(\lambda)$ obey the property $R_k(-\lambda) = (-1)^k R_k(\lambda)$ and can be represented as linear combinations of $\wp$ -function and its derivatives. Coefficients of this expansion are integrals of motion. Fixing values of these integrals, we obtain via the equation $R(z, \lambda) = 0$ the algebraic curve Γ which is a $2N$ -sheet covering of the initial elliptic curve $\mathcal{E}$ realized as a factor of the complex plane with respect to the lattice generated by $2\omega, 2\omega'$.

In a neighborhood of $\lambda = 0$ the matrix $\tilde{L} = G^{-1}LG$ can be written as

$$\tilde{L} = -6\lambda^{-1}(z - \lambda^{-1})(E - I) - 6(z - \lambda^{-1})S + O(1),$$

where E is the same rank 1 matrix with matrix elements $E_{ij} = 1$ as in the previous section and S is the antisymmetric matrix with matrix elements $S_{ij} = \zeta(x_i - x_j)$, $i \neq j$, $S_{ii} = 0$. Therefore, near $\lambda = 0$ the function $R(z, \lambda)$ can be represented in the form

$$\begin{aligned} R(z, \lambda) &= \det(3(z^2 - \lambda^{-2})I + 6\lambda^{-1}(z - \lambda^{-1})(E - I) + 6(z - \lambda^{-1})S + O(1)) \\ &= \det(3(z - \lambda^{-1})^2I + 6\lambda^{-1}(z - \lambda^{-1})E + 6(z - \lambda^{-1})S + O(1)) \\ &= 3^N (z - \lambda^{-1})^{2N} \det\left(I + \frac{2}{z\lambda - 1}E + \frac{2\lambda}{z\lambda - 1}S + O(\lambda^2)\right). \end{aligned}$$

Using the fact that $\det(A + \varepsilon B) = \det A (1 + \varepsilon \operatorname{tr}(A^{-1}B)) + O(\varepsilon^2)$ for any two matrices A, B and the relation $(I - \alpha E)^{-1} = I + \frac{\alpha}{1 - N\alpha}E$, we find

$$\begin{aligned} &\det\left(I + \frac{2}{z\lambda - 1}E + \frac{2\lambda}{z\lambda - 1}S + O(\lambda^2)\right) \\ &= \det\left(I + \frac{2}{z\lambda - 1}E + O(\lambda^2)\right) \left(1 + \frac{2\lambda}{z\lambda - 1} \operatorname{tr}\left(S - \frac{2}{z\lambda + 2N - 1}ES\right) + O(\lambda^2)\right). \end{aligned}$$

For any antisymmetric matrix S we have $\operatorname{tr} S = \operatorname{tr}(ES) = 0$, so we are left with

$$R(z, \lambda) = 3^N (z - \lambda^{-1})^{2N} \det\left(I + \frac{2}{z\lambda - 1}E + O(\lambda^2)\right).$$

Therefore, we can write $R(z, \lambda)$ in the form

$$R(z, \lambda) = 3^N \left(z + (2N - 1)\lambda^{-1} - f_{2N}(\lambda) \right) \left(z - \lambda^{-1} - f_1(\lambda) \right) \prod_{i=2}^{2N-1} \left(z - \lambda^{-1} - f_i(\lambda) \right), \quad (3.24)$$

where f_i are regular functions of λ at $\lambda = 0$. This means that the function z has simple poles on all sheets at the points P_j ($j = 1, \dots, 2N$) located above $\lambda = 0$. The involution ι implies that f_{2N} and f_1 are odd functions: $f_{2N}(-\lambda) = -f_{2N}(\lambda)$, $f_1(-\lambda) = -f_1(\lambda)$. The other sheets can be numbered in such a way that $f_i(-\lambda) = -f_{2N+1-i}(\lambda)$, $i = 2, 3, \dots, N$. We have the following expansions of the function z near the “points at infinity” P_j :

$$\begin{aligned} z &= \lambda^{-1} + f_j(\lambda) \quad \text{near } P_j, \quad j = 1, \dots, 2N - 1, \\ z &= -(2N - 1)\lambda^{-1} + f_{2N}(\lambda) \quad \text{near } P_{2N}. \end{aligned} \quad (3.25)$$

Similarly to the spectral curve of the elliptic Calogero–Moser model (2.41), one of the sheets is distinguished. We call it the upper sheet. There is also another distinguished sheet, where the point P_1 is located. We call it the lower sheet for brevity. The points P_1, P_{2N} are two fixed points of the involution ι .

The genus g of the spectral curve Γ can be found by an argument which is similar to the one in Section 2.5. We have $2g - 2 = \nu$, where ν is the number of ramification points of the covering $\Gamma \rightarrow \mathcal{E}$. The ramification points are zeros on Γ of the function $\partial R / \partial z$. The function $\partial R / \partial z$ has simple poles at the points P_j ($j = 1, \dots, 2N - 1$) on all sheets except the upper one, where it has a pole of order $2N - 1$. Therefore, $\nu = 2(2N - 1)$ and $g = 2N$.

3.5. Analytic properties of the ψ -function on the spectral curve

Similarly to Section 2.6, the coefficients c_i in the pole ansatz for the function ψ are functions on the spectral curve Γ : $c_i = c_i(t, P)$ ($P = (z, \lambda)$ is a point on the curve). Let us normalize them by the condition $c_1(0, P) = 1$. After normalization the components $c_i(0, P)$ become meromorphic functions on Γ outside the points P_j located above $\lambda = 0$. The location of their poles depends on the initial data.

On all sheets except the lower one the leading term of the matrix $\tilde{L}$ as $\lambda \rightarrow 0$ is proportional to $E - I$. Finding explicitly eigenvectors of the matrix $E - I$, one can see that near the “points at infinity” P_j ($j = 2, \dots, 2N$) the functions $c_i(0, P)$ have the form

$$c_i(0, P) = \left(c_i^{0(j)} + O(\lambda) \right) e^{-\zeta(\lambda)(x_i(0) - x_1(0))}, \quad 2 \leq i \leq N, \quad j = 2, \dots, 2N - 1 \quad (3.26)$$

on all sheets except the lower and upper ones. Here $\sum_{i=2}^N c_i^{0(j)} = -1$. On the upper sheet

$$c_i(0, P) = \left(1 + O(\lambda) \right) e^{-\zeta(\lambda)(x_i(0) - x_1(0))}, \quad 2 \leq i \leq N, \quad j = 2N. \quad (3.27)$$

On the lower sheet, the leading term of the matrix $\tilde{L}$ as $\lambda \rightarrow 0$ is $O(1)$. Expanding the matrix $\tilde{L}$ in powers of λ , we have

$$\Lambda I - \tilde{L} = 6f_1'(0)E + \dot{X} - 6D - 6Q + O(\lambda),$$

where Q is the matrix with matrix elements $Q_{ij} = (1 - \delta_{ij})\wp(x_i - x_j)$. Let $c_i^{0(1)}$ be the eigenvector of the matrix $6f_1'(0)E + \dot{X} - 6D - 6Q$ (taken at $t = 0$) with zero eigenvalue normalized by the condition $c_1^{0(1)} = 1$, then in a neighborhood of the point P_1 we have

$$c_i(0, P) = \left(c_i^{0(1)} + O(\lambda) \right) e^{-\zeta(\lambda)(x_i(0) - x_1(0))}, \quad 2 \leq i \leq N, \quad j = 1. \quad (3.28)$$

Let $S(t)$ be the fundamental matrix of solutions to the equation $\partial_t S = MS$, $S(0) = I$. It is a regular function of z, λ for $\lambda \neq 0$. Using the Manakov's triple representation (3.17), we can write

$$\left(\dot{L} + [L, M] + 12D'(L - \Lambda I) \right) \mathbf{c}(t) = 0, \quad \Lambda = 3(z^2 - \wp(\lambda)).$$

Using the relations $\mathbf{c}(t) = S(t)\mathbf{c}(0)$ and $M = \dot{S}S^{-1}$, we rewrite this equation as

$$\left[\partial_t \left(S^{-1}(L - \Lambda I)S \right) + 12S^{-1}D'(L - \Lambda I)S \right] \mathbf{c}(0) = 0.$$

Equivalently, we can represent it in the form of the differential equation

$$\partial_t \mathbf{b}(t) = W(t)\mathbf{b}(t), \quad W(t) = 12S^{-1}D'S,$$

for the vector $\mathbf{b}(t) = S^{-1}(L - \Lambda I)\mathbf{c}(t)$ with the initial condition $\mathbf{b}(0) = 0$. The differential equation with zero initial condition has the unique solution $\mathbf{b}(t) = 0$ for all $t > 0$. It then follows that $\mathbf{c}(t) = S(t)\mathbf{c}(0)$ is the common solution of the equations $\dot{\mathbf{c}} = M\mathbf{c}$ and $\dot{\mathbf{c}} = \Lambda\mathbf{c}$ for all $t > 0$. Therefore, the vector $\mathbf{c}(t, P)$ has the same t -independent poles as $\mathbf{c}(0, P)$.

Similarly to Section 2.6, the next step is to pass to the gauge equivalent pair $\tilde{L}, \tilde{M}$, where

$$\tilde{L} = G^{-1}LG, \quad \tilde{M} = -G^{-1}\partial_t G + G^{-1}MG$$

with the same diagonal matrix $G_{ij} = \delta_{ij}e^{-\zeta(\lambda)x_i}$ as before. The gauge-transformed linear system is

$$\tilde{L}\tilde{\mathbf{c}} = 3(z^2 - \wp(\lambda))\tilde{\mathbf{c}}, \quad \partial_t \tilde{\mathbf{c}} = \tilde{M}\tilde{\mathbf{c}},$$

where $\tilde{\mathbf{c}} = G^{-1}\mathbf{c}$, $\mathbf{c} = (c_1, \dots, c_N)^T$.

It is a straightforward calculation to verify that the following relation holds:

$$\tilde{M} = -\lambda^{-1}\tilde{L} + (3z\lambda^{-2} - 4\lambda^{-3})I + 6(z - \lambda^{-1})(Q - D) + O(1). \quad (3.29)$$

(It should be taken into account that z is of order $O(\lambda^{-1})$, see (3.25), so the terms proportional to z have to be kept in the expansion.) Applying the both sides to the eigenvector $\tilde{\mathbf{c}}$ of $\tilde{L}$ with the eigenvalue $\Lambda = 3(z^2 - \wp(\lambda)) = 3(z^2 - \lambda^{-2}) + O(\lambda^2)$, we get

$$\partial_t \tilde{\mathbf{c}} = -z^3 \tilde{\mathbf{c}} + (z - \lambda^{-1})^3 \tilde{\mathbf{c}} + 6(z - \lambda^{-1})(Q - D)\tilde{\mathbf{c}} + O(1). \quad (3.30)$$

Therefore, since $z = \lambda^{-1} + O(1)$ on all sheets except the upper one, we have

$$\partial_t \tilde{\mathbf{c}}^{(j)} = -(z^3 + O(1))\tilde{\mathbf{c}}^{(j)}, \quad j = 1, \dots, 2N - 1, \quad (3.31)$$

so

$$\tilde{\mathbf{c}}^{(j)}(t, P) = (\mathbf{c}^{0(j)} + O(\lambda))e^{-z^3 t}, \quad j = 1, \dots, 2N - 1.$$

On the upper sheet, the corresponding eigenvector $\tilde{\mathbf{c}}^{2N}$ of the matrix $\tilde{L}$ is proportional to the vector $\mathbf{e} = (1, 1, \dots, 1)^T$ (plus terms of order $O(1)$). Note that $(Q - D)\mathbf{e} = 0$. Therefore, since $z = -(2N - 1)\lambda^{-1} + f_{2N}$ on the upper sheet, we have

$$\partial_t \tilde{\mathbf{c}}^{(2N)} = (-z^3 + k^3(\lambda) + O(1))\tilde{\mathbf{c}}^{(2N)}, \quad (3.32)$$

where

$$k(\lambda) = -2N\lambda^{-1} + f_{2N},$$

so

$$\tilde{\mathbf{c}}^{(2N)}(t, P) = (\mathbf{e} + O(\lambda))e^{(-z^3 + k^3(\lambda))t}.$$

Hence the normalized vector $\mathbf{c}(t, P)$ is of the form

$$c_i^{(j)}(t, P) = c_{ij}(\lambda)e^{-\zeta(\lambda)(x_i(t) - x_1(0)) + v_j(\lambda)t}, \quad (3.33)$$

where $v_j = -z^3$ for $j = 1, \dots, 2N - 1$, $v_{2N} = -z^3 + k^3(\lambda)$ and $c_{ij}(\lambda)$ are regular functions in a neighborhood of $\lambda = 0$. Their values at $\lambda = 0$ are

$$c_{1j}(0) = 1, \quad j = 1, \dots, 2N, \quad c_{ij}(0) = c_i^{0(j)}, \quad i \geq 2, \quad j \neq 2N, \quad c_{i2N}(0) = 1, \quad (3.34)$$

with $\sum_{i=2}^N c_i^{0(j)} = -1$ for $j = 2, \dots, 2N - 1$.

Analytic properties of the function

$$\psi(x, t, P) = \sum_{i=1}^N c_i(t, P)\Phi(x - x_i, \lambda)e^{zx + z^3 t}$$

follow from those of the vector $\mathbf{c}(t, P)$. Eq. (3.33) implies that in the function ψ the essential singularities at $\lambda = 0$ cancel on all sheets except the upper one, where $\psi \propto e^{k(\lambda)x + k^3(\lambda)t} e^{\zeta(\lambda)x_1(0)}$. From (3.34) we see that ψ has simple poles at the points P_1, P_{2N} (the two fixed points of the involution ι) and no poles at the points P_j for $j = 2, \dots, 2N - 1$. The residue at the pole at P_1 is constant as a function of x, t . This is in agreement with the fact that the differential operators B_3, B_5 (3.2) have no free terms, and so the result of their action to a constant vanishes.

The function ψ also has other poles in the finite part of the curve Γ , which do not depend on x, t .

3.6. Self-dual form of the equations of motion

Let us represent the wave function in (3.7) as $\psi = \tilde{\tau}/\tau$, then the linear problem acquires the form

$$\partial_{t_3} \log \frac{\tilde{\tau}}{\tau} = \partial_x^3 \log \frac{\tilde{\tau}}{\tau} + 3\partial_x \log \frac{\tilde{\tau}}{\tau} \partial_x^2 \log(\tau \tilde{\tau}) + \left(\partial_x \log \frac{\tilde{\tau}}{\tau} \right)^3, \quad (3.35)$$

or

$$(D_3 - D_1^3)\tau \cdot \tilde{\tau} = 0. \quad (3.36)$$

By analogy with the KP case, this equation can be called the modified BKP equation. We have $\tilde{\tau} = e^{zx+z^3t_3}\hat{\tau}$, where

$$\hat{\tau}(t_1, t_3, t_5, \dots) = \tau\left(t_1 - \frac{2}{z}, t_3 - \frac{2}{3z^3}, t_5 - \frac{2}{5z^5}, \dots\right).$$

(see (3.5)). Eq. (3.35) becomes

$$\begin{aligned} \partial_{t_3} \log \frac{\hat{\tau}}{\tau} &= \partial_x^3 \log \frac{\hat{\tau}}{\tau} + 3\partial_x \log \frac{\hat{\tau}}{\tau} \partial_x^2 \log(\tau \hat{\tau}) + \left(\partial_x \log \frac{\hat{\tau}}{\tau}\right)^3 \\ &+ 3z^2 \partial_x \log \frac{\hat{\tau}}{\tau} + 3z \partial_x^2 \log(\tau \hat{\tau}) + 3z \left(\partial_x \log \frac{\hat{\tau}}{\tau}\right)^2. \end{aligned} \quad (3.37)$$

We can write

$$\frac{\hat{\tau}}{\tau} = A e^{\alpha x + \beta t} \prod_i \frac{\sigma(x - y_i)}{\sigma(x - x_i)},$$

where A, α, β are constants and y_i are zeros of the function $\hat{\tau}$. Identifying (first order) poles in (3.37) at x_i and y_i , we obtain

$$\begin{cases} \dot{x}_i = 3 \sum_{j \neq i} \wp(x_i - x_j) + 3 \sum_j \wp(x_i - y_j) - 3 \left(\sum_{j \neq i} \zeta(x_i - x_j) - \sum_j \zeta(x_i - y_j) \right)^2 \\ \quad + 6\mu \sum_{j \neq i} \zeta(x_i - x_j) - 6\mu \sum_j \zeta(x_i - y_j) - 3\mu^2 \\ \dot{y}_i = 3 \sum_{j \neq i} \wp(y_i - y_j) + 3 \sum_j \wp(y_i - x_j) - 3 \left(\sum_{j \neq i} \zeta(y_i - y_j) - \sum_j \zeta(y_i - x_j) \right)^2 \\ \quad - 6\mu \sum_{j \neq i} \zeta(y_i - y_j) + 6\mu \sum_j \zeta(y_i - x_j) - 3\mu^2, \end{cases} \quad (3.38)$$

where μ is a constant. This is the self-dual form of equations of motion (1.3). A direct verification of the equations of motion for x_i using calculation of $\ddot{x}_i$ from (3.38) is too complicated to be made explicitly.

3.7. Dynamics of poles of elliptic solutions to the Novikov-Veselov equation

The Novikov-Veselov equation [23] is a close relative of the BKP equation. In fact it is a member of the 2-component BKP hierarchy. The equation reads as

$$v_t = v_{xxx} + v_{\bar{x}\bar{x}\bar{x}} + 6(uv)_x + 6(\bar{u}\bar{v})_{\bar{x}} \quad (3.39)$$

with the additional constraints

$$u_{\bar{x}} = v_x, \quad \bar{u}_x = v_{\bar{x}}. \quad (3.40)$$

These equations are compatibility conditions of the linear problems

$$(\partial_x \partial_{\bar{x}} + 2v)\psi = 0, \quad (3.41)$$

$$\partial_t \psi = (\partial_x^3 + \partial_{\bar{x}}^3 + 6u\partial_x + 6\bar{u}\partial_{\bar{x}})\psi \quad (3.42)$$

for the wave function ψ . The tau-function is connected with $u, \bar{u}, v$ by the formulas

$$v = \partial_x \partial_{\bar{x}} \log \tau, \quad u = \partial_x^2 \log \tau, \quad \bar{u} = \partial_{\bar{x}}^2 \log \tau. \quad (3.43)$$

For the solutions $u, \bar{u}, v$ that are elliptic functions of x we have

$$\tau = C e^{\gamma x \bar{x}} \prod_{i=1}^N \sigma(x - x_i), \quad (3.44)$$

where γ is a constant. The roots x_i depend on $\bar{x}$ and t .

We will be interested in the dynamics of the x_i 's as functions of $\bar{x}$. Substituting the pole ansatz for the wave function

$$\psi = e^{xz + \bar{x}z^{-1}} \sum_i c_i \Phi(x - x_i, \lambda) \quad (3.45)$$

into (3.41), we have:

$$(1 + 2\gamma) \sum_i c_i \Phi(x - x_i) - z \sum_i c_i \dot{x}_i \Phi'(x - x_i) + z \sum_i \dot{c}_i \Phi(x - x_i) + \sum_i \dot{c}_i \Phi'(x - x_i) \\ + z^{-1} \sum_i c_i \Phi'(x - x_i) - \sum_i c_i \dot{x}_i \Phi''(x - x_i) + 2 \left(\sum_i \dot{x}_i \wp(x - x_i) \right) \left(\sum_k c_k \Phi(x - x_k) \right) = 0,$$

where dot denotes the $\bar{x}$ -derivative (in this subsection only). The cancellation of poles leads to the following conditions:

$$(1 + 2\gamma)c_i + 2\dot{x}_i \sum_{k \neq i} c_k \Phi'(x - x_k) + 2c_i \sum_{k \neq i} \dot{x}_k \wp(x_i - x_k) - \wp(\lambda)\dot{x}_i c_i + z\dot{c}_i = 0, \\ z\dot{x}_i c_i - z^{-1}c_i + 2\dot{x}_i \sum_{k \neq i} c_k \Phi(x - x_k) - \dot{c}_i = 0.$$

They can be rewritten in the matrix form as

$$\begin{cases} L\mathbf{c} = 3(z^2 - \wp(\lambda))\mathbf{c} \\ \dot{\mathbf{c}} = \hat{M}\mathbf{c}, \end{cases} \quad (3.46)$$

where

$$L = -6\dot{X}^{-1}(\mathcal{D} + \gamma I) - 6zA - 6B,$$

$$\hat{M} = -z^{-1}I + z\dot{X} + 2\dot{X}A$$

and $\mathcal{D}_{ij} = \delta_{ij} \sum_{k \neq i} \dot{x}_k \wp(x_i - x_k)$. In fact this matrix L should be the same as the matrix L in (3.12). Their off-diagonal parts coincide as written. Equating their diagonal parts, we get the following relations between velocities with respect to t_3 and $\bar{x}$ (which could be denoted as $\bar{t}_1$):

$$\partial_{\bar{x}} x_i \cdot \partial_{t_3} x_i = 6 \sum_{j \neq i} (\partial_{\bar{x}} x_i + \partial_{\bar{x}} x_j) \wp(x_i - x_j) + 6\gamma. \quad (3.47)$$

With the help of identities used in Section 2.3 one can prove the following matrix identity:

$$\dot{L} + [L, \hat{M}] = 2[A, \dot{X}] \left(L - 3(z^2 - \wp(\lambda)) \right) + 6\dot{X}^{-2} \ddot{X} \mathcal{D} - 6\dot{X}^{-1} \dot{\mathcal{D}} + 12\mathcal{D}', \quad (3.48)$$

where $\mathcal{D}'_{ij} = \delta_{ij} \sum_{k \neq i} \dot{x}_k \wp'(x_i - x_k)$. Therefore, the compatibility condition of the linear problems (3.46) is vanishing of the diagonal matrix $\dot{X}^{-2} \ddot{X} \mathcal{D} - \dot{X}^{-1} \dot{\mathcal{D}} + 2\mathcal{D}'$, which leads to the equations of motion

$$\sum_{j \neq i} (\dot{x}_i \ddot{x}_j - \ddot{x}_i \dot{x}_j) \wp(x_i - x_j) - \sum_{j \neq i} \dot{x}_i \dot{x}_j (\dot{x}_i + \dot{x}_j) \wp'(x_i - x_j) = 0. \quad (3.49)$$

They are equivalent to the Manakov's triple representation

$$\dot{L} + [L, \hat{M}] = 2[A, \dot{X}](L - \Lambda I), \quad \Lambda = 3(z^2 - \wp(\lambda)). \quad (3.50)$$

The corresponding spectral curve is the same as for the BKP equation.

The rational degeneration of Eqs. (3.49) reads

$$\sum_{j \neq i} \frac{\dot{x}_i \ddot{x}_j - \ddot{x}_i \dot{x}_j}{(x_i - x_j)^2} + 2 \sum_{j \neq i} \frac{\dot{x}_i \dot{x}_j (\dot{x}_i + \dot{x}_j)}{(x_i - x_j)^3} = 0. \quad (3.51)$$

It was noticed by A.Zotov (see the remark in [14]) that these equations can be resolved with respect to the $\ddot{x}_j$'s and are equivalent to

$$\ddot{x}_i = 2 \sum_{j \neq i} \frac{\dot{x}_i \dot{x}_j}{x_i - x_j}, \quad (3.52)$$

which can be regarded as a limiting case of the rational Ruijsenaars–Schneider system. However, the elliptic system (3.49) hardly admits such a simple resolution.

4. Elliptic solutions to the 2D Toda equation

4.1. The 2D Toda equation

The 2D Toda equation is the first member of the infinite 2DTL hierarchy [30]. It is equivalent to the zero curvature equation $\partial_{\bar{t}_1} C_1 - \partial_{\bar{t}_1} \bar{C}_1 + [C_1, \bar{C}_1] = 0$ for the difference operators

$$C_1 = e^{\eta \partial_x} + b(x), \quad \bar{C}_1 = a(x) e^{-\eta \partial_x}, \quad (4.1)$$

which, in its turn, is the compatibility condition of the linear problems

$$\begin{aligned}\partial_{t_1} \psi(x) &= \psi(x + \eta) + b(x) \psi(x), \\ \partial_{\bar{t}_1} \psi(x) &= a(x) \psi(x - \eta).\end{aligned}\quad (4.2)$$

Writing it explicitly, we obtain the system

$$\begin{cases} \partial_{t_1} \log a(x) = b(x) - b(x - \eta) \\ \partial_{\bar{t}_1} b(x) = a(x) - a(x + \eta). \end{cases}$$

Excluding $b(x)$, we get the second order differential equation for $a(x)$

$$\partial_{t_1} \partial_{\bar{t}_1} \log a(x) = 2a(x) - a(x + \eta) - a(x - \eta) \quad (4.3)$$

which is one of the forms of the 2D Toda equation. In terms of the function $\varphi(x)$ introduced through the relation $a(x) = e^{\varphi(x) - \varphi(x - \eta)}$ it acquires the most familiar form

$$\partial_{t_1} \partial_{\bar{t}_1} \varphi(x) = e^{\varphi(x) - \varphi(x - \eta)} - e^{\varphi(x + \eta) - \varphi(x)}. \quad (4.4)$$

The change of the dependent variables from a, b to the tau-function,

$$a(x) = \frac{\tau(x + \eta)\tau(x - \eta)}{\tau^2(x)}, \quad b(x) = \partial_{t_1} \log \frac{\tau(x + \eta)}{\tau(x)}, \quad (4.5)$$

brings the 2D Toda equation to the bilinear form [11]

$$\frac{1}{2} D_1 D_{\bar{1}} \tau(x) \cdot \tau(x) = \tau^2(x) - \tau(x + \eta)\tau(x - \eta), \quad (4.6)$$

or

$$\partial_{t_1} \partial_{\bar{t}_1} \log \tau(x) = 1 - \frac{\tau(x + \eta)\tau(x - \eta)}{\tau^2(x)}. \quad (4.7)$$

The constant term in the right hand side is chosen from the condition that $\tau(x) = \text{const}$ be a solution.

4.2. Dynamics of poles of elliptic solutions

We are interested in solutions for which $a(x), b(x)$ are elliptic functions of the variable x . For such solutions the tau-function has the form

$$\tau(x) = C e^{c x^2 + r x t_1 + \bar{r} x \bar{t}_1 + \gamma t_1 \bar{t}_1} \prod_{i=1}^N \sigma(x - x_i), \quad (4.8)$$

then

$$a(x) = e^{2\eta^2 c} \prod_k \frac{\sigma(x - x_k + \eta)\sigma(x - x_k - \eta)}{\sigma^2(x - x_k)} = e^{2\eta^2 c} \sigma^{2N}(\eta) \prod_k (\wp(\eta) - \wp(x - x_k)),$$

$$b(x) = \sum_k \dot{x}_k (\zeta(x - x_k) - \zeta(x - x_k + \eta)) + r\eta,$$

where $\dot{x}_k = \partial_{t_1} x_k$.

We begin with investigating the dynamics of poles as functions of the time t_1 . To this end, it is enough to solve the first linear problem in (4.2) with $b(x)$ as above and the following pole ansatz for the ψ -function:

$$\psi = z^{x/\eta} e^{t_1 z + \bar{t}_1 z^{-1}} \sum_{i=1}^N c_i \Phi(x - x_i, \lambda). \quad (4.9)$$

Note that in [17] a slightly different function Φ was used, which differs from the present one by the exponential factor. Substituting (4.9) into (4.2), we get:

$$\begin{aligned} z \sum_i c_i \Phi(x - x_i) + \sum_i \dot{c}_i \Phi(x - x_i) - \sum_i c_i \dot{x}_i \Phi'(x - x_i) &= z \sum_i c_i \Phi(x - x_i + \eta) \\ &+ \left(\sum_k \dot{x}_k (\zeta(x - x_k) - \zeta(x - x_k + \eta)) + r\eta \right) \sum_i c_i \Phi(x - x_i). \end{aligned}$$

The cancellation of poles leads to the conditions

$$\begin{cases} zC_i + \dot{c}_i = r\eta c_i + \dot{x}_i \sum_{k \neq i} c_k \Phi(x - x_k) + c_i \sum_{k \neq i} \dot{x}_k \left(\zeta(x_i - x_k) - \zeta(x_i - x_k + \eta) \right) \\ zC_i - \dot{x}_i \sum_k c_k \Phi(x_i - x_k - \eta) = 0, \end{cases}$$

which can be written in matrix form

$$\begin{cases} Lc = zc \\ \dot{c} = Mc \end{cases} \quad (4.10)$$

with the matrices $L = \dot{X}A^-$, $M = r\eta I + \dot{X}A - \dot{X}A^- + D^0 - D^+$, where A is the same matrix as before and

$$A_{ij}^- = \Phi(x_i - x_j - \eta), \quad D_{ij}^\pm = \delta_{ij} \sum_{k \neq i} \dot{x}_k \zeta(x_i - x_k \pm \eta), \quad D_{ij}^0 = \delta_{ij} \sum_{k \neq i} \dot{x}_k \zeta(x_i - x_k).$$

A direct calculation shows that

$$\dot{L} + [L, M] = \left(\ddot{X}\dot{X}^{-1} + D^+ + D^- - 2D^0 \right) L,$$

so the compatibility condition of the linear problems (4.10) is $\ddot{X}\dot{X}^{-1} + D^+ + D^- - 2D^0 = 0$, which implies equations of motion

$$\begin{aligned} \ddot{x}_i &= - \sum_{k \neq i} \dot{x}_i \dot{x}_k \left(\zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k) \right) \\ &= \sum_{k \neq i} \dot{x}_i \dot{x}_k \frac{\wp'(x_i - x_k)}{\wp(\eta) - \wp(x_i - x_k)} \end{aligned} \quad (4.11)$$

together with their Lax representation. These are equations of motion for the elliptic Ruijsenaars–Schneider N -body system (a relativistic generalization of the Calogero–Moser system).

It can be directly verified that the Ruijsenaars–Schneider system is Hamiltonian with the Hamiltonian

$$H = \sum_i e^{p_i} \prod_{k \neq i} \frac{\sigma(x_i - x_k + \eta)}{\sigma(x_i - x_k)}, \quad (4.12)$$

where p_i, x_i are canonical variables. Clearly,

$$H = \sum_i \dot{x}_i = \text{const tr } L. \quad (4.13)$$

Let us now investigate dynamics of poles as functions of $\bar{t}_1$. Substitution of the pole ansatz into the second linear problem in (4.2) leads to rather cumbersome calculations and necessity to use complicated identities for elliptic functions. Below we follow another way. Substituting the expression (4.8) for the tau-function into Eq. (4.7), we get

$$\gamma - 1 - \sum_i \partial_{\bar{t}_1} \partial_{t_1} x_i \zeta(x - x_i) - \sum_i \partial_{\bar{t}_1} x_i \partial_{t_1} x_i \wp(x - x_i) = -e^{2c\eta^2} \prod_i \frac{\sigma(x - x_i + \eta) \sigma(x - x_i - \eta)}{\sigma^2(x - x_i)}.$$

Equating the coefficients at the first and second order poles, we obtain the relations

$$\partial_{\bar{t}_1} x_i \partial_{t_1} x_i = -e^{2c\eta^2} \sigma^2(\eta) \prod_{k \neq i} \frac{\sigma(x_i - x_k + \eta) \sigma(x_i - x_k - \eta)}{\sigma^2(x_i - x_k)}, \quad (4.14)$$

$$\partial_{\bar{t}_1} \partial_{t_1} x_i = \partial_{\bar{t}_1} x_i \partial_{t_1} x_i \sum_{k \neq i} \left(\zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k) \right) \quad (4.15)$$

(they were mentioned in [17]). Let us differentiate logarithm of Eq. (4.14) with respect to $\bar{t}_1$ and use (4.15). In this way we obtain the equations of motion

$$\begin{aligned} \partial_{\bar{t}_1} \partial_{t_1} x_i &= - \sum_{k \neq i} \partial_{\bar{t}_1} x_i \partial_{t_1} x_k \left(\zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k) \right) \\ &= \sum_{k \neq i} \partial_{\bar{t}_1} x_i \partial_{t_1} x_k \frac{\wp'(x_i - x_k)}{\wp(\eta) - \wp(x_i - x_k)} \end{aligned} \quad (4.16)$$

which are the same as equations of motion (4.11) for t_1 -dynamics. They are Hamiltonian with the Hamiltonian

$$\bar{H} = \sum_i e^{-p_i} \prod_{k \neq i} \frac{\sigma(x_i - x_k - \eta)}{\sigma(x_i - x_k)}. \quad (4.17)$$

Taking into account (4.14) and the fact that

$$\det_{ij} \Phi(x_i - x_j - \eta) = C(\eta, \lambda) \prod_{j < k} \frac{\sigma^2(x_j - x_k)}{\sigma(x_j - x_k + \eta) \sigma(x_j - x_k - \eta)}$$

with a constant $C(\eta, \lambda)$, it is easy to see that

$$\bar{H} = -e^{-2c\eta^2} \sigma^{-2}(\eta) \sum_i \partial_{t_1} x_i = \text{const tr } L^{-1}. \quad (4.18)$$

4.3. Self-dual form of the Ruijsenaars–Schneider equations of motion

Substituting $\psi(x) = z^{\kappa/\eta} e^{t_1 z} \hat{\tau}(x)/\tau(x)$ into the first linear problem in (4.2), we get the equation

$$\partial_{t_1} \log \frac{\hat{\tau}(x)}{\tau(x + \eta)} = z \frac{\hat{\tau}(x + \eta) \tau(x)}{\tau(x + \eta) \hat{\tau}(x)} - z. \quad (4.19)$$

As before, we parametrize the function $\hat{\tau}$ by its zeros y_i , so that

$$\frac{\hat{\tau}(x)}{\tau(x)} = e^{\alpha x + \beta t_1} \prod_i \frac{\sigma(x - y_i)}{\sigma(x - x_i)}$$

with some constants α, β . Eq. (4.19) becomes

$$\sum_i \left(\dot{x}_i \zeta(x - x_i + \eta) - \dot{y}_i \zeta(x - y_i) \right) = e^\mu \prod_i \frac{\sigma(x - x_i) \sigma(x - y_i + \eta)}{\sigma(x - y_i) \sigma(x - x_i + \eta)} + \text{const}$$

with a constant μ . Equating residues at $x = x_i - \eta$ and $x = y_i$, we get the following equations:

$$\begin{cases} \dot{x}_i = -\sigma(\eta) e^\mu \prod_{k \neq i} \frac{\sigma(x_i - x_k - \eta)}{\sigma(x_i - x_k)} \prod_j \frac{\sigma(x_i - y_j)}{\sigma(x_i - y_j - \eta)} \\ \dot{y}_i = -\sigma(\eta) e^\mu \prod_{k \neq i} \frac{\sigma(y_i - y_k + \eta)}{\sigma(y_i - y_k)} \prod_j \frac{\sigma(y_i - x_j)}{\sigma(y_i - x_j + \eta)}. \end{cases} \quad (4.20)$$

This is the self-dual form of the Ruijsenaars–Schneider equations of motion for x_i and y_i . For the proof that the latter follow from (4.20) see [32].

The self-dual form of equations of motion is directly connected with the integrable time discretization of the Ruijsenaars–Schneider system [16,22]. In this interpretation, τ and $\hat{\tau}$ are tau-functions taken at two subsequent values n and $n + 1$ of the discrete time. Accordingly, we denote $x_i = x_i^n$, $y_i = x_i^{n+1}$. It then follows from (4.20) that the discrete time dynamics is given by equations of motion

$$\prod_{k=1}^N \frac{\sigma(x_i^n - x_k^{n-1})}{\sigma(x_i^n - x_k^{n-1} + \eta)} \frac{\sigma(x_i^n - x_k^n + \eta)}{\sigma(x_i^n - x_k^n)} \frac{\sigma(x_i^n - x_k^{n+1} - \eta)}{\sigma(x_i^n - x_k^{n+1})} = -1. \quad (4.21)$$

Remarkably, Eqs. (4.21) coincide with the nested Bethe ansatz equations for the generalized integrable magnet with elliptic R -matrix associated with the root system A_m , with the discrete time n taking values $0, 1, \dots, m + 1$. Eqs. (2.58) are reproduced in the limit $\eta \rightarrow 0$. For the Hamiltonian approach to the time discretization of integrable systems see [28].

5. Conclusion

In this paper we have reviewed double-periodic (elliptic) solutions to integrable non-linear partial differential equations (KP, BKP, 2DTL) and have presented a detailed derivation of equations of motion for their poles. The dynamics of poles for KP and 2DTL equations is given by the known integrable many-body systems (elliptic Calogero–Moser and Ruijsenaars–Schneider models respectively) while for the BKP equation a new many-body system with three-body interaction arises. It is an open question to prove integrability of this system and to find whether or not it is Hamiltonian. We were able to find explicitly only a few non-trivial integrals of motion for this system expressed through coordinates and velocities. We believe that the system is integrable since the equation of the spectral curve depending on the spectral parameter provides a large supply of conserved quantities.

We have also discussed the so-called self-dual form of equations of motion which is intimately connected with the integrable time discretization of the many-body systems. The equations of motion in discrete time mysteriously coincide with the nested Bethe ansatz equations for quantum integrable models with elliptic R -matrix.

There is an open question even for the more familiar KP/Calogero–Moser case. It would be very desirable to extend Shiota's result [27] to the elliptic solutions. Namely, the problem is to establish the correspondence between elliptic solutions to the KP equation and the Calogero–Moser system with elliptic potential on the level of hierarchies, i.e., to prove that the evolution of poles with respect to the higher times t_k of the infinite KP hierarchy is governed by higher Hamiltonians H_k of the elliptic Calogero–Moser system (which are yet to be determined explicitly). In this paper we did this for $k = 2$ and $k = 3$.

Acknowledgments

The author thanks I. Krichever, S. Natanzon, D. Rudneva and P. Wiegmann for discussions. The work was performed at the Steklov Mathematical Institute of Russian Academy of Sciences, Moscow. This work is supported by the Russian Science Foundation under grant 19-11-00062.

Appendix

Matrix identities

Here we prove certain useful identities for the off-diagonal matrices

$$A_{ij} = (1 - \delta_{ij})\Phi(x_i - x_j), \quad B_{ij} = (1 - \delta_{ij})\Phi'(x_i - x_j), \quad C_{ij} = (1 - \delta_{ij})\Phi''(x_i - x_j)$$

and diagonal matrices

$$D_{ij} = \delta_{ij} \sum_{k \neq i} \wp(x_i - x_k), \quad D'_{ij} = \delta_{ij} \sum_{k \neq i} \wp'(x_i - x_k), \quad D''_{ij} = \delta_{ij} \sum_{k \neq i} \wp''(x_i - x_k).$$

We begin with the identity

$$[A, B] + [A, D] = D'. \quad (\text{A.1})$$

To transform the commutators $[A, B] + [A, D]$, we use the identity

$$\Phi(x)\Phi'(y) - \Phi(y)\Phi'(x) = \Phi(x+y)(\wp(x) - \wp(y)) \quad (\text{A.2})$$

which, in turn, directly follows from the easily proved identity

$$\Phi(x, \lambda)\Phi(y, \lambda) = \Phi(x+y, \lambda)(\zeta(x) + \zeta(y) - \zeta(x+y+\lambda) + \zeta(\lambda)). \quad (\text{A.3})$$

With the help of (A.2) we get for $i \neq k$

$$\begin{aligned} ([A, B] + [A, D])_{ik} &= \sum_{j \neq i, k} \Phi(x_i - x_j)\Phi'(x_j - x_k) - \sum_{j \neq i, k} \Phi'(x_i - x_j)\Phi(x_j - x_k) \\ &\quad + \Phi(x_i - x_k) \left(\sum_{j \neq k} \wp(x_j - x_k) - \sum_{j \neq i} \wp(x_i - x_j) \right) = 0, \end{aligned}$$

so $[A, B] + [A, D]$ is a diagonal matrix. To find the diagonal matrix elements, we use the special case of (A.2) at $y = -x$ (obtained as the limit $y \rightarrow -x$):

$$\Phi(x)\Phi'(-x) - \Phi(-x)\Phi'(x) = \wp'(x). \quad (\text{A.4})$$

This leads to

$$\begin{aligned} ([A, B] + [A, D])_{ii} &= \sum_{j \neq i} (\Phi(x_i - x_j)\Phi'(x_j - x_i) - \Phi'(x_i - x_j)\Phi(x_j - x_i)) = \sum_{j \neq i} \wp'(x_i - x_j) = D'_{ii}, \end{aligned}$$

so we finally obtain the matrix identity (A.1).

Combining the derivatives of (A.2) w.r.t. x and y , we obtain the identities

$$\Phi(x)\Phi''(y) - \Phi(y)\Phi''(x) = 2\Phi'(x+y)(\wp(x) - \wp(y)) + \Phi(x+y)(\wp'(x) - \wp'(y)), \quad (\text{A.5})$$

$$\Phi'(x)\Phi''(y) - \Phi'(y)\Phi''(x) = \Phi''(x+y)(\wp(x) - \wp(y)) + \Phi'(x+y)(\wp'(x) - \wp'(y)). \quad (\text{A.6})$$

Their limits as $y \rightarrow -x$ are

$$\Phi(x)\Phi''(-x) - \Phi(-x)\Phi''(x) = 0, \quad (\text{A.7})$$

$$\Phi'(x)\Phi''(-x) - \Phi'(-x)\Phi''(x) = -\frac{1}{6}\wp'''(x) - \wp(\lambda)\wp'(x). \quad (\text{A.8})$$

Using these formulas, it is not difficult to prove the following matrix identities:

$$[A, C] = 2[D, B] + D'A + AD', \quad (\text{A.9})$$

$$[B, C] = [D, C] + D'B + BD' - \frac{1}{6}D''' - \wp(\lambda)D'. \quad (\text{A.10})$$

Finally, we will prove the identity

$$Y + \frac{1}{2}[\dot{X}, C] - \frac{1}{2}\wp(\lambda)[\dot{X}, A] = -\tilde{D}', \quad (\text{A.11})$$

where

$$Y = B\dot{X}A - A\dot{X}B - [A, \tilde{D}], \quad \tilde{D}_{ij} = \delta_{ij} \sum_{k \neq i} \dot{x}_k \wp(x_i - x_k), \quad \tilde{D}'_{ij} = \delta_{ij} \sum_{k \neq i} \dot{x}_k \wp'(x_i - x_k).$$

We have

$$Y_{ii} = \sum_{j \neq i} \left(\Phi'(x_{ij}) \dot{x}_j \Phi(x_{ji}) - \Phi(x_{ij}) \dot{x}_j \Phi'(x_{ji}) \right) = - \sum_{j \neq i} \dot{x}_j \wp'(x_{ij}) = -\tilde{D}'_{ii}$$

due to (A.4). At $i \neq k$ we get, using (A.2):

$$\begin{aligned} Y_{ik} &= \sum_{j \neq i, k} \left(\Phi'(x_{ij}) \dot{x}_j \Phi(x_{jk}) - \Phi(x_{ij}) \dot{x}_j \Phi'(x_{jk}) \right) + \Phi(x_{ik}) \sum_{j \neq i} \dot{x}_j \wp(x_{ij}) - \Phi(x_{ik}) \sum_{j \neq k} \dot{x}_j \wp(x_{kj}) \\ &= \Phi(x_{ik}) \sum_{j \neq i, k} \dot{x}_j \left(\wp(x_{jk}) - \wp(x_{ij}) \right) + \Phi(x_{ik}) \left(\sum_{j \neq i} \dot{x}_j \wp(x_{ij}) - \sum_{j \neq k} \dot{x}_j \wp(x_{kj}) \right) \\ &= -(\dot{x}_i - \dot{x}_k) \Phi(x_{ik}) \wp(x_{ik}). \end{aligned}$$

Now we have for $i \neq k$:

$$Y_{ik} + \frac{1}{2}[\dot{X}, C]_{ik} - \frac{1}{2}\wp(\lambda)[\dot{X}, A]_{ik} = \frac{1}{2}(\dot{x}_i - \dot{x}_k) \left(\Phi''(x_{ik}) - \Phi(x_{ik})(2\wp(x_{ik}) - \wp(\lambda)) \right) = 0$$

due to (A.20), see below in the Appendix. Since $[\dot{X}, A]_{ii} = [\dot{X}, C]_{ii} = 0$, the matrix identity (A.11) is proved.

Proof of the identity used in Section 2.7

Here we prove the identity

$$\begin{aligned} & - \sum_{j \neq i} \left(\sum_{k \neq i} \zeta(x_i - x_k) - \sum_k \zeta(x_i - y_k) - \sum_{k \neq j} \zeta(x_j - x_k) + \sum_k \zeta(x_j - y_k) \right) \wp(x_i - x_j) \\ & + \sum_j \left(\sum_{k \neq i} \zeta(x_i - x_k) - \sum_k \zeta(x_i - y_k) + \sum_{k \neq j} \zeta(y_j - y_k) - \sum_k \zeta(y_j - x_k) \right) \wp(x_i - y_j) \\ & - \sum_{j \neq i} \wp'(x_i - x_j) = 0, \end{aligned} \quad (\text{A.12})$$

where $x_1, \dots, x_N, y_1, \dots, y_N$ are arbitrary variables.

The first non-trivial case is $N = 2$. Put $i = 1$ and consider the left hand side as a function of x_1 . It is easy to see that it is an elliptic function of x_1 . It may have poles at $x_1 = x_2, x_1 = y_1, x_1 = y_2$. Setting $x_1 = x_2 + \varepsilon$, we can directly check that the left hand side is regular at $x_1 = x_2$ and moreover it is $O(\varepsilon)$ as $\varepsilon \rightarrow 0$, so it vanishes at $x_1 = x_2$. In a similar way, one can check that the left hand side is regular at $x_1 = y_1, x_1 = y_2$. It follows from these facts that it is identically equal to zero. The argument for $i = 2$ is the same.

Passing to the general case, let us denote the left hand side of (A.12) by $F_N^{(i)} = F_N^{(i)}(x_1, \dots, x_N, y_1, \dots, y_N)$ and consider it as a function of x_i . It is easy to see that it is an elliptic function of x_i . It may have poles at $x_i = x_{i_0}$ ($i_0 = 1, \dots, N, i_0 \neq i$) and $x_i = y_{i_0}$ ($i_0 = 1, \dots, N$). Setting $x_i = x_{i_0} + \varepsilon, x_i = y_{i_0} + \varepsilon$, it can be checked that $F_N^{(i)}$ is regular, i.e., all singular

contributions cancel and $F_N^{(i)} = O(1)$ as $\varepsilon \rightarrow 0$. Therefore, $F_N^{(i)}$ is a constant independent of x_i . To find the constant, we expand $F_N^{(i)}$ around x_{i_0} up to the constant term in ε :

$$F_N^{(i)} = F_{N-1}^{(i_0)}(x_1, \dots, \hat{x}_i, \dots, x_N, y_1, \dots, \hat{y}_i, \dots, y_N) + G_{N-1}^{(i_0)} + O(\varepsilon), \quad (\text{A.13})$$

where $\hat{x}_i, \hat{y}_i$ means that x_i, y_i are omitted and

$$G_{N-1}^{(i_0)} = G_{N-1}^{(i_0)}(x_1, \dots, \hat{x}_i, \dots, x_N, y_1, \dots, y_N)$$

is given by

$$\begin{aligned} G_{N-1}^{(i_0)} = & \frac{1}{2} \sum_k \wp'(x_{i_0} - y_k) - \frac{1}{2} \sum_{k \neq i, i_0} \wp'(x_{i_0} - x_k) + \zeta(x_{i_0} - y_i) \wp(x_{i_0} - y_i) \\ & - \sum_{j \neq i, i_0} \left(\zeta(x_{i_0} - x_j) - \zeta(x_{i_0} - y_i) + \zeta(x_j - y_i) \right) \wp(x_{i_0} - x_j) \\ & + \sum_{j \neq i} \left(\zeta(x_{i_0} - y_j) - \zeta(x_{i_0} - y_i) + \zeta(y_j - y_i) \right) \wp(x_{i_0} - y_j) \\ & + \left(\sum_{k \neq i, i_0} \zeta(x_{i_0} - x_k) - \sum_{k \neq i} \zeta(x_{i_0} - y_k) + \sum_{k \neq i} \zeta(y_i - y_k) - \sum_{k \neq i, i_0} \zeta(y_i - x_k) \right) \wp(x_{i_0} - y_i). \end{aligned}$$

In the second and the third lines we use the identity

$$\zeta(x) - \zeta(y) - \zeta(x - y) = -\frac{1}{2} \frac{\wp'(x) + \wp'(y)}{\wp(x) - \wp(y)} \quad (\text{A.14})$$

to get that the sum of the second and the third lines is

$$\begin{aligned} & \sum_{j \neq i, i_0} \left(\frac{1}{2} \wp'(x_{i_0} - x_j) - \frac{1}{2} \wp'(x_{i_0} - y_j) \right. \\ & \quad \left. + \left(\zeta(x_{i_0} - y_j) + \zeta(y_j - y_i) - \zeta(x_{i_0} - x_j) - \zeta(x_j - y_i) \right) \wp(x_{i_0} - y_i) \right) \\ & \quad + \left(\zeta(x_{i_0} - y_{i_0}) - \zeta(x_{i_0} - y_i) + \zeta(y_{i_0} - y_i) \right) \wp(x_{i_0} - y_{i_0}). \end{aligned}$$

Substituting this back into the expression for $G_{N-1}^{(i_0)}$, we get after some cancellations:

$$\begin{aligned} G_{N-1}^{(i_0)} = & \frac{1}{2} \wp'(x_{i_0} - y_i) + \frac{1}{2} \wp'(x_{i_0} - y_{i_0}) \\ & + \left(\zeta(x_{i_0} - y_{i_0}) - \zeta(x_{i_0} - y_i) + \zeta(y_{i_0} - y_i) \right) \left(\wp(x_{i_0} - y_{i_0}) - \wp(x_{i_0} - y_i) \right). \end{aligned}$$

Using again the identity (A.14), we see that $G_{N-1}^{(i_0)} = 0$.

Now we are going to use the inductive argument: suppose that $F_{N-1}^{(i)} = 0$ (this is true for $N = 3$), then we see from (A.13) that $F_N^{(i)} = O(\varepsilon)$ as $\varepsilon \rightarrow 0$ and, therefore, $F_N^{(i)} = 0$.

Proof of Eq. (3.15)

Here we prove the identity (3.15). Using the explicit form of the matrices L, M (3.12), (3.13), we write

$$\begin{aligned} \dot{L} + [L, M] = & 36z^2 \left([A, B] + [A, D] \right) \\ & - 6z \left(\dot{A} - [\dot{X}, B] \right) + 36z \left([A, C] - [A, D'] + 2[B, D] \right) \\ & - 6 \left(\dot{B} - [\dot{X}, C] \right) - \ddot{X} + 6\dot{D} \\ & + 36 \left([B, C] - [B, D'] + [C, D] \right). \end{aligned}$$

First of all we notice that $\dot{A}_{ik} = (\dot{x}_i - \dot{x}_k) \Phi'(x_i - x_k)$, $\dot{B}_{ik} = (\dot{x}_i - \dot{x}_k) \Phi''(x_i - x_k)$, and, therefore, we have $\dot{A} = [\dot{X}, B]$, $\dot{B} = [\dot{X}, C]$. Next, we have $[A, B] + [A, D] = D'$ (see (A.1)) and Eqs. (A.9), (A.10) are used to transform $\dot{L} + [L, M]$ to the form (3.15).

Some useful identities

Apart from already mentioned identities for the Φ -function for the calculations in Sections 2 and 3 we need the following ones:

$$\Phi(x)\Phi(-x) = \wp(\lambda) - \wp(x), \quad (\text{A.15})$$

$$\Phi'(x)\Phi(-x) + \Phi'(-x)\Phi(x) = \wp'(\lambda), \quad (\text{A.16})$$

$$\Phi'(x)\Phi'(-x) = \wp^2(x) + \wp(\lambda)\wp(x) + \wp^2(\lambda) - \frac{1}{4}g_2, \quad (\text{A.17})$$

$$\Phi(x)\Phi''(-x) = \wp^2(\lambda) + \wp(\lambda)\wp(x) - 2\wp^2(x), \quad (\text{A.18})$$

$$\Phi'(x)\Phi''(-x) = \left(\wp'(\lambda) - \wp'(x)\right)\left(\wp(x) + \frac{1}{2}\wp(\lambda)\right), \quad (\text{A.19})$$

$$\Phi''(x) = \Phi(x)(2\wp(x) - \wp(\lambda)). \quad (\text{A.20})$$

They eventually follow from the basic identity (A.3). We also need some well known identities for the Weierstrass functions:

$$2\zeta(\lambda) - \zeta(\lambda + x) - \zeta(\lambda - x) = \frac{\wp'(\lambda)}{\wp(x) - \wp(\lambda)}, \quad (\text{A.21})$$

$$\wp'^2(x) = 4\wp^3(x) - g_2\wp(x) - g_3, \quad (\text{A.22})$$

$$\wp(x + \lambda) - \wp(x - \lambda) = -\frac{\wp'(\lambda)\wp'(x)}{(\wp(x) - \wp(\lambda))^2}, \quad (\text{A.23})$$

$$\wp(x + \lambda) + \wp(x - \lambda) = \frac{1}{2} \frac{\wp'^2(x) + \wp'^2(\lambda)}{(\wp(x) - \wp(\lambda))^2} - 2(\wp(x) + \wp(\lambda)). \quad (\text{A.24})$$

References

- [1] A. Abanov, E. Bettelheim, P. Wiegmann, Integrable hydrodynamics of Calogero-Sutherland model: Bidirectional Benjamin-Ono equation, *J. Phys. A* 42 (2009) 135201.
- [2] H. Airault, H.P. McKean, J. Moser, Rational and elliptic solutions of the Korteweg-De Vries equation and a related many-body problem, *Comm. Pure Appl. Math.* 30 (1977) 95–148.
- [3] G. Bonelli, A. Sciarappa, A. Tanzini, P. Vasko, Six-dimensional supersymmetric gauge theories, quantum cohomology of instanton moduli spaces and $gl(N)$ Quantum Intermediate Long Wave Hydrodynamics, *J. High Energy Phys.* 07 (2014) 141.
- [4] F. Calogero, Solution of the one-dimensional N -body problems with quadratic and/or inversely quadratic pair potentials, *J. Math. Phys.* 12 (1971) 419–436.
- [5] F. Calogero, Exactly solvable one-dimensional many-body systems, *Lett. Nuovo Cimento* 13 (1975) 411–415.
- [6] D.V. Chudnovsky, G.V. Chudnovsky, Pole expansions of non-linear partial differential equations, *Nuovo Cimento* 40B (1977) 339–350.
- [7] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Quasi-periodic solutions of the orthogonal KP equation. Transformation groups for soliton equations V, *Publ. RIMS, Kyoto Univ.* 18 (1982) 1111–1119.
- [8] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Transformation groups for soliton equations IV. A new hierarchy of soliton equations of kp type, *Physica D* 4D (1982) 343–365.
- [9] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Transformation Groups for Soliton Equations: Nonlinear Integrable Systems – Classical Theory and Quantum Theory (Kyoto, 1981), World Scientific, Singapore, 1983, pp. 39–119.
- [10] E. Date, M. Jimbo, T. Miwa, Method for generating discrete soliton equations I, II, *J. Phys. Soc. Japan* 51 (1982) 4116–4131.
- [11] M. Jimbo, T. Miwa, Solitons and infinite dimensional Lie algebras, *Publ. Res. Inst. Math. Sci. Kyoto* 19 (1983) 943–1001.
- [12] I.M. Krichever, Rational solutions of the Kadomtsev-Petviashvili equation and integrable systems of N particles on a line, *Funct. Anal. Appl.* 12 (1) (1978) 59–61.
- [13] I.M. Krichever, Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles, *Funk. Anal. Ego Pril.* 14 (4) (1980) 45–54, (in Russian); English translation: *Functional Analysis and Its Applications* 14 (4) (1980) 282–290.
- [14] I. Krichever, A characterization of Prym varieties, *Int. Math. Res. Not.* 2006 (11) (2006) 81476, [arXiv:math/0506238](https://arxiv.org/abs/math/0506238).
- [15] I. Krichever, O. Babelon, E. Billey, M. Talon, Spin generalization of the Calogero-Moser system and the matrix KP equation, *Amer. Math. Soc. Transl. Ser. 2* 170 (1995) 83–119.
- [16] I. Krichever, P. Wiegmann, A. Zabrodin, Elliptic solutions to difference non-linear equations and related many-body problems, *Comm. Math. Phys.* 193 (1998) 373–396.
- [17] I. Krichever, A. Zabrodin, Spin generalization of the Ruijsenaars-Schneider model, non-abelian 2D Toda chain and representations of Sklyanin algebra, *Uspekhi Mat. Nauk* 50 (1995) 3–56, (in Russian) (English translation: *Russ. Math. Surv.*, 50 (1995) 1101–1150).
- [18] I. Loris, R. Willox, Symmetry reductions of the BKP hierarchy, *J. Math. Phys.* 40 (1999) 1420–1431.
- [19] S. Manakov, Method of inverse scattering problem and two-dimensional evolution equations, *Uspekhi Mat. Nauk* 31 (1976) 245–246.
- [20] J. Moser, Three integrable Hamiltonian systems connected with isospectral deformations, *Adv. Math.* 16 (1975) 197–220.
- [21] F.W. Nijhoff, G.D. Pang, A time-discretized version of the Calogero-Moser model, *Phys. Lett. A* 191 (1994) 101–107.
- [22] F.W. Nijhoff, O. Ragnisco, V. Kuznetsov, Integrable time-discretization of the Ruijsenaars-Schneider model, *Comm. Math. Phys.* 176 (1996) 681–700.
- [23] S. Novikov, A. Veselov, Finite-gap two-dimensional periodic Schrödinger operators: exact formulae and evolution equations, *Dokl. AN USSR* 279 (1984) 20–24.
- [24] D. Rudneva, A. Zabrodin, Dynamics of poles of elliptic solutions to BKP equation, [arXiv:1903.00968](https://arxiv.org/abs/1903.00968).

- [25] S.N.M. Ruijsenaars, Integrable particle systems vs solutions of the KP and Toda equations, *Ann. Phys.* 256 (1997) 226–301.
- [26] S.N.M. Ruijsenaars, H. Schneider, A new class of integrable systems and its relation to solitons, *Ann. Physics* 146 (1986) 1–34.
- [27] T. Shiota, Calogero-moser hierarchy and KP hierarchy, *J. Math. Phys.* 35 (1994) 5844–5849.
- [28] Yu. Suris, *The Problem of Integrable Discretization: Hamiltonian Approach*, Springer Basel AG, 2003.
- [29] M.-H. Tu, On the BKP hierarchy: Additional symmetries, fay identity and Adler–Shiota–van Moerbeke formula, *Lett. Math. Phys.* 81 (2007) 93–105.
- [30] K. Ueno, K. Takasaki, Toda lattice hierarchy, *Adv. Stud. Pure Math.* 4 (1984) 1–95.
- [31] S. Wojciechowski, The analogue of the Bäcklund transformation for integrable many-body systems, *J. Phys. A: Math. Gen.* 15 (1982) L653–L657.
- [32] A. Zabrodin, A. Zotov, Self-dual form of Ruijsenaars-Schneider models and ILW equation with discrete Laplacian, *Nuclear Phys. B* 927 (2018) 550–565.