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Hamilton–Jacobi theory for gauge field theories

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A B S T R A C T

Recently, M. de León et al. (Campos et al., 2015) have developed a geometrical description of Hamilton–Jacobi theory for multisymplectic field theory. In our paper we analyze in the same spirit a special kind of field theories which are gauge field theories. The Hamilton–Jacobi theory for this kind of fields is shown.

1. Introduction

The historical beginning of classical field theory comes back to XIX century with the discovery of electromagnetic field by James Clerk Maxwell. However, only in XX century physicists and mathematicians started to look for a proper mathematical description of phenomena described by this theory (see e.g. [2,9,27,28,32,34,36,38,40,41,47,48]). From the mathematical point of view, a classical field is represented by a section of the fiber bundle $\pi : E \to M$ where the base manifold M usually represents spacetime. For instance a real (complex) scalar field is a section of the trivial bundle $M \times \mathbb{R} \to M$ ($M \times \mathbb{C} \to M$), electromagnetic field is a section of the cotangent bundle $\mathsf{T}^*M \to M$ etc. The natural choice for the configuration space of the first order classical field theory is a bundle of first jets J^1E of sections of π. The bundle J^1E plays a similar role in field theory as the tangent bundle $\mathsf{T}M$ in classical mechanics, however, one has to be careful with this comparison since the internal structure of J^1E is much richer than $\mathsf{T}M$.

In mechanics one uses symplectic geometry to derive the dynamics of the mechanical system [1]. The cotangent bundle T^*M associated with the Hamiltonian description of the system has a canonical symplectic structure. On the other hand, the bundle $\mathsf{T}M$ does not have a canonical symplectic form but, at least for regular systems, one can transport the canonical symplectic structure from T^*M to $\mathsf{T}M$ via the Legendre map [33]. In field theory one has to consider a multisymplectic structure which is a generalization of the symplectic one. It turns out that in the presence of a volume form on the $(n+1)$-dimensional manifold M, the phase bundle is isomorphic to the bundle $\Lambda^{n+1}_2 E$ representing two-horizontal $(n+1)$-forms on E and being a canonical multisymplectic manifold. To read more about multisymplectic structures and its applications in field theory one can see for example [10–12,22,25,26,43]. An interesting description of covariant Hamiltonian field theories on manifolds with boundary, and applications to Yang–Mills theories, was developed in [30].

In the very center of interest of physicists are the so called *gauge theories*. Gauge theories are field theories with certain special kind of symmetries called *gauge transformations*. In the modern description of fundamental interactions one usually takes a classical Yang–Mills theory and then quantize it to obtain possible experimental predictions.

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On the other hand, Hamilton–Jacobi theory is a powerful tool in analytical mechanics that provides a method to find the dynamics of the mechanical system. It is also the most clear way to see the classical limit of the quantum mechanics when one takes $\hbar \to 0$. Its geometrical formulation was investigated in many papers, e.g. in [3–7,13–17,19,21,31,35,37] . Recently, a Hamilton–Jacobi for field theories version has been developed too [8,18,20,39,49]. In our paper we would like to investigate the field theoretical version of Hamilton–Jacobi theory for gauge field theories.

The paper is organized as follows. In Section 2 we present the basic features of the geometrical picture of classical field theory of first order. We start with a general definition of a multisymplectic structure. Subsequently, we present basic tools of jet bundle theory and their relations to multisymplectic structures. In 3 we make a brief introduction to principal bundles which provide the most common definition of a gauge field. The main result from this constructions is that a gauge field is just a connection in a principal bundle. In Section 4 we present an equivalent definition of connection which is based on jet bundles and is more useful for our purposes. In particular we present a canonical projection of the configuration space of gauge field theory which is essential in our further work. Section 5 contains the Hamiltonian formalism for gauge theories. In Section 6 we present the main result of our paper which is a Hamilton–Jacobi theory for gauge theories.

Along this paper we will assume that the base manifold M has always dimension $n+1$. If (x^i) are local coordinates in M we introduce notation

$$\mathrm{d}^{n+1}x = \mathrm{d}^0 x \wedge \cdots \wedge \mathrm{d}^n x$$

and

$$\mathrm{d}^n x_i = i_{\frac{\partial}{\partial x^i}} \mathrm{d}^0 x \wedge \cdots \wedge \mathrm{d}^n x$$

for the contraction with the coordinate vector fields. Furthermore, we assume that M is orientable with fixed orientation, together with a given volume form η. Its pullback to any bundle over M will still be denoted η, as for instance $\pi^*\eta$. In addition, local coordinates on M will be chosen compatible with η, which means such that $\eta = \mathrm{d}^{n+1}x$. To great extent, this form η is not needed and our constructions can be generalized, although we are going to make use of η for the sake of simplicity.

2. Multisymplectic structures and jet bundles

We begin reviewing the basic notions of multisymplectic structures and jet bundles.

2.1. Multisymplectic structures

Let V be a finite-dimensional real vector space. A $(k+1)$-form Ω on V is said to be multisymplectic if it is non-degenerate, i.e., if the linear map

$$b_\Omega : V \to \Lambda^k V^*$$

$$v \longmapsto b_\Omega(v) := i_v \Omega$$

is injective. The pair (V, Ω) is called then a *multisymplectic* vector space of order $k+1$. This definition has a straightforward extension to differential manifolds. We say that a pair (P, Ω) where P is a manifold and Ω a closed $(k+1)$-form on P is a multisymplectic manifold if each $(\mathsf{T}_p P, \Omega(p))$ is multisymplectic, for any point $p \in P$.

In a similar way that a cotangent bundle is a canonical example of symplectic manifold, the canonical example of a multisymplectic manifold is the bundle of forms $P = \Lambda^k N$ over a manifold N.

Let N be a smooth manifold of dimension n, $\Lambda^k N$ be the bundle of k-forms on N and $\nu : \Lambda^k N \to N$ be the canonical projection $(1 \leq k \leq n)$. The *Liouville* or *tautological form* of order k is the k-form Θ over $\Lambda^k N$ given by

$$\Theta(\omega)(v_1, \ldots, v_k) = \omega(\mathsf{T}_\omega \nu(v_1), \ldots, \mathsf{T}_\omega \nu(v_k)), \quad \omega \in \Lambda^k N, \quad v_1, \ldots, v_k \in \mathsf{T}_\omega(\Lambda^k N).$$

The canonical multisymplectic form is then defined by

$$\Omega = -\mathrm{d}\Theta.$$

Let us introduce local coordinates (x^i) in N and induced local coordinates $(x^i, p_{i_1 \ldots i_k})$ in $\Lambda^k N$ where $1 \leq i_1 \leq \cdots \leq i_k \leq n$. Then

$$\Theta = \sum_{i_1 < i_2 < \cdots < i_k} p_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \ldots \mathrm{d}x^{i_k}$$

$$\Omega = -\sum_{i_1 < i_2 < \cdots < i_k} \mathrm{d}p_{i_1 \ldots i_k} \wedge \mathrm{d}x^{i_1} \wedge \ldots \mathrm{d}x^{i_k}.$$

It is immediate to check that Ω is indeed multisymplectic.

Another example of multisymplectic manifold comes from the particular case of the previous example. Let $\pi : E \to M$ be a bundle. Let us assume that $\dim M = n+1$ and $\dim E = n+1+m$. Given $1 \leq r \leq m$, we can consider the vector subbundle $\Lambda^k_r E$ of $\Lambda^k E$ that consists of k-forms on E which are r-horizontal with respect to the fibration π, i.e.

$$(\Lambda^k_r E)_e = \{\omega \in \Lambda^k_e E : i_{v_r} \dots i_{v_1}\omega = 0, \quad \forall v_1, \dots, v_r \in \mathsf{V}_e E\}.$$

where $e \in E$ and $\mathsf{V}E \subset \mathsf{T}E$ is the subbundle of vectors tangent to E which are vertical with respect to π.

We denote by $\nu_r, \Theta_r, \Omega_r$ the restrictions to $\Lambda^k_r E$ of ν, Θ, Ω, respectively. It is easy to see that $(\Lambda^k_r E, \Omega_r)$ is a multisymplectic manifold. The most interesting cases from the point of view of field theory are $k = n+1$ and $r = 1, 2$.

Let (x^i, u^α) denote adapted coordinates on E, where $i \in \overline{0, n}$ and $\alpha \in \overline{1, m}$. They induce coordinates $(x^i, u^\alpha, p, p^i{}_\alpha)$ on $\Lambda^{n+1}_2 E$ such that any element $\omega \in \Lambda^{n+1}_2 E$ is locally expressed by

$$\omega = p\mathrm{d}^{n+1}x + p^i{}_\alpha \mathrm{d}u^\alpha \wedge \mathrm{d}^n x_i,$$

where $\mathrm{d}^n x_i = i_{\frac{\partial}{\partial x^i}} \mathrm{d}^{n+1}x$. Therefore Θ_2 and Ω_2 read as

$$\Theta_2 = p\mathrm{d}^{n+1}x + p^i{}_\alpha \mathrm{d}u^\alpha \wedge \mathrm{d}^n x_i$$
$$\Omega = -\mathrm{d}p \wedge \mathrm{d}^{n+1}x - \mathrm{d}p^i{}_\alpha \wedge \mathrm{d}u^\alpha \wedge \mathrm{d}^n x_i.$$

in local coordinates

2.2. First order jet bundles

We will introduce now the notion of first order jet spaces and their duals. We follow the notations in [29]. For a more detailed discussion of the jet bundle geometry see e.g. [46].

Let $\pi : E \to M$ be a bundle with the total space of the dimension $\dim E = n+1+m$. We introduce in a domain $U \in M$ a local coordinate system $(x^i)_{i=0}^n$ on M. In field theory, fields are represented by sections of the fibration π. The total space is a space of values of the field e.g vector fields are sections of the π being a vector bundle, scalar fields are sections of the trivial bundle $E = M \times \mathbb{R}$ or $E = M \times \mathbb{C}$, etc. On an open subset $V \subset E$ such that $\pi(V) = U$ we can introduce local coordinates (x^i, u^α) adapted to the structure of the bundle.

In $\mathsf{T}E$ we have a vector subbundle $\mathsf{V}E$ consisting of those tangent vectors that are vertical with respect to the projection π i.e. $\mathsf{T}\pi(v_p) = 0$ for $v_p \in \mathsf{V}_p E$. We will also need the dual vector bundle V^*E.

The space of first jets of sections of the bundle π will be denoted by J^1E. By definition, the first jet $\mathsf{j}^1_m\phi$ of the section ϕ at the point $m \in M$ is an equivalence class of sections having the same value at the point m and such that the spaces tangent to the graphs of the sections at the point $\phi(m)$ coincide. Therefore, there is a natural projection $\mathsf{j}^1\pi$ from the space J^1E onto the manifold E

$$\mathsf{j}^1\pi : \mathsf{J}^1E \to E : \quad \mathsf{j}^1_m\phi \longmapsto \phi(m).$$

Moreover, every jet $\mathsf{j}^1_m\phi$ may be identified with a linear map $\mathsf{T}\phi : \mathsf{T}_m M \to \mathsf{T}_{\phi(m)}E$. Linear maps coming from jets at the point m form an affine subspace in a vector space $Lin(\mathsf{T}_m M, \mathsf{T}_e E)$ of all linear maps from $\mathsf{T}_m M$ to $\mathsf{T}_e E$. A map belongs to this subspace if composed with $\mathsf{T}\pi$ gives identity. In a tensorial representation we have an inclusion

$$\mathsf{J}^1_e E \subset \mathsf{T}^*_m M \otimes \mathsf{T}_e E.$$

It is easy to check that the affine space $\mathsf{J}^1_e E$ is modeled on the vector space $\mathsf{T}^*_m M \otimes \mathsf{V}_e E$. Summarizing, the bundle $\mathsf{J}^1E \to E$ is an affine bundle modeled on a vector bundle

$$\pi^*(\mathsf{T}^*M) \otimes_E \mathsf{V}E \to E.$$

The symbol $\pi^*(\mathsf{T}^*M)$ denotes the pullback of the cotangent bundle T^*M along to the projection π. In the following we will omit the symbol of the pullback writing simply $\mathsf{T}^*M \otimes_E \mathsf{T}E$ and $\mathsf{T}^*M \otimes_E \mathsf{T}E$.

Using the adapted coordinates (x^i, u^α) in $V \subset E$, we can construct the induced coordinate system $(x^i, u^\alpha, u^\beta{}_j)$ on $\mathsf{j}^1\pi^{-1}(V)$ such that for any section ϕ given by n functions $\phi^a(x^i)$ we have

$$u^\beta{}_j(\phi^\alpha(x^i)) = \frac{\partial \phi^\beta}{\partial x^j}(x^i(m)).$$

In the tensorial representation the jet $\mathsf{j}^1_m\phi$ may be written as

$$\mathrm{d}x^i \otimes \frac{\partial}{\partial x^i} + \frac{\partial \phi^\alpha}{\partial x^j}(x^i(m))\mathrm{d}x^j \otimes \frac{\partial}{\partial u^\alpha},$$

where we have used local bases of sections of T^*M and $\mathsf{T}E$ coming from the chosen coordinates.

We will introduce now the bundle which is dual to the bundle $\mathsf{J}^1E \to E$. Let us recall that each fiber $\mathsf{J}^1_e E$ is an affine space. We can consider a set of affine maps $\mathsf{J}^1_e E \to \mathbb{R}$ which we will denote by $\mathsf{Aff}(\mathsf{J}^1_e E, \mathbb{R})$, for each $e \in E$. Collecting $\mathsf{Aff}(\mathsf{J}^1_e E, \mathbb{R})$ point by point we obtain a bundle of affine maps on J^1E, namely $\mathsf{Aff}(\mathsf{J}^1E, \mathbb{R}) \to E$. From now we will use notation $\mathsf{J}^\dagger E := \mathsf{Aff}(\mathsf{J}^1E, \mathbb{R})$. It is a vector bundle over E. If $(x^i, u^\alpha, u^\beta{}_j)$ are coordinates in J^1E, then we introduce coordinates $(x^i, u^\alpha, r, \varphi^b{}_j)$ in $\mathsf{J}^\dagger E$. The evaluation between J^1E and $\mathsf{J}^\dagger E$ in coordinates reads

$$\mathsf{J}^\dagger E \times_E \mathsf{J}^1E \to \mathbb{R}, \qquad \langle T_e, \mathrm{j}_e\psi\rangle = r + \varphi^b{}_j y^j{}_b$$

We also introduce a vector bundle J^oE over E which is a quotient of $\mathsf{J}^\dagger E$ by constant affine maps, namely

$$\mathsf{J}^oE := \mathsf{J}^\dagger E/\{f : E \to \mathbb{R}\}.$$

It is equipped with adapted coordinates $(x^i, u^\alpha, \varphi^b{}_j)$. The bundle $\mu : \mathsf{J}^\dagger E \to \mathsf{J}^oE$ is a principal $\mathbb{R}$-bundle.

The following theorem justifies the importance of the multisymplectic structures in field theory.

Theorem 1. *There exists an isomorphism $\Psi : \Lambda^{n+1}_2 E \to \mathsf{J}^\dagger E$ given by the formula*

$$\langle\Psi(\omega), \mathrm{j}^1_x\phi\rangle\eta = \phi^*_x(\omega), \qquad \forall \mathrm{j}^1_x\phi \in \mathsf{J}^1_xE, \quad \forall\omega \in \Lambda^{n+1}_2 E.$$

We will therefore identify $\mathsf{J}^\dagger E$ with $\Lambda^{n+1}_2 E$ and J^oE with $\Lambda^{n+1}_2 E/\Lambda^{n+1}_1 E$.

3. Geometry of principal bundles

We introduce now a mathematical framework to analyze gauge theories. It turns out that gauge fields are sections of the bundle of connections in a principal bundle.

3.1. Principal bundles

Let G be a Lie group and P a smooth manifold. We will denote by $\mathfrak{g}$ the Lie algebra of G. We assume that G acts on P from the right-hand side in a smooth, free and proper way. We denote by M the space of orbits P/G. The bundle $\pi : P \to M$ is called a *principal bundle*. It is locally isomorphic to $M \times G$. Let U_α be an open subset in M. We have local trivializations

$$\begin{tikzcd} \pi^{-1}U_\alpha \arrow[rr, "\Psi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times G \arrow[dl, "pr_1"] \\ & U_\alpha & \end{tikzcd}$$

where Ψ_α is a G-equivariant diffeomorphism such that $\Psi_\alpha(p) = (\pi(p), g_\alpha(p))$ for $g_\alpha : P \to G$. Equivariance means that $\Psi_\alpha(pg) = \Psi_\alpha(p)g$ which implies that g_α is also G-equivariant, says $g_\alpha(pg) = g_\alpha(p)g$.

Local trivializations of principal bundles are associated with its local sections. Let $\sigma_\alpha : U_\alpha \to \pi^{-1}U_\alpha$ be a local section of π. Using σ_α we can construct a local trivialization by $\Psi_\alpha(\sigma_\alpha(m)) = (\pi(\sigma_\alpha(m)), e)$, where e is the neutral element of G. Since Ψ_α is equivariant we obtain a trivialization of the whole subset $\pi^{-1}U_\alpha$. Conversely given a trivialization Ψ_α we define value of $\sigma_\alpha(m)$ by condition $\Psi_\alpha(\sigma_\alpha(m)) = (\pi(\sigma_\alpha(m)), e)$. Let us notice that $g_\alpha \circ \sigma_\alpha(m) = e$, for each $m \in M$. Local trivializations (or equivalently local sections) enable us to identify fibers of the bundle π with a group G by choosing an element p which satisfies $g_\alpha(p) = e$.

Let $U_{\alpha\beta} := U_\alpha \cap U_\beta$. For Ψ_α, Ψ_β associated with U_α and U_β, respectively, we have transition conditions such that the following diagram commutes

$$\begin{tikzcd} U_{\alpha\beta} \times G \arrow[dr, "pr_1"'] & \pi^{-1}U_{\alpha\beta} \arrow[l, "\Psi_\alpha"'] \arrow[r, "\Psi_\beta"] \arrow[d, "\pi"] & U_{\alpha\beta} \times G \arrow[dl, "pr_1"] \\ & U_{\alpha\beta} & \end{tikzcd}$$

Let us assume now that m belongs to $U_{\alpha\beta}$. With each trivialization Ψ_α, Ψ_β there are associated functions g_α, g_β. One can show that there exists a function $\bar{g}_{\alpha\beta} : P \to G$ such that

$$\bar{g}_{\alpha\beta}(p) = g_\alpha(p)g_\beta(p)^{-1}$$

In addition, $\bar{g}_{\alpha\beta}$ is constant on each fiber i.e. $\bar{g}_{\alpha\beta}(p) = \bar{g}_{\alpha\beta}(pg)$. Therefore, $\bar{g}_{\alpha\beta}$ defines a function

$$g_{\alpha\beta} : M \to G, \quad g_{\alpha\beta}(\pi(p)) := \bar{g}_{\alpha\beta}(p)$$

The transition functions $g_{\alpha\beta}$ satisfy cocycle conditions

$$g_{\alpha\beta}(m)g_{\beta\alpha}(m) = e \text{ on } U_\alpha \cap U_\beta$$
$$g_{\alpha\beta}(m)g_{\beta\gamma}(m)g_{\gamma\alpha}(m) = e \text{ on } U_\alpha \cap U_\beta \cap U_\gamma.$$

3.2. Adjoint bundle

Let F be a manifold and let G act on F from the left hand side. We assume that G acts on $P \times F$ by

$$g(p, f) = (pg, g^{-1}f).$$

We denote by $N := (P \times F)/G$ the space of orbits of this action. We have a bundle

$$\xi : N \to M, \quad [(p, f)] \to \pi([p]),$$

which is called an *associated bundle* to a principal bundle P. Let $s_\alpha : M \supset \mathcal{U}_\alpha \to P$ be a local section of P and let $N \supset \mathcal{O}_\alpha := \xi^{-1}(\mathcal{U}_\alpha)$. Then for each orbit $y \in \mathcal{O}_\alpha$, where $\xi(y) = m$, there exists a unique element $\chi_s(y) \in F$ such that $(s(m), \chi_s(y))$ belongs to y. A local section of π provides then a local trivialization of N

$$N \supset \mathcal{O}_\alpha \to \mathcal{U}_\alpha \times F, \quad y \longmapsto (\xi(y), \chi_s(y)).$$

The most important examples of associated bundles to P in context of our work are the bundles with fibers $F = \mathfrak{g}$ or $F = G$, i.e. $N = (P \times \mathfrak{g})/G$ and $N = (P \times G)/G$. We will use the notation $\mathrm{ad}(P) := (P \times \mathfrak{g})/G$ and $\mathrm{Ad}(P) := (P \times G)/G$. The action of G on $\mathfrak{g}$ and G on G is the adjoint map given by

$$\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}, \quad (g, X) \longmapsto \mathrm{Ad}_g(X),$$
$$\mathrm{Ad} : G \times G \to G, \quad (g, h) \longmapsto \mathrm{Ad}_g(h),$$

respectively.

Let us take a closer look on the bundle $\xi : \mathrm{ad}(P) \to M$. It has a natural structure of the vector bundle with addition and multiplication given by

$$[p, X_1] + [p, X_2] = [p, X_1 + X_2] \qquad p \in P, \quad X_1, X_2 \in \mathfrak{g} \tag{1}$$
$$\lambda[p, X] = [p, \lambda X] \qquad \lambda \in \mathbb{R} \tag{2}$$

The "zero" vector in the fiber over p is just the element $[p, 0]$. The bundle $\mathrm{ad}P$ is called the *adjoint bundle*. Given a section $s_\alpha : U_\alpha \to P$ we have a local trivialization of $\mathrm{ad}(P)$

$$\mathrm{ad}(P) \to U_\alpha \times \mathfrak{g}, \qquad [p, X] \longmapsto \left(\pi(p), X_{s_\alpha}\right) \quad h \in G.$$

where $X_{s_\alpha} \in \mathfrak{g}$ is the unique element satisfying

$$[p, X] = [s_\alpha\left(\xi(p)\right), X_{s_\alpha}].$$

In the above trivialization the equivalence class is represented by its representative in $e \in G$ i.e. if in a local trivialization $(p, X) = (\pi(p), g_\alpha(p), X)$, then $[p, X] \in \mathrm{ad}P$ reads $(\pi(p), \mathrm{Ad}_{g_\alpha(p)^{-1}}X)$.

The structure of $\mathrm{ad}P$ has its reflection in transition functions between different trivializations of P. Every element (p, X) is in relation with itself, therefore

$$(\pi(p), g_\alpha(p), X) \sim (\pi(p), g_\beta(p), X)$$

where $g_\alpha : \pi^{-1}U_\alpha \to G$ and $g_\beta : \pi^{-1}U_\beta \to G$ are two different trivializations related by the transition function $g_{\alpha\beta}$. Since for each $p \in \pi^{-1}U_{\alpha\beta}$, the relation $g_\alpha(p) = g_{\alpha\beta}(\pi(p))g_\beta(p)$ is satisfied, we obtain

$$(\pi(p), e, \mathrm{Ad}_{g_\alpha(p)^{-1}} \circ X) \sim (\pi(p), e, \mathrm{Ad}_{g_{\alpha\beta}(m)} \circ \mathrm{Ad}_{g_\alpha(p)^{-1}} \circ X).$$

Therefore, the local trivializations of $\mathrm{ad}P$ must satisfy a condition

$$\begin{tikzcd} & \mathrm{ad}(P) \arrow[dl, "triv_1"'] \arrow[dr, "triv_2"] & \\ U_{\alpha\beta} \times \mathfrak{g} \arrow[rr] & & U_{\alpha\beta} \times \mathfrak{g} \end{tikzcd} \qquad \begin{tikzcd} & {[p, X]} \arrow[dl, "triv_1"'] \arrow[dr, "triv_2"] & \\ (m, Y) \arrow[rr] & & (m, \mathrm{Ad}_{g_{\alpha\beta}(m)}Y) \end{tikzcd} \tag{3}$$

on the overlappings $U_{\alpha\beta} = U_\alpha \cap U_\beta$.

3.3. Forms with values in ad$P$

We are ready now to construct the bundle of forms on M with values in $\mathrm{ad}P$. Denote by $\Lambda^k(M, \mathfrak{g})$ the bundle of $\mathfrak{g}$-valued k-forms on M. We assume that G acts on $\mathfrak{g}$ by

$$R_g^* X = \mathrm{Ad}_{g^{-1}}X, \qquad g \in G, \quad X \in \mathfrak{g}$$

Let $\{\xi_\alpha\}$ be a set of local k-forms on M such that for each α, $\xi_\alpha \in \Lambda^k(U_\alpha, \mathfrak{g})$. We also require that for each overlapping $U_{\alpha\beta}$ the condition

$$\xi_\alpha(m) = \mathrm{Ad}_{g_{\alpha\beta}(m)} \circ \xi_\beta(m), \qquad m \in U_{\alpha\beta}, \quad g_{\alpha\beta} : U_{\alpha\beta} \to G \tag{4}$$

is satisfied. We claim that the family of forms $\{\xi_\alpha\}$ defines an element of $\Lambda^k(M, \mathrm{ad}P)$. Indeed, if $(v_1, \ldots, v_k) \in \mathsf{T}_m M$ are tangent vectors at a point $m \in U_{\alpha\beta}$, then $\xi_\alpha(v_1, \ldots, v_k)$ and $\xi_\beta(v_1, \ldots, v_k)$ satisfy (3).

Let us assume now that we have a scalar product on $\mathfrak{g}$ which will be denoted by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. In an obvious way, it defines a scalar product K on each fiber of the bundle $P \times \mathfrak{g} \to P$

$$K : (P \times \mathfrak{g}) \times_P (P \times \mathfrak{g}) \to \mathbb{R}, \qquad K(p)(X, Y) = \langle X, Y \rangle_{\mathfrak{g}}$$

Let us require now that the form K is Ad-invariant i.e.

$$K(p)(X, Y) = K(p)(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) \qquad g \in G.$$

In such a case, it defines a scalar product K_A on the associated bundle $\mathrm{ad}P \to M$, namely

$$K_A : \mathrm{ad}P \times_M \mathrm{ad}P \to \mathbb{R}, \qquad K_A(m)(X_A(p), Y_A(p)) = K(p)(X, Y), \qquad \xi(p) = m.$$

where $X_A(p)$ and $Y_A(p)$ are equivalence classes of the elements (p, X) and (p, Y), respectively. One can easily see that if K is Ad-invariant, then the above definition does not depend on the choice of the representatives.

Let us assume now that the manifold M is equipped with a metric g. It allows us to define a Hodge star operator $\star$. On the other hand, the Hodge star allows us to define a scalar product $(\cdot|\cdot)_k$ on the space $\Lambda^k(U_\alpha)$ given by

$$(\alpha|\beta)_k = \int \alpha \wedge \star\beta$$

The above scalar product may be extended to a scalar product on $\Lambda^k(U_\alpha, \mathfrak{g})$ by

$$(\cdot|\cdot) : \Lambda^k(U_\alpha, \mathfrak{g}) \times \Lambda^k(U_\alpha, \mathfrak{g}) \to \mathbb{R}, \quad (\alpha|\beta) = \int K_{ij}\alpha^i \wedge \star\beta^j$$

where we have used the notation $\alpha = \alpha^i \otimes e_i$, $\beta = \beta^j \otimes e_j$ and $K_{ij} := K(e_i, e_j)$. In physical literature there is a common notation $\mathrm{Tr}(\alpha \wedge \star\beta) := K_{ij}\alpha^i \wedge \star\beta^j$. If we assume again that K (we will omit letter A in K_A) is Ad-invariant then the above formula defines also a scalar product on a space $\Lambda^k(M, \mathrm{ad}P)$.

3.4. Connection in a principal bundle

A connection in a principal bundle $P \to M$ is a G-invariant distribution H in $\mathsf{T}P$ complementary to $\mathsf{V}P$, i.e.

$$\mathsf{T}_p P = \mathsf{V}_p P \oplus H_p, \qquad p \in P$$

and

$$H_p g = H_{pg} \quad g \in G.$$

The above definition is very elegant and general, however when it comes to applications, it is more convenient to use another definition of connection. We will start with introducing some basic mathematical tools. Let X be an element of $\mathfrak{g}$. The group action of G on P defines a vertical vector field σ_X on P associated with the element X, namely

$$\sigma_X(p) := \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} p \exp(tX).$$

The field σ_X is called a *fundamental vector field* corresponding to an element X. The fundamental vector field is equivariant in the sense that

$$\sigma_X(pg) = \sigma_{\mathrm{Ad}_{g^{-1}}(X)}(p).$$

The connection in a principal bundle P is a G-equivariant, $\mathfrak{g}$-valued one-form ω

$$\omega : \mathsf{T}P \to \mathfrak{g},$$

such that $\omega(\sigma_X(p)) = X$ for each $p \in P$ and $X \in \mathfrak{g}$. The G-equivariance means that

$$R_g^* \omega(p) = \mathrm{Ad}_{g^{-1}} \circ \omega(p).$$

Since the connection form is an identity on vertical vectors the difference of two connections is a horizontal form. It follows that the space of connections is an affine subbundle $\mathcal{A} \subset \mathsf{T}^*P \otimes \mathfrak{g}$ over M modeled on a vector bundle of $\mathfrak{g}$-valued horizontal forms on P. Due to the transformation properties, the bundle of horizontal forms may be identified with the bundle $\mathsf{T}^*M \otimes \mathrm{ad}P \to M$. We have the following diagram

$$\begin{tikzcd} \mathcal{A} \subset \mathsf{T}^*P \otimes \mathfrak{g} \arrow[d, "pr_P"] \\ P \arrow[u, bend left, "\omega"] \end{tikzcd}$$

The connection ω provides also a decomposition of the dual bundle T^*P. The annihilator $(\mathsf{V}P)^o$ of the bundle $\mathsf{V}P$ is a vector subbundle in T^*P and by definition it is a bundle of horizontal forms on P. The complementary subbundle of $(\mathsf{V}P)^o$ in T^*P is an annihilator of the horizontal distribution H^o. Therefore, we have a Whitney sum

$$\mathsf{T}^*P = (\mathsf{V}P)^o \oplus_M H^o.$$

We also have identifications $(\mathsf{V}P)^o \simeq P \times \mathsf{T}^*M$ and $H^o \simeq \mathsf{V}^*P$. The decomposition of the cotangent bundle allows us to define a horizontal projection of differential forms. If $\beta \in \Lambda^k P$ is a k-form on P then its horizontal part is given by the map

$$^h : \Lambda^k P \to \Lambda^k_1 P, \quad \beta \longmapsto \beta^h$$
$$\beta^h(v_1, \ldots, v_k) = \beta(v_1^h, \ldots, v_k^h), \quad v_1, \ldots, v_k \in \mathsf{T}P$$

where v_i^h is a horizontal part of the tangent vector v_i.

The curvature of the connection is defined as a horizontal part of the differential $\mathrm{d}\omega$. It is a $\mathfrak{g}$ valued two form on P and it reads

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega],$$

where $[\omega, \omega]$ is the bracket of $\mathfrak{g}$-valued forms on P.

3.5. Gauge fields

The notion of the connection may be equally expressed in terms of a family of $\mathfrak{g}$-valued one forms on M. This approach is widely used by physicists working in classical field theory.

Let $s_\alpha : M \supset U_\alpha \to P$ be a local section of the bundle π. A pull-back of the connection form ω defines a form $A_\alpha := s_\alpha^*\omega$

$$A_\alpha(m) : \mathsf{T}_m M \to \mathfrak{g}, \quad v_m \longmapsto \omega_P(s_\alpha(m))(\mathsf{T}s_\alpha(v_m)). \tag{5}$$

The form A_α is called a *gauge field*. In local coordinates it takes a form

$$A_\alpha(m) = A^a_\alpha(m) \otimes e_a = A^a{}_j(m)\mathrm{d}q^j \otimes e_a \quad m \in M$$

where we skipped the index α in order to simplify the notation. One can show that gauge fields must satisfy a gluing condition

$$A_\alpha = \mathrm{Ad}_{g_{\alpha\beta}} \circ (A_\beta - g^*_{\alpha\beta}\theta)$$

or equivalently

$$A_\alpha = \mathrm{Ad}_{g_{\alpha\beta}} \circ A_\beta + g^*_{\beta\alpha}\theta$$

on the overlapping $U_{\alpha\beta}$. We used here the Maurer–Cartan form

$$\theta : G \to \mathsf{T}^*G \otimes \mathfrak{g}, \quad \theta(g) = \mathsf{T}L_{g^{-1}}.$$

On the other hand, once we have a family of forms A_α we can restore a connection form ω on P. The restriction of ω to $\pi^{-1}U_\alpha$ is given by

$$\omega_\alpha(p) = \mathrm{Ad}_{g_\alpha(p)^{-1}} \circ \pi^* A_\alpha(p) + g^*_\alpha\theta(p). \tag{6}$$

The proof of the above statement is rather technical so we will skip it here.

We can also use s_α to pull-back the curvature form Ω and obtain a $\mathfrak{g}$ valued two-form $F_\alpha = s_\alpha^*\Omega$ on M. It is an easy task to check that F_α and F_β satisfy the condition (4) on the overlappings. Therefore we claim that the family of forms $\{F_\alpha\}$ define a section

$$F : M \to \wedge^2\mathsf{T}^*M \otimes_M \mathrm{ad}P.$$

It turns out that the curvature of the connection ω may be equally represented by a global $\mathfrak{g}$-valued two-form Ω on P or by a global $\mathrm{ad}P$-valued two-form F on M.

4. Gauge fields in terms of jet spaces

In gauge field theories, the Lagrangian density usually depends just on the curvature instead of the full jet of the connection. Therefore it is a natural physical question whether there is a geometrical procedure for describing a reduction of the jet of the connection to the curvature. The answer turns out to be "yes" and it is associated with the notion of the so-called *canonical principal connection* [23,24,44]. We will start with a more precise description of the bundle $C \to M$.

4.1. The jet bundle of a principal bundle

We will consider the bundle of first jets over a principal bundle. Let us take a closer look on the trivialization of the bundle J^1P. From the previous considerations, we have an inclusion $\mathsf{J}^1P \subset \mathsf{T}^*M \otimes_P \mathsf{T}P$ over P. Let $g_\alpha : P \supset \pi^{-1}U_\alpha \to G$ be a local trivialization of P. We can use it to trivialize the tensor bundle $\mathsf{T}^*M \otimes_P \mathsf{T}P$, namely

$$\mathsf{T}^*M \otimes \mathsf{T}P \to \mathsf{T}^*M \otimes (\mathsf{T}M \times G \times \mathfrak{g}) = (\mathsf{T}^*M \otimes \mathsf{T}M) \times G \times (\mathsf{T}^*M \otimes \mathfrak{g}).$$

Notice that using that trivialization the affine bundle J^1P reads that the left bracket is an identity on $\mathsf{T}M$. Therefore, we can skip it and introduce a local trivialization

$$\begin{aligned}&\Psi_\alpha : \mathsf{J}^1P \supset \pi_1^{-1}U_\alpha \to G \times (\mathsf{T}^*M \otimes \mathfrak{g}),\\ &\Psi_\alpha(\mathrm{j}^1_m\phi) \longmapsto \left(g_\alpha(\phi(m)), -\phi^* g_\alpha^*\theta\right),\end{aligned}$$

where θ is a Maurer–Cartan form on G. It can be written as $\theta = \theta^b \otimes e_b$, where θ^b are one-forms on T^*G and $\{e_b\}$ is a basis of $\mathfrak{g}$. Then we obtain a more convenient form of above trivialization $-\phi^*g_\alpha^*\theta = f^b \otimes e_b$, where $f^b = -\phi^*g_\alpha^*\theta^b$ are one forms on M.

Let $g_\alpha : P \supset \pi^{-1}U_\alpha \to G$, $g_\beta : P \supset \pi^{-1}U_\beta \to G$ be two trivializations. One can show that on the overlappings the transition functions must obey a gluing condition

$$-\phi^*g_\beta^*\theta = -\phi^*g_\alpha^*\theta - \mathrm{Ad}_{g_\alpha\circ\phi(m)^{-1}} \circ g_{\beta\alpha}^*\theta.$$

The right action R_g of G on P can be prolonged to the action on the first jet bundle J^1P

$$\mathsf{J}^1R_g : \mathsf{J}^1P \to \mathsf{J}^1P : \mathrm{j}^1_m\phi \to \mathrm{j}^1_m(\phi g),$$

such that the diagram

$$\begin{tikzcd} \mathsf{J}^1P \arrow[r, "\mathsf{J}^1R_g"] \arrow[d, "\pi_{1,0}"'] & \mathsf{J}^1P \arrow[d, "\pi_{1,0}"] \\ P \arrow[r, "R_g"] & P \end{tikzcd}$$

is commutative. So, the action in local trivialization reads

$$\begin{aligned}&\mathsf{J}^1R_h : G \times (\mathsf{T}^*M \otimes \mathfrak{g}) \to G \times (\mathsf{T}^*M \otimes \mathfrak{g}),\\ &(g, j_\alpha) \longmapsto (gh, \mathrm{Ad}_{h^{-1}} \circ j_\alpha), \qquad j_\alpha \in \mathsf{T}^*M \otimes \mathfrak{g}.\end{aligned}$$

The affine dual of the bundle J^1P is the bundle $\mathsf{J}^\dagger P$ over M. We have a trivialization

$$\mathsf{J}^\dagger P \to G \times \mathbb{R} \times \mathsf{T}M \otimes \mathfrak{g}^*.$$

Next, we introduce coordinates $(g_\alpha(p), r, x^i, \varphi^a{}_j)$ in $\mathsf{J}^\dagger P$. If (e^a_*) is a basis of $\mathfrak{g}^*$ then $\varphi^a{}_j$ are coordinates with respect to the basis $\mathrm{d}x^j \otimes e^a_*$. If in a local trivialization $\mathrm{j}^1_m\phi = (g_\alpha \circ \phi(m), x^i, A^a{}_j)$ and $T_m = (g_\alpha(p), r, x^i, \varphi_a{}^j)$, then the evaluation between J^1P and $\mathsf{J}^\dagger P$ is given by

$$\langle \mathrm{j}^1_m\phi, T_m\rangle = r + \varphi_a{}^j A^a{}_j.$$

The bundle J^oP is locally isomorphic to $G \times \mathsf{T}M \otimes \mathfrak{g}^*$. It has adapted coordinates $(g_\alpha(p), x^i, \varphi_a{}^j)$.

4.2. Connection in a principal bundle as sections of the jet bundle

We will present now an equivalent approach to the notion of connections in a principal bundle. Let $\pi_{1,0} : \mathsf{J}^1P \to P$ be the bundle of first jets of P. Let us also consider a section $\sigma : P \to \mathsf{J}^1P$. For each $p \in P$ the value $\sigma(p)$ is a first jet of a certain section of the bundle π. Let us denote it by $\sigma(p) = \mathrm{j}^1_m\phi$ where $\phi(m) = p$. There is a tangent map $\mathsf{T}\phi : \mathsf{T}_mM \to \mathsf{T}_{\phi(m)}P$ associated with $\mathrm{j}^1_m\phi$. The image $\mathsf{T}\phi(\mathsf{T}_mM) \subset \mathsf{T}_{\phi(m)}P$ is a horizontal complement to V_mP in the tangent space $\mathsf{T}_{\phi(m)}P$, namely

$$\mathsf{T}_{\phi(m)}P = \mathsf{V}_{\phi(m)}P \oplus \mathsf{T}\phi(\mathsf{T}_mM).$$

Therefore, a global section of the bundle $\pi_{1,0}$ provides a horizontal distribution in the bundle $\pi : P \to M$.

Since a principal connection has to be equivariant with respect to the action of G we can divide the bundle $\pi_{1,0} : \mathsf{J}^1P \to P$ by this action and obtain a bundle

$$\pi_C : C \to M$$

where $C := \mathsf{J}^1P/G$. Each section of π_C provides a principal connection of the bundle $\pi : P \to M$. The bundle π_C is an affine bundle over M modeled on a vector bundle $\mathsf{T}^*M \otimes_P \mathrm{ad}P$. The trivialization of C comes from the trivializations of J^1P. We have

$$C \to \mathsf{T}^*M \otimes \mathfrak{g} : [\mathrm{j}^1_m\phi] \to (j_\alpha).$$

The transition functions between different trivializations read

$$\begin{tikzcd} & C \arrow[dl, "triv_1"'] \arrow[dr, "triv_2"] & \\ \mathsf{T}^*M \otimes \mathfrak{g} \arrow[rr] & & \mathsf{T}^*M \otimes \mathfrak{g} \end{tikzcd} \qquad \begin{tikzcd} & {[\mathrm{j}^1_m\phi]} \arrow[dl, "triv_1"'] \arrow[dr, "triv_2"] & \\ (j_\alpha) \arrow[rr] & & (\mathrm{Ad}_{g_{\beta\alpha}(m)} \circ j_\alpha + g^*_{\alpha\beta}\theta) \end{tikzcd}$$

There is a one-to-one correspondence between sections of the bundle π_C and G-invariant sections of the bundle $\pi_{1,0}$ given by the formula

$$[\sigma] : M \to C, \quad [\sigma](m) = [\sigma(p)], \quad \sigma : P \to \mathsf{J}^1P$$

where we assume that σ is G-invariant and $\pi(p) = m$. A glance at the expression shows that the above definition does not depend on the choice of the representatives.

The $\mathfrak{g}$-valued one forms j_α can be identified with gauge fields. Therefore, we will introduce coordinates (x^i, j^a_k) in C such that

$$j_\alpha = j^a{}_k \mathrm{d}x^k \otimes e_a.$$

Since the forms j_α may be identified with gauge fields, from now on we will write A_α and $A^a{}_k$ instead of j_α and $j^a{}_k$, respectively.

The bundle J^1C may be equipped with local coordinates $(x^i, A^a{}_k, A^a{}_{kl})$. Similarly, in the dual bundle $\mathsf{J}^\dagger C$ we have coordinates $(x^i, A^a{}_k, r, \varphi^a{}_{kl})$. In the bundle $\mathsf{J}^\dagger C$ we can distinguish a vector subbundle of constant affine maps on the fibers. In coordinates it is given by $(x^i, A^a{}_k, r, 0)$. Since each such a map depends only on its projection on C it can be associated with a function $f : C \to \mathbb{R}$. We introduce the bundle

$$\mathsf{J}^oC := \mathsf{J}^\dagger C/\{f : C \to \mathbb{R}\}$$

with coordinates $(x^i, A^a{}_k, \varphi^a{}_{kl})$.

4.3. Canonical principal connection

It is well-known that if $P \to M$ is a principal bundle, then there is no a canonical choice of a connection in it. However, it turns out that in a case of the bundle $P \times C \to C$ such a canonical choice exists [42,44]. As it was stated before there is a natural embedding $\mathsf{J}^1P \subset \mathsf{T}^*M \otimes_P \mathsf{T}P$. Furthermore, there also exists an embedding of J^1P in $\mathsf{T}^*P \otimes_P \mathsf{V}P$ given by the formula

$$\mathsf{J}^1P \to \mathsf{T}^*P \otimes_P \mathsf{V}P, \quad \mathrm{j}^1_m\phi \longmapsto (id_{\mathsf{T}P} - \mathsf{T}(\phi \circ \pi)).$$

This embedding induces a map

$$\theta : \mathsf{J}^1P \times_P \mathsf{T}P \to \mathsf{V}P, \quad (\mathrm{j}^1_m\phi, v) \longmapsto (id_{\mathsf{T}P} - \mathsf{T}(\phi \circ \pi))(v),$$

and if we take the quotients by G we obtain

$$\theta_G : C \times_M \mathsf{A} \to \mathrm{ad}P,$$
$$(x^i, A^a{}_j) \times (x^i, \dot{x}^k, X^b) \longmapsto (x^i, X^b - A^a{}_j\dot{x}^j).$$

The connection in the bundle $P \times C \to C$ is given by a splitting of the short exact sequence

$$0 \to \mathsf{V}(P \times_M C) \to \mathsf{T}(P \times_M C) \to P \times_M \mathsf{T}C \to 0$$

We can rewrite the above construction by using these identifications

$$\mathsf{V}(P \times_M C) \simeq C \times \mathsf{V}P, \tag{7}$$
$$\mathsf{T}(P \times_M C) \simeq \mathsf{T}P \times_M \mathsf{T}C, \tag{8}$$

and taking the quotient by G. Then, we obtain a short exact sequence

$$0 \to C \times_M \mathrm{ad}P \to \mathsf{A} \times_M \mathsf{T}C \to \mathsf{T}C \to 0.$$

One can see that the embedding θ_G provides a splitting of this sequence

$$\begin{tikzcd} 0 \arrow[r] & C \times_M \mathrm{ad}P \arrow[r] & \mathsf{A} \times_M \mathsf{T}C \arrow[r] \arrow[d, "id\times\tau_C"] & \mathsf{T}C \arrow[r] & 0. \\ & & \mathsf{A} \times_M C \arrow[ul, "\theta"] & & \end{tikzcd}$$

It follows that, if $P \to M$ is a principal bundle then the bundle $P \times_M C \to C$ admits a canonical connection. A form θ_G defines a horizontal lift

$$Z_C : \mathsf{T}C \to \mathsf{T}C \times_M \mathsf{A}.$$
$$(x^i, A^a_j, \dot{x}^k, \dot{A}^b_l) \longmapsto (x^i, A^a_j, \dot{x}^k, \dot{A}^b_l) \times (x^i, \dot{x}^j, A^a_k \dot{x}^k),$$

which is a section of the tensor bundle

$$Z_C : C \to \mathsf{T}^*C \otimes (\mathsf{T}C \times_M A),$$
$$Z_C = \mathrm{d}x^i \otimes \partial_{x^i} + \mathrm{d}A^a_j \otimes \partial_{A^a_j} + A^a_j \mathrm{d}x^j \otimes e_a.$$

We can define now the strength F_C of the canonical connection Z_C

$$F_C = \frac{1}{2}[Z_C, Z_C]_{FN}, \quad F_C \in \wedge^2 \mathsf{T}^*C \otimes_M \mathrm{ad}P$$
$$F_C = (\mathrm{d}A^a{}_j \wedge \mathrm{d}x^j + \frac{1}{2} c^a_{mn} \mathrm{d}x^m \wedge \mathrm{d}x^n) \otimes e_a$$

where $[\cdot,\cdot]_{FN}$ is the Frölicher–Nijenhuis bracket. The strength form F_C is called a *canonical strength* because if $\omega : M \to C$ is any principal connection then $\omega^* F_C = F_\omega$. The form F_C allows us to define an affine surjective map

$$\mathcal{F} : \mathsf{J}^1 C \to C \times \wedge^2 \mathsf{T}^* M \otimes_M \mathrm{ad}P, \qquad \mathcal{F}(\mathrm{j}^1\omega) = \omega^* F_C = F_\omega.$$

There exists a canonical splitting over C [45]

$$\mathsf{J}^1 C = C_+ \oplus_C C_-,$$
$$C_+ = \mathsf{J}^2 P/G, \quad C_- = C \times_M \wedge^2 \mathsf{T}^* M \otimes_M \mathrm{ad}P$$

with projections

$$\mathcal{F} : \mathsf{J}^1 C \to C_-,$$
$$(x^i, A^a{}_j, A^b{}_{jk}) \longmapsto \left(x^i, A^a{}_j, \frac{1}{2}(A^l{}_{jk} - A^l{}_{kj} + c^l_{ab} A^a{}_j A^b{}_k)\right),$$
$$\mathcal{S} : \mathsf{J}^1 C \to C_+,$$
$$(x^i, A^a{}_j, A^b{}_{jk}) \longmapsto \left(x^i, A^a{}_j, \frac{1}{2}(A^l{}_{jk} + A^l{}_{kj} - c^l_{ab} A^a{}_j A^b{}_k)\right).$$

One can see that the result of the first projection is just the curvature form of the connection

$$F_\omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega],$$
$$F_\omega = \frac{1}{2} F^a{}_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j \otimes e_b$$

where

$$F^a{}_{ij} = \partial_j A^a{}_i - \partial_i A^a{}_j + c^a_{mn} A^m{}_i A^n{}_j.$$

Therefore, we can introduce coordinates

$$(x^i, A^a{}_j, F^b{}_{jk}) \ in\ C_-, \tag{9}$$
$$(x^i, A^a{}_j, S^b{}_{jk}) \ in\ C_+ \tag{10}$$

such that $F^b{}_{jk} = -F^b{}_{kj}$ and $S^b{}_{jk} = S^b{}_{kj}$.

4.4. The phase bundle

In the previous section we have described a procedure for reducing the bundle $\mathsf{J}^1 C$ to the bundle $\wedge^2 \mathsf{T}^* M \otimes_M \mathrm{ad}P$. We will show now that $\mathcal{F}$ defines a similar reduction on the dual side. We recall that the bundle $\mathsf{J}^\dagger C$ consists of affine maps on $\mathsf{J}^1 C$. The bundle $\mathsf{J}^1 C$ can be decomposed on subbundles C_+ and C_- where the first one is affine and the second one is of vector type. Therefore each affine map on J^1C may be similarly decomposed on an affine map on C_+ and a linear map on C_-. Let us recall that the bundle $\mathrm{J}^\dagger C$ is isomorphic to $\Lambda_2^{n+1}C$ via the formula

$$\begin{aligned}&\Psi : \Lambda_2^{n+1}C \to \mathrm{J}^\dagger C\\&\langle \Psi(\alpha), \mathrm{j}^1\omega\rangle\eta = \omega^*\alpha, \qquad \alpha \in \Lambda_2^{n+1}C\end{aligned}$$

The pull-back $\omega^*\alpha$ reads in local coordinates as

$$\begin{aligned}&\alpha(x^i, A^a{}_j, p, p^{ik}{}_a) = p\mathrm{d}^{n+1}x + p^{ik}{}_a \mathrm{d}A^a{}_k \wedge \mathrm{d}^n x_i, \qquad \omega(x^i) = (x^i, A^a{}_k(x^i))\\&\omega^*\alpha = p\mathrm{d}^{n+1}x + p^{ik}{}_a \mathrm{d}A^a{}_k(x^i) \wedge \mathrm{d}^n x_i = p\mathrm{d}^{n+1}x + p^{ik}{}_a \partial_l A^a{}_k(x^i)\mathrm{d}x^l \wedge \mathrm{d}^n x_i\\&\quad = p\mathrm{d}^{n+1}x + p^{lk}{}_a \partial_l A^a{}_k(x^i)\mathrm{d}^{n+1}x = (p + p^{lk}{}_a \partial_l A^a{}_k(x^i))\mathrm{d}^{n+1}x.\end{aligned}$$

Therefore the evaluation between J^1C and $\mathrm{J}^\dagger C$ in given by

$$\langle \Psi(\alpha), \mathrm{j}^1\omega\rangle = p + p^{lk}{}_a A^a{}_{kl}.$$

In the same way, we can show that the evaluation between $\mathrm{J}^oC \simeq \Lambda_2^{n+1}C/\Lambda_1^{n+1}C$ and J^1C reads as

$$\langle \Psi(\alpha), \mathrm{j}^1\omega\rangle = p^{lk}{}_a A^a{}_{kl}.$$

Let us recall that the coordinates $F^b{}_{jk}$ and $S^b{}_{jk}$ in C_- and C_+ are antisymmetric and symmetric in j, k respectively. It means that the dual bundle to C_- in J^oC is a subbundle $C_-^* \subset \mathrm{J}^oC$ such that each map from C_-^* vanishes on C_+. Similarly, the dual bundle of C_+ is a subbundle $C_+^* \subset \mathrm{J}^oC$ such that each map from C_+^* vanishes on C_-. We have a decomposition

$$\begin{aligned}&\mathrm{J}^oC = C_+^* \oplus_C C_-^*\\&p^{ik}{}_a = \frac{1}{2}(p^{ik}{}_a + p^{ki}{}_a) + \frac{1}{2}(p^{ik}{}_a - p^{ki}{}_a)\end{aligned}$$

and coordinates

$$\begin{aligned}&(x^i, A^a{}_j, \varphi^{ik}{}_a) \text{ in } C_-^*, \qquad \varphi^{ik}{}_a = -\varphi^{ki}{}_a\\&(x^i, A^a{}_j, \xi^{ik}{}_a) \text{ in } C_+^*, \qquad \xi^{ik}{}_a = \xi^{ki}{}_a.\end{aligned}$$

One can show that there is an isomorphism

$$C_-^* = C \times (\wedge^2 \mathsf{T}M \otimes_M \mathrm{ad}^*P).$$

Given a volume form η on M, we have the following decomposition

$$\mathrm{J}^\dagger C = C_+^* \oplus_C C_-^* \oplus \mathbb{R}.$$

Since the bundle $\mathrm{J}^\dagger C$ is isomorphic to the bundle $\Lambda_2^n(C)$, then each element of $C_-^* \oplus \mathbb{R}$ may be represented as follows

$$\alpha(x^i, A^a{}_j, p, \varphi^{ik}{}_a) = p\mathrm{d}^{n+1}x + \varphi^{ik}{}_a \mathrm{d}A^a{}_k \wedge \mathrm{d}^n x_i$$

Similarly, we can identify the elements of C_+^* and C_-^* with forms

$$\begin{aligned}&\Lambda_2^{n+1}C/\Lambda_1^{n+1}C \ni \quad \varphi^{ik}{}_a \mathrm{d}A^a{}_k \wedge \mathrm{d}^n x_i \text{ in } C_-^*,\\&\Lambda_2^{n+1}C/\Lambda_1^{n+1}C \ni \quad \xi^{ik}{}_a \mathrm{d}A^a{}_k \wedge \mathrm{d}^n x_i \text{ in } C_+^*.\end{aligned}$$

Finally, there is a decomposition

$$\Lambda_2^{n+1}(\mathrm{J}^\dagger C) = \Lambda_2^{n+1}(C_+^* \oplus C_-^* \oplus \mathbb{R}) = \Lambda_2^{n+1}(C_+^* \oplus \mathbb{R}) \oplus \Lambda_2^{n+1}(C_-^* \oplus \mathbb{R})$$

which means that the canonical multisymplectic form on $\mathrm{J}^\dagger C$ may be written as

$$\Omega = \Omega_+ + \Omega_-, \qquad \text{where} \quad \Omega_+ \in \Lambda_2^{n+1}(C_+^* \oplus \mathbb{R}), \quad \Omega_- \in \Lambda_2^{n+1}(C_-^* \oplus \mathbb{R})$$

and

$$\Omega_- = -\mathrm{d}p \wedge \mathrm{d}^{n+1}x - \mathrm{d}\varphi^{jk}{}_a \wedge \mathrm{d}A^a{}_j \wedge \mathrm{d}^n x_k.$$

We can easily check that submanifold $(C_+^* \oplus \mathbb{R}, \Omega_-)$ is still a multisymplectic manifold. If we take a Hamiltonian section $h : C_-^* \to C_-^* \oplus \mathbb{R}$ then the pull-back of the multisymplectic form reads

$$h^*\Omega = h^*\Omega_- = \Omega_{-h} \in \Lambda_2^{n+1}(C_-^*).$$

5. Hamiltonian formalism for gauge theories

We will describe now the Hamiltonian formalism for field theories which configuration space is C_{-} instead of full J^1C. For example, this is a case of Yang–Mills theories. Since the bundle of gauge fields is the bundle $\pi_C : C \to M$, then the role of the configuration space is played by its bundle of first jets J^1C. Therefore, the Lagrangian density is a map

$$\mathcal{L} : \mathrm{J}^1C \to \Lambda^{n+1}M.$$

Let us assume now that $\mathcal{L}$ does not depend on the full jet of connection but only on its projection on C_{-}. Then we have

$$\mathcal{L} : C \times_M \wedge^2 \mathrm{T}^*M \otimes_M \mathrm{ad}P \to \Lambda^{n+1}M.$$

In presence of a volume form the Lagrangian density can be written as $\mathcal{L} = L\eta$, where L is called *Lagrangian function*. In coordinates we have

$$\mathcal{L}(x^i, A^a{}_j, F^b{}_{jk}) = L(x^i, A^a{}_j, F^b{}_{jk})\eta.$$

The Hamiltonian side of the theory is usually obtained from the Lagrangian density $\mathcal{L}$ via Legendre transform. Since the Lagrangian depends only on C_{-} and

$$A^b{}_{jk} = F^b{}_{jk} + S^b{}_{jk}$$
$$\frac{\partial}{\partial A^b{}_{jk}} = \frac{\partial}{\partial F^b{}_{jk}} + \frac{\partial}{\partial S^b{}_{jk}}$$

then the Legendre map $\mathrm{J}^1C \to \mathrm{J}^\dagger C$ reduces to

$$Leg_{\mathcal{L}} : C_{-} \to C_{-}^* \oplus \mathbb{R}, \tag{11}$$

$$Leg_{\mathcal{L}}(x^i, A^a{}_j, F^b{}_{jk}) = \left(x^i, A^a{}_j, L - \frac{\partial L}{\partial F^b{}_{jk}} F^b{}_{jk}, \frac{\partial L}{\partial F^b{}_{jk}}\right) \tag{12}$$

Using the projection $\mu : \mathrm{J}^\dagger C \to \mathrm{J}^o C$ we can now introduce the map

$$leg_{\mathcal{L}} = \mu \circ Leg_{\mathcal{L}}, \quad leg_{\mathcal{L}} : C_{-} \to C_{-}^*, \tag{13}$$

$$leg_{\mathcal{L}}(x^i, A^a{}_j, F^b{}_{jk}) = \left(x^i, A^a{}_j, \frac{\partial L}{\partial F^b{}_{jk}}\right). \tag{14}$$

The Hamiltonian section is then defined as

$$h : C_{-}^* \to C_{-}^* \oplus \mathbb{R}, \quad h = Leg_{\mathcal{L}} \circ leg_{\mathcal{L}}^{-1},$$
$$h(x^i, A^a{}_j, \varphi_b{}^{jk}) = (x^i, A^a{}_j, L(x^i, A^a{}_j, F^b{}_{jk}) - \varphi_b{}^{jk} F^b{}_{jk}, \varphi_b{}^{jk})$$

with the associated Hamiltonian density

$$\mathcal{H} : C_{-}^* \oplus \mathbb{R} \to \Lambda^{n+1}M,$$
$$\mathcal{H}(x^i, A^a{}_j, p, \varphi_b{}^{jk}) = (p + \varphi_b{}^{jk} F^b{}_{jk} - L)\mathrm{d}^{n+1}x$$

and the Hamiltonian function

$$H(x^i, A^a{}_j, \varphi_b{}^{jk}) = \varphi_b{}^{jk} F^b{}_{jk} - L. \tag{15}$$

Let us consider now a section

$$\tau : M \to C_{-}^*, \quad \tau(x^i) = \left(x^i, \tau^a{}_j(x^i), \tau_a{}^{jk}(x^i)\right).$$

The Hamilton–De Donder–Weyl equations on C_{-}^* read

$$\frac{\partial \tau^a{}_j}{\partial x^i} = \frac{\partial H}{\partial \varphi_a{}^{ij}} \circ \tau, \qquad \frac{\partial \tau_a{}^{ij}}{\partial x^i} = -\frac{\partial H}{\partial A^a{}_j} \circ \tau$$

6. Hamilton–Jacobi theory for gauge theories

In this section we will apply the formalism derived in [8] for the case of gauge fields. Let us consider the diagram

$$\begin{tikzcd}
\mathsf{J}^o C \arrow[dr, "\pi_1^o"] & \\
\mathsf{J}^\dagger C \arrow[u, "\mu"] \arrow[d] & M \\
C \arrow[u, bend left, "\gamma"] \arrow[ur, "\pi"'] &
\end{tikzcd}$$

where γ is a section. We will assume now that we have a connection in the bundle $\mathsf{J}^o C \to M$ and that $\mathbf{h}$ is its horizontal projector. The connection in π_1^o can be projected to a connection in the bundle $C \to M$ in such a way that its horizontal projector reads

$$\begin{aligned}
&\mathbf{h}^\gamma(\omega) : \mathsf{T}_\omega C \to \mathsf{T}_\omega C, \\
&X \longmapsto \mathbf{h}^\gamma(\omega)(X) = \mathsf{T}_f \pi^\circ_{1,0}(\mathbf{h}(f)(Y)),
\end{aligned}$$

where $f = (\mu \circ \gamma)(\omega)$ and Y is any tangent vector of $\mathsf{T}_f \mathsf{J}^o C$ which projects onto X by $\mathsf{T}\pi^\circ_{1,0}$. The Hamilton–Jacobi equation can be expressed as follows.

Theorem 2. *Assume that γ is closed and that the induced connection $\mathbf{h}^\gamma$ on $C \to M$ is flat. Then the following conditions are equivalent:*
(i) If σ is an integral section of $\mathbf{h}$ then $\mu \circ \gamma \circ \sigma$ is a solution of the Hamilton–De Donder–Weyl equations
(ii) The $(n+1)$-form $h \circ \mu \circ \gamma$ is closed

In our case, the above diagram may be reduced to the diagram

$$\begin{tikzcd}
C^*_- \arrow[d, bend right, "h"'] \arrow[dr, "\pi_1^o"] & \\
C^*_- \oplus \mathbb{R} \arrow[u, "\mu"'] \arrow[d] & M \\
C \arrow[u, bend left, "\gamma_-"] \arrow[uu, bend left=60, "\lambda"] \arrow[ur, "\pi"'] &
\end{tikzcd}$$

Let us denote by $\lambda := \mu \circ \gamma_-$ a section of the bundle $C^*_- \to C$. We have the following theorem.

Theorem 3. *Assume that γ_- is closed and that the induced connection $\mathbf{h}^\gamma$ on $C \to M$ is flat. Then the following conditions are equivalent:*
(i) If σ is an integral section of $\mathbf{h}$ then $\lambda \circ \sigma$ is a solution of the Hamilton–De Donder–Weyl equations
(ii) The $(n+1)$-form $h \circ \lambda$ is closed

Let us assume now that $\lambda = \mathrm{d}S$ where

$$S = S^i(x^i, A^a{}_j)\mathrm{d}^n x_i$$

is a 1-semibasic form on C such that $\frac{\partial S_i}{\partial A^a{}_j} = -\frac{\partial S_j}{\partial A^a{}_i}$. Then in local coordinates the equation $\mathrm{d}(h \circ \lambda) = 0$ is equivalent to

$$\frac{\partial S^i}{\partial x^i} + H\left(x^i, A^a{}_j, \frac{\partial S_i}{\partial A^a{}_j}\right) = f(x^i). \tag{16}$$

7. Example: Yang–Mills theories

Let us consider now a Hamilton–Jacobi theory for Yang–Mills theory. A free Yang–Mills theory is described by the Lagrangian

$$\mathcal{L} : \mathsf{J}^1 C \supset C \times_M \wedge^2 \mathsf{T}^* M \otimes_M \mathrm{ad}P \to \Omega^{n+1}, \quad \mathcal{L}(\omega, F) = \frac{1}{4} K_{ab} F^a \wedge \star F^b,$$

where $F = F^a \otimes e_a = F^a{}_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j \otimes e_a$. One can easily show that the associated Lagrangian function is

$$L(q^i, A^a{}_j, F^b{}_{jk}) = \frac{1}{4} K_{ab} g^{im} g^{jn} F^a{}_{ij} F^b{}_{mn} \sqrt{|g|} = \frac{1}{4} F^a{}_{ij} F_a{}^{ij} \sqrt{|g|}$$

where g is a metric on M and K is an Ad-invariant scalar product on $\mathfrak{g}$. To write the Hamilton–Jacobi equation we need the Hamiltonian side of the theory. Using (12) and (14) we have

$$leg_{\mathcal{L}}(x^i, A^a{}_j, F^a{}_{ij}) = \left(x^i, A^a{}_j, \frac{\partial L}{\partial F^a{}_{ij}}\right) = \left(x^i, A^a{}_j, \frac{1}{2}F_a{}^{ij}\sqrt{|g|}\right)$$
$$Leg_{\mathcal{L}}(x^i, A^a{}_j, F^a{}_{ij}) = \left(x^i, A^a{}_j, -\frac{1}{4}F^a{}_{ij}F_a{}^{ij}\sqrt{|g|}, \frac{1}{2}F_a{}^{ij}\sqrt{|g|}\right)$$

The inverse map of $leg_{\mathcal{L}}{}^{-1}$ reads

$$leg_{\mathcal{L}}^{-1}(x^i, A^a{}_j, \varphi_a{}^{ij}) = \left(x^i, A^a{}_j, \frac{2}{\sqrt{|g|}}\varphi^a{}_{ij}\right).$$

from which we have that

$$F^a{}_{ij} = \frac{2}{\sqrt{|g|}}\varphi^a{}_{ij}$$

Therefore, from (15) we obtain that the Yang–Mills Hamiltonian reads

$$H(x^i, A^a{}_j, \varphi_a{}^{ij}) = \frac{2}{\sqrt{|g|}}\varphi_a{}^{ij}\varphi^a{}_{ij} - L = \frac{2}{\sqrt{|g|}}\varphi_a{}^{ij}\varphi^a{}_{ij} - \frac{1}{4}\cdot\frac{4}{|g|}\varphi_a{}^{ij}\varphi^a{}_{ij}\sqrt{|g|} = \frac{1}{\sqrt{|g|}}\varphi_a{}^{ij}\varphi^a{}_{ij}$$

We can find now the Hamilton–De Donder–Weyl equations

$$\frac{\partial \tau^a{}_j}{\partial x^i} = \frac{\partial H}{\partial \varphi_a{}^{ij}} \circ \tau, \qquad \frac{\partial \tau_a{}^{ij}}{\partial x^i} = -\frac{\partial H}{\partial A^a{}_j} \circ \tau$$

for the above Hamiltonian. Let us consider a section

$$\tau(x^i) = \left(x^i, \tau^a{}_j(x^i), \tau_a{}^{jk}(x^i)\right)$$

The derivatives of the Hamiltonian read

$$\frac{\partial H}{\partial \varphi_a{}^{ij}} = \frac{2}{\sqrt{|g|}}\varphi^a{}_{ij}$$
$$\frac{\partial H}{\partial A^a{}_j} = \frac{\partial H}{\partial \varphi_b{}^{kl}} \cdot \frac{\partial \varphi_b{}^{kl}}{\partial A^a{}_j} = \frac{2}{\sqrt{|g|}} \cdot \varphi^b{}_{kl} \cdot \frac{\sqrt{|g|}}{2} \cdot \frac{\partial F_b{}^{kl}}{\partial A^a{}_j} = \varphi^b{}_{kl} \cdot \frac{\partial F_b{}^{kl}}{\partial A^a{}_j}$$

where

$$\frac{\partial F_b{}^{kl}}{\partial A^a{}_j} = c_{bma}(A^{mk}\delta^l{}_j - A^{ml}\delta^k{}_j)$$

so that

$$\frac{\partial H}{\partial A^a{}_j} = \varphi^b{}_{kl}c_{bma}(A^{mk}\delta^l{}_j - A^{ml}\delta^k{}_j) = 2\varphi^b{}_{kl}c_{bma}A^{mk}\delta^l{}_j = 2\varphi_b{}^{kj}c^{bma}A^m{}_k.$$

In the above calculations c^k_{ab} are the structure constants of the Lie algebra $\mathfrak{g}$. Therefore, the Hamilton–De Donder–Weyl equations for Yang–Mills theory read

$$\frac{\partial \tau^a{}_j}{\partial q^i} = \frac{2}{\sqrt{|g|}}\varphi^a{}_{ij},$$
$$\frac{\partial \tau_a{}^{ij}}{\partial q^i} = -2\varphi_b{}^{kj}c^b_{ma}A^m{}_k.$$

We can also write the Hamilton–Jacobi equation

$$\frac{\partial S_i}{\partial x^i} + \frac{1}{\sqrt{|g|}}K^{ab}g_{ki}g_{lj}\frac{\partial S_i}{\partial A^a{}_j}\frac{\partial S_k}{\partial A^b{}_l} = f(x^i).$$

8. Hamilton–Jacobi theory for gauge theories in a Cauchy data space

The Cauchy data space itself allows to relate the finite-dimensional and the infinite-dimensional formulation of field theory. It was shown in [8] that the Hamilton–Jacobi theory for classical fields may be formulated in a setting of the Cauchy space. We will show here a similar construction of the Hamilton–Jacobi theory for gauge field theories. We will base on the notation and constructions from [8].

8.1. A space of Cauchy data

We will introduce now a notion of a Cauchy data space which is the fundamental tool to consider infinite-dimensional version of our field theory. Let M be an $n+1$-dimensional manifold and Σ an n-dimensional, compact, oriented and embedded submanifold of M. We say that Σ is a *Cauchy surface*. Let us assume that Σ has a volume form η_Σ such that

$$\int_\Sigma \eta_\Sigma = 1.$$

We define a slicing of M which is a diffeomorphism

$$\chi_M : \mathbb{R} \times \Sigma \to M, \qquad (t,p) \longmapsto \chi_M(t,p).$$

We will assume here that M admits such a slicing and omit a detailed discussion of topological properties that ensure it. For each fixed $t \in \mathbb{R}$ the slicing defines an embedding of Σ in M

$$(\chi_M)_t : \Sigma \to M, \quad p \longmapsto \chi_M(t,p)$$

We will also assume that there exists such $t_0 \in \mathbb{R}$ that $\Sigma = (\chi_M)_{t_0}(\Sigma)$. The infinitesimal generator of χ_M will be denoted by ξ_M:

$$\xi_M : M \to \mathfrak{X}(M), \quad \xi_M := (\chi_M)_*\left(\frac{\partial}{\partial t}\right)$$

where $\frac{\partial}{\partial t}$ is the vector field tangent to the curve $(t,x) \to (t+s,x)$.

Let $\pi : E \to M$ be a fiber bundle and let us define the set

$$\widetilde{E} := \{\sigma : \Sigma \to E, \quad \text{such that} \quad \pi \circ \sigma = (\chi_M)_t\}.$$

This is a line bundle over $\mathbb{R}$ and there is a one-to-one correspondence between sections of $E \to M$ and sections of $\widetilde{E} \to \mathbb{R}$. If $\phi : M \to E$ is a section than the section of $\widetilde{E} \to \mathbb{R}$ is given by

$$t \longmapsto \sigma_t = \phi \circ (\chi_M)_t.$$

Furthermore, the standard bundle structures define by composition the bundle structures on the space of χ-sections. For instance, one can construct a bundle

$$\widetilde{\pi}^o_{1,0} : \widetilde{\mathsf{J}^o E} \to \widetilde{E}, \quad \sigma_{\mathsf{J}^o E} \longmapsto \pi^o_{1,0} \circ \sigma_{\mathsf{J}^o E}.$$

One can also define tangent vectors and forms on $\widetilde{E}$. A vector tangent to the curve $t \to \sigma_t$ is given by map a $\widetilde{X} : \Sigma \to \mathsf{T}E$ such that the diagram

$$\begin{tikzcd} \Sigma \arrow[r, "\widetilde{X}"] \arrow[dr, "\sigma_0"'] & \mathsf{T}E \arrow[d, "\tau_E"] \\ & E \end{tikzcd}$$

is commutative and such that there exists $k \in \mathbb{R}$ that

$$\mathsf{T}\pi(\widetilde{X}(p)) = k\xi_M\Big(\pi(\sigma_E(p))\Big), \quad \forall p \in N$$

Forms on $\widetilde{E}$ can be constructed from forms on E. Let α be a $(k+n)$-form on E. The k-form $\widetilde{\alpha}$ on $\widetilde{E}$ is given by

$$\widetilde{\alpha}(\sigma_E)(\widetilde{X}_1, \ldots, \widetilde{X}_k) = \int_\Sigma \sigma_E^*(i_{\widetilde{X}_1,\ldots,\widetilde{X}_k}\alpha)$$

The most important examples of the above construction are forms $\widetilde{\Omega}_h$ and $\widetilde{\eta}$ obtained from the forms Ω_h and η, respectively.

8.2. Hamilton–Jacobi theory on a bundle $\widetilde{C}$

Let us consider now the case when $E = C$. The bundle $\widetilde{C}$ inherits an affine structure from $C \to M$ where the model bundle is the vector bundle $\mathsf{T}^*M \otimes \mathrm{ad}P \to \mathbb{R}$. The correspondence between $C \to M$ and $\widetilde{C} \to \mathbb{R}$ constitutes the relation between the infinite-dimensional and finite-dimensional picture of gauge field theory.

For a section $\tau : M \to \mathsf{J}^o C$ there is an associated section

$$c : \mathbb{R} \to \widetilde{\mathsf{J}^o C}, \qquad t \longmapsto \tau_{|N_t} \circ (\chi_M)_t.$$

One can show that τ satisfies the Hamilton equations if and only if

$$i_{\dot{c}(t)}\widetilde{\Omega}_h = 0.$$

The above formula is an infinite-dimensional version of the Hamilton's equations on the bundle C.

Let us move now to the Hamilton–Jacobi equation. Let us assume now that γ is a solution of the Hamilton–Jacobi equation and that we have a connection in the bundle $\mathsf{J}^{\dagger}C \to M$ which satisfies Hamilton's field equations.

The section γ defines a section

$$\widetilde{\gamma} : \widetilde{C} \to \widetilde{\mathsf{J}^{o}C},$$
$$\sigma_C \longmapsto \mu \circ \gamma \circ \sigma_C.$$

On the other hand we can construct vector fields $\widetilde{X}^{\mathbf{h}}$ and $\widetilde{X}^{\mathbf{h}^{\gamma}}$ from the connection $\mathbf{h}$ and $\mathbf{h}^{\gamma}$ by

$$\widetilde{X}^{\mathbf{h}} : \widetilde{\mathsf{J}^{o}C} \to \mathsf{T}\widetilde{\mathsf{J}^{o}C},$$
$$\sigma_{\mathsf{J}^o} \to \widetilde{X}^{\mathbf{h}}(\sigma) \quad : \Sigma \to \mathsf{T}\mathsf{J}^{o}C$$
$$p \longmapsto \widetilde{X}^{\mathbf{h}}(\sigma)(p) = \mathrm{Hor}\Big(\xi((\chi_M)_t(p))\Big)$$

where $\mathrm{Hor}(X)(y)$ represents the horizontal lift of the tangent vector X to the point y.

In the same way we can construct the vector field $\widetilde{X}^{\mathbf{h}^{\gamma}}$ on $\widetilde{C}$ using the horizontal lift of the connection $\mathbf{h}^{\gamma}$.

Now we can introduce an infinite-dimensional version of the Hamilton–Jacobi theorem [8]. Before we will do that let us notice that there is no notion of Hamilton–Jacobi equation neither a solution of this equation in an infinite-dimensional case. Therefore, our idea is to extrapolate this definition from the finite-dimensional case through the properties that admit such an extrapolation.

First of all, let us notice that in a finite-dimensional case given the pair $(\mathsf{T}^*Q \times \mathbb{R}, H)$ where H is a Hamiltonian on $\mathsf{T}^*Q \times \mathbb{R}$ then the pair of forms $(\omega_Q, \mathrm{d}t)$ defines a cosymplectic structure on $\mathsf{T}^*Q \times \mathbb{R}$. However, $(\mathsf{T}^*Q \times \mathbb{R}, H)$ admits also another cosymplectic structure given by a pair $(\Omega_Q, \mathrm{d}t)$ where Ω_Q is a two-form such that

$$\Omega_Q = \omega_Q + \mathrm{d}H \wedge \mathrm{d}t.$$

The dynamics of the system is given by a Reeb vector field R_H satisfying $i_{R_H}\Omega_Q = 0$ and $i_{R_H}\mathrm{d}t = 1$. Secondly, the solution of a Hamilton–Jacobi equation for a classical Hamiltonian system $(\mathsf{T}^*Q, \omega_Q, H)$ is a closed one-form $\gamma : Q \to \mathsf{T}^*Q$ such that $H \circ \gamma = const$. However, the condition $\mathrm{d}\gamma = 0$ is equivalent to the condition $\gamma^*\omega_Q = 0$ and both of them just say that $\gamma(Q)$ is a Lagrangian submanifold in T^*Q.

It follows then that $(\widetilde{\mathsf{J}^{o}C}, \widetilde{\Omega}_h, \widetilde{\eta})$ is a precosymplectic system. In our infinite-dimensional case the vector field $\widetilde{X}^{\mathbf{h}}$ satisfies properties $i_{\widetilde{X}^{\mathbf{h}}}\widetilde{\Omega}_h = 0$ and $i_{\widetilde{X}^{\mathbf{h}}}\widetilde{\eta} = 1$. For dimensional reasons it fails to be cosymplectic. What is more, we have a following theorem

Theorem 4. *The section $\widetilde{\gamma}$ satisfies:*

(i) $\widetilde{\gamma}^*\widetilde{\Omega}_h = 0$

(ii) $i_{\mathsf{T}\widetilde{\gamma}(\sigma_C)(\widetilde{X}^{h^{\gamma}})}\widetilde{\Omega}_h = 0$ *for all* $\sigma_C \in \widetilde{C}$*which is an integral submanifold of* $\mathbf{h}^{\gamma}_{\widetilde{\pi}(\sigma_C)}$.

Due to these properties we call $\widetilde{\gamma}$ a solution of a Hamilton–Jacobi equation in an infinite dimensional setting.

8.3. Reduced Hamilton–Jacobi in a space of Cauchy data

In the previous subsection we presented the infinite-dimensional formulation of Hamilton–Jacobi equation for gauge theories. Now we will show how to reduce it when Hamiltonian depends only on the part C^*_-.

First of all, let us notice that many of our constructions from Section 4 have their reflections in the bundles of χ-sections. For example, the decomposition $\mathsf{J}^1C = C_- \oplus_C C_+$ implies a decomposition $\widetilde{\mathsf{J}^1C} = \widetilde{C_+} \oplus_{\widetilde{C}} \widetilde{C_-}$. The same happens with the other bundles, namely $\widetilde{\mathsf{J}^oC} = \widetilde{C^*_+} \oplus_{\widetilde{C}} \widetilde{C^*_-}$ and $\widetilde{\mathsf{J}^{\dagger}C} = \widetilde{C^*_+} \oplus_{\widetilde{C}} \widetilde{C^*_-} \oplus \mathbb{R}$. Let us consider now the restriction of Ω to C^*_-. An infinite-dimensional counterpart of (C^*_-, Ω_{-h}) is a pair $(\widetilde{C^*_-}, \widetilde{\Omega}_{-h})$. Let us remind that (C^*_-, Ω_{-h}) is a multisymplectic manifold while $(\widetilde{C^*_-}, \widetilde{\Omega}_{-h})$ is presymplectic.

Let us assume now that γ is a solution of the reduced Hamilton–Jacobi equation (16). We define a section

$$\widetilde{\gamma}_- : \widetilde{C} \to \widetilde{C^*_-} \oplus \mathbb{R},$$

$$\sigma_C \longmapsto \mu \circ \gamma_- \circ \sigma_C.$$

One can easily see that

$$\widetilde{\gamma}^*_-\widetilde{\Omega}_h = \widetilde{\gamma}^*_-(\widetilde{\Omega}_{+h} + \widetilde{\Omega}_{-h}) = \widetilde{\gamma}^*_-\widetilde{\Omega}_{-h}.$$

On the other hand $\mathsf{T}\widetilde{\gamma}_-(\sigma_C)(\widetilde{X}^{h^{\gamma}_-}) \in \widetilde{C^*_-}$ so that $i_{\mathsf{T}\widetilde{\gamma}_-(\sigma_C)(\widetilde{X}^{h^{\gamma}_-})}\widetilde{\Omega}_h = i_{\mathsf{T}\widetilde{\gamma}_-(\sigma_C)(\widetilde{X}^{h^{\gamma}_-})}\widetilde{\Omega}_{-h}$. Hence, we have a following conclusion

Theorem 5. *The section $\widetilde{\gamma} : \widetilde{C} \to \widetilde{C^*_-} \oplus \mathbb{R}$ satisfies:*

(i) $\widetilde{\gamma}^*\widetilde{\Omega}_{-h} = 0$,

(ii) $i_{\mathsf{T}\widetilde{\gamma}(\sigma_E)(\widetilde{X}^{h^{\gamma}})}\widetilde{\Omega}_{-h} = 0$ *for all* $\sigma_E \in \widetilde{E}$*which is an integral submanifold of* $\mathbf{h}^{\gamma}_{\widetilde{\pi}(\sigma_C)}$.

Similarly as in Theorem 4 we call $\widetilde{\gamma}$ a solution of the reduced Hamilton–Jacobi equation in an infinite dimensional setting.

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References

[1] R. Abraham, J.E. Marsden, Foundations of Mechanics, second ed., Benjamin–Cummings, New York, 1978.
[2] V. Aldaya, J.A. de Azcárraga, Geometric formulation of classical mechanics and field theory, Riv. Nuovo. Cimento 3 (10) (1980) 1–66.
[3] P. Balseiro, J.C. Marrero, D. Martín de Diego, E. Padrón, A unified framework for mechanics: Hamilton–Jacobi equation and applications, Nonlinearity 23 (8) (2010) 1887–1918.
[4] M. Barbero-Linán, M. Delgado-Téllez, D. Martín de Diego, A geometric framework for discrete Hamilton–Jacobi equation, AIP Conf. Proc., Vol. 1460, 2012, pp. 164–168.
[5] M. Barbero-Linán, M. de León, D. Martín de Diego, Lagrangian submanifolds and Hamilton–Jacobi equation, Monatsh. Math. 171 (3–4) (2013) 269–290.
[6] L.M. Bates, F. Fasso, N. Sansonetto, The Hamilton–Jacobi equation, integrability, and nonholonomic systems, J. Geom. Mech. 6 (4) (2014) 441–449.
[7] A.M. Bloch, T. Ohsawa, Nonholomic Hamilton–Jacobi equation and integrability, J. Geom. Mech. 1 (4) (2011) 461–481.
[8] C.M. Campos, D.M. de Diego, M. de León, M. Vaquero, Hamilton–Jacobi theory in Cauchy data space, Rep. Math. Phys. 76 (3) (2015) 359–387.
[9] F. Cantrijn, M. de Leon, M. de Diego, J. Vankershaver, Geometric aspects of nonholonomic field theories, Rep. Math. Phys. 46 (2005) 387–411.
[10] F. Cantrijn, L.A. Ibort, M. de Leon, Hamiltonian structures on multisymplectic manifolds, Rend. Semin. Mat. Univ. Politec. Torino 54 (1996) 225–236.
[11] F. Cantrijn, A. Ibort, M. de León, On the geometry of multisymplectic manifolds, J. Aust. Math. Soc. Ser. A 66 (3) (1999) 303–330.
[12] J.F. Cariñena, M. Crampin, L.A. Ibort, On the multisymplectic formalism for first order theories, Differential Geom. Appl. 1 (1991) 354–374.
[13] J.F. Cariñena, X. Gracia, G. Marmo, E. Martínez, M.C. Munoz-Lecanda, N. Román-Roy, Geometric Hamilton–Jacobi theory, Int. J. Geom. Methods Mod. Phys. 3 (7) (2006) 1417–1458.
[14] J.F. Cariñena, X. Gracia, G. Marmo, E. Martínez, M.C. Munoz-Lecanda, N. Román-Roy, Geometric Hamilton–Jacobi theory for nonholonomic dynamical systems, Int. J. Geom. Methods Mod. Phys. 7 (3) (2010) 431–454.
[15] J.F. Cariñena, X. Gracia, G. Marmo, E. Martínez, M.C. Munoz-Lecanda, N. Román-Roy, Structural aspects of Hamilton–Jacobi theory, Int. J. Geom. Methods Mod. Phys. 13 (2) (2015) 1650017.
[16] L. Colombo, M. de León, P.D. Prieto-Martínez, N. Román-Roy, Geometric Hamilton–Jacobi theory for higher-order autonomous systems, J. Phys. A 47 (23) (2014) 235203.
[17] G.C. Constantelos, On the Hamilton–Jacobi theory with derivatives of higher order, Nuovo Cimento B 1184 (1) (1984) 91–101.
[18] M. de León, J.C. Marrero, D.M. de Diego, A geometric Hamilton–Jacobi theory for classical field theories, in: Variations, Geometry and Physics, Nova Sci. Publ., New York, 2009, pp. 129–140.
[19] M. de León, J.C. Marrero, D.M. de Diego, Linear almost Poisson structures and Hamilton–Jacobi equation. Applications to nonholonomic mechanics, J. Geom. Mech. 2 (2010) 159–198.
[20] M. de León, J.C. Marrero, D. Martín de Diego, M. Salgado, S. Vilarino, Hamilton–Jacobi theory in k-symplectic field theories, Int. J. Geom. Methods Mod. Phys. 7 (8) (2010) 1491–1507.
[21] D. Dominici, J. Gomis, G. Longhi, J.M. Pons, Hamilton–Jacobi theory for constrained systems, J. Math. Phys. 25 (8) (1984) 2439–2460.
[22] A. Echeverria-Enriquez, M.C. Muñoz-Lecanda, Geometry of multisymplectic Hamiltonian first order theory, J. Math. Phys. 41 (2000) 7402–7444.
[23] P.L. García, Connections and 1-jets fibre bundles, Rend. Semin. Mat. Univ. Padova 47 (1972) 227–242.
[24] P.L. García, Gauge algebras, curvature and symplectic structure, J. Differential Geom. 12 (2) (1977) 209–227.
[25] M.J. Gotay, A multisymplecitc framework for classical field theory and the calculus of variations I: Covariant Hamiltonian formalism, in: M. Francaviglia (Ed.), Mechanics, Analysis, and Geometry: 200 Years After Lagrange, North Holland, Amsterdam, 1991, pp. 203–235.
[26] M.J. Gotay, A multisymplectic framework for classical field theory and the calculus of variations II. Space + time decomposition, Differential Geom. Appl. 1 (1991) 375–390.
[27] M.J. Gotay, J. Isenberg, J.E. Marsden, Momentum maps and classical relativistic fields, Part I: Covariant field theory, 2004, arXiv:physics/9801019v2.
[28] M.J. Gotay, J. Isenberg, J.E. Marsden, Momentum maps and classical relativistic fields, Part II: Canonical analysis of field theories, 2004, arXiv:math-ph/0411032v1.
[29] K. Grabowska, A Tulczyjew triple for classical fields, J. Phys. A 45 (2012) 145207.
[30] A. Ibort, A. Spivak, Covariant Hamiltonian field theories on manifolds with boundary: Yang–Mills theories, J. Geom. Mech. 9 (2017) 47–82.
[31] D. Iglesias-Ponte, M. de León, D. Martín de Diego, Towards a Hamilton–Jacobi theory for nonholonomic mechanical systems, J. Phys. A 41 (1) (2008) 015205.
[32] J. Kijowski, W.M. Tulczyjew, A Symplectic Framework for Field Theories, in: Lecture Notes in Physics, vol. 107, Springer-Verlag, Berlin-New York, 1979.
[33] J. Klein, Espaces variationnels et mécanique, Ann. Inst. Fourier 12 (1962) 1–124.
[34] O. Krupková, Hamiltonian field theory, J. Geom. Phys. 43 (2002) 93–132.
[35] M. Leok, D. Sosa, Dirac structures and Hamilton–Jacobi theory for Lagrangian mechanics on Lie algebroids, J. Geom. Mech. 4 (4) (2012) 421–442.
[36] M. de León, D. Martín de Diego, A. Santamaría-Merino, Tulczyjews triples and Lagrangian submanifolds in classical field theories, in: W. Sarlet, F. Cantrijn (Eds.), Applied Differential Geometry and Mechanics, Univ. of Gent, Gent, Academia Press, 2003, pp. 21–47.
[37] M. de León, D. Martín de Diego, M. Vaquero, On the Hamilton–Jacobi theory for singular Lagrangian systems, J. Math. Phys. 54 (3) (2013) 032902.
[38] M. de León, J. Marin-Solano, J.C. Marrero, A geometrical approach to classical field theories: a constraint algorithm for singular theories, in: New Developments in Differential Geometry (Debrecen, 1994), in: Math. Appl., vol. 350, Kluwer Acad. Publ, Dordrecht, 1996, pp. 291–312.
[39] M. de León, P.D. Prieto-Martí nez, N. Román-Roy, S. Vilarino, Hamilton–Jacobi theory in multisymplectic classical field theories, J. Math. Phys. 58 (2017) 092901.
[40] M. de León, P.R. Rodrigues, Generalized Classical Mechanics and Field Theory: A Geometrical Approach of Lagrangian and Hamiltonian Formalisms Involving Higher Order Derivatives, Vol. 112, North Hollanda, Amsterdam, 1985.
[41] A.M. Rey, N. Roman-Roy, M. Salgado, S. Vilariño, On the k-symplectic, k-cosymplectic and multisymplectic formalism of classical field theories, J. Geom. Mech. 3 (2011) 113–137.

[42] G. Sardanashvily, Gauge Theory in Jet Manifolds, in: Hadronic Press Monographs in Applied Mathematics, Hadronic Press, Inc, Palm Harbor, FL, 1993, p. viii+159.
[43] G. Sardanashvily, Generalized Hamiltonian Formalism for Field Theory. Constraint Systems, World Scientific Publishing Co., Inc., River Edge, NJ, 1995, p. viii+155.
[44] G. Sardanashvily, Advanced Differential Geometry for Theoreticians: Fiber Bundles, Jet Manifolds and Lagrangian Theory, LAP Lambert Academic Publishing, 2013, May 1.
[45] G. Sardanashvily, Noether'S Theorems. Applications in Mechanics and Field Theory, in: Atlantis Studies in Variational Geometry, vol. 3, Atlantis Press, Paris, 2016.
[46] D.J. Saunders, The Geometry of Jet Bundles, in: London Mathematical Society Lecture Note Series, vol. 142, Cambridge University Press, Cambridge, 1989.
[47] W.M. Tulczyjew, A symplectic framework for linear field theories, Ann. Mat. Pura Appl. 130 (1982) 177–195.
[48] L. Vitagliano, The Hamilton–Jacobi formalism for higher order field theories, Int. J. Geom. Meth. Mod. Phys 07 (2010) 1413–1436.
[49] L. Vitagliano, Geometric Hamilton–Jacobi field theory, Int. J. Geom. Methods Mod. Phys. 9 (2) (2012) 1260008.