

# A class of Einstein–Weyl spaces associated to an integrable system of hydrodynamic type

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## Abstract

HyperCR Einstein–Weyl equations in  $2 + 1$  dimensions reduce to a pair of quasi-linear PDEs of hydrodynamic type. All solutions to this hydrodynamic system can in principle be constructed from a twistor correspondence, thus establishing the integrability. Simple examples of solutions including the hydrodynamic reductions yield new Einstein–Weyl structures.

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## 1. The equation

Let us consider a pair of quasi-linear PDEs

$$u_t + w_y + uw_x - wu_x = 0, \quad u_y + w_x = 0 \quad (1.1)$$

for two real functions  $u = u(x, y, t)$ ,  $w = w(x, y, t)$ . This system of equation has recently attracted a lot of attention in the integrable systems literature [9,10,17,18]. In [3] it arose in a different context, as a symmetry reduction of the heavenly equation.

The system (1.1) shares many properties with two more prominent dispersionless integrable equations: the dispersionless Kadomtsev–Petviashvili equation (dKP), and the  $SU(\infty)$  Toda equation, but it is simpler in some ways:

- Its Lax representation

$$[L, M] = 0, \quad \text{where } L = \partial_t - w\partial_x - \lambda\partial_y, \quad M = \partial_y + u\partial_x - \lambda\partial_x \quad (1.2)$$

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does not contain derivatives with respect to the spectral parameter  $\lambda$  (the Lax pairs for  $SU(\infty)$  Toda, and dKP contain such terms).

- Consider a one-form

$$e(\lambda) = dx - u dy + w dt + \lambda(dy - u dt) + \lambda^2 dt.$$

The system (1.1) is equivalent to the Frobenius integrability condition

$$e(\lambda) \wedge de(\lambda) = 0, \quad (1.3)$$

where  $d$  keeps  $\lambda$  constant. This formulation is dual to the Lax representation (1.2), because the distribution spanned by  $L$  and  $M$  can be defined as the kernel of  $e(\lambda)$ . The analogous dual formulations of dKP and  $SU(\infty)$  Toda involve distributions defined by two-forms [5,16,21], and are considerably more complicated.

One of the aims of this paper is to provide a twistor description of (1.1) given by the following theorem.

**Theorem 1.1.** *There is a one-to-one correspondence between the equivalence classes of solutions to (1.1) under point transformations, and complex surfaces (twistor spaces)  $\mathcal{Z}$  such that:*

- *There exists a holomorphic fibration  $\pi : \mathcal{Z} \rightarrow \mathbb{CP}^1$ .*
- *There exists a three-parameter family of holomorphic sections of  $\pi$  with normal bundle  $\mathcal{O}(2)$  invariant under an anti-holomorphic involution  $\tau : \mathcal{Z} \rightarrow \mathcal{Z}$  which fixes an equator of each section.*

The existence of the anti-holomorphic map  $\tau$  is required to construct real solutions to (1.1). If one is merely interested in complex solutions, then the holomorphic fibration, and its  $\mathcal{O}(2)$  sections are all one needs. Theorem 1.1 provides a parameterisation of local solutions to a non-linear equation (1.1) by a holomorphic data unconstrained by any equations. In this sense it resembles the *inverse scattering transform*. It remains to be seen whether this theorem can be implemented in practice to construct explicit new solutions to (1.1).

The proof of Theorem 1.1 will be postponed to Section 2.2. In the next section we shall demonstrate that solutions of (1.1) can be used to construct Lorentzian Einstein–Weyl (EW) structures in three dimensions (formulae (2.3)). All EW structures which admit a hyperboloid of Cauchy–Riemann structures locally arise from solutions to (1.1). We shall give examples of new Einstein–Weyl spaces which arise in that way. In Section 3 we review the hydrodynamic reductions of (1.1), and use them to construct another class of Einstein–Weyl spaces. Finally, in Section 4 we study a hierarchy of commuting flows associated to (1.1).

## 2. The geometry

Let  $W$  be a three-dimensional manifold with a torsion-free connection  $D$ , and a conformal structure  $[h]$  of signature  $(+ + -)$  which is compatible with  $D$  in a sense that

$$Dh = \omega \otimes h$$

for some one-form  $\omega$ . Here  $h \in [h]$  is a representative metric in a conformal class. If we change this representative by  $h \rightarrow \psi^2 h$ , then  $\omega \rightarrow \omega + 2 d \ln \psi$ , where  $\psi$  is a non-vanishing function on  $W$ . A triple  $(W, [h], D)$  is called a Weyl structure. The conformally invariant Einstein–Weyl equations are

$$R_{(ab)} = \frac{1}{3} R h_{ab}, \quad a, b, \dots = 1, 2, 3. \quad (2.1)$$

Here  $R_{(ab)}$  is the symmetrised Ricci tensor of  $D$ , and  $R$  is the Ricci scalar. One can regard  $h$  and  $\omega$  as the unknowns in these equations. Once they have been found, the covariant differentiation w.r.t.  $D$  is given by

$$D\chi = \nabla\chi - \frac{1}{2}(\chi \otimes \omega + (1 - m)\omega \otimes V - h(\omega, \chi)h),$$

where  $\chi$  is a one-form of conformal weight  $m$ , and  $\nabla$  is the Levi-Civita connection of  $h$ .

It is well known [2,13,19] that the EW equations are equivalent to the existence of a two-dimensional family of surfaces  $Z \subset W$  which are null with respect to  $h$ , and totally geodesic with respect to  $D$ . This condition has been used in [5] to construct a Lax representation for EW equation. The details are as follows: let  $V_1, V_2, V_3$  be three independent vector fields on  $W$ , and let  $e_1, e_2, e_3$  be the dual one-forms. Assume that

$$h = e_2 \otimes e_2 - 2(e_1 \otimes e_3 + e_3 \otimes e_1)$$

and some one-form  $\omega$  give an EW structure. Let  $V(\lambda) = V_1 - 2\lambda V_2 + \lambda^2 V_3$ , where  $\lambda \in \mathbb{CP}^1$ . Then  $h(V(\lambda), V(\lambda)) = 0$  for all  $\lambda \in \mathbb{CP}^1$  so  $V(\lambda)$  determines a sphere of null vectors. The vectors  $V_1 - \lambda V_2$  and  $V_2 - \lambda V_3$  form a basis of the orthogonal complement of  $V(\lambda)$ . For each  $\lambda \in \mathbb{CP}^1$  they span a null two surface. Therefore the Frobenius theorem implies that the horizontal lifts

$$L = V_1 - \lambda V_2 + l\partial_\lambda, \quad M = V_2 - \lambda V_3 + m\partial_\lambda \quad (2.2)$$

of these vectors to  $T(W \times \mathbb{CP}^1)$  span an integrable distribution, and (2.1) is equivalent to

$$[L, M] = \alpha L + \beta M$$

for some  $\alpha, \beta$  which are linear in  $\lambda$ . The functions  $l$  and  $m$  are third order in  $\lambda$ , because the Möbius transformations of  $\mathbb{CP}^1$  are generated by vector fields quadratic in  $\lambda$ .

Let  $W_1, \dots, W_4$  be linearly dependent vector fields which span  $TW$ . Given a Lax representation  $[W_1 - \lambda W_2, W_3 - \lambda W_4] = 0$ , it is always possible to put it in the form (2.2) with  $m = l = 0$ . Therefore the Lax pair (1.2) for Eq. (1.1) is a special case of the Einstein–Weyl Lax pair (2.2). One finds that

$$V_1 = \partial_t + u\partial_y + (u^2 - w)\partial_x, \quad V_2 = \partial_y + u\partial_x, \quad V_3 = \partial_x$$

and

$$[V_1 - \lambda V_2, V_2 - \lambda V_3] = -(V_2(u) - \lambda V_3(u))(V_2 - \lambda V_3)$$

is equivalent to Eq. (1.1).

The dual one-forms  $(e_1, e_2, e_3)$  give a metric in the EW conformal class. The associated one-form can now be found such that the resulting Einstein–Weyl structure is

$$h = (dy - u dt)^2 - 4(dx - u dy + w dt) dt, \quad \omega = u_x dy + (uu_x + 2u_y) dt. \quad (2.3)$$

The Ricci scalar of  $D$  is  $R = (3/8)(u_x)^2$ , and the one-form  $e(\lambda)$  used in the dual formulation (1.3) is  $e(\lambda) = e_1 + 2\lambda e_2 + \lambda^2 e_3$ .

The absence of the vertical terms in the Lax pair implies that the Einstein–Weyl structure belongs to the Lorentzian analogue of the so-called hyperCR class [11]. The (Lorentzian) hyperCR EW spaces arise on the space of trajectories of tri-holomorphic conformal Killing vectors in four-dimensional manifolds with (pseudo)hyper-complex structure.<sup>1</sup> Readers unfamiliar with the details of these pseudo-hyper-complex geometries in four dimensions should note that the condition  $l = m = 0$  in (2.2) can be used as an equivalent definition of the hyperCR class, and can go directly to the statement of Theorem 2.1.

Any pseudo-hyper-complex conformal structure  $([g], I, S, T)$  in four dimensions with a conformal Killing vector  $K$  gives rise to an EW structure [5,14] defined by

$$h := |K|^{-2}g - |K|^{-4}K \otimes K, \quad \omega := 2|K|^{-2} *_g (K \wedge dK), \quad g \in [g], \quad (2.4)$$

and all EW spaces arise from this construction. If the Killing vector  $K$  preserves the endomorphisms  $I, S, T$ , then the resulting EW structure (2.4) is called hyperCR. Conversely, given a hyperCR Einstein–Weyl structure  $(h, \omega)$  one can construct the representative  $g$  of a pseudo-hyper-complex conformal class  $[g]$  by

$$g = e^T (Vh - V^{-1}(dT + \beta)^2), \quad (2.5)$$

where  $T$  is a group parameter, and the function  $V$  and the one-form  $\beta$  solve the monopole equation

$$*(dV + \frac{1}{2}\omega V) = d\beta \quad (2.6)$$

(here  $*$  is taken with respect to  $h$ ). There exists a special solution to (2.6) with  $\beta = -\omega/2$ , such that the resulting metric is pseudo-hyper-Kähler (the two-forms associated to  $I, S, T$  are closed). The details of all that are in [11]. Minor sign changes are needed to apply the theory in signature  $(++-)$ . The hyperCR EW structure were constructed out of symmetry reductions of heavenly equations in [6].

We shall now show that the system (1.1) arises as a symmetry reduction of the pseudo-hyper-Kähler condition with a general homothetic Killing vector, thus establishing the following result.

<sup>1</sup> A smooth real four-dimensional manifold  $\mathcal{M}$  equipped with three real endomorphisms  $I, S, T : T\mathcal{M} \rightarrow T\mathcal{M}$  of the tangent bundle satisfying the algebra of pseudo-quaternions

$$-I^2 = S^2 = T^2 = 1, \quad IST = 1,$$

is called pseudo-hyper-complex iff the almost complex structure  $\mathbf{J} = aI + bS + cT$  is integrable for any point of the hyperboloid  $a^2 - b^2 - c^2 = 1$ . A choice of a vector  $X \in T\mathcal{M}$  defines a  $(++-)$  conformal structure  $[g]$  with an orthonormal frame  $X, IX, SX, TX$ . If this conformal structure admits a Killing vector  $K$  which preserves  $I, S, T$ , then the hyperboloid of complex structures  $\mathbf{J}$  descends to a hyperboloid of Cauchy–Riemann (CR) structures on the space of orbits  $W$  of  $K$ . This justifies the terminology.

**Theorem 2.1.** *All Lorentzian hyperCR Einstein–Weyl structures are locally of the form (2.3), where  $u, w$  satisfy (1.1).*

**Proof.** It follows from the work of Plebański [20] that all pseudo-hyper-Kähler (or ASD vacuum) metrics are locally of the form

$$g = 2(dZ dY + dW dX - \Theta_{XX} dX^2 - \Theta_{YY} dY^2 + 2\Theta_{XY} dW dZ), \quad (2.7)$$

where  $(W, Z, X, Y)$  are the local coordinates on the open ball in  $\mathbb{R}^4$ , and  $\Theta = \Theta(W, Z, X, Y)$  satisfies the second heavenly equation

$$\Theta_{ZY} + \Theta_{WX} + \Theta_{XX}\Theta_{YY} - \Theta_{XY}^2 = 0. \quad (2.8)$$

In an ASD vacuum the most general tri-holomorphic homothetic Killing vector  $K$  satisfies  $\mathcal{L}_K \Sigma_i = c \Sigma_i$ , where  $\mathcal{L}_K$  is the Lie derivative along  $K$  and  $\Sigma_1, \Sigma_2, \Sigma_3$  are three closed self-dual two-forms corresponding to the complex structures. We can set  $c = 1$  without loss of generality. In the coordinate system adopted to (2.7)

$$\Sigma_1 = dW \wedge dZ, \quad \Sigma_2 = dW \wedge dX + dZ \wedge dY$$

and the residual freedom in the choice of coordinates can be used to set

$$K = Z \frac{\partial}{\partial Z} + X \frac{\partial}{\partial X}.$$

The Killing equations yield

$$\mathcal{L}_K(\Theta_{XX}) = -\Theta_{XX}, \quad \mathcal{L}_K(\Theta_{XY}) = 0, \quad \mathcal{L}_K(\Theta_{YY}) = \Theta_{YY}.$$

Let  $U$  and  $T$  be functions on  $\mathbb{R}^4$  such that  $K = \partial/\partial T$  and  $\mathcal{L}_K(U) = 0$ . We can take

$$T = \ln(Z), \quad U = -\frac{X}{Z}.$$

The compatibility conditions for the Killing equations imply the existence of  $G = G(Y, W, U)$  such that

$$\Theta_{XX} = -e^{-T} G_{UU}, \quad \Theta_{XY} = G_{YU}, \quad \Theta_{YY} = -e^T G_{YY}.$$

The heavenly equation (2.8) becomes

$$-(G_Y - U G_{YU}) + G_{UW} + G_{YY} G_{UU} - G_{YU}^2 = 0,$$

or (in terms of differential forms)

$$\begin{aligned} & -G_Y dY \wedge dU \wedge dW + U dG_U \wedge dU \wedge dW + dG_U \wedge dY \wedge dU \\ & + dG_Y \wedge dG_U \wedge dW = 0. \end{aligned} \quad (2.9)$$

Define

$$x = G_U, \quad y = Y, \quad t = -W, \quad H(x, y, t) = xU(x, y, t) - G(Y, W, U(x, y, t))$$

and perform a Legendre transform

$$dH = d(xU - G) = U dx - G_Y dY - G_W dW = H_x dx + H_y dy + H_t dt.$$

Therefore

$$U = H_x, \quad G_Y = -H_y, \quad G_W = H_t.$$

Differentiating these relations we find

$$G_{UU} = \frac{1}{H_{xx}}, \quad G_{YU} = -\frac{H_{xy}}{H_{xx}}, \quad G_{YY} = -H_{yy} + \frac{H_{xy}^2}{H_{xx}}.$$

The differential equation for  $H(x, y, t)$  is obtained from (2.9)

$$H_{xt} - (H_{xy}H_x - H_yH_{xx}) = H_{yy}. \quad (2.10)$$

This equation is equivalent to the system (1.1) which can be seen by setting  $u = H_x$ ,  $w = -H_y$ .

The metric (2.7) can be written in the form (2.5) where  $h, \omega$  are given by (2.3), and  $V = u_x/2$ ,  $\beta = -\omega/2$  satisfy the monopole equation (2.6). We deduce that (2.3) is the most general EW space which arises on the space of orbits of tri-holomorphic homothety in pseudo-hyper-Kähler four manifold, and so it is the most general hyperCR EW space.  $\square$

## 2.1. Simple solutions

Simple classes of solutions to (1.1) yield non-trivial Einstein–Weyl structures, some of which appear to be new:

- Let us assume that  $u$  and  $w$  in (1.1) do not depend on  $y$ . One needs to consider the two cases  $w = 0$  and  $w = w(t) \neq 0$  separately. The corresponding equations can now be easily integrated to give (in the  $w \neq 0$  case one needs to change variables)

$$h = (dy + A dt)^2 - 4 dx dt, \quad \omega = A' dy + AA' dt, \quad (2.11)$$

where  $A = A(x)$  is an arbitrary function. Some interesting complete solutions belong to this class. For example  $A = a^2x$ , where  $a$  is a non-zero constant leads to the Einstein–Weyl structure on Thurston's nil manifold  $S^1 \times \mathbb{R}^2$  [19]: setting  $\hat{x} = a^2x$ , and rescaling  $h$  by a constant factor gives

$$h = a^2(dy + \hat{x} dt)^2 - 4 d\hat{x} dt, \quad \omega = a^2(dy + \hat{x} dt).$$

In this simple case we can find a kernel of the Lax vector fields (1.2) (the twistor functions) to be  $\lambda, \psi = y + \lambda t - a^{-2} \ln(\lambda - a^2x)$ .

- Looking for  $t$ -independent solutions reduces (1.1) to a linear equation. Rewriting the resulting system as

$$dx \wedge du - dy \wedge dw = 0, \quad dx \wedge dw - (udw - wdu) \wedge dy = 0$$

and regarding  $x$  and  $y$  as functions of  $u$  and  $w$  yields a system of linear equation. One of these equations implies that  $y = -F_w$ ,  $x = F_u$  for some  $F = F(u, w)$ , while the other equation yields

$$F_{uu} + uF_{uw} + wF_{ww} = 0.$$

- The constraint  $u_x = 0$  leads to trivial EW spaces. One finds that  $u$  has to be linear in  $y$ , and the EW one-form  $\omega$  is closed. It can therefore be set to 0 by the conformal rescaling, and the EW structure is conformal to an Einstein metric.

## 2.2. Proof of Theorem 1.1

Given a real-analytic solution to (1.1) we can complexify it, and regard  $u$  and  $w$  as holomorphic functions of local complex coordinates  $(x, y, t)$  on a complex three-manifold  $W^{\mathbb{C}}$ . The twistor space  $\mathcal{Z}$  for such solution is obtained by factoring  $F = W^{\mathbb{C}} \times \mathbb{CP}^1$  by the distribution  $L, M$  (1.2). This clearly has a projection  $q : F \mapsto \mathcal{Z}$  and we have a double fibration

$$\begin{array}{ccc} & W^{\mathbb{C}} \times \mathbb{CP}^1 & \\ r \swarrow & & \searrow q \\ W^{\mathbb{C}} & & \mathcal{Z}. \end{array}$$

The absence of vertical terms in  $L, M$  shows that  $\lambda$  descends from  $F$  to  $\mathcal{Z}$  thus giving the holomorphic projection  $\pi : \mathcal{Z} \rightarrow \mathbb{CP}^1$ . Each point  $p \in W^{\mathbb{C}}$  determines a sphere  $l_p$  (a section of  $\pi$ ) made up of all the integral surfaces of  $L, M$  through  $p$ . The normal bundle of  $l_p$  in  $\mathcal{Z}$  is  $N = T\mathcal{Z}|_{l_p}/Tl_p$ . This is a rank 1 vector bundle over  $\mathbb{CP}^1$ , therefore it has to be one of the standard line bundles  $\mathcal{O}(n)$ . To see that  $n = 2$ , note that  $N$  can be identified with the quotient  $r^*(T_p W^{\mathbb{C}})/\{\text{span } L, M\}$ . In their homogeneous form the operators  $L, M$  have weight 1, so the distribution spanned by them is isomorphic to the bundle  $\mathbb{C}^2 \otimes \mathcal{O}(-1)$ . The definition of the normal bundle as a quotient gives a sequence of sheaves over  $\mathbb{CP}^1$ .

$$0 \rightarrow \mathbb{C}^2 \otimes \mathcal{O}(-1) \rightarrow \mathbb{C}^3 \rightarrow N \rightarrow 0$$

and we see that  $N = \mathcal{O}(2)$ , because the last map, is given explicitly by  $(V_1, V_2, V_3) \mapsto V(\lambda) = V_1 - 2\lambda V_2 + \lambda^2 V_3$  clearly projecting onto  $\mathcal{O}(2)$ .

If  $u, w$  is a real solutions defined on a real slice  $W \subset W^{\mathbb{C}}$ , then one has an additional structure on  $\mathcal{Z}$ . The real structure  $\tau(x, y, t) = (\bar{x}, \bar{y}, \bar{t})$  maps integral surfaces of  $L, M$  to integral surfaces, and therefore induces an anti-holomorphic involution  $\tau : \mathcal{Z} \rightarrow \mathcal{Z}$ . The fixed points of this involution correspond to real integral surfaces in  $W$ , and  $\tau$ -invariant  $\mathcal{O}(2)$  sections correspond to points in  $W$ .

Conversely, let us assume that we are given a complex manifold  $\mathcal{Z}$  with additional structures described in Theorem 1.1. The general construction of Hitchin [13] equips the moduli space  $W$  of  $\mathcal{O}(2)$  rational curves with a real-analytic Einstein–Weyl structure: the Kodaira theorems imply that  $W$  is three-dimensional. Two points in  $W$  are null-separated if the corresponding sections intersect at one point in  $\mathcal{Z}$ . This defines the conformal structure  $[h]$ . To define a connection note that a direction at  $p \in W$  corresponds to a one-dimensional

space of  $\mathcal{O}(2)$  curves in  $\mathcal{Z}$  which vanish at two points  $Z_1$  and  $Z_2$ . This gives distinguished curves in  $W$  which pass through null surfaces in  $W$  corresponding to  $Z_1, Z_2$ . There is one such curve through  $p$  and Hitchin defines it to be a geodesic. He moreover shows that the resulting connection is torsion-free, and that the Einstein–Weyl equations hold.

This works for arbitrary complex surface with an embedded  $\mathcal{O}(2)$  rational curve. The additional structure in the statement of [Theorem 1.1](#) is the holomorphic projection  $\pi$ . Its existence implies that the resulting EW space is hyperCR. Any holomorphic line bundle  $L \rightarrow \mathcal{O}(2)$  with  $c_1(L) = 0$  inherits the holomorphic projection to  $\mathbb{CP}^1$ . Lifts of holomorphic sections of  $\mathcal{Z} \rightarrow \mathbb{CP}^1$  to  $L$  are rational curves with normal bundle  $\mathcal{O}(1) \oplus \mathcal{O}(1)$ . Therefore  $L$  is a twistor space of a pseudo-hyper-complex four manifold  $\mathcal{M}$  [4]. This pseudo-hyper-complex structure is preserved by a Killing vector which gives rise to a hyperboloid of CR structures on  $W$ . [Theorem 2.1](#) implies that  $(h, \omega)$  are locally given by (2.3), and we can read off  $(u, w)$  which solve [Eq. \(1.1\)](#).

### 2.3. Geodesic congruences

Let  $(W, [h], D)$  be a  $2+1$  EW structure. A geodesic congruence  $\Gamma$  in a region in  $\hat{W} \subset W$  is a set of geodesic, one through each point of  $\hat{W}$ . Let  $\chi$  be a generator of  $\Gamma$  (a vector field tangent to  $\Gamma$ ). The geodesic condition  $\chi^a D_a \chi^b \sim \chi^b$  implies  $D_a \chi^b = M_a^b + A_a \chi^b$  for some  $A_a$ , where  $M_a^b$  is orthogonal to  $\chi^a$  on both indices. Consider the decomposition of  $M_{ab}$

$$M_{ab} = \Omega_{ab} + \Sigma_{ab} + \frac{1}{2}\theta\hat{h}_{ab}.$$

The shear  $\Sigma_{ab}$  is trace-free and symmetric. The twist  $\Omega_{ab}$  is anti-symmetric, and the divergence  $\theta$  is a weighted scalar. Here  $\hat{h}_{ab} = \|\chi\|^2 h_{ab} - \chi_a \chi_b$  is an orthogonal projection of  $h_{ab}$ . The shear-free geodesics congruences (SFC) exist on any Einstein–Weyl space. This follows from a three-dimensional version of Kerr’s theorem which states that SFCs correspond to holomorphic curves in the twistor space  $\mathcal{Z}$ . On the other hand, imposing the vanishing of the divergence of a congruence gives restrictions on EW structures, and implies that the EW space is hyperCR [1]. In the local coordinate system adopted in [Theorem 2.1](#)  $(h, \omega)$  are given by (2.3), and the shear-free, divergence-free geodesic congruence is generated by a one-form  $\chi = dt$ . In accordance with the general theory of SFC on Einstein–Weyl spaces [1], the preferred monopole proportional to the scalar twist  $\kappa = *(\chi \wedge D\chi) = -u_x/4$  will lead to a pseudo-hyper-Kähler metric with a tri-holomorphic homothety in four dimensions. This metric is explicitly given by (2.5), where  $V = -(1/2)\kappa$ ,  $\beta = -\omega/2$ . Any other monopole yields a general pseudo-hyper-complex conformal structure with a tri-holomorphic symmetry.

### 3. The hydrodynamic reductions

[Eq. \(1.1\)](#) can be cast in a general quasi-linear vector form

$$\mathbf{u}_y + A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_t = 0, \quad (3.1)$$



where  $\mathbf{u} = (u, w)^T$  is a vector whose components depend on  $(x, y, t)$ , and

$$A(\mathbf{u}) = \begin{pmatrix} 0 & 1 \\ -w & u \end{pmatrix}, \quad B(\mathbf{u}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

A class of solutions to any equation of the form (3.1) can be generated by assuming that  $\mathbf{u} = \mathbf{u}(R^1, \dots, R^N)$ , where  $R^i = R^i(x, y, t)$  (the so-called Riemann invariants) satisfy a pair of commuting systems of hydrodynamic type

$$R_y^i = \gamma^i(R) R_x^i, \quad R_t^i = \mu^i(R) R_x^i, \quad i = 1, 2, \dots, N \quad (3.2)$$

(the summation convention has been suspended in this section). The compatibility conditions for the system (3.2) yield

$$\frac{\partial_j \gamma^i}{\gamma^j - \gamma^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j, \quad \partial_j = \frac{\partial}{\partial R^j}. \quad (3.3)$$

It turns out that the additional relations

$$\partial_k \frac{\partial_j \gamma^i}{\gamma^j - \gamma^i} = \partial_j \frac{\partial_k \gamma^i}{\gamma^k - \gamma^i}$$

(and analogous relations for  $\mu^i$ ) hold. These conditions imply the existence of a diagonal metric  $g = \sum_i g_{ii}(R) d(R^i)^2$  such that  $\Gamma_{ji}^i = \partial_j \ln(\sqrt{g_{ii}}) = \partial_j \gamma^i / (\gamma^j - \gamma^i)$  are the contracted components of the Levi-Civita connection of  $g$ .

If conditions (3.3) are satisfied, the general solutions to (3.2) are implicitly given by the generalised hodograph formula of Tsarev [22]

$$v^i(R) = x + \gamma^i(R)y + \mu^i(R)t, \quad i = 1, \dots, N.$$

Once  $\gamma^i$  have been found, the functions  $v^i$  (called the characteristic speeds) should be determined from the linear relations  $\Gamma_{ji}^i = \partial_j v^i / (v^j - v^i)$ ,  $i \neq j$ . Substituting this expression in (3.1) shows that  $\partial_i \mathbf{u}$  are eigenvectors of  $(A - \gamma I - \mu B)$  with zero eigenvalue. Therefore  $\gamma^i$  and  $\mu^i$  satisfy the dispersion relation

$$\det(A - \gamma I - \mu B) = 0. \quad (3.4)$$

Solutions to (3.1) obtained from this algorithm are known as the non-linear interactions of  $N$  planar simple waves. The procedure explained in this section has been applied in [7,8,12] to construct explicit solutions to various PDEs which admit a representation (3.1).

Ferapontov and Khusnutdinova [9] define a hyperbolic system of the form (3.1) to be integrable if it possesses non-linear interactions of  $N$  planar simple waves parameterised by  $N$  arbitrary functions of one variable. They have demonstrated [10] that this definition of integrability is equivalent to the existence a scalar pseudo-potential formulation of the form

$$\psi_y = P(\psi_x, u, w), \quad \psi_t = Q(\psi_x, u, w),$$

where  $\psi = \psi(x, y, t)$  and  $P, Q$  are rational in  $\psi_x$ . This then implies that the integrable equations (3.1) arise as dispersionless (or quasi-classical) limits of non-linear PDEs solvable

by inverse scattering transform [23]. These ‘dispersive’ PDEs are compatibility conditions for the overdetermined linear system

$$\Psi_Y = P \left( \frac{\partial}{\partial X} \right) \Psi, \quad \Psi_T = Q \left( \frac{\partial}{\partial X} \right) \Psi,$$

where now  $P$  and  $Q$  are linear differential operators, and the dispersionless limits can be obtained by setting

$$\frac{\partial}{\partial X^a} \rightarrow \varepsilon \frac{\partial}{\partial x^a}, \quad \Psi(X^a) = \exp \left( \psi \left( \frac{x^a}{\varepsilon} \right) \right)$$

and taking the limit  $\varepsilon \rightarrow 0$ . Finding a dispersive analogue of (1.1) is an interesting open problem.

### 3.1. Example

According to Pavlov [18] the hydrodynamic reductions of (1.1) are characterised in a sense that explicit formulae for  $\gamma^i(R)$ , and  $\mu^i(R)$  can be found. This does not, however, lead to explicit (or even implicit) solutions to (1.1). The constraints on a solution to (1.1) imposed by the existence of  $N$ -component reductions are not known. To this end, we shall work out the constraint, and the corresponding solution which arise from a one-component reduction.

For  $N = 1$  we have  $u = u(R)$ ,  $w = w(R)$ , where the scalar variable  $R = R^1$  satisfies a pair of PDEs  $R_y = \gamma(R)R_x$ ,  $R_t = \mu(R)R_x$ . All integrability conditions hold automatically, and the dispersion relation (3.4) yields

$$\mu = w + \gamma u + \gamma^2.$$

Implicit differentiation of  $u$ ,  $w$  with respect to  $(x, y, t)$ , and eliminating  $(u', w', R_x)$  gives a constraint

$$u_x w_y - u_y w_x = 0, \tag{3.5}$$

which characterises solutions to (1.1) arising from one-component hydrodynamic reductions. Using the relations

$$\begin{aligned} *dt &= dt \wedge dy, & *dy &= 2dt \wedge dx - u dt \wedge dy, \\ *dx &= 2w dy \wedge dt + dy \wedge dx + u dt \wedge dx, \end{aligned}$$

where  $*$  :  $\Lambda^a(W) \rightarrow \Lambda^{3-a}(W)$  is the Hodge operator associated to the EW structure (2.3) we find that the constraint (3.5) is equivalent to the relation

$$|du|^2 := du \wedge *du = 0.$$

The solution can now easily be found by applying the Legendre transform. Regarding  $u$  and  $y$  as functions of  $(w, t, x)$  gives

$$u_x = 0, \quad u_w = y_x, \quad u_w y_t - u_t y_w - 1 + u y_x - w(u_w y_x - u_x y_w) = 0,$$

where the first relation arises from the constraint (3.5). These equations can be integrated to give two classes of solutions

$$\begin{aligned} u_1 &= at + aw + \frac{1}{a}, & y_1 &= ax - \frac{at^2}{2} + f_1(w+t), & u_2 &= 2\sqrt{w+t}, \\ y_2 &= \frac{x-t}{\sqrt{w+t}} + f_2(w+t), \end{aligned}$$

where  $f_1$  and  $f_2$  are arbitrary functions of one variable (one arbitrary function of  $t$  has been eliminated in each class by a redefinition of coordinates), and  $a$  is a non-zero constant. The resulting Einstein–Weyl spaces (2.3) can be written down explicitly, and are completely characterised by the condition  $|du| = 0$ .

#### 4. The hierarchy

Consider a sphere of one-forms on an open set in  $\mathbb{R}^{n+1}$

$$\begin{aligned} e(\lambda) &= dt_0 + (\lambda - H_0) dt_1 + (\lambda^2 - \lambda H_0 - H_1) dt_2 + \cdots \\ &\quad + (\lambda^n - \lambda^{n-1} H_0 - \cdots - \lambda H_{n-2} - H_{n-1}) dt_n, \end{aligned}$$

where  $H = H(t_0, t_1, \dots, t_n)$ ,  $H_a = \partial H / \partial t_a$  and  $\lambda \in \mathbb{CP}^1$ . The system of PDEs

$$e(\lambda) \wedge d(e(\lambda)) = 0 \tag{4.1}$$

coincides with (1.1) if  $n = 2$ ,  $t_0 = x$ ,  $t_1 = y$ ,  $t_2 = t$  and  $u = H_x$ ,  $w = -H_y$ . If  $n > 2$  then (4.1) is highly overdetermined, and the Cauchy data can be specified on a surface of co-dimension  $n - 1$  (rather than on a hypersurface). We shall call this system a truncated hierarchy associated to (1.1). Allowing infinite sums in  $e(\lambda)$  would lead to the full hierarchy. The Frobenius theorem implies that an  $n$ -dimensional distribution of vector fields on  $\mathbb{R}^{n+1} \times \mathbb{CP}^1$  annihilating  $e(\lambda)$  is in involution. This gives rise to the Lax representation. The vector fields

$$L_a = \frac{\partial}{\partial t_{a+1}} + \frac{\partial H}{\partial t_a} \frac{\partial}{\partial t_0} - \lambda \frac{\partial}{\partial t_a}, \quad a = 0, \dots, n-1 \tag{4.2}$$

satisfy  $L_a \lrcorner e(\lambda) = 0$ , and the relations

$$[L_a, L_b] = 0$$

yield the commuting flows of the hierarchy

$$H_{(a+1)b} - H_{(b+1)a} + H_a H_{0b} - H_b H_{0a} = 0. \tag{4.3}$$

The Lax representation (4.2) fits into a general class of Lax formulations recently introduced in [17]. Theorem 1.1 should generalise to solutions of (4.3). The twistor space  $\mathcal{Z}$  is a surface which arises as a factor space of  $\mathbb{C}^n \otimes \mathbb{CP}^1$  by a complexified distribution (4.2). Repeating the steps of the proof of Theorem 1.1 shows that the holomorphic fibration  $\mathcal{Z} \rightarrow \mathbb{CP}^1$  admits an  $(n+1)$  family of holomorphic sections with normal bundle  $\mathcal{O}(n)$ . The converse

(recovering  $H(t_0, t_1, \dots, t_n)$ ) form  $\mathcal{Z}$  is, however, more difficult, because the vital relation (2.3) with Einstein–Weyl geometry is missing for  $n > 2$ . This interesting problem and its connection with the quasi-classical  $\bar{\partial}$  approach [15] will be addressed elsewhere.

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