



From local to global in F-theory model building

Björn Andreas^{a,*}, Gottfried Curio^b

^a Institut für Mathematik, Humboldt Universität zu Berlin, Rudower Chaussee 25, 12489 Berlin, Germany

^b Arnold-Sommerfeld-Center for Theoretical Physics, Department für Physik, Ludwig-Maximilians-Universität München, Theresienstr. 37, 80333 München, Germany

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ABSTRACT

When locally engineering F-theory models some $D7$ -branes for the gauge group factors are specified and matter is localized on the intersection curves of the compact parts of the world-volumes. In this note, we discuss to what extent one can draw conclusions about F-theory models by just restricting the attention locally to a particular seven-brane. Globally, the possible $D7$ -branes are not independent from each other and the (compact part of the) $D7$ -brane can have unavoidable intrinsic singularities. Many special intersecting loci which were not chosen by hand occur inevitably, notably codimension-three loci which are *not* intersections of matter curves. We describe these complications specifically in a global $SU(5)$ model and also their impact on the tadpole cancellation condition.

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1. Inevitable effects in global F-theory

We consider F-theory on $\mathbf{R}^{3,1} \times X$ where X is an elliptically fibered Calabi–Yau fourfold over a complex threefold B_3 . This set-up is used to describe in an effective four-dimensional theory certain gauge theories with matter.

1.1. The local point of view

Engineering a GUT model in local F-theory models essentially the following procedure is chosen

- *Codimension one:* One demands the existence of a $D7$ -brane (the compact part of its world-volume is a divisor D_G in B_3) which encodes the GUT group G . This is assumed to give just the gauge group.
- *Codimension two:* Then one demands the existence of further $D7$ -branes (this again refers to codimension one) such that at the intersection curves in D_G (we are speaking here of the compact parts of the world-volumes) those enhancements occur which give the matter multiplets (quarks, leptons, Higgses) one wants to encode.
- *Codimension three:* The matter curves in D_G intersect at points. One tries to arrange things such that at certain points intersections occur which encode suitable Yukawa couplings.

So these three steps refer, geometrically speaking, to more and more specialized choices in (complex) codimension one, two and three in B_3 . The underlying philosophy is that by restricting attention to what happens inside D_G , one stays decoupled from ‘global complications’. Technically, the decoupling of the surrounding B_3 (and of gravity) is achieved by taking the size of B_3 to infinity while holding the size of D_G fixed, or, reversing it, by making sure that D_G is shrinkable in a fixed B_3 (this leads to D_G being a del Pezzo surface) [1]. This suggests the idea that the effective theory has decoupled itself

* Corresponding author.

E-mail address: andreas@math.hu-berlin.de (B. Andreas).

thereby from the ‘complications of global geometry’ and can that one can focus on local considerations. This philosophy came from the D -brane models where string theory is used to actually doing field theory. The reason for the whole procedure is that the engineered field theory models are ‘better’ than ordinary field theory models; this is because at the end one nevertheless still wants to participate at the benefits of having a field theory which is thought to be embeddable in a full string theory. So the justification of the whole enterprise stands and falls with this embeddability in a consistent global string model, i.e., the hope is a possible passage from local to global.

1.2. The global point of view

There were however always signs that global consistency has to be taken into account even in a local procedure. The $D3$ -brane charge tadpole cancellation condition showed already that one has to have always an eye on *all* global contributions; cf. Section 4 below.

One feature which makes F-theory models particularly interesting is that a similar story occurs also for the $D7$ -branes: the condition that the $D7$ -branes with compact parts of world-volumes D_i (and multiplicities m_i) satisfy

$$\sum m_i D_i = 12c_1(B_3). \quad (1)$$

This is the cohomological decomposition of the discriminant $\{\Delta = 0\}$.

So the following has to be taken into account. First there is the global binding of $D7$ -branes Eq. (1). Furthermore, when engineering the matter curves that one wants to have from certain surface components, all topological unavoidable intersections with D_G must be taken into account. Especially important in this respect is the divisor D_{I_1} of an I_1 -fiber singularity. This is the locus where just the fiber degenerates simply and thus makes itself not felt as a singularity of the total space.¹ As we will recall below in our main example of $G = SU(5)$ it is the intersection of D_G with this $D7$ -brane which leads to the matter we want to engineer.

- **Codimension one:** Starting with the $D7$ -brane wrapping D_G gives effectively the gauge group G . Because of the global consistency relation Eq. (1) it is usually *not* possible to encode just a *pure* gauge theory! This means that even if one does not want to encode further non-abelian gauge group factors nevertheless one has to satisfy Eq. (1), at least via the contribution of the I_1 -surface D_{I_1} . The latter will usually for topological reasons inevitably intersect D_G and give matter curves. Let us consider for illustration the case where B_3 is itself \mathbf{P}^1 fibered over a surface B_2 (such that X is $K3$ fibered, the case with a heterotic dual; the fibration type of the \mathbf{P}^1 over B_2 is encoded by a cohomology class t in the base²). If G is in the E series only *one* matter curve occurs, whose cohomology can be ‘turned off’ by choosing t related in a specific way to $c_1(B_2)$; this means that the two surfaces can be geometrically separated. In the A and D series, however, *two* matter curves occur with different cohomology class which can not be ‘turned off’ both at the same time; that is the two surfaces can not be separated from each other, we get *inevitably* one matter curve at least just from consistency.

Furthermore, this unavoidable D_{I_1} component turns out to be a *singular* surface, again something which was not chosen but which rather just occurs in the detailed consideration. It has always a curve of intrinsic cusp singularities, for example, which is (accidentally) at the same time a curve of cuspidal fiber singularities, i.e., the locus in D_{I_1} where the fiber type changes from nodal I_1 to cuspidal II .

In certain cases, like $SU(n)$ for $n = 4$ or $5, 6$, there is a further curve of intrinsic singularities built by tacnodes or higher double points; ‘accidentally’ it happens that this curve lies even in D_G , so it is *actually one of the matter curves* which thus occurs here in D_G in a collision with a singular locus of the second $D7$ -brane.

- **Codimension two:** Therefore we have the following situation in codimension two: although one might choose specializations which give other components of the discriminant and lead to matter curves, we will usually have the remaining amount of the discriminant divisor (Δ) which represents the I_1 -surface D_{I_1} . Except for a very special “separation case” (where D_{I_1} and D_G are disjoint) this leads to matter curves at the intersection loci in D_G . We may have chosen at will certain $D7$ -branes already; our point here is that because of the ‘global binding’ Eq. (1) we usually get further $D7$ -branes, i.e. components D_i of (Δ), which we possibly did not want to have and which nevertheless lead to further intersection curves in D_G , i.e. further matter multiplets in the effective four-dimensional effective theory. That is, even by restricting attention to D_G and to some intersection curves we want to engineer, we are not protected against further intersections from further Δ -components; rather we have the overall global binding Eq. (1) to satisfy. The minimalistic way is then the further D_{I_1} component (which luckily often gives just the relevant matter already).

If one wants a matter curve not contained in D_G this must arise from further surface components D_i of (Δ) which in turn will intersect D_G .

D_{I_1} has singular curves (even in D_G) which signify further special behaviour; cf. above.

For phenomenological reasons (as we will see in the $SU(5)$ example) we may want to have besides the fermionic matter of quarks and leptons also Higgses; these come from the same type (fundamental) multiplet type as one of the

¹ In the eight-dimensional case of F-theory on $K3$ the collision of n nodal fibers in the base \mathbf{P}^1 leads to an $I_n = A_{n-1}$ singularity of the total space which results in a non-abelian gauge group in an effective theory.

² Cf. the classification by a number n when having a \mathbf{P}^1 instead of B_2 and a Hirzebruch surface instead of B_3 .

matter curves; so the corresponding curve has to be reducible, a condition in complex structure moduli which has to be stabilized.

- *Codimension three:* Finally there are points of special further enhancements. Among them are the intersections of two matter curves in D_G (whereas a meeting of three curves usually has to be stabilized), but *there are more special enhancement points as the global analysis reveals* (notably the $P = Q = 0$ locus in the $SU(n)$ cases). This is an important global feature. Furthermore the intrinsic cusp curve C of D_{I_1} intersects D_G in some points³.

The message of these details is simple. If one wants to build a GUT model with a specific matter content and wants to draw specific conclusions, one has reason to care about the global structure of the discriminant (the $D7$ -brane components) and the finer specialization structures of this locus. Surely when it comes to tadpole cancellation a detailed overview of all possible contributions to the Euler number is required. If we assume that actually the resolution of the singular model is concerned here (and if that exists), one has to do all the needed blow-up processes; cf. the cases in [2]. These matters are not yet fully elucidated. One case is discussed in [3] and we will give also a discussion below.

Note added: As this note was prepared for final publication the paper [4] appeared in which also codimension-three loci in the $SU(5)$ model were investigated. Related papers are [5].

2. The discriminant equation and singularity loci

In this section we recall some details of the geometry of the F-theory models. We assume the existence of a section $\sigma: B_3 \rightarrow X$ and that X can be described by a Weierstrass model

$$y^2 = x^3 + fx + g \quad (2)$$

where f and g are sections of $K_{B_3}^{-4}$ and $K_{B_3}^{-6}$, respectively. The elliptic fiber degenerates over the discriminant locus $D = \{\Delta = 0\}$ of the above equation, where

$$\Delta = 4f^3 + 27g^2. \quad (3)$$

We will denote the cohomology classes of the vanishing divisors (f) and (g) by $F := 4c_1(B_3)$ and $G := 6c_1(B_3)$; similarly $D := 12c_1(B_3)$ for (Δ) . For $p \in D \subset B_3$ the type of singular fiber is determined by the orders of vanishing $a := \text{ord}(f)$, $b := \text{ord}(g)$ and $c := \text{ord}(\Delta)$ according to the Kodaira list of singularities of elliptic fibrations.

a	b	c	Fiber	Singularity
≥ 0	≥ 0	0	Smooth	None
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	none
≥ 1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
2	≥ 3	$n+6$	I_n^*	D_{n+4}
≥ 2	3	$n+6$	I_n^*	D_{n+4}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

So for the singular fiber type I_1 (nodal) or type II (cuspidal) no singularity of the total space arises. This list originated in the case corresponding to an F-theory model on an elliptically fibered $K3$ -surface, i.e., a compactification to 8 dimensions; indicated are the vanishing orders in the coordinate z in the base \mathbf{P}^1 of the fibration. This type of structure will be prolonged adiabatically in the following to compactification models of (6 or) 4 dimensions. The complex three-dimensional base B_3 of the elliptically fibered Calabi–Yau fourfold X will be assumed to be \mathbf{P}^1 fibered over an own base surface B_2 (equivalently X is assumed to be fibered by elliptic $K3$ -surfaces over B_2 ; this case has a heterotic dual).

So we consider B_3 being a \mathbf{P}^1 bundle which is the projectivization $\mathbf{P}(Y)$ of a vector bundle $Y = \mathcal{O} \oplus \mathcal{T}$ with \mathcal{T} a line bundle over B_2 and $\mathcal{O} = \mathcal{O}_{B_2}$. Furthermore, let $\mathcal{O}(1)$ be a line bundle on the total space of $\mathbf{P}(Y) \rightarrow B_2$ which restricts on each \mathbf{P}^1 fiber to the corresponding line bundle over \mathbf{P}^1 . With $r = c_1(\mathcal{O}(1))$, $t = c_1(\mathcal{T})$ and $c_1(\mathcal{O} \otimes \mathcal{T}) = r + t$ then the cohomology ring of B_3 is generated over the cohomology ring of B_2 by the element r with the relation $r(r+t) = 0$. The total Chern class $c(B_3) = c(B_2)(1+r)(1+r+t)$ gives (we set $c_1 := c_1(B_2)$; here c_1 and t are understood as pullbacks to B_3)

$$c_1(B_3) = c_1 + 2r + t, \quad c_2(B_3) = c_2 + c_1 t + 2c_1 r. \quad (4)$$

³ Over C lie also more complicated (cuspidal) fibers which collide at these points with the G singularity; also various complicated point singularities of D_{I_1} , detected by an analysis of the discriminant equation, can occur.

2.1. Codimension-one loci

If we engineer an ADE gauge group G in four dimensions we just demand a corresponding surface component in D . Let us call this surface component B_2 and denote its cohomology class by r . The next step in the engineering process is to demand some loci where matter charged under G is located; so let us look what happens inevitably in a global model when one starts and demands just the component B_2 . We will have automatically the decomposition $D = D_1 + D_2$ where D_1 denotes the component with generic I_1 fibers and D_2 has G fibers. This leads to the cohomological relations $F_2 = ar$, $G_2 = br$, $D_2 = cr$ (for $f = f_1 f_2$ and $g = g_1 g_2$ with $f_2 = z^a$, $g_2 = z^b$ where z is the coordinate on the fiber \mathbf{P}^1 of B_3). For the remaining locus D_1 of I_1 fibers and the other terms one gets

$$\begin{aligned} F_1 &= 4c_1 + (8 - a)r + 4t \\ G_1 &= 6c_1 + (12 - b)r + 6t \\ D_1 &= 12c_1 + (24 - c)r + 12t. \end{aligned} \quad (5)$$

For E_k singularities the I_1 -surface component D_1 is given by the equation $4f_1^3 - 27g_1^2 = 0$. For $SU(n)$, however, we have $f = f_1$, $g = g_1$ and one can split off in Δ a z^n -factor

$$\{4f_1^3 + 27g_1^2 = 0\} = D_1 + nr. \quad (6)$$

For $SO(2(n+4))$ we define $f_2 = z^2$, $g_2 = z^3$ and get again Eq. (6).

2.2. Codimension-two loci

The intersection curves between the G -surface D_2 and the inevitably occurring I_1 -surface D_1 have an interpretation as matter locations because the collision of singularity types will lead to a gauge group enhancement: for example, if $G = SU(n)$ is enhanced to $SU(n+1)$ we get a fundamental representation from the decomposition $\mathbf{ad}_{SU(n+1)} = \mathbf{V} \oplus \bar{\mathbf{V}} \oplus \mathbf{ad}_{SU(n)} \oplus \mathbf{C}$ under $SU(n) \times U(1) \subset SU(n+1)$, or an antisymmetric one from an $SO(2n)$ enhancement $\mathbf{ad}_{SO(2n)} = \Lambda^2 \mathbf{V} \oplus \Lambda^2 \bar{\mathbf{V}} \oplus \mathbf{ad}_{SU(n)} \oplus \mathbf{C}$. Similarly we get the \mathbf{V} and the \mathbf{S} of $SO(10)$ from the decompositions of enhancements $\mathbf{ad}_{SO(12)} = \mathbf{V} \oplus \bar{\mathbf{V}} \oplus \mathbf{ad}_{SO(10)} \oplus \mathbf{C}$ and $\mathbf{ad}_{E_6} = \mathbf{S} \oplus \bar{\mathbf{S}} \oplus \mathbf{ad}_{SO(10)} \oplus \mathbf{C}$. Likewise an E_7 enhancement of E_6 will provide the $\mathbf{27}$.

Let us look at the first few non-trivial cases (we give in the last three entries the dual heterotic data where an $H_V \times E_8$ bundle (V, V_2) is given with $H_V = SU(n)$; cf. Section 3)

G	a	b	c	Matter curve(s)	fib_{enh}	Matter	H_{V_1}	Het	Het. loc.
E_7	3	5	9	f_{4c_1-t}	" E_8 "	$(\frac{1}{2})\mathbf{56}$	$SU(2)$	$H^1(Z, V)$	a_2
E_6	3	4	8	q_{3c_1-t}	E_7	$\mathbf{27}$	$SU(3)$	$H^1(Z, V)$	a_3
$SO(10)$	2	3	7	h_{2c_1-t}	E_6	$\mathbf{16}$	$SU(4)$	$H^1(Z, V)$	a_4
				q_{3c_1-t}	$SO(12)$	$\mathbf{10}$		$H^1(Z, \Lambda^2 V)$	a_3
$SU(5)$	0	0	5	h_{c_1-t}	$SO(10)$	$\mathbf{10}$	$SU(5)$	$H^1(Z, V)$	a_5
				p_{8c_1-3t}	$SU(6)$	$\bar{\mathbf{5}}$		$H^1(Z, \Lambda^2 V)$	$R(a_i)$

Some further matter curves are given here in the following table where the polynomial, including multiplicities, giving the defining equation of $D_1 r$ is displayed

G	Equation of $D_1 r$
A_1	$H_{2c_1-2t}^2 P_{8c_1-6t}$
A_n	$h_{c_1-t}^4 P_{8c_1-(7-n)t}$
D_4	$\prod_{i=0}^2 (h_{2c_1-t}^2 + \omega^i P_{2c_1-t}^2)$
D_5	$h_{2c_1-t}^3 q_{3c_1-t}^2$
D_6	$h_{2c_1-t}^2 P_{4c_1-t}^2$
E_k	$q_{\frac{12}{k}c_1-t}^k$

where $n = 2, 3, 4, 5$, $\omega = e^{2\pi i/3}$, $k = 6, 7, 8$ and $k' = 10 - k$. In all cases we get the correct sum for the total cohomology class

$$D_1 r = (12c_1 - (12 - c)t)r. \quad (7)$$

2.3. Separation cases and pseudo-separation cases

If we want to obtain a degeneration which is purely in codimension one, we must arrange things such that D_1 and r do not intersect. This can be achieved by adjusting the Chern class t which specifies how the \mathbf{P}^1 is fibered over B_2 . The table shows

that for D_n and A_n there is more than one matter curve, so we can not ‘turn off’ cohomologically all of them simultaneously. However, for the E series this is possible as there only one matter curve appears. From (7) the separation of D_1 and r can be achieved for t given by

$$t = \frac{12}{12-c}c_1, \quad (8)$$

provided the right hand side is an integral class. Therefore for the E_n series with $c = 8, 9, 10$ we can adjust a ‘separation case’ between D_1 and r , i.e. a matter free situation, by setting $t = 3c_1, 4c_1, 6c_1$ for E_6, E_7, E_8 (the case E_8 is somewhat special; cf. [6]). In the D series we find only for D_4 a realizable codimension-one case, with $t = 2c_1$. In the A series only a pseudo-separation case can be established by setting $t = c_1$, i.e. the matter can not be completely ‘turned off’, only one of the two matter curves is turned off cohomologically, say (h) .

2.4. The cusp curve and further singular curves

Inevitably occurs another relevant codimension-two locus: the cusp curve. The naive cusp locus is $C_{naive} = \{f_1 = 0 = g_1\}$. In case (6) applies this naive locus will contain also higher singularities over the matter curve h such that the true cusp set is

$$C = C_{naive} - x(h), \quad (9)$$

where x is the intersection multiplicity of f_1 and g_1 along (h) (computed via their resultant). The cuspidality means here that not only C is a locus of intrinsic cusp singularities of D_1 but the singularity type of the elliptic fiber over points in C is also cuspidal ($y^2 - x^3 = 0$).

We will have further curves of intrinsic singularities of D_1 [2]: for $SU(4)$ a curve of tacnodes, for $SU(5)$ and $SU(6)$ a curve of higher double points. There one needs two or more blow ups.

2.5. Codimension-three loci

For $G = SU(n)$ or $SO(2n)$ one gets two matter curves which intersect in some points of B_2 (considered in detail below for $SU(5)$). Two other possible types of codimension-three loci are point singularities of D_1 and intersection points of the cusp curve C with B_2 . Further there occurs a codimension-three locus $(P) \cap (Q)$; cf. below, for global reasons.

3. An $SU(5)$ GUT model

We start from the Weierstrass model (2) and expand f and g in the section given by z , the coordinate of the \mathbf{P}^1 fiber of B_3 over B_2 ; so $z = 0$ corresponds to the locus B_2 of $SU(5)$ GUT group (the divisor r). Note that the cohomology class $4c_1(B_3)$ of the bundle of which f is a section reads on r just $4(c_1 - t)$. Now develop f in a polynomial in z with coefficient functions given by suitable sections over B_2 . The constant term has precisely the mentioned cohomological ‘degree’ $4(c_1 - t)$; each z -power then consumes one $-t$ from this class because the vanishing divisor of the section z is again r and we have $r|_r = -t|_r$; therefore the coefficient of z^i is some $f_{4c_1-4t+it}$

$$f = \frac{1}{2^4 \cdot 3} \sum_{i=0}^7 f_{4c_1-(4-i)t} z^i + \mathcal{O}(z^8), \quad (10)$$

$$g = \frac{1}{2^5 \cdot 3^3} \sum_{j=0}^7 g_{6c_1-(6-j)t} z^j + \mathcal{O}(z^8), \quad (11)$$

where the $f_{4c_1-(4-i)t}, g_{6c_1-(6-j)t}$ are sections of line bundles over B_2 with Chern classes indicated by the subscripts. As we will be interested only in the development of the discriminant Δ up to the order z^7 we keep only the terms shown. Actually we will take as highest terms for f and g the terms $f_{4c_1}z^4$ and $g_{6c_1}z^6$, respectively (note that these terms already correspond heterotically to the complex structure moduli of the heterotic elliptic Calabi–Yau threefold, whereas the lower terms will be relevant for description of the first E_8 bundle). So the actual starting point will be

$$f = \frac{1}{2^4 \cdot 3} (f_{4c_1-4t} + f_{4c_1-3t}z + f_{4c_1-2t}z^2 + f_{4c_1-t}z^3 + f_{4c_1}z^4), \quad (12)$$

$$g = \frac{1}{2^5 \cdot 3^3} (g_{6c_1-6t} + g_{6c_1-5t}z + g_{6c_1-4t}z^2 + g_{6c_1-3t}z^3 + g_{6c_1-2t}z^4 + g_{6c_1-t}z^5 + g_{6c_1}z^6). \quad (13)$$

The discriminant expression Δ in Eq. (3) will now also be expanded as a polynomial in z where for an $I_5 = A_4$ singularity the coefficients of z^i for $i = 0, 1, 2, 3, 4$ have to cancel, giving expressions for $f_{4c_1-(4-i)t}$ and $g_{6c_1-(6-j)t}$ (subscripts indicate the ‘cohomological degrees’)⁴

⁴ In general one has to invoke a generalised Weierstrass/Tate equation [7]; the dictionary to the coefficients of [4], for example, is $a_5 = h, -4a_4 = H, 12a_3 = q, 48a_2 = f_{4c_1-t}, 48f_0 = f_{4c_1}, -288a_4f_0 + 864a_0 = g_{6c_1-t}, 864g_0 = g_{6c_1}$.

$$f_{4c_1-4t} = -h^4 \quad (14)$$

$$f_{4c_1-3t} = 2h^2H \quad (15)$$

$$f_{4c_1-2t} = 2hq - H^2 \quad (16)$$

$$g_{6c_1-6t} = h^6 \quad (17)$$

$$g_{6c_1-5t} = -3h^4H \quad (18)$$

$$g_{6c_1-4t} = 3h^2(H^2 - hq) \quad (19)$$

$$g_{6c_1-3t} = \frac{3}{2}h(2Hq - hf_{4c_1-t}) - H^3 \quad (20)$$

$$g_{6c_1-2t} = \frac{3}{2}(f_{4c_1-t}H + q^2 - h^2f_{4c_1}). \quad (21)$$

Here we introduced arbitrary sections of the following cohomological degrees

$$h_{c_1-t}, \quad H_{2c_1-t}, \quad q_{3c_1-t}, \quad f_{4c_1-t}, \quad g_{6c_1-t} \quad (22)$$

and similarly also f_{4c_1} and g_{6c_1} . The discriminant has then the following structure

$$\Delta = cz^5 \Delta_1 = cz^5 \left(h^4 P + h^2 \left[-2HP + hQ \right] z + \left[-3q^2 H^3 + \mathcal{O}(h) \right] z^2 + \mathcal{O}(z^3) \right) \quad (23)$$

where $c = (2^{10} \cdot 3^3)^{-1}$ and

$$P = P_{8c_1-3t} = -3Hq^2 - 3f_{4c_1-t}qh + \left[2g_{6c_1-t} - 3f_{4c_1}H \right] h^2 \quad (24)$$

$$Q = Q_{9c_1-3t} = -q^3 - \left(\frac{3}{4}f_{4c_1-t}^2 + \left[2g_{6c_1-t} - 3f_{4c_1}H \right] H \right) h + \left(2g_{6c_1}h - 3f_{4c_1}q \right) h^2. \quad (25)$$

Let us now read off the various relevant subloci from the discriminant Eq. (23).

- **Codimension one:** We encoded the divisor of $SU(5)$ singularity type by the factor z^5 ; this is just the surface B_2 (of class r and with multiplicity 5). The remaining factor (in curly brackets) gives the defining polynomial of D_{I_1} .
- **Codimension two:** There are two matter curves, corresponding to singularity enhancements in codimension two in B_3 , given by the loci $h = 0$ ($SO(10)$ enhancement), leading to antisymmetric matter in the **10** and **$\overline{10}$** , and $P = 0$ ($SU(6)$ enhancement), leading to fundamental matter in the **5** and **$\overline{5}$**

$$h = 0 \implies (a, b, c) = (2, 3, 7) \stackrel{\wedge}{=} A_4 \rightarrow D_5 \quad (26)$$

$$P = 0 \implies (a, b, c) = (0, 0, 6) \stackrel{\wedge}{=} A_4 \rightarrow A_5. \quad (27)$$

As the Higgs fields H_u and H_d of the MSSM sit in the **5** and **$\overline{5}$** , respectively, we may want to have further independent curves giving an $SU(6)$ enhancement like P . So we have to tune the complex structure moduli of P in such a way that the locus $P = 0$ in r becomes reducible and decomposes actually in three curves (this may or may not come from a reducibility of the I_1 -surface D_1 itself). Whether such a locus in the complex structure moduli space is somehow stabilized remains an open question.

- **Codimension three:** The singularity type is enhanced even further in codimension three. Various such point loci occur in the $SU(5)$ -surface r (by Eq. (24)) the intersection locus $h = P = 0$ of the matter curves is contained in either $h = H = 0$ or $h = q = 0$

$$h = H = 0 \implies (a, b, c) = (3, 4, 8) \stackrel{\wedge}{=} A_4 \rightarrow E_6 \quad (28)$$

$$h = q = 0 \implies (a, b, c) = (2, 3, 8) \stackrel{\wedge}{=} A_4 \rightarrow D_6 \quad (29)$$

$$P = Q = 0 \quad (\text{but not } h = q = 0) \implies (a, b, c) = (0, 0, 7) \stackrel{\wedge}{=} A_4 \rightarrow A_6. \quad (30)$$

In Eq. (30) the conditions $P = Q = 0$ have to be taken in a generic sense (the conditions $h = q = 0$ in Eq. (29) imply also $P = Q = 0$, but this is a non-generic solution). Note that while the E_6 and D_6 enhancement points are expected from a local ansatz as intersection of the matter curves h and P , the A_6 enhancement locus $(P) \cap (Q) - (h) \cap (q)$ arises from the precise structure of the discriminant Eq. (23) in the global set-up.

3.1. Intrinsic singularities of the I_1 -surface: the cusp curve C

The surface D_1 has some intrinsic singularities. First Eq. (3) suggests that the curve $\{f = g = 0\}$ in $\{\Delta = 0\}$ is a curve of intrinsic cusp singularities of D_1 (resolvable by one blow-up). Actually by Eq. (2) over the corresponding points also lie cuspidal fibers (type II).

Actually the curve $\{f = g = 0\}$ is reducible as $\{f = 0\}$ and $\{g = 0\}$ have the curve $\{h = 0\}$ in B_2 as common component; on the level of divisors we have $(f)|_r = 4(h)$ and $(g)|_r = 6(h)$ from

$$f = -h^4 + 2h^2 Hz + (2hq - H^2)z^2 + f_{4c_1-t}z^3 + f_{4c_1}z^4 \quad (31)$$

$$g = h^6 - 3h^4 Hz + 3h^2(H^2 - hq)z^2 + \left(\frac{3}{2}h(2Hq - hf_{4c_1-t}) - H^3\right)z^3 \\ + \frac{3}{2}(f_{4c_1-t}H + q^2 - h^2f_{4c_1})z^4 + g_{6c_1-t}z^5 + g_{6c_1}z^6. \quad (32)$$

The intersection multiplicity of $(f) = \{f = 0\}$ and $(g) = \{g = 0\}$ at the curve $(h) = \{h = 0\}$ can be computed as the h -order of the resultant of the polynomials in z given by f and g ; this gives the order 15 (where $15 = 3n = \text{ord}_h \text{Res}(f, g)$ for the case of I_n with $n \in \{4, 5, 6\}$). Thus from the naive locus $(f)(g)$ a component $15(h)r$ has to be split off to get the true cusp curve

$$C = (f)(g) - 15(h)r \quad (33)$$

(on the divisorial level in D_1). We find that cohomologically

$$Cr = 24[h]^2 + 15[h]t = 3[h](8[h] + 5t) = 3[h][P] = 3[h]([H] + 2[q]). \quad (34)$$

where the factor $3 = \gcd(24, 15)$ is a divisorial multiplicity in $C|_r$ such that $\#(C \cap r) = Cr/3$ as cardinality. Whether actually also $C \cap r \subset (h) \cap (P)$ as point sets is less clear at first as we did in Eq. (34) just a cohomological computation. That $C \cap r \subset (h)$, however, is clear as the cardinality is turned off by setting the cohomology class $[h]$ of (h) in (34) to zero (or because actually the divisor (h) occurs as factor in $C|_r$). We will show now that even $C \cap r \subset (h) \cap (P) = ((h) \cap (H)) \cup ((h) \cap (q))$; possibly one has even $C \cap r = (h) \cap (P)$.

3.2. Investigation of the locus $C \cap B_2$

Concerning the point locus $C \cap r$ think of H (or q) and h as local functions in the r -plane around a point $p \in C \cap r$; we know that $h(p) = 0$ and we want to show that either $H(p) = 0$ or $q(p) = 0$. If x and y are local coordinates near p in B_2 then C is parametrized as $(x(\tau), y(\tau), z(\tau))$ for some parameter τ ; we will take h, H and q locally as functions of (x, y) and make the normalisation that $\tau = 0$ gives $z = 0$ (this means that $z_0 = 0$ below). We saw above that the points in Cr come with multiplicity 3 and not 1 so we cannot take z itself as a parameter. Therefore we make the following ansatz for the parametrization of C near p (we have $h_0 = 0$, i.e. $j > 0$, from $p \in (h)$; similarly $p \in (H)$ just if $H_0 = 0$, and correspondingly for (q))

$$h = h(\tau) = h_j \tau^j + h_{j+1} \tau^{j+1} + \dots \quad (35)$$

$$H = H(\tau) = H_0 + H_1 \tau + H_2 \tau^2 + \dots \quad (36)$$

$$z = z(\tau) = z_k \tau^k + z_{k+1} \tau^{k+1} + \dots \quad (37)$$

Here the ansatz for $z(\tau)$ comes with $k \geq 3$ from the multiplicity 3. With this ansatz f, g and Δ_1 (the defining polynomial of D_1 , i.e. the factor in curly brackets in Eq. (23)) become expressions in τ which by $C \subset (f) \cap (g)$ and $C \subset (\Delta_1)$ must vanish identically in τ . As the coefficients of the individual τ -powers must vanish let us look for the lowest-order term. Note first that

$$\Delta_1 = [-3Hq^2 + \mathcal{O}(h)](h^4 - 2h^2 Hz + H^2 z^2) + \mathcal{O}(z^3) \quad (38)$$

from $P = -3Hq^2 + \mathcal{O}(h)$. We want to show that at an intersection point in $C \cap r$ either $H_0 = 0$ or $q_0 = 0$; so let us assume that both are non-zero. Then the lowest-order term in Δ_1 is $-3H_0 q_0^2$ times one of the three terms $h_j^4 \tau^{4j}$ or $-2H_0 h_j^2 z_k \tau^{2j+k}$ or $H_0^2 z_k^2 \tau^{2k}$, which gives in turn $H_0 q_0 = 0$.

3.3. Intrinsic singularities of the I_1 -surface: the curve of higher double points

Besides the cusp curve C the surface D_1 has also a curve of higher double points (resolved by a process of three blow-ups) which turns out to be just the curve (h) . Eq. (23) shows that the defining polynomial for D_1 (given in the curly brackets)

can be written near h as follows (keeping for each z -power just the leading h -power)

$$-3Hq^2(h^4 - 2h^2Hz + H^2z^2) + \mathcal{O}(z^3). \quad (39)$$

We want to look at the leading terms near $(h, z) = (0, 0)$. Written in the variable $w := Hz - h^2$ the terms up to third-order here become structurally (i.e. everything up to coefficients and where $H \neq 0$) $w^2 + z^3 \rightarrow h^6 + h^4w + w^2$ near $(h, w) = (0, 0)$; this gives the normal form $h^6 + v^2$ with $v := w + \frac{3}{2}h^4$, i.e. the curve (h) is actually a singular curve of higher double points of D_1 . So this matter curve does not arise in the standard framework of the collision rules where two smooth surfaces intersect transversally. Note that the prefactor in Eq. (39) shows that at the points of (h) of E_6 and D_6 enhancements, where in addition to h also H or q vanishes, respectively, the singularity of D_1 will be even worse.

3.4. Intrinsic singularities of the (P) -curve: its double point locus

As Eq. (24) shows the locus $(h) \cap (q)$ of points p of D_6 -enhancement lies in (P) . If we consider h and q as local functions in B_2 near p we find from Eq. (24) the double point structure $q^2 + qh + h^2$. This holds, strictly speaking, only for the case where (h) is not turned off by going to the pseudo-separation case $t = c_1$, as then a constant h cannot serve as a local coordinate.

3.5. Comparison with heterotic string theory

For the case that B_3 is \mathbf{P}^1 fibered over B_2 one can compare with the heterotic side [8]. There a vector bundle V of structure group H is specified which breaks the gauge group E_8 (we may assume that the second E_8 is completely broken). If $H = SU(N)$ for $N = 3, 4$ or 5 what remains in the effective four-dimensional gauge theory is the commutator E_6 , $SO(10)$ or $SU(5)$. The heterotic Calabi–Yau space Z is elliptically fibered over the surface B_2 which is visible to both sides of the duality, i.e. the duality arises by expanding adiabatically the eight-dimensional duality. On a generic elliptic fiber the $SU(N)$ bundle decomposes as a sum of line bundles; each of these is characterized by a fiber point. Globally over B_2 these points trace out a surface, the spectral cover C . Cohomologically $C = N\sigma + \eta$ where σ is the class of the base B_2 and η is (the pullback of) a class in B_2 . The dictionary to the F-theory side is implemented by setting $\eta = 6c_1 - t$. The equation for C is given (in affine fiber coordinates) for an $SU(5)$ bundle by

$$a_0 + a_2x + a_3y + a_4x^2 + a_5xy = 0 \quad (40)$$

(here the a_i are certain sections over B_2 of cohomology class $\eta - ic_1 = (6 - i)c_1 - t$). The matter localized on curves in B_2 arises as follows. In F-theory the $\mathbf{10}$ arose on the $SO(10)$ enhancement curve (h) , and the $\bar{\mathbf{5}}$ similarly from the $SU(6)$ enhancement curve (P) ; in the heterotic theory a larger gauge group means having a reduced structure group. This means, for (h) , that one of the fiber points of C becomes zero (in the group law), which takes place where the surface C intersects B_2 : this happens at the curve defined by $a_5 = 0$ where the structure group is reduced to $SU(4)$. So h corresponds to a_5 which has the right cohomological degree $c_1 - t$. Similarly the $\mathbf{5}$ is supported on the curve $(R) = \{R = 0\}$ for the resultant $R = \text{Res}(a_0 + a_2x + a_4x^2, a_3 + a_5x) = a_0a_5^2 - a_2a_3a_5 + a_3^2a_4$, corresponding to P in Eq. (24) and of the right cohomological degree $8c_1 - 3t$. The complete dictionary is (here the degrees match)

$$h_{c_1-t} = a_5 \quad (41)$$

$$-3H_{2c_1-t} = a_4 \quad (42)$$

$$q_{3c_1-t} = a_3 \quad (43)$$

$$3f_{4c_1-t} = a_2 \quad (44)$$

$$2g_{6c_1-t} - 3f_{4c_1}H = a_0. \quad (45)$$

Gauge group enhancements to E_6 , $SO(12)$ and $SU(7)$ are localized at the intersections $(h)(H)$, $(h)(q)$ and $(P)(Q)$ in F-theory, that is, on $(a_5)(a_4)$, $(a_5)(a_3)$ and $(R)(S)$ where $S_{9c_1-3t} = -(\frac{3}{4}a_2^2 + 2a_0a_4)a_5 - a_3^3$. At $\{a_5 = a_4 = 0\}$ the structure group H is reduced to $SU(3)$ with commutator $G = E_6$; a corresponding reasoning can be applied to the other points.

The results on $e(\bar{X})$ can be compared [2] via the relation $n_3 = n_5$ with a corresponding heterotic computation where an $SU(N) \times E_8$ bundle (V_1, V_2) is given

$$24n_5 = 288 + (1200 + 107N - 18N^2 + N^3)c_1^2 + (1080 - 36N + 3N^2)c_1t + (360 + 3N)t^2 \quad (46)$$

$$\rightarrow 288 + 1410c_1^2 + 975c_1t + 375t^2 \quad (47)$$

$$\rightarrow 288 + 2760c_1^2 \quad (48)$$

with $N = 0, 2, 3, 4, 5$ for a gauge group $G = E_8, E_7, E_6, SO(10), SU(5)$, where we also indicated the specializations for $SU(5)$ and for the pseudo-separation case $t = c_1$. In the separation case (pseudo-separation for $N = 5$) of $t = (6 - N)c_1$ we get that the important expression, $\eta - Nc_1 = (6 - N)c_1 - t$, related to heterotic matter; cf. [9,10], vanishes.

4. Tadpole cancellation

If D -brane charges do not cancel and leave a net RR charge in the vacuum a tadpole arises. This tadpole is not seen in local models as any excess RR charge can escape to infinity; however, in global models this issue cannot be ignored. As the cancellation condition involves an Euler number computation the considerations about globally consistent packages of special degeneration loci in various codimensions have an application here.

An F-theory background contains a number of spacetime filling D3-branes which are located at points in B_3 . The condition contains further contributions from supersymmetric fluxes associated to either the bulk supergravity fields on B or to the world-volume gauge fields on Δ . Tadpole cancellation requires the various contributions to satisfy the condition

$$\frac{e(X)}{24} = n_3 + \frac{1}{2} \int_X G \wedge G + \int_{D_i} c_2(E_i). \quad (49)$$

This formula applies for X being smooth. Which formula has to be applied for the physically relevant case of a singular fourfold has not been worked out fully. Also the meaning of the Euler number, if in the correct formula still relevant, has to be clarified. We will restrict to the computation of the Euler characteristic of the resolved Calabi–Yau fourfold assuming that such a resolution exists globally and arises from resolving fiberwise (inserting the Hirzebruch trees for the respective Kodaira singularity type of the fiber); clearly this is only the simplest assumption (this computation agrees in some cases where it can be checked with a toric computation). The Euler number of a resolved model might still get correction terms in its function as a contribution to n_3 ; nevertheless we find agreement with a corresponding heterotic computation for the number of five-branes in some cases which can be compared. In any case this serves as an illustration of the phenomena which occur in considering the contributions of the various relevant subloci.

It is instructive to recall first the *smooth case*, i.e. X has no singularities and is described by a smooth Weierstrass model. As a second step, we consider the case where X only develops an *ADE* singularity along a codimension-one subvariety r (in itself a problematic assumption as we recalled above in connection with such ‘separation cases’ and as we will develop further below). The general case will be discussed again for our $SU(5)$ model.

4.1. Smooth case

For a smooth X we expect only contributions from type I_1 singular fibers over D_1 minus C and type II singular fibers over the cusp curve C

$$e(X) = e(I_1)(e(D_1) - e(C)) + e(II)e(C) = 288 + 360c_1(B_3)^3 \quad (50)$$

where $e(D)$ is the Euler characteristic of the I_1 -surface, which itself is singular along the cusp curve C ; we get [2] a “Plücker-like” formula

$$e(D) = c_2(B_3)D - c_1(B_3)D^2 + D^3 + \Delta_C \quad (51)$$

where Δ_C is understood as a correction term to the smooth case, i.e. to the Euler characteristic of a smooth surface inside B_3 and we find $\Delta_C = 2(e(C) - DC)$ [2] but Δ_C receives further corrections when specifying a section of G singularities along r (cf. below). The expression for the Euler characteristic of C can be easily evaluated in the smooth case by simply noting that $C = FG$ and using the fact that the normal bundle of C in B_3 is given by $N_C|_{B_3} = (\mathcal{O}(F) \oplus \mathcal{O}(G))|_C$ and restricting the short exact sequence $0 \rightarrow T_D \rightarrow T_{B_3|D} \rightarrow N_{D|B_3} \rightarrow 0$ to C , we find

$$e(C) = c_1(B_3)FG - (F + G)FG. \quad (52)$$

4.2. Singular case – codimension one

In [2] an Euler number formula was derived for the case with singularities only over the codimension-one locus B_2 (with r_G and c_G the rank and Coxeter number, resp., of the gauge group G). Applying the stratification method as above we get

$$e(\bar{X}) = 288 + 360 \int_{B_3} c_1^3(B_3) - r_G c_G (c_G + 1) \int_{B_2} c_1^2(B_2) \quad (53)$$

$$\rightarrow 288 + (180(12 + n^2) - r_G c_G (c_G + 1)) c_1^2 \quad (54)$$

(\bar{X} the fiberwise resolved model). Here we indicated also the specialization $B_3 = \mathbf{F}_{k,m,n}$ over $B_2 = \mathbf{F}_k$ with the \mathbf{P}^1 fibration $t = mb + nf$, with actually $k = 0$ and $m = n$ from the pure codimension-one (separation) condition. The case of having a degeneration purely in codimension one, i.e., a ‘separation case’ between the two discriminant components B_2 and D_1 (without matter curve) is realizable for $G = E_8, E_7, E_6, D_4$ over $B_2 = F_0$ with $n = m = 12, 8, 6, 4$.

Thereby we find agreement between Eq. (47) and [6] over $B_2 = \mathbf{F}_0$ for true separation cases E_k and D_4 . For the pseudo-separation cases we find also agreement with our formula. Via a computation using toric geometry and computer analysis [6] one finds the following table

G	$e(\bar{X})$
D_4	$288 + 4872c_1^2$
E_6	$288 + 7704c_1^2$
E_7	$288 + 11286c_1^2$
E_8	$288 + 20640c_1^2$

In the A series the pseudo-separation specialization $t = c_1$ is used such that the h curve is ‘turned off’ cohomologically and only the P curve remains.

4.3. General case – $SU(5)$ singularity

To illustrate the general case (i.e. where no special choice of t is made and all matter curves contribute as well as the codimension-three singularities in B_3 are present) we will consider the example of having an $SU(5)$ singularity (in the fiber) along $r = B_2$ in B_3 . Following the procedure above, we decompose the discriminant D into $D_1 + D_2$ where again D_1 denotes the component with I_1 fibers. With $D_2 = 5r$, $F_2 = 0$ and $G_2 = 0$ we get from Eq. (5) expressions for D_1 , F_1 and G_1 . To determine the Euler characteristic of \bar{X} we have to compute first the Euler characteristic of the singular surface D_1 taking its singularity structure into account. We showed that D_1 is singular along the cusp curve C and also along a curve of higher double points which we identified as the matter curve h . Thus when computing the Euler characteristic of D_1 we actually expect correction terms Δ_C and Δ_h to $e(D_1^{\text{smooth}})$. Now as the cusp and higher double point curve intersect we expect a further correction term $\Delta_{C \cap h}$. Moreover, the singularity structure along h will change if at special loci coefficient functions vanish so that we get degenerations of the structure of Eq. (39) which happens at the loci $h \cap q$ and $h \cap H$ (we also include a term Δ_p corresponding to possible corrections from other point singularities of D_1). In summary, we get

$$e(D_1) = c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_C + \Delta_h + \Delta_{C \cap h} + \Delta_{h \cap q} + \Delta_{h \cap H} + \Delta_p \quad (55)$$

where $\Delta_C = 2(e(C) - CD) + 25Cr$ (here one has to work with C redefined by Eq. (33); for the expressions of the other correction terms Δ_i ; cf. [2]). Summarizing all contributions we find (where $\#$ denotes the cardinality of a set and polynomials stand for their zero-divisors)

$$\begin{aligned} e(\bar{X}) = & +e(I_1)[e(D_1) - e(C) - e(h) - e(P) + \#(h \cap P) + \#(C \cap r)] + e(II)[e(C) - \#(C \cap r)] \\ & + e(A_4)[e(B_2) - e(h) - e(P) + \#(h \cap P)] \\ & + e(D_5)[e(h) - \#(h \cap P)] + e(A_5)[e(P) - \#(h \cap P) - (\#(P \cap Q) - \#(h \cap q))] \\ & + e(E_6)[\#(h \cap H)] + e(D_6)[\#(h \cap q)] + e(A_6)[\#(P \cap Q) - \#(h \cap q)] \end{aligned} \quad (56)$$

$$= e(D_1) + e(C) - \#(C \cap r) + 5e(B_2) + \#(h \cap P) + \#(P \cap Q) - \#(h \cap q) \quad (57)$$

(note that $h \cap P = (h \cap H) \cup (h \cap q)$ which is a disjoint decomposition, so the cardinalities add). If actually $C \cap r = (h) \cap (P)$ one gets $e(D_1) + e(C) + 5e(B_2) + (\#(P \cap Q) - \#(h \cap q))$. This formula can be compared with Eq. (47). In the general case the singular structure of D_1 along (h) has to be taken into account. Here we restrict us to the pseudo-separation case of $t = c_1$ where the matter curve (h) is turned off and $D_1 \cap r = (P)$, $C \cap r = \emptyset$. Then

$$e(\bar{X}) = e(D_1) + e(C) + 5e(B_2) + \#(P \cap Q). \quad (58)$$

So for this pseudo-separation case we find

$$e(D_1) = c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_C = 228 + 4493c_1^2 \quad (59)$$

and using $e(B_2) = 12 - c_1^2$ and $e(C) = (c_1(B_3) - (F_1 + G_1))F_1G_1 = -1728c_1^2$ we find

$$e(\bar{X}) = 288 + 2760c_1^2 + \#(P \cap Q). \quad (60)$$

The $n_3 = e(\bar{X})/24$ matches not immediately the corresponding number of heterotic five-branes; cf. Eq. (48). So either there is a further singular contribution (in $e(D_1)$ or $e(C)$) or here the contribution to the number of three-branes n_3 is not derived from the Euler number of the fiberwise resolved model \bar{X} . This shows again the special status of the codimension-three locus $P \cap Q$ which does not arrive as intersection of matter curves (which themselves arise as intersections of $D7$ -brane components).

4.4. Hodge numbers and $e(\bar{X})$

A comparison of moduli spaces for F-theory and a dual heterotic string theory on a Calabi–Yau threefold Z , elliptically fibered over B_2 , gives the following expressions [11]

$$h^{11}(\bar{X}) = h^{11}(Z) + 1 + rk = 12 - c_1^2 + rk \quad (61)$$

$$h^{31}(\bar{X}) = h^{21}(Z) + I + n_o + 1 = 12 + 29c_1^2 + I + n_o \quad (62)$$

$$h^{21}(\bar{X}) = n_o \quad (63)$$

for the Hodge numbers of a resolved \bar{X} . Here $I = I_{SU(N)} + I_{E_8}$ denote the number of moduli of the $SU(N)$ resp. E_8 bundle in the heterotic model with

$$I_{SU(N)} = (N - 1) + \left(\frac{N^3 - N}{6} - 3N^2 + 18N + 6 \right) c_1^2 + \left(\frac{N^2}{2} - 6N - 1 \right) c_1 t + \frac{N}{2} t^2 \quad (64)$$

$$I_{E_8} = 8 + 166c_1^2 + 181c_1 t + 60t^2. \quad (65)$$

Further n_o refers to the total number of odd bundle moduli, $rk = 16 - (N - 1) - 8$, and we have $\eta_{SU(N)} = 6c_1 - t$ and $\eta_{E_8} = 6c_1 + t$. The Euler characteristic of \bar{X} can be expressed as [12]

$$e(\bar{X}) = 48 + 6(h^{11}(\bar{X}) - h^{21}(\bar{X}) + h^{31}(\bar{X})) \quad (66)$$

and inserting the above expressions gives

$$e(\bar{X}) = 288 + (1200 + 107N - 18N^2 + N^3)c_1^2 + (1080 - 36N + 3N^2)c_1 t + (360 + 3N)t^2. \quad (67)$$

Thus for $N = 5$, $t = c_1$ we get $e(\bar{X}) = 288 + 2760c_1^2$ matching the heterotic result Eq. (48).

5. The other $SU(n)$ cases

Here we give corresponding information for the other $SU(n)$ cases; cf. appendix and also [2].

5.1. The locus $C \cap B_2$

In all I_n cases we have $C \cap r \subset (h)$, resp. $\subset (H)$ for $n = 2$. For the cohomological intersection one finds the following. For $n = 4, 5, 6$ where $C = (f)(g) - 3n(h)r$ one has again

$$Cr = 3[h](8[h] + nt) = 3[h][P]. \quad (68)$$

Here the (cohomological) degree (i.e., cohomology class $[P]$) of the second matter curve P is generally computed as follows: from $D = 12(c_1 + 2t + r)$ one has $Dr = 12(c_1 - t) = 12[h]$ and from $D = nr + D_1$ one finds that $Dr = -nt + 4[h] + [P]$, or $[P] = 8[h] + nt$.

Similarly for $n = 2$ and 3 one gets the following: for $n = 2$ where $(f)|_r = 2(H)$, $(g)|_r = 3(H)$ and $C = (f)(g) - 3(H)r$ one has

$$Cr = 3[H](2[H] + t) = \frac{3}{2}[H][P], \quad (69)$$

as one has from $Dr = 6[H]$ and $D_1 r = 2[H] + [P]$ that $6[H] = -2t + 2[H] + [P]$, or that $[P] = 4[H] + 2t$; for $n = 3$ where $C = (f)(g) - 8(h)r$ one has

$$Cr = 8[h](3[h] + t) = \frac{8}{3}[h][P] \quad (70)$$

as one gets from $D_1 r = 3[h] + [P]$ that $12[h] = -3t + 3[h] + [P]$, or $[P] = 9[h] + 3t$.

5.2. Euler characteristic for the pseudo-separation case $t = c_1$

The pseudo-separation case $t = c_1$ gives for the $SU(n)$ series

$$\begin{aligned} e(\bar{X}) &= e(I_1) \left[e(D_1) - e(C) - e(P) \right] + e(II) e(C) \\ &\quad + e(A_{n-1}) \left[e(B_2) - e(P) \right] + e(A_n) \left[e(P) - \#(P \cap Q) \right] + e(A_{n+1}) \left[\#(P \cap Q) \right] \\ &= e(D_1) + e(C) + ne(B_2) + \#(P \cap Q) \\ &= 288 + (2880 - n(n^2 - 1))c_1^2 + \#(P \cap Q). \end{aligned} \quad (71)$$

This shows a correction $2872c_1^2 + \#(P \cap Q)$ relative to the proper separation case of Eq. (53). For the discussion of the special role of the locus $P \cap Q$; cf. the remark after Eq. (60).

G	$e(\bar{X}) - \#(P \cap Q)$
A_1	$288 + 2874c_1^2$
A_2	$288 + 2856c_1^2$
A_3	$288 + 2820c_1^2$
A_4	$288 + 2760c_1^2$
A_5	$288 + 2670c_1^2$

Here $e(\bar{X}) - \#(P \cap Q)$ matches the number of heterotic five-branes; cf. [2].

Appendix

Here we collect some useful results for the other $SU(n)$ cases which parallel those in the main text for our main example $SU(5)$.

A.1. Case $SU(2)$

The discriminant has the following structure

$$\Delta = z^2 \left(H^2 P + (-f_3 P + Q)z + g_4^2 z^2 \right) \quad (72)$$

with

$$P = -\frac{3}{4}f_3^2 + 2g_4 H + 3f_2 H^2 \quad (73)$$

$$Q = f_3 \left(\frac{1}{4}f_3^2 - g_4 H - 3f_2 H^2 \right) + 2g_3 H^3 \quad (74)$$

and

$$f = \frac{1}{48}(-H^2 + f_3 z + f_2 z^2), \quad g = \frac{1}{864} \left(H^3 - \frac{3}{2}f_3 H z + g_4 z^2 + g_3 z^3 \right). \quad (75)$$

Let us now read off the various relevant subloci from the discriminant Eq. (72)

- **Codimension one:** The divisor of the $SU(2)$ singularity, i.e. the surface B_2 of divisor r with multiplicity 2, is represented by the factor z^2 ; the other factor is the equation for D_{I_1}
- **Codimension two:** There are two matter curves (singularity enhancements in codimension two in B_3)

$$H = 0 \implies (a, b, c) = (1, 2, 3) \hat{=} A_1 \rightarrow A_1(III) \quad (76)$$

$$P = 0 \implies (a, b, c) = (0, 0, 3) \hat{=} A_1 \rightarrow A_2. \quad (77)$$

- **Codimension three:** The singularity type is enhanced even further in codimension three (the intersection of the matter curves (H) and (P) is here the locus where $H = f_3 = 0$)

$$H = f_3 = 0 \implies (a, b, c) = (2, 2, 4) \hat{=} A_1 \rightarrow A_2(IV) \quad (78)$$

$$P = Q = 0 \quad (\text{but not } H = f_3 = 0) \implies (a, b, c) = (0, 0, 4) \hat{=} A_1 \rightarrow A_3. \quad (79)$$

A.2. Case $SU(3)$

Here the discriminant is

$$\Delta = z^3 \left(h^3 P + (-q P + Q)z + \left[h(-3f_2 g_3 h + 6f_2^2 q) + 3g_3 q^2 \right] z^2 + (f_2^3 + g_3^2) z^3 \right) \quad (80)$$

with

$$P = -q^3 - 3f_2 h^2 q + 2g_3 h^3 \quad (81)$$

$$Q = \frac{5}{4}q^4 + \frac{9}{2}f_2 h^2 q^2 - 4g_3 h^3 q - \frac{3}{4}f_2^2 h^4 \quad (82)$$

and

$$f = \frac{1}{48}(-h^4 + 2hqz + f_2z^2), \quad g = \frac{1}{864}\left(h^6 - 3h^3q + \frac{3}{2}(q^2 - f_2h^2)z^2 + g_3z^3\right). \quad (83)$$

Again we read off from (80) the various subloci

- *Codimension one:* The divisor of the $SU(2)$ singularity, i.e. the surface B_2 of divisor r with multiplicity 2, is represented by the factor z^2 ; the other factor is the equation for D_{I_1}
- *Codimension two:* Here we have again two matter curves

$$h = 0 \implies (a, b, c) = (2, 2, 4) \stackrel{\wedge}{=} A_2 \rightarrow A_2(IV) \quad (84)$$

$$P = 0 \implies (a, b, c) = (0, 0, 4) \stackrel{\wedge}{=} A_2 \rightarrow A_3. \quad (85)$$

- *Codimension three:* The singularity type is enhanced even further in codimension three (the intersection of the matter curves (h) and (P) is here the locus where $h = q = 0$)

$$h = q = 0 \implies (a, b, c) = (2, 3, 6) \stackrel{\wedge}{=} A_2 \rightarrow D_4 \quad (86)$$

$$P = Q = 0 \quad (\text{but not } h = q = 0) \implies (a, b, c) = (0, 0, 5) \stackrel{\wedge}{=} A_2 \rightarrow A_4 \quad (87)$$

(note that the D_4 point is a triple or quadruple point of P or Q , respectively).

A.3. Case $SU(4)$

Here the discriminant equation looks as follows

$$\Delta = z^4\left(h^4P + h^2[-2HP + Q]z + \mathcal{O}(z^2)\right) \quad (88)$$

with (where $e := f_2 + H^2$, $k := 2g_2 - 3f_1H$)

$$P = h^2k - \frac{3}{4}e^2 \quad (89)$$

$$Q = -h^2\left(kH - 2g_1h^2 + \frac{3}{2}f_1e\right) \quad (90)$$

and

$$f = \frac{1}{48}\left(-h^4 + 2h^2Hz + f_2z^2 + f_1z^3\right) \quad (91)$$

$$g = \frac{1}{864}\left(h^6 - 3h^4Hz + \frac{3}{2}h^2(H^2 - f_2)z^2 + \left[\frac{1}{2}H(H^2 + 3f_2) - \frac{3}{2}f_1h^2\right]z^3 + g_2z^4 + g_1z^5\right). \quad (92)$$

Now we read off the various subloci

- *Codimension one:* The divisor of the $SU(4)$ singularity, represented by the factor z^4 , is the surface B_2 of divisor r with multiplicity 2; the other factor is the equation for D_{I_1}
- *Codimension two:* We have again two matter curves

$$h = 0 \implies (a, b, c) = (2, 3, 6) \stackrel{\wedge}{=} A_3 \rightarrow D_4 \quad (93)$$

$$P = 0 \implies (a, b, c) = (0, 0, 5) \stackrel{\wedge}{=} A_3 \rightarrow A_4. \quad (94)$$

- *Codimension three:* Finally the further singularity enhancement in codimension three (the intersection of the matter curves (h) and (P) is here the locus where $h = f_2 = 0$)

$$h = f_2 = 0 \implies (a, b, c) = (3, 3, 6) \stackrel{\wedge}{=} A_3 \rightarrow D_4 \quad (95)$$

$$h = H = 0 \implies (a, b, c) = (2, 4, 6) \stackrel{\wedge}{=} A_3 \rightarrow D_4 \quad (96)$$

$$P = Q = 0 \implies (a, b, c) = (0, 0, 6) \stackrel{\wedge}{=} A_3 \rightarrow A_5. \quad (97)$$

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