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Gerbes on G_2 manifolds

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ABSTRACT

On a projective complex manifold, the Abelian group of divisors maps surjectively onto that of holomorphic line bundles (the Picard group). On a G_2-manifold we use coassociative submanifolds to define an analogue of the divisors, and a gauge theoretical equation for a connection on a gerbe to define an analogue of the Picard group. Then, we construct a map from the former to the later. We also prove that the canonical map from our analogue of the Picard group to the third cohomology group with integer coefficients is surjective. As a side remark we make an observation relating the topological type of coassociative submanifolds and the cohomology classes they represent.

1. Introduction

In a complex manifold X one defines the Abelian group of divisors, $Div(X)$, as formal sums of complex codimension 1 submanifolds. On the other hand the holomorphic line bundles also form an Abelian group, known as the Picard group $Pic(X)$. It is a classical fact that $Pic(X) \cong Div(X)/\sim$, where $\sim$ denotes linear equivalence of divisors. If X is supposed to have a Kähler form, it follows from Hodge theory that each holomorphic line bundle has a unique Hermitian–Yang–Mills (HYM) connection, equivalently a unique connection with harmonic curvature. In this short note, inspired by Hitchin's work on the moduli of special Lagrangian submanifolds [1], we imitate part of these ideas from complex geometry to the case of G_2 manifolds.

Let (X^7, g) be a compact G_2-manifold, i.e. a compact, Riemannian 7-manifold with holonomy contained in G_2. Equivalently, one can think of the metric g has being induced from a certain 3-form φ, which is harmonic with respect to g. Appealing to this point of view, we will also refer to a G_2-manifold as the pair (X^7, φ). In addition, a G_2-manifold is said to be irreducible if its holonomy representation is irreducible, i.e. $Hol(g) = G_2$. For future reference we shall use the notation $\psi = *\varphi \in \Omega^4(X, \mathbb{R})$, where $*$ is the Hodge-$*$ operator of the Riemannian metric g.

On a G_2-manifold there are very interesting calibrated submanifolds called associatives and coassociates. Being the later ones are the main subject of this paper, we now focus on them. These are 4-dimensional submanifolds N^4 calibrated by ψ, i.e. $\psi|_N = dvol_{g|_N}$, where $g|_N$ denotes the restriction of the metric g to N. We use them to define the following G_2 analogue of the group of divisors.

Definition 1. Let $CDiv(X, \varphi)$ denote the Abelian group of finite formal sums

$$\sum_{i=1}^{k} q_i N_i,$$

where the $q_i \in \mathbb{Z}$ and the N_i are coassociative submanifolds.

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In [2], Joyce conjectured that it may be possible to define an enumerative invariant of G_2-manifolds by "counting" coassociative submanifolds. In [3], Donaldson and Segal suggested it may be easier, to define an invariant from solutions of a gauge theoretical equation, the G_2-monopole equation. The authors have further suggested that such monopoles may be somehow related to coassociative submanifolds, and this have been further investigated in [4]. This short note contains a detour, which considers gauge theoretical objects of another nature, which we show to also be related to coassociative submanifolds. In order to further motivate our construction, we now mention some natural questions regarding coassociative submanifolds: Given a class $\alpha \in H^3(X, \mathbb{Z})$, is there a coassociative representative of the Poincaré dual of α? What are the possible topological obstructions for a class α to be represented by a coassociative submanifold? (as on G_2-manifolds $H^4_7 = 0$ we do not know any such). Given a class α, are there any topological/geometric restrictions of possible coassociative representatives of α? Some of these problems seems to be currently out of reach and is certainly not the author's goal to attempt it. Nevertheless, we believe our results may be of interest and we hope may motivate further research along similar lines. We give some possibilities for these at the end of this note.

We turn now to the definitions of the gauge theoretical objects, which are our G_2 analogues of the Picard group. Recall that we want these to be related to coassociative submanifolds, which are of codimension 3, this is where gerbes with connection come into play. For the sake of simplicity, in this introduction it is enough to think of a gerbe $\mathcal{G}$ as being determined by a representative of a Čech cocycle $\check{H}^2(X, C^\infty(S^1))$, where $C^\infty(S^1)$ is the sheaf of smooth functions with values in S^1. This should be compared with the way a cocycle in $\check{H}^1(X, C^\infty(S^1))$ determines a circle bundle. Gerbes, as bundles, can be equipped with connections. There is the notion of a trivialization of a gerbe, and if $\{U_i\}_{i\in I}$ is an open cover of X, such that on each U_i we have gerbe trivializations, then a connection is determined by connection 2-forms F_i satisfying some extra conditions, see Section 2 for more details. These conditions ensure that the exterior derivatives dF_i agree on double intersections and so define a global, closed 3-form H called the curvature of the connection. The third cohomology class represented by H is integral and independent of the connection. This class is denoted by $c_1(\mathcal{G})$ and called the first Chern class of the gerbe $\mathcal{G}$. In fact, $c_1(\mathcal{G})$ may also be defined as the image under the Bockstein morphism of the exponential sequence, i.e.

$$c_1 : \check{H}^2(X, C^\infty_X(S^1)) \xrightarrow{\sim} H^3(X, \mathbb{Z}).$$

We are now ready to define our G_2-analogue of the Picard group.

Definition 2. A **monopole gerbe** is a pair $(\mathcal{G}, F)$, where $\mathcal{G}$ is a gerbe and F a connection on $\mathcal{G}$, such that

- There is an open cover $\{U_i\}_{i\in I}$ of X equipped with local trivializations of $\mathcal{G}$, and connection 2-forms F_i satisfying

$$*(F_i \wedge \psi) = d\phi_i,$$

  for some $\phi_i : U_i \to \mathbb{R}$.
- The curvature of F is the harmonic representative of $c_1(\mathcal{G})$.

The monopole gerbes form an Abelian group, which we denote $MPic(X, \varphi)$.

The operation under which the monopole gerbes form an Abelian group is the tensor product of the underlying gerbes and the canonically induced connections, see Section 3.

Remark 1. We compare each of the conditions above with the definition of a holomorphic line bundle, or rather a complex line bundle with a connection inducing a holomorphic structure. In this comparison:

- The first condition is the analogue of the existence of holomorphic trivializations where the connection 1-forms are of type $(1, 0)$.
- The second condition is analogous to changing the Hermitian structure so that the connection has harmonic curvature. In the Kähler case there is a unique such connection (up to gauge) and this is the so called Hermitian–Yang–Mills one (HYM).

In the general context of a Riemannian manifold, Hitchin defined in [1] a way to canonically associate a gerbe with connection to a codimension 3 submanifold. We review this construction in Proposition 2. We are now ready to state our first result.

Theorem 1. *Let (X^7, φ) be a compact, irreducible G_2-manifold, then Hitchin's construction yields a group homomorphism*

$$m : CDiv(X, \varphi) \to MPic(X, \varphi). \tag{1.1}$$

There is a forgetful map $MPic(X, \varphi) \to H^3(X, \mathbb{Z})$, which associates to $(\mathcal{G}, F)$ the topological type of the gerbe $c_1(\mathcal{G})$. Our second result is the following analogue of the hard-Lefschetz theorem for $(1, 1)$ classes.

Theorem 2. *Let (X, φ) be a compact, irreducible G_2-manifold, then the map $MPic(X, \varphi) \to H^3(X, \mathbb{Z})$ is surjective.*

Understanding the image of the map resulting from composing that of Theorem 2 together with that of Theorem 1 would give an answer to the first question mentioned above. Namely, which classes can be represented by coassociative submanifolds. For instance, if the map (1.1) is surjective, then a G_2-analogue of the Hodge conjecture would be true. However, this may well not be the case and, as said before, it is not our goal to attempt an answer to that question.

A more practical goal with the tools at hand, is to obtain restrictions on the geometry/topology of possible coassociative representatives of certain classes. This may be of interest for better understanding the global geometry of coassociative fibrations on G_2 manifolds. One further possible reason of interest is in giving topological obstructions for the convergence of the mean curvature flow of a 4 dimensional submanifold to a coassociative one. Our final result, which is not an application of the previous ones, is aligned with that direction. It is an observation that yields topological restrictions on coassociative submanifolds in terms of the class they represent. This follows from a coassociative analogue of the adjunction formula and the fact that on a compact, non-flat G_2-manifold $p_1(X) \cup [\varphi] > 0$. Namely we prove that

Proposition 1. *Let (X, φ) be a compact G_2-manifold, $\alpha \in H^3(X, \mathbb{Z})$, and N a coassociative representative of α. If τ and χ respectively denote the signature and Euler characteristic of N, then $p_1(X) \cup \alpha = 6\tau - 2\chi$. In particular,*

- $\tau > \frac{1}{3}\chi$ *if and only if* $p_1(X) \cup \alpha > 0$;
- $\tau = \frac{1}{3}\chi$ *if and only if* $p_1(X) \cup \alpha = 0$;
- $\tau < \frac{1}{3}\chi$ *if and only if* $p_1(X) \cup \alpha < 0$

and there are integral classes α satisfying the first and third inequalities in the right hand side.

Remark 2. In this result I have used the convention that my 3-forms are modeled on

$$\varphi_0 = e^{123} + e^1 \wedge (e^{45} - e^{67}) + e^2 \wedge (e^{46} - e^{75}) + e^3 \wedge (e^{47} - e^{56}),$$

on $\mathbb{R}^7$. In this way, $N = 0 \times \mathbb{R}^4$ is coassociative and the map $e_i \mapsto \iota_{e_i}\varphi$ yields an isomorphism of $TN^{\perp}$ with $\Lambda^2_- N$, the anti-self-dual 2-forms on N. For such irreducible G_2-manifolds locally modeled on these we have $p_1(X) \cup [\varphi] > 0$, see Remark 6. However, many authors use G_2-structures locally modeled on $\varphi_0 = e^{123} + e^1 \wedge (e^{45} + e^{67}) + e^2 \wedge (e^{46} + e^{75}) - e^3 \wedge (e^{47} + e^{56})$, in which case the normal bundle of a coassociative submanifold N is isomorphic to $\Lambda^2_+ N$. For such structures, we have $p_1(X) \cup [\varphi] < 0$ on any irreducible G_2-manifold. Moreover, the result of Proposition 1 should then be stated as $p_1(X) \cup \alpha = 6\tau + 2\chi$ and so

- $\tau > -\frac{1}{3}\chi$ if and only if $p_1(X) \cup \alpha > 0$;
- $\tau = -\frac{1}{3}\chi$ if and only if $p_1(X) \cup \alpha = 0$;
- $\tau < -\frac{1}{3}\chi$ if and only if $p_1(X) \cup \alpha < 0$.

In fact, these two statements are related by changing the orientation of the coassociative, in which case is calibrated by $-\psi$.

An immediate consequence of this result is that given any irreducible G_2-manifold (X, φ) and a compact 4-manifold N^4, there are classes in X, such that N cannot be embedded in X as a coassociative representative of such classes. For example, given that a 4-torus has $\tau = \frac{\chi}{3} = 0$ we have

Corollary 1. *Any coassociative T^4 of a compact G_2-manifold must represent a class α such that $p_1(X) \cup \alpha = 0$. In particular, if $b^3(X) = 1$, then (X, φ) has no coassociative tori.*

We must however remark that up to the author's knowledge there are no known examples of G_2-manifolds with $b^3(X) = 1$.

This note is organized as follows. Section 2 gives a self contained introduction to gerbes, which recalls Hitchin's working definition of gerbes and connections on them. Then, we recall Hitchin's construction of a gerbe with connection associated with a codimension-3 submanifold. In Section 3 we give the proofs of the results mentioned in this introduction. Section 4 gives an illustrative example of the construction in the case when the G_2-structure is reducible to $SU(3)$. Finally, in section, 5 we mention some possible future directions related to our work.

2. Gerbes

In the introduction we worked with a **gerbe** as being determined by a representative of a Cêch cocycle $\mathcal{G} \in \check{H}^2(X, C^\infty(S^1))$. Alternatively, as in [1], one can take an open cover $\{U_\alpha\}_{\alpha \in I}$ and a gerbe $\mathcal{G}$ can be defined by the following data

- A Hermitian line bundle $L_{\alpha\beta}$ over each $U_{\alpha\beta} = U_\alpha \cap U_\beta$, and isomorphisms $L_{\beta\alpha} \cong L_{\alpha\beta}^{-1}$.
- Hermitian trivializations $\theta_{\alpha\beta\gamma} : L_{\alpha\beta}L_{\beta\gamma}L_{\gamma\alpha} \cong \underline{\mathbb{C}}$, such that $\delta\theta = 1$ on $U_{\alpha\beta\gamma\delta}$.

One recovers the cocycle $\mathcal{G} \in \check{H}^2(X, C^\infty(S^1))$ as follows. First, refine the trivialization so that the line bundles become trivializable in the double intersections. Then, the trivializations $\theta : \underline{\mathbb{C}} \cong \underline{\mathbb{C}}$ in the triple intersections are simply S^1-valued functions. Moreover, the condition that $\delta\theta = 1$ in the quadruple intersections means they form a cocycle in $\check{H}^2(X, C^\infty(S^1))$. This is the original point of view taken in the introduction. Turning back to Hitchin's point of view, a **connection** F on $\mathcal{G}$ is determined by the following data

- Hermitian connections $\nabla_{\alpha\beta}$ on the $L_{\alpha\beta}$, such that $\nabla_{\alpha\beta\gamma}\theta_{\alpha\beta\gamma} = 0$ on $U_{\alpha\beta\gamma}$.
- 2-forms F_α on the U_α, such that on $U_{\alpha\beta}$

$$F_{\alpha\beta} = F_\alpha - F_\beta,$$

  is the curvature of $\nabla_{\alpha\beta}$ on $L_{\alpha\beta}$.

Then, in the double intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$

$$dF_\beta = dF_\alpha + dF_{\alpha\beta} = dF_\alpha,$$

by the Bianchi identity. Therefore, there is a well defined 3-form H such that

$$H|_{U_\alpha} = dF_\alpha,$$

for all $\alpha \in I$. This H is called the **curvature** of the gerbe connection.

Remark 3. If $(\mathcal{G}, F)$ is a monopole gerbe, then it is easy to see that the connections $\nabla_{\alpha\beta}$ on the line bundles satisfy

$$F_{\alpha\beta} \wedge \psi = *d\phi_{\alpha\beta},$$

where $F_{\alpha\beta} = F_\alpha - F_\beta$ and $\phi_{\alpha\beta} = \phi_\alpha - \phi_\beta$. In other words $(\nabla_{\alpha\beta}, \phi_{\alpha\beta})$ form an Abelian monopole on $L_{\alpha\beta}$. This justifies our nomenclature.

Codimension-3 submanifolds and gerbes

In this section we shall consider a more general setup where (X^n, g) is a real n-dimensional Riemannian manifold and N a codimension 3 (embedded) submanifold. The next proposition is an analogue of the construction of the map $Div \to Pic$ in complex geometry and I learned it for gerbes from Hitchin's paper [1]. For completeness, we shall include the construction.

Proposition 2. *Let N be a codimension 3, connected and embedded submanifold of X and $H \in PD[N] \in H^3(X, \mathbb{Z})$. Then, there is a Gerbe with connection $(\mathcal{G}_H, F)$ whose curvature is H. In particular, $c_1(\mathcal{G}_H) = PD\,[N]$.*

Proof. To construct the gerbe take a finite open cover given by $U_0 = X \backslash N$ and $\{U_\alpha\}_{\alpha\in I}$, such that $N \subset \cup_{\alpha\in I} U_\alpha$ and each $U_\alpha \cap N$, $U_\alpha \cap U_\beta$ is contractible. We shall use the indices $\{i, j, k\}$ to refer to either 0 or $\alpha \in I$. To define the Gerbe $\mathcal{G}_H$, one must give line bundles L_{ij} on the double intersections $U_{ij} = U_i \cap U_j$ satisfying a cocycle condition on the triple intersections. Using the notation $L_{ij} = L_{ji}^{-1}$, this is given by fixing a trivialization $L_{ij} \otimes L_{jk} \otimes L_{ki} \cong \underline{\mathbb{C}}$ on the triple intersections $U_{ijk} = U_i \cap U_j \cap U_k$.

Notice that up to homotopy $U_{0\alpha} \cong (U_\alpha \cap N) \times \mathbb{S}^2$, then we let $L_{\alpha 0}$ be the pullback of Hopf bundle on the $\mathbb{S}^2$ factor. On $U_{\alpha\beta}$ we let $L_{\alpha\beta}$ be the trivial bundle. Then the cocycle condition is trivially satisfied on the triple intersection $U_{0\alpha\beta}$ by fixing a trivialization $\underline{\mathbb{C}} \cong L_{0\alpha} \otimes L_{\beta 0}|_{U_{0\alpha\beta}}$.

We turn now to the definition of the connection. This requires giving the connection 2-forms F_i on each U_i, such that

$$F_{ij} = F_i - F_j,$$

is the curvature of a connection on L_{ij}. We weakly solve the PDE for currents

$$\Delta H_0 = H - \delta_N$$

which is possible, since $[H - \delta_N] = 0$ in de Rham cohomology for currents. Then $dH_0 = 0$, as $\Delta dH_0 = d\Delta H_0 = 0$, so that dH_0 is both exact and harmonic and so vanishes. Also notice that H_0 is only unique up to a harmonic 3-form. Then, we solve

$$\Delta H_\alpha = H, \quad dH_\alpha = 0,$$

on each open set U_α. Using the solutions H_0, H_α to these equations, one can define the connection 2-forms by

$$F_0 = d^* H_0 \text{ on } U_0$$
$$F_\alpha = d^* H_\alpha \text{ on } U_\alpha.$$

Notice that F_0 is indeed uniquely defined as any other H_0 will differ by a global harmonic three form, which is then coclosed and give rise to the same F_0. One still needs to check that the 2-forms F_{ij} are the curvature of a connection on L_{ij}. For $F_{\alpha\beta}$ this is obvious as $dF_{\alpha\beta} = d(F_\alpha - F_\beta) = 0$ on $U_{\alpha\beta}$ and the Poincaré lemma gives a primitive to $F_{\alpha\beta}$ which we take to be our connection on $L_{\alpha\beta}$.

Over $U_{0\alpha}$, there is a unique nontrivial 2 cycle, namely the one generated by the 2-spheres S^2 in the normal bundle. These do bound a 3-dimensional disk D^3 in U_α but not in $U_{0\alpha}$. Indeed any such disk D^3 does need to intersect N. Since $dF_{\alpha 0} = d(F_\alpha - F_0) = \delta_N$, Stokes' theorem gives

$$\int_{S^2} F_{0\alpha} = \int_{D^3} dF_{0\alpha} = \int_{D^3} \delta_N = 1.$$

This shows that the 2-forms F_i do define a connection on the gerbe $\mathcal{G}_N$. To check that H is its curvature we compute $dF_0 = dd^*H_0 = H$ in U_0, and $dF_\alpha = dd^*H_\alpha = H$ in U_α. □

Remark 4. Hitchin's construction described above can possibly also be phrased in the language of Hodge sparks, as in section 12 of [5]. For instance, the connection 2-form F_0 constructed above can be extended as a current to all of X and that is the spark of the cycle N (see definition 12.2 in [5]).

Let Y denote the disjoint union of a collection of open sets covering X. Then, the connections on the gerbes $\mathcal{G}_N$ above were defined using 2-forms $F \in \Omega^2(Y)$ satisfying some compatibility conditions, also known as curvings [6]. Given two open coverings $Y_1 = \{U^1_\alpha\}_{\alpha\in I}$ and $Y_2 = \{U^2_\beta\}_{\beta\in J}$, we can define a refined open cover

$$Y_{12} = \{U^1_\alpha \cap U^2_\beta\}_{(\alpha,\beta)\in I\times J}.$$

Then, we can add two connections $F^1 \in \Omega^2(Y_1)$ and $F^2 \in \Omega^2(Y_2)$ to define a new one $F^{12} \in \Omega^2(Y_{12})$, such that

$$F^{12}|_{U^1_\alpha\cap U^2_\beta} = F^1|_{U^1_\alpha} + F^2|_{U^2_\beta}.$$

Proceeding inductively, given connections $F_i \in \Omega^2(Y_i)$, for $i = 1, \dots, k$. We define $Y_{1\dots k}$, for any k-tuple of open covers, as well as the sum of the connections $F^{1\dots k} \in \Omega^2(Y_{1\dots k})$.

Definition 3. Let $N = N_1 \cup \dots \cup N_k$ with each N_i as in Proposition 2, and $H_i \in PD[N_i]$ be 3-forms representing the respective cohomology classes. Denote the gerbes constructed via Proposition 2 by $(\mathcal{G}_{N_i}, F_i)$, where the $F_i \in \Omega^2(Y_i)$ are connections on $\mathcal{G}_{N_i}$ with curvature H_i. Then, we define the gerbe with connection $(\mathcal{G}_N, F)$ to be given by $\mathcal{G}_N = \mathcal{G}_{N_1} \otimes \dots \otimes \mathcal{G}_{N_k}$ and the connection F by $F^{1\dots k} \in \Omega^2(Y_{1\dots k})$.

Notice that the gerbe constructed in the previous Definition 3 has first Chern class $c_1(\mathcal{G}_N) = c_1(\mathcal{G}_{N_1}) + \dots + c_1(\mathcal{G}_{N_k}) = PD[N]$. Its curvature is indeed $H = H_1 + \dots + H_k$.

Lemma 1. *In the setup of Definition 3, let all N_i's be embedded and such that $N = N_1 \cup \dots \cup N_k$ remains embedded. Then, given $H \in PD[N]$ the gerbe with connection $(\mathcal{G}_N, F)$ from Proposition 2 coincides with the one constructed via Definition 3, for any choice of $H_i \in PD[N_i]$, such that $H = \sum_{i=1}^k H_i$.*

Proof. For simplicity we shall only do the case $k = 2$, the gerbes in question are defined via line bundles on the double intersections of the open cover given by the sets $U^1_0 = X\backslash N_1$, $U^2_0 = X\backslash N_2$ and $\{U^1_\alpha\}_{\alpha\in I_1}$, $\{U^2_\alpha\}_{\alpha\in I_2}$ such that $N_i \subset \cup_{\alpha\in I_i} U^i_\alpha$, for $i = 1, 2$ and we suppose the U^1_α's are disjoint from the U^2_α's. As before, let $U_0 = X\backslash N$ and weakly solve the PDE's for the following 3-forms (currents)

$$\Delta H^i_\alpha = H_i, \quad dH^i_\alpha = 0,$$

on each U^i_α and

$$\Delta H^i_0 = H_i - \delta_{P_i},$$

on U^i_0 for $i = 1, 2$. Then, the 2-forms defining the connection are given by

$$\begin{aligned} F_0 &= d^*(H^1_0 + H^2_0), \text{ on } U_0 \\ F^1_\alpha &= d^*(H^1_\alpha + H^2_0), \text{ on } U^1_\alpha \\ F^2_\alpha &= d^*(H^2_\alpha + H^1_0), \text{ on } U^2_\alpha. \end{aligned}$$

It follows that $dF_0 = H_1 + H_2$ on U_0 and for $i = 1, 2$ one has $dF^i_\alpha = H_1 + H_2$ on each U^i_α, which shows that the curvature of the connection on $\mathcal{G}_{N_1} \otimes \mathcal{G}_{N_2}$ is $H = H_1 + H_2$. To check that the connection does not depend on the splitting $H = H_1 + H_2$ take instead the splitting given by the 3-forms $H_1 + d\omega$ and $H_2 - d\omega$ for some $\omega \in \Omega^2(X)$. These are obviously cohomologous to the initial ones and do add to H. The forms H^i_0, H^i_α change by $+\alpha, -\alpha$, for $i = 1, 2$ respectively, where α satisfies

$$dd^*\alpha = d\omega, \quad d\alpha = 0.$$

So the forms $F^i_0 = d^*H^i_0$ and $F^i_\alpha = d^*H^i_\alpha$ change by $\pm d^*\alpha$ showing that $F_0 = F^1_0 + F^2_0$ do not change. Exactly the same argument proves that the F_α remain unchanged and therefore so does the connection. □

3. Proof of the main results

In this section we return to the case when (X, φ) is a G_2-manifold and denote by $\psi = *\varphi$ the calibrating 4-form. The 3-forms in a G_2 manifold, pointwise split into G_2-irreducible representations as

$$\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27},$$

where the subscripts in the right hand side denote the respective dimension. The respective projections will be denoted by π_1, π_7, π_{27}.

Recall from Definition 2 that a gerbe with connection $(\mathcal{G}, F)$ is said to be a monopole gerbe if its curvature $H \in \Omega^3(X, \mathbb{R})$ is the harmonic representative of $c_1(\mathcal{G}) \in H^3(X, \mathbb{Z})$ and there is a trivialization $\{U_\alpha\}_{\alpha \in I}$ and 2-forms F_α satisfying

$$*(F_\alpha \wedge \psi) = d\phi_\alpha,$$

where the ϕ_α's are real valued functions. It is immediate from the discussion at the end of the previous section that the monopole gerbes form an Abelian group, which we have denoted by $MPic(X, \varphi)$. The group operation is the tensor product of the gerbes and addition of the connection, as described at the end of last section (see Definition 3, Lemma 1 and the discussion preceding these). We shall now turn to the proof of the first main result, Theorem 1. Namely, that associated to a coassociative submanifold there is a canonical monopole gerbe.

3.1. Proof of Theorem 1

We start by proving an easy but key lemmata.

Lemma 2. *Let N be a coassociative submanifold and δ_N the current it generates, then $\delta_N \wedge \varphi = 0$.*

Proof. An equivalent way to define a coassociative submanifold is to say that $\varphi|_N = 0$. Then for all $\eta \in \Omega^1(X)$,

$$\delta_N \wedge \varphi(\eta) := \int_N \eta \wedge \varphi = 0. \quad \square$$

Lemma 3. *Let (F, ϕ) be a 2-form and a function on a contractible open set U of a G_2-manifold. Then, if $\pi_7 dF = 0$ and $F \wedge \psi = *d\phi$, we have $F = d^*G$ where G is a closed 3-form.*

Proof. We write $F = *(f \wedge \psi) + g$ for some 1-form f and $g \in \Lambda^2_{14}$. Then, as $F \wedge \psi = *d\phi$, we conclude that $f = \frac{d\phi}{3}$ and using table 3 in [7] we compute $dF = -\frac{1}{7}\Delta(\frac{\phi}{3})\varphi + \pi_7 dg + \pi_{27} dg$. Moreover, as $\pi_7(dF) = 0$ by assumption, we conclude that $\pi_7 dg = 0$.

Now, we compute $d * F = *(\pi_7 dg \wedge \varphi) = 0$ and so F is coclosed. As U is contractible we can write $F = d^*G''$ on U and extend G'' to a 3-form G' on X, for example by multiplying G'' by a bump function supported on a slightly larger open set $U' \supset U$ and letting it vanish on its complement. Then, by de Rham's theorem $G' = da_2 + d^*a_4 + a_3$ for some $a_i \in \Omega^i$ with a_3 harmonic. We now redefine $G = da_2|_U$, which is therefore closed (in fact exact) and on U

$$d^*G = d^*da_2 = d^*G' = d^*G'' = F,$$

as we wanted to show. $\square$

We now prove Theorem 1 which we also restate here for convenience.

Theorem 3. *Let (X, φ) have holonomy strictly equal to G_2, N be a connected, embedded, coassociative submanifold. Then, the gerbe with connection associated with N is a monopole gerbe.*

Proof. Let H be the harmonic representative of $PD[N]$ and recall the construction of the connection F on $\mathcal{G}_N$. On the open cover $\{U_i\}_{i \in \{0\} \cup I}$ it is given by a collection of 2-forms F_i. These are defined such that each $F_i = d^*H_i$, with each H_i being a closed 3-form on U_i such that $\Delta H_0 = H - \delta_N$ and $\Delta H_\alpha = H$, for $\alpha \in I$.

We shall first work on the open set U_0. As X is supposed to have full G_2 holonomy, there can be no parallel 1-forms, [8]. This, together with the fact that a G_2-manifold is Ricci flat, implies through the Böchner-formula that there are no harmonic 1-forms, and so no harmonic 3-forms of type Λ^3_7. Hence, as H is the harmonic representative of $PD[N]$, and the Laplacian preserves the type decomposition, H has no component in Λ^3_7. Moreover, Lemma 2 guarantees that δ_N also has no component in Λ^3_7. Putting this together with $\Delta H_0 = H - \delta_N$ we then have that

$$\Delta \pi_7 H_0 = \pi_7 \Delta H_0 = 0.$$

Again, the hypothesis that g has full G_2 holonomy then yields that $\pi_7 H_0 = 0$. Then, the equation $dH_0 = 0$ turns into $d\pi_1(H_0) = -d\pi_{27}(H_0)$ and if one writes $\pi_1(H_0) = -a\varphi$, for some function a, this is

$$d\pi_{27}(H_0) = da \wedge \varphi. \tag{3.1}$$

Then we compute $d^*\pi_{27}H_0 = \pi_7 d^*\pi_{27}H_0 + \pi_{14}d^*\pi_{27}H_0$, using that $\pi_7 d^*\pi_{27}H_0 = -\frac{1}{3} * (*(*d\pi_{27}H_0 \wedge \varphi) \wedge \psi)$ together with Eq. (3.1) gives

$$\begin{aligned} d^*H_0 &= \pi_{14}d^*\pi_{27}H_0 - \frac{1}{3} * (*(*d\pi_{27}H_0 \wedge \varphi) \wedge \psi) - d^*(a\varphi) \\ &= \pi_{14}d^*\pi_{27}H_0 + \frac{4}{3} * (da \wedge \psi) + *(da \wedge \psi) \\ &= \pi_{14}d^*\pi_{27}H_0 + \frac{7}{3} * (da \wedge \psi), \end{aligned}$$

where we used that $*(*(da \wedge \varphi) \wedge \varphi) = -4da$. Now we put $F_0 = d^*H_0$, $\phi = 7a$ and compute $*(F_0 \wedge \psi)$. Since Λ^2_{14} is the kernel of wedging with ψ and $*(*(d\phi \wedge \psi) \wedge \psi) = 3d\phi$, we obtain

$$* (F_0 \wedge \psi) = d\phi. \tag{3.2}$$

We now need to define the connection 2-forms for our monopole gerbe on the remaining U_α's. Since H is closed we can locally find, on each U_α, a 2-form F'_α such that $dF'_\alpha = H$. Restricting to each U_α, we shall seek a connection 2-form $F_\alpha = F'_\alpha + da_\alpha$ such that the monopole equation $*(F_\alpha \wedge \psi) = d\phi_\alpha$ holds on U_α for some ϕ_α.

To do this we set $(a_\alpha, \phi_\alpha) = (*(db_\alpha \wedge \psi), -d^*b_\alpha)$ and solve for b_α instead. Using $3d^7 \cdot = *(*(d \cdot \wedge \psi) \wedge \psi)$ and that $3d^*d^7 = d^*d$, the monopole equation turns into

$$- * (F'_\alpha \wedge \psi) = 3d^*d^7 b_\alpha + dd^*b_\alpha = \Delta b_\alpha.$$

Moreover, as g is Ricci flat, on 1-forms $\Delta = \nabla^*\nabla$ and so we need to solve $\nabla^*\nabla b_\alpha = - * (F_\alpha \wedge \psi)$. This can be done by solving the Dirichlet problem

$$\begin{aligned} \nabla^*\nabla b_\alpha &= - * (F'_\alpha \wedge \psi), \ \text{on } U_\alpha \\ b_\alpha|_{\partial U} &= 0, \ \text{on } \partial U_\alpha. \end{aligned}$$

This follows from minimizing the functional $J(u) = \int_U |\nabla u|^2 + \langle u, f\rangle$, where $f = *(F'_\alpha \wedge \psi)$. To prove it is coercive and bounded by below we proceed as follows

$$\begin{aligned} J(u) &= \int_U |\nabla u|^2 + \langle u, f\rangle \\ &\geq \int_U |\nabla |u||^2 - \frac{\varepsilon}{2}\int_U |u|^2 - \frac{1}{2\varepsilon}\int_U |f|^2 \\ &\geq \left(c_U - \frac{\varepsilon}{2}\right)\int_U |u|^2 - \frac{1}{2\varepsilon}\int_U |f|^2 \end{aligned}$$

where $c_U > 0$ is some constant and $\varepsilon > 0$ is to be chosen small enough to ensure the first term is positive. We also remark that in the computation above, the first inequality follows from Kato's and Young's inequalities, while the second one makes use of Poincaré's inequality, as u has vanishing boundary values. This shows that the functional J is coercive and bounded by blow, so we can find a solution to the Dirichlet problem above by taking a minimizing sequence of J. Standard elliptic estimates yield the regularity of the solution b_α.

Then set $F_\alpha = F'_\alpha + da_\alpha$ with $(a_\alpha, \phi_\alpha) = (*(db_\alpha \wedge \psi), -d^*b_\alpha)$, and it follows by construction that (F_α, ϕ_α) satisfy both: the monopole equation, and $dF_\alpha = H$. Moreover, it follows from Lemma 3 that the connection 2-forms constructed in this way agree with the one from the construction in Proposition 2. □

In fact, as remarked to the author by an anonymous referee, the second part of the previous proof also proves that given a harmonic 3-form H on a compact, irreducible G_2-manifold, then there is a monopole gerbe whose curvature is H. The author, does not know whether a gerbe with harmonic curvature, on a compact, irreducible G_2-manifold can be written as a monopole gerbe without changing the connection (up to gauge). If true, this is analogous to the fact that in an irreducible Calabi–Yau manifold of complex dimension greater than 2, any line bundle with harmonic curvature is holomorphic. A slightly more general version of this "integrability" theorem is stated as Problem 1.

The second part of the previous proof is, in fact also the key for proving Theorem 2

Proof. We prove that the map $MPic(X, \varphi) \to H^3(X, \mathbb{Z})$, given by $(\mathcal{G}, F) \mapsto c_1(\mathcal{G})$ is surjective. To do this we start with a gerbe $\mathcal{G}$ and construct a connection F on $\mathcal{G}$ satisfying the required conditions so that $(\mathcal{G}, F)$ is a monopole gerbe. Let $H \in c_1(\mathcal{G})$ be the harmonic representative and $\{U_\alpha\}_{\alpha\in I}$ be a good open cover of X, i.e. each U_α and $U_{\alpha\beta}$ is contractible.

Then, following the last step in the proof of Theorem 3 we let F'_α be such that $dF'_\alpha = H$. Then, correct each of these to $F_\alpha = F'_\alpha + da_\alpha$, with $(a_\alpha, \phi_\alpha) = (*(db_\alpha \wedge \psi), -d^*b_\alpha)$, such that $F_\alpha \wedge \psi = *d\phi_\alpha$. As before, this holds if and only if $\nabla^*\nabla b_\alpha = - * (F_\alpha \wedge \psi)$ which can be solved by standard minimization techniques as in the previous proof. □

Remark 5. Theorem 2 and its proof show that any gerbe with harmonic curvature can be tensored with a flat gerbe so that the resulting gerbe with connection is a monopole gerbe.

Proof of Proposition 1

Let $p_1(X)$ denote the first Pontryagin class of the tangent bundle of X. If $N \subset X$ is a coassociative submanifold, then $TX|_N \cong TN \oplus \Lambda^2_-(N)$ and so $p_1(X)|_N = p_1(N) + p_1(\Lambda^2_-(N))$. Moreover, if N represents a class $\alpha \in H^3(X, \mathbb{Z})$, we have

$$\langle p_1(X) \cup \alpha, [X]\rangle = \int_N p_1(N) + \int_N p_1(\Lambda^2_-(N)). \tag{3.3}$$

The first term is 3τ and we shall now compute the second. Fix a local orthonormal framing $\{e_0, e_1, e_2, e_3\}$. Then in this framing, the curvature of the Levi-Civita connection on N acts via the matrix

$$R = \begin{pmatrix} 0 & \Omega^0_1 & \Omega^0_2 & \Omega^0_3 \\ -\Omega^0_1 & 0 & \Omega^1_2 & \Omega^1_3 \\ -\Omega^0_2 & -\Omega^1_2 & 0 & \Omega^2_3 \\ -\Omega^0_3 & -\Omega^1_3 & -\Omega^2_3 & 0 \end{pmatrix} \in \Omega^2(N, \mathfrak{so}(TN)),$$

where $\Omega^i_j(X, Y) = e^i(R(X, Y)e_j)$. Using this, the Gauss–Bonnet formula and Chern–Weil theory give

$$\begin{aligned} \chi &= \frac{1}{2^4\pi^2 2!}\sum_{ijkl} \varepsilon_{ijkl}\Omega^i_j \wedge \Omega^k_l = \frac{1}{2^3\pi^2}\sum_{ijk} \varepsilon_{ijk}\Omega^0_i \wedge \Omega^j_k \\ &= \frac{1}{4\pi^2}\left(\Omega^0_1 \wedge \Omega^2_3 + \Omega^0_2 \wedge \Omega^3_1 + \Omega^0_3 \wedge \Omega^1_2\right). \\ p_1(N) &= -\frac{1}{8\pi^2}\mathrm{tr}(R \wedge R) \\ &= \frac{1}{4\pi^2}\left(\Omega^0_1 \wedge \Omega^0_1 + \Omega^0_2 \wedge \Omega^0_2 + \Omega^0_3 \wedge \Omega^0_3 + \Omega^1_2 \wedge \Omega^1_2 + \Omega^1_3 \wedge \Omega^1_3 + \Omega^2_3 \wedge \Omega^2_3\right). \end{aligned}$$

Now let $\omega_1 = e_{01} - e_{23}, \omega_2 = e_{02} - e_{31}, \omega_3 = e_{03} - e_{12}$ be a local basis of Λ^2_-. The curvature of the induced connection on Λ^2_- acts on this basis by

$$R_{\Lambda^2_-} = \begin{pmatrix} 0 & -(\Omega^0_3 - \Omega^1_2) & \Omega^0_2 + \Omega^1_3 \\ \Omega^0_3 - \Omega^1_2 & 0 & -(\Omega^0_1 - \Omega^2_3) \\ -(\Omega^0_2 + \Omega^1_3) & \Omega^0_1 - \Omega^2_3 & 0 \end{pmatrix} \in \Omega^2(N, \mathfrak{so}(\Lambda^2_-)).$$

Then, we compute

$$\begin{aligned} p_1(\Lambda^2_- N) &= -\frac{1}{8\pi^2}\mathrm{tr}(R_{\Lambda^2_-} \wedge R_{\Lambda^2_-}) \\ &= \frac{1}{4\pi^2}\left(\Omega^0_1 \wedge \Omega^0_1 + \Omega^0_2 \wedge \Omega^0_2 + \Omega^0_3 \wedge \Omega^0_3 + \Omega^1_2 \wedge \Omega^1_2 + \Omega^1_3 \wedge \Omega^1_3 + \Omega^2_3 \wedge \Omega^2_3\right) \\ &\quad - \frac{1}{2\pi^2}\left(\Omega^0_1 \wedge \Omega^2_3 + \Omega^0_2 \wedge \Omega^3_1 + \Omega^0_3 \wedge \Omega^1_2\right) \\ &= p_1(N) - 2\chi. \end{aligned}$$

Hence, inserting this into equality (3.3) and using the signature theorem we have

$$\int_X p_1(X) \cup \alpha = \langle (2p_1(N) - 2\chi), [N]\rangle = 6\tau - 2\chi \tag{3.4}$$

as we wanted to prove.

Remark 6. 1. Using our conventions for the 3-form φ, see Remark 2, we have $p_1(X) \cup [\varphi] > 0$ for any compact nonflat G_2-manifold. To see this we notice that $p_1(X) \cup \varphi = -\frac{1}{8\pi^2}\mathrm{tr}(F_R \wedge F_R) \wedge \varphi$. Moreover, as X has holonomy G_2, F_R takes values in $\Lambda^2_{14} \cong \mathfrak{g}_2$ by the Ambrose–Singer theorem. Then, using our conventions we have $F_R \wedge \varphi = *F_R$, so that

$$\langle p_1(X) \cup \varphi, [X]\rangle = -\int_X \frac{1}{8\pi^2}\mathrm{tr}(F_R \wedge *F_R) = \|F_R\|^2_{L^2} > 0,$$

as (X, φ) is not flat.

Using the other convention for φ we have $F_R \wedge \varphi = - * F_R$ and so this sign gets reversed. In any case, there do exist classes $\alpha^+, \alpha^- \in H^3(X, \mathbb{R})$ such that $p_1(X) \cup \alpha^+ > 0$ and $p_1(X) \cup \alpha^- < 0$.

2. In [9], corollary $4-3$, McLean observed a particular case of this equality. Namely, the torus $\mathbb{T}^7$ has trivial tangent bundle, hence $p_1(\mathbb{T}^7) = 0$, and any coassociative submanifold of $\mathbb{T}^7$ must have $\tau = \chi/3$. For instance $N = \mathbb{T}^4 \times 0 \subset \mathbb{T}^7$ has $\tau = 0 = \chi$, which does satisfy such equality.
3. On a G_2-manifold there is one other class of very interesting submanifolds known as associatives. These are defined by requiring that the restriction of φ to them agrees with the volume form of the induced metric. Then, there are no associative submanifolds M of a compact, irreducible G_2-manifold (X, φ) representing the class $p_1(X)$. Otherwise we would have $vol(M) = \int_M \varphi = \int_X p_1(X) \cup [\varphi] < 0$, which is clearly impossible. If one uses the other convention for φ, then the same argument shows that there are no associatives representing the class $-p_1(X)$. I was informed by Henrique Sá Earp that in joint work with Johannes Nordström they have constructed associative submanifolds representing the negative of such classes, i.e. $-p_1(X)$ in my convention, $p_1(X)$ in theirs.

4. A toy example

In this section we explore the construction above in a case when the holonomy representation is actually reducible and in order to adapt the definition of monopole gerbe to this case we require that $\pi_7(H) = 0$, where H is the harmonic curvature. Let $X = S^1 \times M^6$ with (M, ω, Ω) being an irreducible Calabi–Yau with Kähler form ω and holomorphic volume form $\Omega = \Omega_1 + i\Omega_2$. In this case the G_2-structure is

$$\varphi = d\theta \wedge \omega - \Omega_1, \ \psi = -d\theta \wedge \Omega_2 + \frac{\omega^2}{2},$$

where θ is a periodic coordinate on $\mathbb{S}^1$. From, the last of these formulas it is easy to identify two types of coassociative submanifolds of $S^1 \times M$. Namely, those calibrated by either $-d\theta \wedge \Omega_2$, or $\frac{\omega^2}{2}$. These are of the form $S^1 \times SL^3$ and $p \times \mathcal{D}^4$, where SL, $\mathcal{D}$ and p are respectively a special Lagrangian submanifold of M, a divisor in M, and a point in S^1.

In order to analyze some features of our construction in this case we need to explain how to get a complex line bundle $\pi_* \mathcal{G}$ over X from a gerbe $\mathcal{G}$ over X. This is most naturally seen by regarding $\mathcal{G}$ as a line bundle over the loop space $L(M \times S^1)$ as in [1] and chapter 6 in [10]. Each element of $L(X)$ is a map $\gamma : S^1 \to X$. As S^1 is 1-dimensional, the pulled back gerbe $\gamma^* \mathcal{G}$ has a flat trivialization given by a flat line bundle $L_\gamma \to S^1$, and trivializations whose difference is a flat bundle with trivial holonomy are regarded as equivalent. Then, the moduli space of flat connections on S^1, i.e. $H^1(S^1, \mathbb{Z}) = S^1$ acts on these trivializations by tensoring with a flat connection. This shows that the space

$$P = \{(\gamma, L_\gamma) \mid \gamma \in L(X) \text{ and } L_\gamma \text{ is a trivialization of } \gamma^* \mathcal{G}\},$$

together with the S^1 action described above is a circle bundle over $L(X)$. Now we consider the map

$$M \xrightarrow{\Gamma} L(X), \ p \mapsto \gamma_p, \tag{4.1}$$

where $\gamma_p : S^1 \to X$ is the constant loop at p, i.e. $\gamma_p(\theta) = (\theta, p)$ for all $\theta \in S^1$.

Definition 4. We define $\pi_* \mathcal{G}$ to be the complex line bundle associated with $\Gamma^* P$ over X. When, $X = S^1 \times M^6$ we shall also denote by $\pi_* \mathcal{G}$ the pullback of this line bundle to M.

Proposition 3. *Let $D^4 \subset M$ be a 4-dimensional submanifold and $\mathcal{G}$ the gerbe associated with $p \times D^4 \subset S^1 \times M$. If $\mathcal{G}$ is a monopole gerbe, then D is a divisor. In particular $\pi_* \mathcal{G}$ comes equipped with an HYM connection.*

Proof. Let H be the harmonic representative of $N = p \times D$. As the Poincaré dual of N is $[d\theta] \cup PD[D]$, this is of the form $H = d\theta \wedge h$ where $h \in \Omega^2(M)$ is the harmonic representative of $PD[D]$. Similarly $\delta_M = \delta(\theta - pt) d\theta \wedge \delta_D$ and we solve the equation

$$\Delta H_0 = (h - \delta_D \delta(\theta - pt)) \wedge d\theta, \tag{4.2}$$

which implies $dH_0 = 0$. Using the splitting $\Lambda^3 X = \Lambda^3 M \oplus (\Lambda^1 S^1 \otimes \Lambda^2 M)$ we write $H_0 = g(\theta) + d\theta \wedge f(\theta)$ and compute

$$d_M f = \frac{\partial g}{\partial \theta} \tag{4.3}$$

$$d_M g = 0$$

$$\Delta H_0 = d\theta \wedge \left(\Delta_M f - \frac{\partial^2 f}{\partial \theta^2} \right) + \Delta_M g - \frac{\partial^2 g}{\partial \theta^2}. \tag{4.4}$$

So the component of Eq. (4.2) in $\Lambda^3 M$ turns into $\Delta_M g - \frac{\partial^2 g}{\partial \theta^2} = 0$. Separation of variables plus periodicity in θ implies that g must be constant (in θ) and g_M-harmonic. Then, Eq. (4.3) turns into $d_M f = 0$.

Now define the map $\pi_* : \Omega^{*+1}(M \times S^1) \to \Omega^*(M)$, given by $\pi_*(\omega) = 0$ and $\pi_*(d\theta \wedge \omega) = \int_{S^1} d\theta \wedge \omega$, for $\omega \in \Lambda^k M$. Let $h_0 = \pi_* H_0$, then applying π_* to Eqs. (4.2)–(4.4) leads to

$$\Delta f_0 = h_0 - \delta_D, \quad d_M f_0 = 0,$$

where $f_0 = \int_{S^1} f \in \Omega^2(M)$. So in order to prove that D is a divisor, i.e. the current δ_D is of type $(1, 1)$ we just need to show that Δf_0 is of type $(1, 1)$, as h_0 certainly is as it is harmonic and $H^{2,0}(M) = 0$ for any irreducible Calabi–Yau. Note that such statement does not follow neither from the Kähler identities, neither from the $\partial\overline{\partial}$-lemma, as we do not know the type decomposition of f_0 neither whether it is exact. However, if $\mathcal{G}$ is a monopole gerbe, then $d^* H_0 \wedge \psi = *d\phi$ where ϕ is the function such that $\pi_1(H_0) = \phi(d\theta \wedge \omega - \Omega_1)$. Then, this equation yields

$$d\theta \wedge \left(\frac{\partial f}{\partial \theta} \wedge \Omega_2 - d_M^* f \wedge \frac{\omega^2}{2} \right) - \frac{\partial f}{\partial \theta} \wedge \frac{\omega^2}{2} = *_M \frac{\partial \phi}{\partial \theta} - d\theta \wedge *_M d_M \phi.$$

Applying π_* to it and using the fundamental theorem of calculus gives $d_M^* f_0 \wedge \frac{\omega^2}{2} = - *_M d_M \Phi$, where $\Phi = \int_{S^1} \phi$. This equation is easily seen to be equivalent to $d_M^* f_0 = I d_M \Phi$ and so

$$\Delta f_0 = 2i\partial\overline{\partial}\Phi,$$

which is clearly of type $(1, 1)$ and so D is a divisor. $\square$

5. Some problems and questions

We now state some directions and conjectures for further research along these lines. The first of these goes back to Hitchin's work on the moduli of special Lagrangian submanifolds, [1].

1. In order to understand what is the image of the map m one could hope to proceed as in the case of line bundles. In our setting this would be something along the following lines: start with a gerbe with connection $(\mathcal{G}, F)$. Then, construct a rank 3-real vector bundle V such that the vanishing locus of a generic section lies in the Poincaré dual of $c_1(\mathcal{G})$. Finally, consider sections of V satisfying a partial differential equation, such that their vanishing locus is coassociative.

   The first steps in this construction would give a notion of a "section" of a gerbe. I am unaware if something like this already exists in the literature.
2. The map $MPic(X, \varphi) \to H^3(X, \mathbb{Z})$, from Theorem 2 has a kernel which can be used to define an analogue of the Jacobian

$$\begin{array}{ccccc} & & CDiv(X,\varphi) & & \\ & & m\downarrow & & \\ Jac(X,\varphi) & \to & MPic(X,\varphi) & \xrightarrow{c_1} & H^3(X,\mathbb{Z}). \end{array}$$

   In this way $MJac(X, \varphi) \subseteq H^2(X, \mathbb{Z})$ and this can be used to define linear equivalence of coassociatives. Namely, two coassociatives N_1 and N_2 are linearly equivalent if and only if the holonomies of the flat monopole gerbe associated with $N_1 - N_2$ vanish.
3. Given a G_2-manifold (X, φ), we can say it is polarizable (or prequantizable) if $[\varphi] \in H^3(X, \mathbb{Z})$, i.e. is integral. In this case, by fixing a good open cover $\{U_i\}_{i \in I}$, we can define a monopole gerbe $(\mathcal{G}, F)$ with curvature H. This is somewhat similar in spirit to a polarized complex manifold, or a prequantizable symplectic manifold. It remains to see what can we do with this G_2-version.
4. In the case of a complex manifold and a complex line bundle L, the standard integrability theorem guarantees that if A is a connection on L whose curvature has no $(0, 2)$ component, then the bundle is holomorphic, i.e. there are trivializations where the $(0, 1)$-components of the connection forms vanish. We conjecture that the following analogue of this works for closed G_2-structures.

Problem 1. Let $U \subset \mathbb{R}^7$ be a contractible precompact set, equipped with a closed G_2-structure φ and H be a closed 3-form on U with $H_7 = 0$. Is there a 2-form F and a function ϕ, such that $H = dF$ and $F \wedge \psi = *d\phi$?

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