Journal of Geometry and Physics 134 (2018) 38–47

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Transversely holomorphic branched Cartan geometry

Indranil Biswas [a,*], Sorin Dumitrescu [b]

[a] School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India
[b] Université Côte d'Azur, CNRS, LJAD, France

ARTICLE INFO

Article history:
Received 16 March 2018
Accepted 7 August 2018
Available online xxxx

MSC:
53C05
53C12
55R55

Keywords:
Holomorphic foliation
Transverse structure
Cartan geometry

ABSTRACT

In Biswas and Dumitrescu (2018), we introduced and studied the concept of holomorphic *branched Cartan geometry*. We define here a foliated version of this notion; this is done in terms of Atiyah bundle. We show that any complex compact manifold of algebraic dimension d admits, away from a closed analytic subset of positive codimension, a nonsingular holomorphic foliation of complex codimension d endowed with a transversely flat branched complex projective geometry (equivalently, a $\mathbb{C}P^d$-geometry). We also prove that transversely branched holomorphic Cartan geometries on compact complex projective rationally connected varieties and on compact simply connected Calabi–Yau manifolds are always flat (consequently, they are defined by holomorphic maps into homogeneous spaces).

1. Introduction

In the recent article [1], the authors introduced and studied the concept of *branched Cartan geometry* in the complex setting. This concept generalizes to higher dimension the notion of branched (flat) complex projective structure on a Riemann surface introduced and studied by Mandelbaum in [2,3]. This new framework is much more flexible than that of the usual holomorphic Cartan geometries; for example, all compact complex projective manifolds admit branched holomorphic projective structures.

In this paper we deal with a foliated version of branched Cartan geometry. More precisely, we give a definition, in terms of Atiyah bundle, of a branched holomorphic Cartan geometry transverse to a holomorphic foliation. There is a natural curvature tensor which vanishes exactly when the transversely branched Cartan geometry is flat. When this happens, away from the branching divisor, the foliation is transversely modeled on a homogeneous space in the classical sense (see, for example, [4]). The local coordinates with values in the homogeneous space extend through the branching divisor as a ramified holomorphic map (the branching divisor correspond to the ramification set). It should be mentioned that transversely holomorphic affine as well as projective structures for (complex) codimension one foliations are studied extensively (see [5–7] and references therein); such structures are automatically flat.

In Section 3, we use the formalism of Atiyah bundle, to deduce, in the flat case, the existence of a developing map which is a holomorphic map ρ from the universal cover of the foliated manifold into the homogeneous space; the differential $d\rho$ of ρ is surjective on an open dense set of the universal cover, and the foliation on it is given by the kernel of $d\rho$. We also show that any complex compact manifold of algebraic dimension d admits, away from a closed analytic subset of positive codimension, a nonsingular holomorphic foliation of complex codimension d, endowed with a transversely flat branched complex projective geometry (which is same as a $\mathbb{C}P^d$-geometry).

* Corresponding author.
E-mail addresses: indranil@math.tifr.res.in (I. Biswas), dumitres@unice.fr (S. Dumitrescu).

https://doi.org/10.1016/j.geomphys.2018.08.005
0393-0440/

In Section 4 we use characteristic classes to prove a criterion for a holomorphic foliation to admit a branched transversely Cartan geometry. In particular, the criterion asserts that, on compact Kähler manifolds, foliations $\mathcal{F}$ with strictly negative conormal bundle do not admit any branched transversely holomorphic Cartan geometry whose model is the complex affine space (which is same as a holomorphic affine connection).

In Section 5 we consider holomorphic foliations $\mathcal{F}$ on two classes of special manifolds $\widehat{X}$: projective rationally connected manifolds, and simply connected Calabi–Yau manifolds. In both cases, we show that all transversely branched holomorphic Cartan geometries (on the open dense set X of $\widehat{X}$ where the foliation is nonsingular) are necessarily flat and come from a holomorphic map into a homogeneous space with surjective differential at the general point.

2. Foliation and transversely branched Cartan geometry

2.1. Partial connection along a foliation

Let X be a connected complex manifold equipped with a nonsingular holomorphic foliation $\mathcal{F}$; so, $\mathcal{F}$ is a holomorphic subbundle of the holomorphic tangent bundle TX such that the sheaf of holomorphic sections of $\mathcal{F}$ is closed under the Lie bracket operation of vector fields. Let

$$\mathcal{N}_{\mathcal{F}} := TX/\mathcal{F} \longrightarrow X$$

be the normal bundle to the foliation. Let

$$q : TX \longrightarrow N_{\mathcal{F}} \tag{2.1}$$

be the quotient map. There is a natural flat holomorphic partial connection $\nabla^{\mathcal{F}}$ on $N_{\mathcal{F}}$ in the direction of $\mathcal{F}$. We will briefly recall the construction of $\nabla^{\mathcal{F}}$. Given locally defined holomorphic sections s and t of $\mathcal{F}$ and $\mathcal{N}_{\mathcal{F}}$ respectively, choose a locally defined holomorphic section $\widetilde{t}$ of TX that projects to t. Now define

$$\nabla^{\mathcal{F}}_s t = q([s, \widetilde{t}]),$$

where q is the projection in (2.1). it is easy to see that this is independent of the choice of the lift $\widetilde{t}$ of t. Indeed, if $\widehat{t}$ is another lift of t, then $[s, \widetilde{t}-\widehat{t}]$ is a section of $\mathcal{F}$, because $\widetilde{t}-\widehat{t}$ is a section of $\mathcal{F}$. From the Jacobi identity for Lie bracket it follows that the curvature of $\nabla^{\mathcal{F}}$ vanishes identically.

We will define partial connections in a more general context.

Let H be a complex Lie group. Its Lie algebra will be denoted by $\mathfrak{h}$. Let

$$p : E_H \longrightarrow X \tag{2.2}$$

be a holomorphic principal H-bundle on X. This means that E_H is a complex manifold equipped with a holomorphic action

$$p' : E_H \times H \longrightarrow E_H$$

of H, and p is a holomorphic surjective submersion, such that

- $p \circ p' = p \circ p_E$, where $p_E : E_H \times H \longrightarrow E_H$ is the natural projection, and
- the map $p_E \times p' : E_H \times H \longrightarrow E_H \times_X E_H$ is an isomorphism; note that the first condition ensures that the image of $p_E \times p'$ is contained in $E_H \times_X E_H \subset E_H \times E_H$.

Let

$$\mathrm{d}p : TE_H \longrightarrow p^*TX \tag{2.3}$$

be the differential of the map p in (2.2). This homomorphism $\mathrm{d}p$ is surjective because p is a submersion. The kernel of $\mathrm{d}p$ is identified with the trivial vector bundle $E_H \times \mathfrak{h}$ using the action of H on E_H (equivalently, by the Maurer–Cartan form). Consider the action of H on TE_H given by the action of H on E_H. It preserves the sub-bundle kernel($\mathrm{d}p$). Define the quotient

$$\mathrm{ad}(E_H) := \mathrm{kernel}(\mathrm{d}p)/H \longrightarrow X\,.$$

This $\mathrm{ad}(E_H)$ is a holomorphic vector bundle over X. In fact, it is identified with the vector bundle $E_H \times^H \mathfrak{h}$ associated to E_H for the adjoint action of H on $\mathfrak{h}$; this identification is given by the above identification of kernel($\mathrm{d}p$) with $E_H \times \mathfrak{h}$. This vector bundle $\mathrm{ad}(E_H)$ is known as the adjoint vector bundle for E_H. Since the adjoint action of H on $\mathfrak{h}$ preserves its Lie algebra structure, for any $x \in X$, the fiber $\mathrm{ad}(E_H)_x$ is a Lie algebra isomorphic to $\mathfrak{h}$. In fact, $\mathrm{ad}(E_H)_x$ is identified with $\mathfrak{h}$ uniquely up to a conjugation.

The direct image p_*TE_H is equipped with an action of H given by the action of H on TE_H. Note that p_*TE_H is a locally free quasi-coherent analytic sheaf on X. Its H-invariant part

$$(p_*TE_H)^H \subset p_*TE_H$$

is a locally free coherent analytic sheaf on X. The corresponding holomorphic vector bundle is denoted by $\mathrm{At}(E_H)$; it is known as the Atiyah bundle for E_H [8]. It is straight-forward check that the quotient

$$(TE_H)/H \longrightarrow X$$

is identified with $\mathrm{At}(E_H)$. Consider the short exact sequence of holomorphic vector bundles on E_H

$$0 \longrightarrow \mathrm{kernel}(\mathrm{d}p) \longrightarrow \mathrm{T}E_H \xrightarrow{\mathrm{d}p} p^*TX \longrightarrow 0\,.$$

Taking its quotient by H, we get the following short exact sequence of vector bundles on X

$$0 \longrightarrow \mathrm{ad}(E_H) \xrightarrow{\iota''} \mathrm{At}(E_H) \xrightarrow{\widehat{\mathrm{d}p}} TX \longrightarrow 0\,, \tag{2.4}$$

where $\widehat{\mathrm{d}p}$ is constructed from $\mathrm{d}p$; this is known as the Atiyah exact sequence for E_H. Now define the subbundle

$$\mathrm{At}_{\mathcal{F}}(E_H) := (\widehat{\mathrm{d}p})^{-1}(\mathcal{F}) \subset \mathrm{At}(E_H)\,. \tag{2.5}$$

So from (2.4) we get the short exact sequence

$$0 \longrightarrow \mathrm{ad}(E_H) \longrightarrow \mathrm{At}_{\mathcal{F}}(E_H) \xrightarrow{\mathrm{d}'p} \mathcal{F} \longrightarrow 0\,, \tag{2.6}$$

where $\mathrm{d}'p$ is the restriction of $\widehat{\mathrm{d}p}$ in (2.4) to the subbundle $\mathrm{At}_{\mathcal{F}}(E_H)$.

A partial holomorphic connection on E_H in the direction of $\mathcal{F}$ is a holomorphic homomorphism

$$\theta\,:\,\mathcal{F} \longrightarrow \mathrm{At}_{\mathcal{F}}(E_H)$$

such that $\mathrm{d}'p \circ \theta = \mathrm{Id}_{\mathcal{F}}$, where $\mathrm{d}'p$ is the homomorphism in (2.6). Giving such a homomorphism θ is equivalent to giving a homomorphism $\varpi\,:\,\mathrm{At}_{\mathcal{F}}(E_H) \longrightarrow \mathrm{ad}(E_H)$ such that the composition

$$\mathrm{ad}(E_H) \hookrightarrow \mathrm{At}_{\mathcal{F}}(E_H) \xrightarrow{\varpi} \mathrm{ad}(E_H)$$

is the identity map of $\mathrm{ad}(E_H)$, where the inclusion of $\mathrm{ad}(E_H)$ in $\mathrm{At}_{\mathcal{F}}(E_H)$ is the injective homomorphism in (2.6). Indeed, the homomorphisms ϖ and θ uniquely determine each other by the condition that the image of θ is the kernel of ϖ.

Given a partial connection $\theta\,:\,\mathcal{F} \longrightarrow \mathrm{At}_{\mathcal{F}}(E_H)$, and any two locally defined holomorphic sections s_1 and s_2 of $\mathcal{F}$, consider the locally defined section $\varpi([\theta(s_1),\ \theta(s_2)])$ of $\mathrm{ad}(E_H)$ (since $\theta(s_1)$ and $\theta(s_2)$ are H-invariant vector fields on E_H, the Lie bracket $[\theta(s_1),\ \theta(s_2)]$ is also an H-invariant vector field). This defines an $\mathcal{O}_X$-linear homomorphism

$$\mathcal{K}(\theta) \in H^0(X,\ \mathrm{Hom}(\textstyle\bigwedge^2 \mathcal{F},\ \mathrm{ad}(E_H))) = H^0(X,\ \mathrm{ad}(E_H) \otimes \textstyle\bigwedge^2 \mathcal{F}^*)\,,$$

which is called the *curvature* of the connection θ. The connection θ is called flat if $\mathcal{K}(\theta)$ vanishes identically.

A partial connection on E_H induces a partial connection on every bundle associated to E_H. In particular, a partial connection on E_H induces a partial connection on the adjoint bundle $\mathrm{ad}(E_H)$.

Since $\mathrm{At}_{\mathcal{F}}(E_H)$ is a subbundle of $\mathrm{At}(E_H)$, any partial connection $\theta\,:\,\mathcal{F} \longrightarrow \mathrm{At}_{\mathcal{F}}(E_H)$ produces a homomorphism $\mathcal{F} \longrightarrow \mathrm{At}(E_H)$; this homomorphism will be denoted by θ'. Note that from (2.4) we have an exact sequence

$$0 \longrightarrow \mathrm{ad}(E_H) \xrightarrow{\iota'} \mathrm{At}(E_H)/\theta'(\mathcal{F}) \xrightarrow{\widehat{\mathrm{d}p}} TX/\mathcal{F} = \mathcal{N}_{\mathcal{F}} \longrightarrow 0\,, \tag{2.7}$$

where ι' is given by ι'' in (2.4).

Lemma 2.1. *Let θ be a flat partial connection on E_H. Then θ produces a flat partial connection on $\mathrm{At}(E_H)/\theta'(\mathcal{F})$ that satisfies the condition that the homomorphisms in the exact sequence (2.7) are connection preserving.*

Proof. The image of θ defines an H-invariant holomorphic foliation on E_H; let

$$\widetilde{F} \subset TE_H \tag{2.8}$$

be this foliation. Note that the differential dp in (2.3) produces an isomorphism of $\widetilde{F}$ with $p^*\mathcal{F}$. The natural connection on the normal bundle $TE_H/\widetilde{F}$ in the direction of $\widetilde{F}$ is evidently H-invariant (recall that $\mathrm{At}(E_H) = (TE_H)/H$). On the other hand, we have $(TE_H/\widetilde{F})/H = \mathrm{At}(E_H)/\theta'(\mathcal{F})$. Therefore, the above connection on $TE_H/\widetilde{F}$ in the direction of $\widetilde{F}$ descends to a flat partial connection on $\mathrm{At}(E_H)/\theta'(\mathcal{F})$ in the direction on $\mathcal{F}$.

Let s be a holomorphic section of $\mathcal{F}$ defined on an open subset $U \subset X$. Let s' be the unique section of $\widetilde{F}$ over $p^{-1}(U) \subset E_H$ such that $\mathrm{d}p(s') = s$. Let t be a holomorphic section of $\mathrm{kernel}(\mathrm{d}p) \subset TE_H$ over $p^{-1}(U)$. Then the Lie bracket $[s',\ t]$ has the property that $\mathrm{d}p([s',\ t]) = 0$, meaning $[s',\ t]$ is a section of $\mathrm{kernel}(\mathrm{d}p)$. Since $\mathrm{ad}(E_H) = \mathrm{kernel}(\mathrm{d}p)/H$, it now follows that the inclusion of $\mathrm{ad}(E_H)$ in $\mathrm{At}(E_H)/\theta'(\mathcal{F})$ in (2.7) preserves the partial connections on $\mathrm{ad}(E_H)$ and $\mathrm{At}(E_H)/\theta'(\mathcal{F})$ in the direction of $\mathcal{F}$. Since $[s',\ t]$ is a section of $\mathrm{kernel}(\mathrm{d}p)$, it also follows that the projection $\widehat{\mathrm{d}p}$ in (2.7) is also partial connection preserving. □

2.2. Transversely branched Cartan geometry

Let G be a connected complex Lie group and $H \subset G$ a closed complex Lie subgroup. The Lie algebra of G will be denoted by $\mathfrak{g}$. As in (2.2), E_H is a holomorphic principal H-bundle on X. Let

$$E_G = E_H \times^H G \longrightarrow X \tag{2.9}$$

be the principal G-bundle on X obtained by extending the structure group of E_H using the inclusion of H in G. The inclusion of $\mathfrak{h}$ in $\mathfrak{g}$ produces a fiber-wise injective homomorphism of Lie algebras

$$\iota : \mathrm{ad}(E_H) \longrightarrow \mathrm{ad}(E_G), \tag{2.10}$$

where $\mathrm{ad}(E_G) = E_G \times^G \mathfrak{g}$ is the adjoint bundle for E_G. Let θ be a flat partial connection on E_H in the direction of $\mathcal{F}$. So θ induces flat partial connections on the associated bundles E_G, $\mathrm{ad}(E_H)$ and $\mathrm{ad}(E_G)$.

A transversely branched holomorphic Cartan geometry of type (G, H) on the foliated manifold $(X, \mathcal{F})$ is

- a holomorphic principal H-bundle E_H on X equipped with a flat partial connection θ, and
- a holomorphic homomorphism

$$\beta : \mathrm{At}(E_H)/\theta'(\mathcal{F}) \longrightarrow \mathrm{ad}(E_G), \tag{2.11}$$

such that the following three conditions hold:

(1) β is partial connection preserving,
(2) β is an isomorphism over a nonempty open subset of X, and
(3) the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ad}(E_H) & \xrightarrow{\iota'} & \mathrm{At}(E_H)/\theta'(\mathcal{F}) & \longrightarrow & \mathcal{N}_{\mathcal{F}} & \longrightarrow & 0 \\ & & \| & & \downarrow\beta & & \downarrow\overline{\beta} & & \\ 0 & \longrightarrow & \mathrm{ad}(E_H) & \xrightarrow{\iota} & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{ad}(E_G)/\mathrm{ad}(E_H) & \longrightarrow & 0 \end{array} \tag{2.12}$$

where the top exact sequence is the one in (2.7), and ι is the homomorphism in (2.10).

From the commutativity of (2.12) it follows immediately that the homomorphism $\overline{\beta} : \mathcal{N}_{\mathcal{F}} \longrightarrow \mathrm{ad}(E_G)/\mathrm{ad}(E_H)$ in (2.12) is an isomorphism over a point $x \in X$ if and only if $\beta(x)$ is an isomorphism.

Let n be the complex dimension of $\mathfrak{g}$. Consider the homomorphism of nth exterior products

$$\bigwedge^n \beta : \bigwedge^n (\mathrm{At}(E_H)/\theta'(\mathcal{F})) \longrightarrow \bigwedge^n \mathrm{ad}(E_G)$$

induced by β. The homomorphism β fails to be an isomorphism precisely over the divisor of the section $\bigwedge^n \beta$ of the line bundle $\mathrm{Hom}(\bigwedge^n(\mathrm{At}(E_H)/\theta'(\mathcal{F})), \bigwedge^n \mathrm{ad}(E_G))$. This divisor $\mathrm{div}(\bigwedge^n \beta)$ will be called the *branching divisor* for $((E_H, \theta), \beta)$. We will call $((E_H, \theta), \beta)$ a holomorphic Cartan geometry if β is an isomorphism over X.

Take a holomorphic principal H-bundle E_H on X equipped with a flat partial connection θ in the direction of $\mathcal{F}$. Giving a homomorphism β as in (2.11) satisfying the above conditions is equivalent to giving a holomorphic $\mathfrak{g}$-valued one-form ω on E_H satisfying the following conditions:

(1) ω is H-equivariant for the adjoint action of H on $\mathfrak{g}$,
(2) ω vanishes on the foliation $\widetilde{F} \subset TE_H$ in (2.8) given by the image of θ,
(3) the resulting homomorphism $\omega : (TE_H)/\widetilde{F} \longrightarrow E_H \times \mathfrak{g}$ is an isomorphism over a nonempty open subset of E_H, and
(4) the restriction of ω to any fiber of p (see (2.2)) coincides with the Maurer–Cartan form for the action of H on the fiber.

To see that the two descriptions of a transversely branched holomorphic Cartan geometry are equivalent, first recall that $p^*\mathrm{At}(E_H) = TE_H$, and the pullback of $p^*\mathrm{ad}(E_G)$ is identified with the trivial vector bundle $E_H \times \mathfrak{g} \longrightarrow E_H$. Given a homomorphism $\beta : \mathrm{At}(E_H)/\theta'(\mathcal{F}) \longrightarrow \mathrm{ad}(E_G)$ that satisfying the above conditions, the composition

$$TE_H = p^*\mathrm{At}(E_H) \longrightarrow p^*(\mathrm{At}(E_H)/\theta'(\mathcal{F})) \xrightarrow{p^*\beta} p^*\mathrm{ad}(E_G) = E_H \times \mathfrak{g}$$

defines a holomorphic $\mathfrak{g}$-valued one-form ω on E_H that satisfies the above conditions. Conversely, any holomorphic $\mathfrak{g}$-valued one-form ω on E_H that satisfying the above conditions, produces a homomorphism

$$(TE_H)/\widetilde{F} \longrightarrow E_H \times \mathfrak{g}$$

because it vanishes on $\widetilde{F}$. This homomorphism is H-equivariant, so descends to a homomorphism

$$\mathrm{At}(E_H)/\theta'(\mathcal{F}) = ((TE_H)/\widetilde{F})/H \longrightarrow (E_H \times \mathfrak{g})/H = \mathrm{ad}(E_G)$$

over X. This descended homomorphism satisfies the conditions needed to define a transversely branched holomorphic Cartan geometry.

If $\mathcal{F}$ is the trivial foliation (by points) then the previous definition is exactly that of a branched Cartan geometry on X, as given in [1].

3. Connection and developing map

3.1. Holomorphic connection on E_G

Let $((E_H,\ \theta),\ \beta)$ be a transversely branched Cartan geometry of type $(G,\ H)$ on the foliated manifold $(X,\ \mathcal{F})$. We will show that this data produces a holomorphic connection on the principal G-bundle E_G defined in (2.9).

Consider the homomorphism

$$\mathrm{ad}(E_H) \longrightarrow \mathrm{ad}(E_G) \oplus \mathrm{At}(E_H)\,, \quad v \longmapsto (\iota(v),\ -\iota''(v)) \tag{3.1}$$

(see (2.10) and (2.4) for ι and ι'' respectively). The corresponding quotient $(\mathrm{ad}(E_G) \oplus \mathrm{At}(E_H))/\mathrm{ad}(E_H)$ is identified with the Atiyah bundle $\mathrm{At}(E_G)$. The inclusion of $\mathrm{ad}(E_G)$ in $\mathrm{At}(E_G)$ as in (2.4) is given by the inclusion $\mathrm{ad}(E_G) \hookrightarrow \mathrm{ad}(E_G) \oplus \mathrm{At}(E_H)$, $w \longmapsto (w,\ 0)$, while the projection $\mathrm{At}(E_G) \longrightarrow TX$ is given by the composition

$$\mathrm{At}(E_G) \hookrightarrow \mathrm{ad}(E_G) \oplus \mathrm{At}(E_H) \xrightarrow{(0,\widehat{\mathrm{d}p})} TX\,,$$

where $\widehat{\mathrm{d}p}$ is the projection in (2.4).

Consider the subbundle $\theta'(\mathcal{F}) \subset \mathrm{At}(E_H)$ in (2.7). The composition

$$\mathrm{At}(E_H) \longrightarrow \mathrm{At}(E_H)/\theta'(\mathcal{F}) \xrightarrow{\beta} \mathrm{ad}(E_G)\,,$$

where the first homomorphism is the quotient map, will be denoted by β'. The homomorphism

$$\mathrm{ad}(E_G) \oplus \mathrm{At}(E_H) \longrightarrow \mathrm{ad}(E_G)\,, \quad (v,\ w) \longmapsto v + \beta'(w) \tag{3.2}$$

vanishes on the image of $\mathrm{ad}(E_H)$ by the map in (3.1). Therefore, the homomorphism in (3.2) produces a homomorphism

$$\varphi\ :\ \mathrm{At}(E_G) = (\mathrm{ad}(E_G) \oplus \mathrm{At}(E_H))/\mathrm{ad}(E_H) \longrightarrow \mathrm{ad}(E_G)\,. \tag{3.3}$$

The composition

$$\mathrm{ad}(E_G) \hookrightarrow \mathrm{At}(E_G) \xrightarrow{\varphi} \mathrm{ad}(E_G)$$

clearly coincides with the identity map of $\mathrm{ad}(E_G)$. Hence φ defines a holomorphic connection on the principal G-bundle E_G [8]. Note that θ is not directly used in the construction of the homomorphism φ. Let

$$\mathrm{Curv}(\varphi) \in H^0(X,\ \mathrm{ad}(E_G) \otimes \Omega^2_X)$$

be the curvature of the connection φ.

Lemma 3.1. *The curvature* $\mathrm{Curv}(\varphi)$ *lies in the image of the homomorphism*

$$H^0(X,\ \mathrm{ad}(E_G) \otimes \bigwedge\nolimits^2 \mathcal{N}^*_{\mathcal{F}}) \hookrightarrow H^0(X,\ \mathrm{ad}(E_G) \otimes \Omega^2_X)$$

given by the inclusion $q^*\ :\ \mathcal{N}^*_{\mathcal{F}} \hookrightarrow \Omega^1_X$ *(the dual of the projection in* (2.1)*).*

Proof. Let $\widetilde{\theta}$ be the partial connection on E_G induced by the partial connection θ on E_H. Note that $\widetilde{\theta}$ is flat because θ is flat. Since the homomorphism β in (2.11) is partial connection preserving, it follows that the restriction of the connection φ in the direction of $\mathcal{F}$ coincides with $\widetilde{\theta}$. Hence the restriction of φ to $\mathcal{F}$ is flat.

In fact, since β is connection preserving, the contraction of $\mathrm{Curv}(\varphi)$ by any tangent vector of TX lying in $\mathcal{F}$ vanishes. This implies that $\mathrm{Curv}(\varphi)$ is actually a section of $\mathrm{ad}(E_G) \otimes \bigwedge^2 \mathcal{N}^*_{\mathcal{F}}$. $\square$

The transversely branched Cartan geometry $((E_H,\ \theta),\ \beta)$ will be called *flat* if the curvature $\mathrm{Curv}(\varphi)$ vanishes identically.

3.2. The developing map

Assume that $((E_H,\ \theta),\ \beta)$ is flat and X is simply connected. Fix a point $x_0 \in X$ and a point $z_0 \in (E_H)_{x_0}$ in the fiber of E_H over x_0. Using the flat connection φ on E_G and the trivialization of $(E_G)_{x_0}$ given by z_0, the principal G-bundle E_G gets identified with $X \times G$. Using this identification, the inclusion of E_H in E_G produces a holomorphic map

$$\rho\ :\ X \longrightarrow G/H\,. \tag{3.4}$$

If the base point z_0 is replaced by $z_0h \in (E_H)_{x_0}$, where $h \in H$, then the map ρ in (3.4) gets replaced by the composition

$$X \xrightarrow{\rho} G/H \xrightarrow{y \mapsto hy} G/H\,.$$

The map ρ will be called a developing map for $((E_H,\ \theta),\ \beta)$.

The differential of ρ is surjective outside the branching divisor for $((E_H,\ \theta),\ \beta)$. Indeed, the differential $d\rho\ :\ TX \longrightarrow \rho^*T(G/H)$ of ρ is given by the homomorphism $\overline{\beta}$ in (2.12). It was noted earlier that $\overline{\beta}$ fails to be an isomorphism exactly over the branching divisor for $((E_H,\ \theta),\ \beta)$.

Note that ρ is a constant map when restricted to a connected component of a leaf for $\mathcal{F}$, because the connection φ restricted to such a connected component is induced by a connection on E_H (it is induced by the partial connection θ on E_H). In particular, ρ is a constant map if there is a dense leaf for $\mathcal{F}$. In that case, $\mathrm{rank}(\mathcal{N}_{\mathcal{F}}) = \dim \mathfrak{g} - \dim \mathfrak{h} = 0$, so X is the unique leaf.

If X is not simply connected, fix a base point $x_0 \in X$, and let $\psi : \widetilde{X} \longrightarrow X$ be the corresponding universal cover. Considers the pull-back $\widetilde{\mathcal{F}}$ of the foliation $\mathcal{F}$, as well as the pull-back of the transversely branched flat Cartan geometry $((E_H, \theta), \beta)$, to $\widetilde{X}$ using ψ. Then the developing map of the transversely flat Cartan geometry on $(\widetilde{X}, \widetilde{\mathcal{F}})$ is a holomorphic map $\rho : \widetilde{X} \longrightarrow G/H$ (as before, we need to fix a point in $(\psi^* E_H)_{x'_0}$, where $x'_0 \in \widetilde{X}$ is the base point), which is a submersion away from the inverse image, under ψ, of the branching divisor). Moreover, the monodromy of the flat connection on E_G produces a group homomorphism (called monodromy homomorphism) from the fundamental group $\pi_1(X, x_0)$ of X into G, and ρ must be equivariant with respect to the action of $\pi_1(X, x_0)$ by deck-transformation on $\widetilde{X}$ and through the image of the monodromy morphism on G/H. The reader will find more details about this construction in [4].

3.3. Fibrations over a homogeneous space

The standard (flat) Cartan geometry on the homogeneous space $X = G/H$ is given by the following tautological construction.

Let F_H be the holomorphic principal H-bundle on X defined by the quotient map $G \longrightarrow G/H$ (we use the notation F_H instead of E_H because it is a special case which will play a role later). Identify the Lie algebra $\mathfrak{g}$ with the Lie algebra of right-invariant vector fields on G. This produces an isomorphism

$$\beta_{G,H} : \mathrm{At}(F_H) \longrightarrow \mathrm{ad}(F_G) \tag{3.5}$$

and hence a Cartan geometry of type G/H on X (the foliation on G/H (leaves are points) is trivial and there is no branching divisor).

The principal G-bundle

$$F_G := F_H \times^H G \longrightarrow X = G/H ,$$

obtained by extending the structure group of E_H using the inclusion of H in G, is canonically identified with the trivial principal G-bundle $X \times G$. To see this, consider the map

$$G \times G \longrightarrow G \times G , \ (g_1, g_2) \longmapsto (g_1, g_1 g_2) . \tag{3.6}$$

Note that E_G is the quotient of $G \times G$ where any $(g_1 h, g_2)$ is identified with (g_1, g_2), where $g_1, g_2 \in G$ and $h \in H$. Therefore, the map in (3.6) produces an isomorphism of E_G with $X \times G$. The connection on F_G given by the above Cartan geometry of type G/H on $X = G/H$ is the trivial connection on $X \times G$. In particular, the Cartan geometry of type G/H on X is flat.

The above holomorphic $\mathfrak{g}$-valued 1-form on $G = F_H$ will be denoted by $\beta_{G,H}$.

Let X be a connected complex manifold and

$$\gamma : X \longrightarrow G/H$$

a holomorphic map such that the differential

$$d\gamma : TX \longrightarrow T(G/H)$$

is surjective over a nonempty subset of X.

Consider the foliation on X given by the kernel of $d\gamma$. It is a singular holomorphic foliation, which is regular on the dense open set of X where the homomorphism $d\gamma$ is surjective. It extends to a regular holomorphic foliation

$$\mathcal{F} \subset TX' \tag{3.7}$$

on an open subset X' of X of complex codimension at least two (containing the open set where $d\gamma$ is surjective).

Set E_H to be the pullback $\gamma^* F_H$.

Note that we have a holomorphic map $\eta : E_H \longrightarrow F_H$ which is H-equivariant and fits in the commutative diagram

$$\begin{array}{ccc} E_H & \xrightarrow{\eta} & F_H \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\gamma} & G/H \end{array}$$

Notice that, by construction, the H-bundle E_H is trivial along the leaves of $\mathcal{F}$ and hence it inherits a flat partial connection θ along the leafs of the foliation $\mathcal{F}$ constructed in (3.7). Let

$$\theta' : \mathcal{F} \longrightarrow \mathrm{At}(E_H) = \mathrm{At}(\gamma^* F_H) \tag{3.8}$$

be the homomorphism giving this partial connection.

We will show that $(E_H,\ \eta^*\beta_{G,H})$ defines a transversely holomorphic branched flat Cartan geometry of type G/H on the foliated manifold $(X',\ \mathcal{F})$, where $X' \subset X$ is the dense open subset introduced earlier. It is branched over points $x \in X'$ where $d\gamma(x)$ is not surjective.

To describe the above branched Cartan geometry in terms of the Atiyah bundle, first note that $\mathrm{At}(E_H) = \mathrm{At}(\gamma^* F_H)$ coincides with the subbundle of the vector bundle $\gamma^*\mathrm{At}(F_H) \oplus TX$ given by the kernel of the homomorphism

$$\gamma^*\mathrm{At}(F_H) \oplus TX \longrightarrow \gamma^* T(G/H)\,,\ \ (v,\ w) \longmapsto \gamma^* p_{G,H}(v') - d\gamma(w)\,,$$

where $p_{G,H} : \mathrm{At}(F_H) \longrightarrow T(G/H)$ is the natural projection (see (2.4)), while v' is the image of v under the natural map $\gamma^*\mathrm{At}(F_H) \longrightarrow \mathrm{At}(F_H)$, and

$$d\gamma : TX \longrightarrow \gamma^* T(G/H)$$

is the differential of γ.

Notice that the restriction of the homomorphism

$$\gamma^*\mathrm{At}(F_H) \oplus TX \longrightarrow \gamma^*\mathrm{ad}(F_G)\,,\ \ (a,\ b) \longmapsto \gamma^*\beta_{G,H}(a)$$

(see (3.5) for $\beta_{G,H}$) to $\mathrm{At}(\gamma^* F_H) \subset \gamma^*\mathrm{At}(F_H) \oplus TX$ is a homomorphism

$$\mathrm{At}(\gamma^* F_H) \longrightarrow \mathrm{ad}(\gamma^* F_G) = \gamma^*\mathrm{ad}(F_G) = \mathrm{ad}(E_G)\,,$$

which vanishes on $\theta'(\mathcal{F})$, where θ' is constructed in (3.8).

It defines a transversely branched holomorphic Cartan geometry of type G/H on $(X', \mathcal{F})$.

The divisor of X' over which the above branched transversely Cartan geometry of type G/H on X' fails to be a Cartan geometry coincides with the divisor over which the differential $d\gamma$ fails to be surjective.

It was observed earlier that the model Cartan geometry defined by $\beta_{G,H}$ in (3.5) is flat. As a consequence of it, the above branched transversely Cartan geometry of type G/H on X' is flat.

The developing map for this flat branched Cartan geometry on X', is the map γ itself restricted to X'.

The following proposition is proved similarly.

Proposition 3.2. *Let X be connected complex manifold, and let M be a complex manifold endowed with a holomorphic Cartan geometry of type (G, H). Suppose that there exists a holomorphic map $f : X \longrightarrow M$ such that the differential df is surjective on an open dense subset of X. Then the kernel of df defines a holomorphic foliation $\mathcal{F}$ on an open dense subset X' of X of complex codimension at least two. Moreover $\mathcal{F}$ admits a transversely branched holomorphic Cartan geometry of type (G, H), which is flat if and only if the Cartan geometry on M is flat.*

Proof. The proof is the same as above if one considers, instead of $\gamma^* F_H$ and $\gamma^*\beta_{G,H}$, the pull back of the Cartan geometry of M through f. □

3.4. Transversely affine and transversely projective geometry

Let us recall two standard models G/H which are of particular interest: the complex affine and the complex projective geometries.

Consider the semi-direct product $\mathbb{C}^d \rtimes \mathrm{GL}(d, \mathbb{C})$ for the standard action of $\mathrm{GL}(d, \mathbb{C})$ on $\mathbb{C}^d$. This group $\mathbb{C}^d \rtimes \mathrm{GL}(d, \mathbb{C})$ is identified with the group of all affine transformations of $\mathbb{C}^d$. Set $H = \mathrm{GL}(d, \mathbb{C})$ and $G = \mathbb{C}^d \rtimes \mathrm{GL}(d, \mathbb{C})$.

By definition, a given regular holomorphic foliation $\mathcal{F}$ of complex codimension d admits a transversely (branched) *holomorphic affine connection* if it admits a transversely (branched) holomorphic Cartan geometry of type G/H. When the transversely Cartan geometry is flat, we say that $\mathcal{F}$ admits a transversely (branched) *complex affine geometry*.

We also recall that a holomorphic foliation $\mathcal{F}$ of complex codimension d admits a transversely (branched) *holomorphic projective connection* if it admits a (branched) holomorphic Cartan geometry of type $\mathrm{PGL}(d+1, \mathbb{C})/Q$, where $Q \subset \mathrm{PGL}(d+1, \mathbb{C})$ is the maximal parabolic subgroup that fixes a given point for the standard action of $\mathrm{PGL}(d+1, \mathbb{C})$ on $\mathbb{C}P^d$ (the space of lines in $\mathbb{C}^{d+1}$). If the transversely Cartan geometry is flat, we say that $\mathcal{F}$ admits a transversely (branched) *complex projective geometry*.

We have seen in Section 3.3 that any holomorphic map $X \longrightarrow \mathbb{C}P^d$ which is a submersion on an open dense set gives rise to a holomorphic foliation with transversely branched complex projective geometry. Conversely, we have seen in Section 3.2 that on simply connected manifolds, any foliation with transversely branched complex projective geometry is given by a holomorphic map $X \longrightarrow \mathbb{C}P^d$ which is a submersion on an open dense set.

Consider now a complex manifold X of algebraic dimension $a(X) = d$. Recall that the algebraic dimension is the degree of transcendence over $\mathbb{C}$ of the field $\mathcal{M}(X)$ of meromorphic functions on X. It is known that $a(X)$ is at most the complex dimension of X with equality if and only if X is birational to a complex projective manifold (see [9]), known as Moishezon manifolds.

Proposition 3.3. *Suppose that X is a compact complex manifold of algebraic dimension $a(X) = d$. Then, away from an analytic subset of positive codimension, X admits a nonsingular holomorphic foliation of complex codimension d, endowed with a transversely branched complex projective geometry.*

Proof. This is a direct application of the algebraic reduction theorem (see [9]) which asserts that X admits a modification $\widehat{X}$ such that there exists a holomorphic surjective map $f : \widehat{X} \longrightarrow Y$ to a compact complex projective manifold Y of complex dimension d such that

$$f^* : \mathcal{M}(Y) \longrightarrow \mathcal{M}(\widehat{X}) = \mathcal{M}(X)$$

is an isomorphism. Moreover, since Y is projective, there exists a finite algebraic map $\pi : Y \longrightarrow \mathbb{C}P^d$ (see a short proof of this classical fact in [1] Proposition 3.1). Hence we get a holomorphic surjective fibration $\pi \circ f : \widehat{X} \longrightarrow \mathbb{C}P^d$. Now Proposition 3.2 applies. □

4. A topological obstruction

Let X be a compact connected Kähler manifold of complex dimension d equipped with a Kähler form ω. Chern classes will always mean ones with real coefficients. For a torsionfree coherent analytic sheaf V on X, define

$$\text{degree}(V) := (c_1(V) \cup \omega^{d-1}) \cap [X] \in \mathbb{R}. \tag{4.1}$$

The degree of a divisor D on X is defined to be $\text{degree}(\mathcal{O}_X(D))$.

Fix an effective divisor D on X. Fix a holomorphic principal H-bundle E_H on X.

Proposition 4.1. *Let $\mathcal{F}$ be a holomorphic nonsingular foliation on the Kähler manifold X. Assume that $\mathcal{F}$ admits a transversely branched Cartan geometry of type G/H with principal H-bundle E_H and branching divisor D. Then* $\text{degree}(\mathcal{N}^*_{\mathcal{F}}) - \text{degree}(D) = \text{degree}(\text{ad}(E_H))$.

In particular, if $D \neq 0$, then $\text{degree}(\mathcal{N}^*_{\mathcal{F}}) > \text{degree}(\text{ad}(E_H))$.

Proof. Let k be the complex dimension of the transverse model geometry G/H.

Recall that the homomorphism $\overline{\beta} : \mathcal{N}_{\mathcal{F}} \longrightarrow \text{ad}(E_G)/\text{ad}(E_H)$ in (2.12) is an isomorphism over a point $x \in X$ if and only if $\beta(x)$ is an isomorphism.

The branching divisor D coincides with the vanishing divisor of the holomorphic section $\bigwedge^k \overline{\beta}$ of the holomorphic line bundle $\bigwedge^k(\mathcal{N}^*_{\mathcal{F}}) \otimes \bigwedge^k(\text{ad}(E_G)/\text{ad}(E_H))$. We have

$$\begin{aligned} \text{degree}(D) &= \text{degree}(\bigwedge^k(\text{ad}(E_G)/\text{ad}(E_H)) \otimes \bigwedge^k(\mathcal{N}^*_{\mathcal{F}})) \\ &= \text{degree}(\text{ad}(E_G)) - \text{degree}(\text{ad}(E_H)) + \text{degree}(\mathcal{N}^*_{\mathcal{F}}). \end{aligned} \tag{4.2}$$

Recall that E_G has a holomorphic connection ϕ (see (3.3)). It induces a holomorphic connection on $\text{ad}(E_G)$. Hence we have $c_1(\text{ad}(E_G)) = 0$ [8, Theorem 4], which implies that $\text{degree}(\text{ad}(E_G)) = 0$. Therefore, from (4.2) it follows that

$$\text{degree}(\mathcal{N}^*_{\mathcal{F}}) - \text{degree}(D) = \text{degree}(\text{ad}(E_H)). \tag{4.3}$$

If $D \neq 0$, then $\text{degree}(D) > 0$. Hence in that case (4.3) yields $\text{degree}(\mathcal{N}^*_{\mathcal{F}}) > \text{degree}(\text{ad}(E_H))$. □

Corollary 4.2.

(i) If $\text{degree}(\mathcal{N}^*_{\mathcal{F}}) < 0$, *then there is no branched transversely holomorphic affine connection on X transversal to $\mathcal{F}$.*
(ii) If $\text{degree}(\mathcal{N}^*_{\mathcal{F}}) = 0$, *then for every branched transversely holomorphic affine connection on X transversal to $\mathcal{F}$ the branching divisor on X is trivial.*

Proof. Recall that a transversely branched holomorphic affine connection on X transversal to $\mathcal{F}$ is a transversely branched holomorphic Cartan geometry on X of type G/H, where $H = \text{GL}(d, \mathbb{C})$ and $G = \mathbb{C}^d \rtimes \text{GL}(d, \mathbb{C})$. The homomorphism

$$\text{M}(d, \mathbb{C}) \otimes \text{M}(d, \mathbb{C}) \longrightarrow \mathbb{C}, \quad A \otimes B \longmapsto \text{trace}(AB)$$

is nondegenerate and $\text{GL}(d, \mathbb{C})$-invariant. In other words, the Lie algebra $\mathfrak{h}$ of $H = \text{GL}(d, \mathbb{C})$ is self-dual as an H-module. Hence we have $\text{ad}(E_H) = \text{ad}(E_H)^*$, in particular, the equality

$$\text{degree}(\text{ad}(E_H)) = 0$$

holds. Hence from Proposition 4.1,

$$\text{degree}(\mathcal{N}^*_{\mathcal{F}}) = \text{degree}(D). \tag{4.4}$$

As noted before, for a nonzero effective divisor D we have $\text{degree}(D) > 0$. Therefore, the corollary follows from (4.4). □

5. Flatness of the transverse geometry on some special varieties

In this section we consider holomorphic foliations $\mathcal{F}$ on projective rationally connected manifolds and on simply connected Calabi–Yau manifolds $\widehat{X}$. In both cases, we show that the only transversely branched holomorphic Cartan geometries, on the open dense set X of $\widehat{X}$ where the foliation is nonsingular, are necessarily flat and come from a holomorphic map into a homogeneous space (as described in Section 3.3).

5.1. Rationally connected varieties

Let $\widehat{X}$ be a smooth complex projective rationally connected variety. Let $X \subset \widehat{X}$ be a Zariski open subset such that the complex codimension of the complement $\widehat{X} \setminus X$ is at least two. Take a nonsingular foliation

$$\mathcal{F} \subset TX$$

on X. Let $((E_H, \theta), \beta)$ be a transversely branched holomorphic Cartan geometry of type (G, H) on the foliated manifold $(X, \mathcal{F})$.

There is a nonempty open subset of X which can be covered by smooth complete rational curves C such that the restriction $(TX)|_C$ is ample. On a curve any holomorphic connection is flat. Further, since a rational curve is simply connected, any holomorphic bundle on it equipped with a holomorphic connection is isomorphic to the trivial bundle equipped with the trivial connection. If $(TX)|_C$ is ample, then $H^0(C, (\Omega^2_X)|_C) = 0$. Therefore, any holomorphic bundle on X with a holomorphic connection has the property that the curvature vanishes identically. In particular, the transversely branched holomorphic Cartan geometry $((E_H, \theta), \beta)$ must be flat. Further, X is simply connected because $\widehat{X}$ is so. Therefore, the transversely branched holomorphic Cartan geometry $((E_H, \theta), \beta)$ is the pullback, of the standard Cartan geometry on G/H of type (G, H), by a developing map $f : X \longrightarrow G/H$. The foliation is given by kernel(df).

This yields the following:

Corollary 5.1. *Let $\widehat{X}$ be a smooth complex projective rationally connected variety, and let $\mathcal{F}$ be a holomorphic nonsingular foliation of positive codimension defined on a Zariski open subset X of complex codimension at least two in $\widehat{X}$. Then there is no transversely branched Cartan geometry, with model a nontrivial analytic affine variety G/H, on X transversal to $\mathcal{F}$. In particular, there is no transversely holomorphic affine connection on X transversal to $\mathcal{F}$.*

Proof. Assume, by contradiction, that there is a transversely branched Cartan geometry on X whose model is an analytic affine variety G/H. By the above observations, the branched Cartan geometry is necessarily flat and is given by a holomorphic developing map $f : X \longrightarrow G/H$. Since the target G/H is an affine analytic variety, Hartog's theorem says that f extends to a holomorphic map $\widehat{f} : \widehat{X} \longrightarrow G/H$. Now, as $\widehat{X}$ is compact, and G/H is affine, $\widehat{f}$ must be constant: a contradiction; indeed, as for f, the differential of $\widehat{f}$ is injective on $TX/\mathcal{F}$ at a general point of X. $\square$

Notice that if G is a complex linear algebraic group and H a closed reductive algebraic subgroup, then G/H is an affine analytic variety (see Lemma 3.32 in [10]).

5.2. Simply connected Calabi–Yau manifolds

Let $\widehat{X}$ be a simply connected compact Kähler manifold with $c_1(\widehat{X}) = 0$. As before, $X \subset \widehat{X}$ is a dense open subset such that the complement $\widehat{X} \setminus X$ is a complex analytic subset of complex codimension at least two. Take a nonsingular foliation

$$\mathcal{F} \subset TX$$

on X. Take a complex Lie group G such that there is a holomorphic homomorphism $G \longrightarrow \mathrm{GL}(n, \mathbb{C})$ with the property that the corresponding homomorphism of Lie algebras is injective.

Let $((E_H, \theta), \beta)$ be a transversely branched holomorphic Cartan geometry of type (G, H) on the foliated manifold $(X, \mathcal{F})$. Consider the holomorphic connection φ on E_G over X (see (3.3)). The principal G-bundle E_G extends to a holomorphic principal G-bundle $\widehat{E}_G$ over $\widehat{X}$, and the connection φ extends to a holomorphic connection $\widehat{\varphi}$ on $\widehat{E}_G$ [11, Theorem 1.1]. We know that $\widehat{E}_G$ is the trivial holomorphic principal G-bundle, and $\widehat{\varphi}$ is the trivial connection [1, Theorem 6.2]. Also, X is simply connected because $\widehat{X}$ is so. Therefore, the transversely branched holomorphic Cartan geometry $((E_H, \theta), \beta)$ is the pullback, of the standard Cartan geometry on G/H of type (G, H), by a developing map $f : X \longrightarrow G/H$. The foliation is given by kernel(df).

As before we have the following:

Corollary 5.2. *Let $\widehat{X}$ be a simply connected Calabi–Yau manifold, and let $\mathcal{F}$ be a holomorphic nonsingular foliation of positive codimension defined on a Zariski open subset X of complex codimension at least two in $\widehat{X}$. Then there is no transversely branched Cartan geometry, with model a nontrivial analytic affine variety G/H, on X transversal to $\mathcal{F}$. In particular, there is no transversely holomorphic affine connection on X transversal to $\mathcal{F}$.*

Its proof is identical to that of Corollary 5.1.

References

[1] I. Biswas, S. Dumitrescu, Branched holomorphic Cartan geometries and Calabi–Yau manifolds, Int. Math. Res. Not. (2018). http://dx.doi.org/10.1093/imrn/rny003.
[2] R. Mandelbaum, Branched structures on Riemann surfaces, Trans. Amer. Math. Soc. 163 (1972) 261–275.
[3] R. Mandelbaum, Branched structures and affine and projective bundles on Riemann surfaces, Trans. Amer. Math. Soc. 183 (1973) 37–58.
[4] P. Molino, Riemannian Foliations, in: Progress in Mathematics, vol. 73, Birkhäuser Boston, Inc., Boston, MA, 1988. Translated from the French by Grant Cairns. With appendices by Cairns, Y. Carrière, É. Ghys, E. Salem and V. Sergiescu.
[5] B.A. Scárdua, Transversely affine and transversely projective holomorphic foliations, Ann. Sci. Éc. Norm. Supér. 30 (1997) 169–204.
[6] F. Loray, J.V. Pereira, F. Touzet, Representations of quasi-projective groups, flat connections and transversely projective foliations, J. Éc. Polytech. Math. 3 (2016) 263–308.
[7] G. Cousin, J.V. Pereira, Transversely affine foliations on projective manifolds, Math. Res. Lett. 21 (2014) 985–1014.
[8] M.F. Atiyah, Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. 85 (1957) 181–207.
[9] K. Ueno, Classification theory of algebraic varities and compact complex spaces, in: Lect. Notes Math., vol. 439, Springer-Verlag, 1975.
[10] B. McKay, Extension phenomena for holomorphic geometric structures, SIGMA 5 (2009) 58.
[11] I. Biswas, A criterion for holomorphic extension of principal bundles, Kodai Math. J. 34 (2011) 71–78.