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Diophantine equations, Platonic solids, McKay correspondence, equivelar maps and Vogel's universality

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ABSTRACT

We notice that one of the Diophantine equations, $knm = 2kn + 2km + 2nm$, arising in the universality originated Diophantine classification of simple Lie algebras, has interesting interpretations for two different sets of signs of variables. In both cases it describes "regular polyhedra" with k edges in each vertex, n edges of each face, with total number of edges $|m|$, and Euler characteristics $\chi = \pm 2$. In the case of negative m this equation corresponds to $\chi = 2$ and describes true regular polyhedra, Platonic solids. The case with positive m corresponds to Euler characteristic $\chi = -2$ and describes the so called equivelar maps (charts) on the surface of genus 2. In the former case there are two routes from Platonic solids to simple Lie algebras—abovementioned Diophantine classification and McKay correspondence. We compare them for all solutions of this type, and find coincidence in the case of icosahedron (dodecahedron), corresponding to E_8 algebra. In the case of positive k, n and m we obtain in this way the interpretation of (some of) the mysterious solutions (Y-objects), appearing in the Diophantine classification and having some similarities with simple Lie algebras.

1. Introduction

Since ancient time natural numbers have been suggested as a basic notion in the construction of our knowledge about nature. However, it is rare when they are the part of the basis of construction of a given mathematical or physical theory. In this paper we consider several such cases and observe that they are based on the same Diophantine equation. Moreover, since two of these cases are connected to simple Lie algebras, we are naturally led to a comparison of these two superficially disjoint theories.

The focus of the present paper is the following Diophantine equation

$$\frac{1}{k}+\frac{1}{n}+\frac{1}{m}=\frac{1}{2}, \quad k, n, m \in \mathbb{Z} \setminus 0 \tag{1}$$

or in more general form, which allows zero values of integers k, n, m:

$$knm = 2kn + 2km + 2nm. \tag{2}$$

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Table 1
McKay correspondence and Diophantine classification.

Solutions (k, n, m)	Platonic solids	Subgroups of $SU(2)$	McKay correspondence	Diophantine classification
$(5, 3, -30)$	Icosahedron			
$(3, 5, -30)$	Dodecahedron	$2I$, $\|2I\| = 120$	E_8	E_8
$(4, 3, -12)$	Cube			
$(3, 4, -12)$	Octahedron	$2O$, $\|2O\| = 48$	E_7	E_6
$(3, 3, -6)$	Tetrahedron	$2T$, $\|2T\| = 24$	E_6	$SO(8)$
$(2, n, -n)$	n-polygon	C_n, $\|C_n\| = n$	A_{n-1}	A_n
		C_{2n}, $\|C_{2n}\| = 2n$	A_{2n-1}	
		BD_{2n}, $\|BD_{2n}\| = 4n$	D_{n-2}	
$(0, 0, 0)$				$D_{2,1,\lambda}$

We would like to point out that this equation appears in three circumstances, depending particularly on the signs of integers k, n, m. In two of them there are (different) routes from this equation to simple Lie algebras.

One route is the famous McKay correspondence [1]. It is well-known that solutions of Eq. (1) with $(+ + -)$ signs of variables describe Platonic solids (see below in Section 2). Take the invariance subgroup of a given Platonic solid (it is finite subgroup of the group $SO(3)$). Lift it to the group $SU(2)$ by double-covering map

$$1 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 1, \tag{3}$$

and assign to this subgroup of $SU(2)$ by McKay procedure the simple Lie algebra from the list of ADE algebras (see Section 3). Note that one has to consider also degenerate "Platonic solids", and take into account different liftings of groups. All that is briefly described in Section 3.

An other route from Diophantine equation (1), with the same set of signs, to simple Lie algebras is given by recently developed [2] Diophantine classification of simple Lie algebras, based on Vogel's universality [3,4] and Deligne's conjecture on exceptional simple Lie algebras [5]. This is briefly described in Section 4.

In Section 5 we compare these two routes from solutions of Diophantine equation (1) to simple Lie algebras, and find several common features and differences.

Finally, we discuss relation of Diophantine equations (1) with the theory of equivelar maps [6–8] on orientable surfaces of genus two. They appear to correspond to the same Eq. (1) with $(+ + +)$ signs of variables. In Diophantine classification this case corresponds to mysterious Y-objects, which have certain similarity with simple Lie algebras, but up to now were not identified with any known objects. This is discussed in Section 6.

2. Platonic solids' Diophantine equation

Consider Platonic solid with number of edges of any face r, number of edges at any vertex n, total number of edges E, total number of vertices V, and total number of faces F. We have

$$nV = 2E\,, \quad rF = 2E\,.$$

Then Euler's theorem

$$V - E + F = 2$$

can be rewritten as

$$\frac{1}{r} + \frac{1}{n} - \frac{1}{E} = \frac{1}{2}. \tag{4}$$

This is the particular case of Diophantine equation (1) with the special choice $(+ + -)$ of signs of integers k, n, m.

Solutions (r, n, E) of Eq. (4) are:

- $(5, 3, 30)$ or $(3, 5, 30)$—dodecahedron or icosahedron,
- $(4, 3, 12)$ or $(3, 4, 12)$—cube (hexahedron) or octahedron,
- $(3, 3, 6)$—tetrahedron,
- $(2, n, n)$ (or, the same, $(r, 2, r)$)—regular n-polygon.

This information is listed in the first and second column of Table 1.

3. McKay correspondence

McKay correspondence assigns to finite subgroups of $SU(2)$ group Dynkin diagrams of some simple Lie algebras in the following way. Let G be an arbitrary finite subgroup of the group $SU(2)$ and let V be the restriction of 2-dimensional

Table 2
Simple Lie algebras on Vogel's plane.

Algebra	α	β	γ
$sl(N)$	-2	2	N
$so(N)$	-2	4	$N-4$
$sp(N)$	-2	1	$N/2+2$
$Exc(n)$	-2	$n+4$	$2n+4$

representation of $SU(2)$ on that subgroup G. Let $\{V_i\}$ be the set of all irreducible representations of group G, including trivial one. Then consider decomposition

$$V \otimes V_i = \sum_j m_{ij} V_j. \tag{5}$$

One can prove that for all pairs (ij), m_{ij} are symmetric, i.e. $m_{ij} = m_{ji}$, and that coefficients m_{ij} are equal to 0 or 1. Thus one comes to graphs Γ_G with vertices corresponding to spaces V_i and edges between vertices V_i, V_j iff $m_{ij} \neq 0$. These graphs appear to be Dynkin diagrams of all affine untwisted Kac–Moody algebras of types $\hat{A}$, $\hat{D}$, $\hat{E}$, (McKay, 1980, [1]).

Particularly, among finite groups G there are subgroups of $SU(2)$ which are double coverings (through the double covering $SU(2) \to SO(3)$) of finite subgroups of $SO(3)$ which are groups of rotational symmetries of Platonic solids. These are

- $2I$ —binary icosahedral group, which is double covering of icosahedral group I—the group of rotational symmetries of icosahedron and dodecahedron. It has $2 \times 60 = 120$ elements.
- $2O$ —binary octahedral group, double covering of octahedral group O—the group of rotational symmetries of cube and octahedron. It has $2 \times 24 = 48$ elements.
- $2T$ —binary tetrahedral group, double covering (extension) of tetrahedral group T—the group of rotational symmetries of tetrahedron. It has $2 \times 12 = 24$ elements.

Besides these three groups, there are only two families of finite subgroups of $SU(2)$: C_n and BD_{2n}. They correspond to degenerate Platonic solids, namely regular n-polygons, which can be considered as "polyhedra" with two faces, n vertices and n edges.

- C_n —cyclic group, preimage of cyclic group C_n of rotations of regular n-polygon, if n is even, or C_n—preimage of cyclic group C_n of rotations of regular n-polygon, if n is odd, or C_{2n} —preimage of cyclic group C_n of rotations of regular n-polygon, if n is odd.
- $2D_{2n} = BD_{2n}$, binary dihedral group BD_{2n}, double covering of dihedral group D_{2n} of rotations and reflections of n-polygon. (Reflections of n-polygon can be considered as rotation in 3-dimensional space.) Group D_{2n} has $n+n = 2n$ elements and group BD_{2n} has $2 \times 2n = 4n$ elements.

McKay correspondence assigns to all these subgroups simple Lie algebras whose Dynkin diagrams are defined by relation (5). They are listed in third and fourth columns of Table 1.

4. Diophantine classification of simple Lie algebras

Now turn to the Diophantine classification of simple Lie algebras obtained in [2]. The idea is based on the analysis of the so called universal character of adjoint representation.

Recall briefly the main ideas of the universality approach to simple Lie algebras [3,4]. Vogel plane is a space which provides coordinatization of simple Lie algebras. It is 2-dimensional projective space $\mathbb{C}P^2$ factorized by the action of group S_3 of permutations of homogeneous coordinates (α, β, γ) on $\mathbb{C}P^2$. Functions on Vogel plane are functions on coordinates (α, β, γ), which are scaling invariant and symmetric. To every simple Lie algebra corresponds the separate point on Vogel plane. The values of Vogel parameters (defined up to a rescaling and permutations) for all simple Lie algebras are given in Table 2. All exceptional algebras belong to the line $Exc(n)$ and are parameterized by numbers $n = -2/3, 0, 1, 2, 4, 8$ for exceptional algebras G_2, D_4, F_4, E_6, E_7 and E_8, respectively.

We say that a function f universalizes some quantity if f is an 'reasonable' function on Vogel plane which is equal to this quantity at the points of Vogel plane corresponding to given simple Lie algebras (see Table 2). For example consider such a quantity as dimension of Lie algebra. One can see that function

$$d(\alpha, \beta, \gamma) = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma}\,, \quad \text{where} \quad t = \alpha + \beta + \gamma\,,$$

is universal function for dimension of Lie algebra. The value of this functions at coordinates of an arbitrary simple Lie algebra is equal to the dimension of that simple Lie algebra (see Table 2).

We come to another very important universal function considering such a quantity as the values of the character of adjoint representation at the Weyl line. Namely, let $\mathfrak{g}$ be a simple Lie algebra, and let $\chi_{ad}^{(\mathfrak{g})}$ be character of its adjoint representation. Let

Table 3
Points in Vogel's plane: series.

$\mathfrak{g}$	α, β, γ	k, n, m
$\mathfrak{sl}(n+1)$	$-2, 2, n+1$	$-n, n, 2$
$D_{2,1,\lambda}$	$\alpha+\beta+\gamma=0$	0,0,0

$\{\mu\}$ be a set of roots of Lie algebra $\mathfrak{g}$. Then consider Weyl vector which is equal to half-sum of all positive roots: $\rho = \frac{1}{2}\sum_{\mu>0}\mu$. Consider a function $f(x) = f_{\mathfrak{g}}(x) = \chi_{ad}^{(\mathfrak{g})}(x\rho)$. One can see that for this function the following relation holds:

$$f_{\mathfrak{g}}(x) = \chi_{ad}^{(\mathfrak{g})}(x\rho) = r + \sum_{\mu} e^{x(\mu,\rho)}, \tag{6}$$

where r is rank of algebra $\mathfrak{g}$.

In the papers [9,10] it was suggested the universalization of function (6):

$$f(x) = f(x|\alpha,\beta,\gamma) = \frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(x\frac{\alpha}{4})}\frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})}\frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})}. \tag{7}$$

The values of this function at points of Vogel plane corresponding to an arbitrary simple Lie algebra $\mathfrak{g}$ are equal to character of adjoint representation of the algebra $\mathfrak{g}$ on Weyl line:

$$\chi_{ad}^{(\mathfrak{g})}(x\rho) = f(x|\alpha,\beta,\gamma). \tag{8}$$

This function is very important for construction of universal expression for volumes of groups and partition function of Chern–Simons theory (see [11]). Moreover it turns out that this function can be used for Diophantine classification of simple Lie algebras, [2]. Let us briefly recall results of [2].

Left hand side of Eq. (8) is the finite sum of exponents, which happens if all possible poles of right hand side of this equation are cancelled by the zeros of numerator at the points of Vogel plane corresponding to simple Lie algebra. One can put things upside down and seek all points on the Vogel's plane for which this happens. One of the possible patterns of cancellation is the following

$$\begin{cases} 2t-\alpha = (k-1)\alpha \\ 2t-\beta = (n-1)\beta, \\ 2t-\gamma = (m-1)\gamma \end{cases} \qquad t = \alpha+\beta+\gamma, \tag{9}$$

where k, n, m are integers. Generally there are seven such patterns. The pattern (9) requires that determinant of corresponding 3×3 matrix vanishes. Thus we come to the condition

$$knm = 2kn + 2km + 2nm. \tag{10}$$

We obtain our main Eq. (1) in a more general form (2). If condition (10) is obeyed and all integers are non-zero, then solutions of Eqs. (9) are

$$\alpha = \frac{2t}{k},\ \beta = \frac{2t}{n},\ \gamma = \frac{2t}{m}. \tag{11}$$

Besides that, the non-singular form of Diophantine equation (1), Eq. (10), has solution $(0, 0, m)$, with an arbitrary m. Solution with $m = 0$ is particularly interesting, and it is the most general in a certain sense. It corresponds to an object with $E = 0$, and hence $V + F = 2$. Corresponding solution for Vogel's parameters is an arbitrary triple α, β, γ with only restriction $\alpha+\beta+\gamma = 0$. This solution corresponds to superalgebra $D_{2,1,\lambda}$, see [3].

The list of solutions, with corresponding Vogel's parameters and simple Lie algebras, is given in Tables 3 and 4, series solutions in Table 3, and isolated solutions in Table 4. Simple Lie (super)algebras, corresponding, in Diophantine classification, to solutions from the first column of Table 1, are presented in the last, fifth column of Table 1.

5. Comparison of McKay correspondence and Diophantine classification

Now we turn to comparison of connection between solutions and simple Lie algebras in McKay correspondence and in Diophantine classification. All necessary information is already combined in Table 1, so we have to look on its rows.

We first notice that both approaches combine Platonic solids in dual pairs: dodecahedron with icosahedron, cube with octahedron, and tetrahedron is alone as it is self-dual. Duality is standard, including vertex $\leftrightarrow$ face correspondence, etc.

The corresponding algebra is the same in both routes, in the first case of dodecahedron/icosahedron solution $(5, 3, -30)$, and is the largest of exceptional algebras, E_8. Two other cases give different outputs in McKay and Diophantine classification: solution $(4, 3, -12)$ gives E_6 (instead of E_7 in McKay correspondence), solution $(3, 3, -6)$ gives $SO(8)$ (instead of E_6 in McKay correspondence).

Table 4
Isolated solutions.

$k\,n\,m$	$\alpha\beta\gamma$	Dim	Rank	Algebra
5 3 −30	−6 −10 1	248	8	$\mathfrak{e}_8$
4 3 −12	−3 −4 1	78	6	$\mathfrak{e}_6$
3 3 −6	−2 −2 1	28	4	$\mathfrak{so}(8)$
1 −4 −4	4 −1 −1	0	0	$0d_3$
1 −3 −6	6 −2 −1	0	0	$0d_4$
6 6 6	1 1 1	−125	−19	Y_1
10 5 5	1 2 2	−144	−14	Y_{10}
8 8 4	1 1 2	−147	−17	Y_{11}
12 6 4	1 2 3	−165	−13	Y_{15}
20 5 4	1 4 5	−228	−10	Y_{29}
12 12 3	1 1 4	−242	−18	Y_{31}
15 10 3	2 3 10	−252	−8	Y_{35}
18 9 3	1 2 6	−272	−14	Y_{38}
24 8 3	1 3 8	−322	−12	Y_{43}
42 7 3	1 6 14	−492	−10	Y_{47}

Next we see that Diophantine classification choose one of two series corresponding to polygon solution $(2, n, -n)$ in McKay correspondence, namely A_n series (i.e. $sl(n+1)$ algebras). In McKay correspondence for that degenerate "Platonic solid" we get either A_{n-1}, A_{2n-1} or D_{2n}. So, there is no exact coincidence for polygon solution.

6. Equivelar maps and solutions with positive k, n, m.

In Eq. (4) we considered solutions of Diophantine equation (1) with signs $(++-)$. These solutions correspond to Platonic solids. There is a number of solutions of Eq. (1) with 'wrong', i.e. all positive, signs. They are presented in the last 10 lines of Table 4, and correspond to the so called Y-objects [2]. E.g. triple $(6, 6, 6)$ is a solution of Eq. (1), and it cannot be interpreted in terms of Platonic solids. However, we can get a reasonable interpretation of these solutions assigning to them so called equivelar maps on double torus (compact Riemannian surface of genus $g = 2$, and Euler characteristics $\chi = -2$).

Equivelar map on compact Riemann surface is [6] polyhedral map with faces all having equal (say p) edges and vertices all having equal (say q) number of edges. Initially theory of equivelar maps were developed under the name of *regular maps* in [7,8]. Another names for equivelar maps are: *equivelar* $\{p, q\}$ *maps*, or simply $\{p, q\}$ *maps*, where $\{p, q\}$ is Schläfli symbol, and *locally regular maps* (see below).

In the same way as for Eq. (4) one immediately derives for an arbitrary equivelar map on double torus an equation

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{E} = \frac{1}{2}. \tag{12}$$

Thus we come to Eq. (1) with positive k, n, m. Correspondingly, we get a connection of Y-objects (which have some features of simple Lie algebras, see [2]) with equivelar maps on genus two surfaces.

Note that there is a notion of *regular maps*, which are equivelar maps with additional transitivity requirement on automorphism group of the graph of the map. Actually transitivity itself already implies that the map is equivelar. (See [12,13] for definitions and discussion.) Accordingly, another name for equivelar maps is *locally regular maps*. Among solutions of Eq. (12), listed in Table 4, only Y_1, Y_{10}, Y_{11}, Y_{15} and Y_{43} give rise to regular maps (see Table 9 in [12]).

7. Conclusion

In this paper we present some observations, which connect Diophantine equations (1), (2) with Platonic solids, McKay correspondence, equivelar maps and Diophantine classification of simple Lie algebras. Latter, in turn, is based on Vogel's universality approach. It appears that there are two routes from these equations to simple Lie algebras, and they have some similar features and some differences. Particularly, maximal exceptional E_8 algebra has the same Diophantine solution origin in both routes. We also observe the connection of these equations with equivelar and regular maps on surface of genus 2 (double tori), and in this way get some interpretation of Y-objects. These objects share some features with simple Lie algebras.

One can ask on the similar interpretation of other six equations [2] of Diophantine classification:

$$kmn = mn + 2kn + 2km \tag{13}$$

$$kmn = mn + 2kn + 2km + 2n - 2k \tag{14}$$

$$kmn = mn + kn + 2km + 3n + 2k \tag{15}$$

$$kmn = mn + 2kn + 2km + 2n + 2m - 3k - 5 \tag{16}$$

$$kmn = 2mn + 2kn + 2km - 2n - 3m \tag{17}$$

$$kmn = 2mn + 2kn + 2km - 2n - 2m - 2k + 5. \tag{18}$$

Origin of these equations are different patterns of cancellation of zeros in denominator and numerator in universal character (7), so that final answer is regular function in the entire complex x plane, as is the case for all simple Lie algebras. One can try to compare them with Arthur Cayley's [14] form of Euler theorem for some polyhedra:

$$d_v V - E + d_f F = 2D, \tag{19}$$

where d_v, d_f, D are called vertex figure density, face density and density, respectively.

Another possibly relevant remark is that for a given number of edges, vertices and faces, one can construct another polyhedra, sometimes with the same Euler characteristics, so called Kepler–Poinsot polyhedra [14,15], which however are not convex.

Hopefully on the bases of observations in present work one can extend the area of the common Diophantine equations origin of different objects, such as simple Lie algebras, equivelar maps on surfaces of different genus, etc. Another important direction is an interpretation (identification) of Y-objects, and their deeper understanding, which is an interesting challenge.

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