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On cohomology of filiform Lie superalgebras

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Abstract: Suppose the ground field $\mathbb{F}$ is an algebraically closed field of characteristic different from 2, 3. We determine the Betti numbers and make a decomposition of the associative superalgebra of the cohomology for the model filiform Lie superalgebra. We also describe the associative superalgebra structures of the (divided power) cohomology for some low-dimensional filiform Lie superalgebras.

Keywords: filiform Lie superalgebra; Betti number; associative superalgebra

Mathematics Subject Classification 2010: 17B30, 17B56

0 Introduction

In 1970, in the study of the reducibility of the varieties of nilpotent Lie algebras, Vergne introduced the concept of filiform Lie algebras and showed the fact that every filiform Lie algebra can be obtained by an infinitesimal deformation of the model filiform Lie algebra L_n (see [1]). Since then, the study of the filiform Lie algebras, especially the model filiform Lie algebra, has become an important subject. Many conclusions on cohomology of the model filiform Lie algebra with coefficients in the trivial module have been obtained. For example, the Betti numbers for L_n with coefficients in the trivial module over a field of characteristic zero have been calculated in [2–4]. A result, in [5], states that the filiform Lie algebras L_n and $\mathfrak{m}_2(n)$ have the same Betti numbers over a field of characteristic two, which is different from the case of characteristic zero. Moreover, the first three Betti numbers of L_n and $\mathfrak{m}_2(n)$ over $\mathbb{Z}_2$ have been calculated in [6]. As what happens in the Lie case, every filiform Lie superalgebra can be obtained by an infinitesimal deformation of the model filiform Lie superalgebra $L_{n,m}$. Many conclusions on cohomology of the model filiform Lie superalgebra with coefficients in the adjoint module have been obtained. For example, Khakimdjanov and Navarro gave a complete description of the second cohomology of $L_{n,m}$ with coefficients in the adjoint module in [7–11]. The first cohomology of $L_{n,m}$ with coefficients in the adjoint module has been described in [12] by calculating the derivations. However, in the trivial module case, less of work is done for $L_{n,m}$.

Throughout this paper, the ground field $\mathbb{F}$ is an algebraically closed field of characteristic different from 2, 3 and all vector spaces, algebras are over $\mathbb{F}$. In the characteristic zero case, for any non-negative integer k, we make a decomposition of $\mathrm{H}^k(L_{n,m})$ by the Hochschild-Serre spectral sequences, moreover, we can describe completely the Betti number of $\mathrm{H}^\bullet(L_{n,m})$ and the superalgebra structures of the cohomology for some

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low-dimensional filiform Lie superalgebras. We also describe the graded superalgebra structures of the (divided power) cohomology for some low-dimensional filiform Lie superalgebras by the Hochschild-Serre spectral sequences to a certain ideal, in characteristic different from 2, 3.

1 Filiform Lie superalgebras

A *Lie superalgebra* is a $\mathbb{Z}_2$-graded algebra whose multiplication satisfies the skew-supersymmetry and the super Jacobi identity (see [13]). For a Lie superalgebra L, we inductively define two sequences:

$$L_{\bar{0}}^0 = L_{\bar{0}}, \qquad L_{\bar{0}}^{i+1} = [L_{\bar{0}}, L_{\bar{0}}^i]$$

and

$$L_{\bar{1}}^0 = L_{\bar{1}}, \qquad L_{\bar{1}}^{i+1} = [L_{\bar{0}}, L_{\bar{1}}^i].$$

If there exists $(m,n) \in \mathbb{N}^2$ such that $L_{\bar{0}}^m = 0$, $L_{\bar{0}}^{m-1} \neq 0$ and $L_{\bar{1}}^n = 0$, $L_{\bar{1}}^{n-1} \neq 0$, the pair (m,n) is called the *super-nilindex* of L. In particular, A Lie superalgebra L is called *filiform* if its super-nilindex is $(\mathrm{dim}L_{\bar{0}} - 1, \mathrm{dim}L_{\bar{1}})$ (see [7]).

Denote by $\mathcal{F}_{n,m}$ the set of filiform Lie superalgebras of super-nilindex (n,m). Let $\mathcal{F}$ be a filiform Lie superalgebra over $\mathbb{C}$. If $\mathcal{F} \in \mathcal{F}_{n,m}$, there exists a basis $\{X_0, X_1, \ldots, X_n \mid Y_1, \ldots, Y_m\}$ of $\mathcal{F}$ such that:

$$\begin{aligned}
&[X_0, X_i] = X_{i+1},\ 1 \le i \le n-1, \quad [X_0, X_n] = 0;\\
&[X_1, X_2] \in \mathbb{C}X_4 + \cdots + \mathbb{C}X_n;\\
&[X_0, Y_i] = Y_{i+1},\ 1 \le i \le m-1, \quad [X_0, Y_m] = 0.
\end{aligned}$$

Recall the following classifications up to isomorphism (see [14]):

The classification of $\mathcal{F}_{1,2}$:
(1) $\mathcal{F}_{1,2}^1 : [X_0, Y_1] = Y_2$;
(2) $\mathcal{F}_{1,2}^2 : [X_0, Y_1] = [X_1, Y_1] = Y_2$;
(3) $\mathcal{F}_{1,2}^3 : [X_0, Y_1] = Y_2,\ [Y_1, Y_1] = X_1$.

The classification of $\mathcal{F}_{2,2}$:
(1) $\mathcal{F}_{2,2}^1 : [X_0, X_1] = X_2,\ [X_0, Y_1] = Y_2$;
(2) $\mathcal{F}_{2,2}^2 : [X_0, X_1] = 2[Y_1, Y_2] = X_2,\ [X_0, Y_1] = Y_2,\ [Y_1, Y_1] = X_1$;
(3) $\mathcal{F}_{2,2}^3 : [X_0, X_1] = [Y_1, Y_1] = X_2,\ [X_0, Y_1] = Y_2$;
(4) $\mathcal{F}_{2,2}^4 : [X_0, X_1] = X_2,\ [X_0, Y_1] = [X_1, Y_1] = Y_2$;
(5) $\mathcal{F}_{2,2}^5 : [X_0, X_1] = [Y_1, Y_1] = X_2,\ [X_0, Y_1] = [X_1, Y_1] = Y_2$.

Let $L_{n,m}$ be the filiform Lie superalgebra with a homogeneous basis

$$\{X_0, X_1, \ldots, X_n \mid Y_1, \ldots, Y_m\}$$

and Lie super-brackets are given by

$$[X_0, X_i] = X_{i+1},\ 1 \le i \le n-1; \quad [X_0, Y_j] = Y_{j+1},\ 1 \le j \le m-1.$$

We call $L_{n,m}$ the *model filiform Lie superalgebra* and $\{X_0, X_1, \ldots, X_n \mid Y_1, \ldots, Y_m\}$ the *standard basis* of $L_{n,m}$.

Obviously, we have:

$$\mathcal{F}_{1,2}^1 = L_{1,2}, \quad \mathcal{F}_{2,2}^1 = L_{2,2}.$$

2 (Divided power) cohomology

In this section, we introduce the definitions of cohomology and divided power cohomology of $\mathfrak{g}$ with coefficients in the trivial module. For more details, the reader is referred to [13], [15] and [16].

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra, denote by $\mathfrak{g}^*$ the dual superspace of $\mathfrak{g}$. Fix an ordered basis of $\mathfrak{g}$

$$\{x_1, \dots, x_m \mid x_{m+1}, \dots, x_{m+n}\}, \tag{2.0.1}$$

where $|x_1| = \cdots = |x_m| = 0$, $|x_{m+1}| = \cdots = |x_{m+n}| = 1$, and write

$$\{x_1^*, \dots, x_m^* \mid x_{m+1}^*, \dots, x_{m+n}^*\},$$

for the dual basis.

For $k \in \mathbb{Z}$, we let $\bigwedge^k \mathfrak{g}^*$ be the k-th super-exterior product of $\mathfrak{g}^*$. Let $\bigwedge^\bullet \mathfrak{g}^* = \bigoplus_{k\in\mathbb{N}_0} \bigwedge^k \mathfrak{g}^*$. Then $\bigwedge^\bullet \mathfrak{g}^*$ can be viewed as a $\mathfrak{g}$-module in a natural manner. Note that $\bigwedge^\bullet \mathfrak{g}^*$ also has a $\mathbb{Z}$-grading structure given by setting $\|x_1\| = \ldots = \|x_{m+n}\| = 1$. Hereafter $\|x\|$ denotes the $\mathbb{Z}$-degree of a $\mathbb{Z}$-homogeneous element x in a $\mathbb{Z}$-graded superspace. Let $d : \bigwedge^\bullet \mathfrak{g}^* \longrightarrow \bigwedge^\bullet \mathfrak{g}^*$ be the linear operator induced by the dual of the Lie superalgebra bracket map $\mathfrak{g}^* \longrightarrow \bigwedge^2 \mathfrak{g}^*$. Then

$$\begin{aligned}
&d(1) = 0,\\
&d(x_i^*) = \sum_{1\leq k<l\leq m+n} (-1)^{|x_k^*||x_l^*|} a_{kl}^i x_k^* x_l^* - \frac{1}{2} \sum_{m+1\leq k\leq m+n} a_{kk}^i {x_k^*}^2,\ 1 \leq i \leq m+n,\\
&d(x\wedge y) = d(x)\wedge y + (-1)^{\|x\|} x \wedge d(y),\ x, y \in \bigwedge^\bullet \mathfrak{g}^*,
\end{aligned}$$

where a_{kl}^i, $1 \leq i, k, l \leq m+n$, are the structure constants of $\mathfrak{g}$ with respect to the basis (2.0.1). Then d is a $\mathfrak{g}$-module homomorphism and

$$d^2 = 0, \quad |d| = 0, \quad \|d\| = 1.$$

Denote by $\mathrm{H}^\bullet(\mathfrak{g})$ the *cohomology* of $\mathfrak{g}$ defined by the cochain complex $(\bigwedge^\bullet \mathfrak{g}^*, d)$. Note that $\bigwedge^\bullet \mathfrak{g}^*$ is a $\mathbb{Z}$-graded associative superalgebra in a natural manner, which induces a $\mathbb{Z}$-graded associative superalgebra structure on $\mathrm{H}^\bullet(\mathfrak{g})$. In particular, the dimension of $\mathrm{H}^k(\mathfrak{g})$ is called the k-th *Betti number*.

Over a field of prime characteristic $p > 2$, we introduce the definition of divided power cohomology of $\mathfrak{g}$. For a multi-index $\underline{r} = (r_1, \dots, r_{m+n})$, where $r_1, \dots, r_m$ are 0 or 1, and $r_{m+1}, \dots, r_{m+n}$ are non-negative integers, we set

$$u_i^{(r_i)} = \frac{x_i^{r_i}}{r_i!} \quad \text{and} \quad u^{(\underline{r})} = \prod_{i=1}^{m+n} u_i^{(r_i)}.$$

Clearly, their multiplication relations are

$$u^{(\underline{r})} u^{(\underline{s})} = \left(\prod_{i=1}^{m} \min(1, 2 - r_i - s_i)\right) (-1)^{\sum_{j=1}^{m}\sum_{k=m+1}^{m+n} r_k s_j + \sum_{1\leq j<k\leq m} r_k s_j} \begin{pmatrix} \underline{r}+\underline{s} \\ \underline{r} \end{pmatrix} u^{\underline{r}+\underline{s}} \tag{2.0.2}$$

where

$$\left(\begin{array}{c} \underline{r}+\underline{s} \\ \underline{r} \end{array}\right) = \prod_{i=m+1}^{m+n} \left(\begin{array}{c} r_i+s_i \\ r_i \end{array}\right)$$

Fix two n-tuples of positive integers $\underline{t} = (t_1, \ldots, t_n)$ and $\pi = (\pi_1, \ldots, \pi_n)$, where $\pi_i = p^{t_i}$, $1 \leq i \leq n$. Denote

$$\mathcal{O}(m,n;\underline{t}) = \mathcal{O}(\mathfrak{g}^*;\underline{t}) = \mathrm{span}_{\mathbb{F}}\left(u^{(\underline{r})} \mid r_i \begin{cases} < \pi_i, & m+1 \leq i \leq m+n; \\ = 0 \text{ or } 1, & 1 \leq i \leq m. \end{cases} \right)$$

From Eq. (2.0.2), $\mathcal{O}(\mathfrak{g}^*;\underline{t})$ is a finite dimensional submodule and a graded subalgebra of $\bigwedge^\bullet \mathfrak{g}^*$. In particular,

$$\mathcal{O}(\mathfrak{g}^*;\underline{t}) = \bigoplus_{i=0}^{m-n+\sum_{j=1}^{n} p^{t_j}} \mathcal{O}_i(\mathfrak{g}^*;\underline{t}),$$

where $\mathcal{O}_i(\mathfrak{g}^*;\underline{t}) = \mathrm{span}_{\mathbb{F}}\{u^{(\underline{r})} \mid \sum_{k=1}^{m+n} r_k = i\}$. Let $d : \mathcal{O}(\mathfrak{g}^*;\underline{t}) \longrightarrow \mathcal{O}(\mathfrak{g}^*;\underline{t})$ be the linear operator induced by the dual of the Lie superalgebra bracket map. Then

$$\begin{aligned} & d(1) = 0, \\ & d(x_i^*) = \sum_{1 \leq k<l \leq m+n} (-1)^{|x_k^*||x_l^*|} a_{kl}^i x_k^* x_l^* - \sum_{m+1 \leq k \leq m+n} a_{kk}^i x_k^{*(2)}, \ 1 \leq i \leq m+n, \\ & d(u^{(\underline{r})}) = \sum_{j=1}^{m+n} (-1)^{r_1+\ldots+r_{j-1}+|x_j^*|(r_{m+1}+\ldots+r_{j-1})} d(x_j^*) u^{(\underline{r}-\varepsilon_j)}, \end{aligned}$$

where $\varepsilon_j = (\delta_{j,1}, \ldots, \delta_{j,m+n})$, a_{kl}^i, $1 \leq i,j,k,l \leq m+n$, are the structure constants of $\mathfrak{g}$ with respect to the basis (2.0.1). Then d is a $\mathfrak{g}$-module homomorphism and

$$d^2 = 0, \quad |d| = 0, \quad \|d\| = 1.$$

Denote by $\mathrm{DPH}^\bullet(\mathfrak{g})$ the *divided power cohomology* of $\mathfrak{g}$ defined by the cochain complex $(\mathcal{O}(\mathfrak{g}^*;\underline{t}), d)$. Note that $\mathcal{O}(\mathfrak{g}^*;\underline{t})$ is a $\mathbb{Z}$-graded associative superalgebra in a natural manner, which induces a $\mathbb{Z}$-graded associative superalgebra structure on $\mathrm{DPH}^\bullet(\mathfrak{g})$.

Finally, we close this section with some remarks on the Hochschild-Serre spectral sequence (see [13]). Suppose I is an ideal of $\mathfrak{g}$. Then there is a convergent spectral sequence called the Hochschild-Serre spectral sequence such that

$$\mathrm{E}_2^{k,s} = \mathrm{H}^k(\mathfrak{g}/I, \mathrm{H}^s(I)) \Longrightarrow \mathrm{H}^{k+s}(\mathfrak{g}) \tag{2.0.3}$$

3 Characteristic zero

Throughout this section the ground field $\mathbb{F}$ is an algebraically closed field of characteristic zero.

3.1 Model filiform Lie superalgebras

Let $\mathcal{I}$ be a subspace of $L_{n,m}$ spanned by

$$\{X_1, \dots, X_n \mid Y_1, \dots, Y_m\}.$$

Obviously, $\mathcal{I}$ is an abelian ideal of $L_{n,m}$. Moreover, $\mathrm{H}^\bullet(L_{n,m})$ is described by using the Hochschild-Serre spectral sequence relative to $\mathcal{I}$.

Lemma 3.1. Let

$$D : \bigwedge^{\bullet} \mathcal{I}^* \longrightarrow \bigwedge^{\bullet} \mathcal{I}^*,$$
$$x \longmapsto -X_0 \cdot x.$$

Let $D_k = D\mid_{\bigwedge^k \mathcal{I}^*}$. Then $\mathrm{H}^k(L_{n,m}) = \mathrm{Ker}\ D_k \bigoplus \big(\mathbb{F}X_0^* \bigwedge \big(\bigwedge^{k-1} \mathcal{I}^*/\mathrm{Im}\ D_{k-1}\big)\big)$.

Proof. From Eq. (2.0.3), we have

$$\mathrm{E}_\infty^{p,k-p} \cong \begin{cases} \mathrm{H}^0\big(\mathbb{F}X_0, \bigwedge^k \mathcal{I}^*\big), & p = 0; \\ \mathrm{H}^1\big(\mathbb{F}X_0, \bigwedge^{k-1} \mathcal{I}^*\big), & p = 1; \\ 0, & p \neq 0, 1. \end{cases}$$

Then we have

$$\mathrm{H}^k(L_{n,m}) = \bigoplus_{p+q=k} \mathrm{E}_\infty^{p,q} = \mathrm{H}^0\Big(\mathbb{F}X_0, \bigwedge^k \mathcal{I}^*\Big) \bigoplus \mathrm{H}^1\Big(\mathbb{F}X_0, \bigwedge^{k-1} \mathcal{I}^*\Big).$$

By the definitions of the low cohomology (see [13]), we have

$$\mathrm{H}^0\Big(\mathbb{F}X_0, \bigwedge^k \mathcal{I}^*\Big) = \mathrm{Ker}\ D_k,$$
$$\mathrm{Der}\Big(\mathbb{F}X_0, \bigwedge^{k-1} \mathcal{I}^*\Big) = \mathbb{F}X_0^* \bigwedge \bigwedge^{k-1} \mathcal{I}^*,$$
$$\mathrm{Inder}\Big(\mathbb{F}X_0, \bigwedge^{k-1} \mathcal{I}^*\Big) = \mathbb{F}X_0^* \bigwedge \mathrm{Im}\ D_{k-1}.$$

Moreover, we obtain that

$$\mathrm{H}^k(L_{n,m}) = \mathrm{Ker}\ D_k \bigoplus \Big(\mathbb{F}X_0^* \bigwedge \Big(\bigwedge^{k-1} \mathcal{I}^*/\mathrm{Im}\ D_{k-1}\Big)\Big).$$

□

Theorem 1. As a $\mathbb{Z}$-graded superalgebra, we have

$$\mathrm{H}^\bullet(L_{n,m}) = \mathrm{Ker}\ D \ltimes \Big(\mathbb{F}X_0^* \bigwedge \Big(\bigwedge^{\bullet} \mathcal{I}^*/\mathrm{Im}\ D\Big)\Big).$$

Proof. For any $x, y \in \mathrm{Ker}\ D$, $\overline{z} \in \bigwedge^{\bullet}\mathcal{I}^*/\mathrm{Im}\ D$, we have

$$\begin{aligned} D(x \wedge y) = -X_0 \cdot (x \wedge y) &= -[(X_0 \cdot x) \wedge y + x \wedge (X_0 \cdot y)], \\ &= D(x) \wedge y + x \wedge D(y), \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} (X_0^* \wedge \overline{z}) \wedge x &= X_0^* \wedge (\overline{z} \wedge x), \\ &= X_0^* \wedge (\overline{z \wedge x}) \in \mathbb{F}X_0^* \bigwedge \left(\dot{\bigwedge} \mathcal{I}^*/\mathrm{Im}\ D \right). \end{aligned}$$

Thus, Ker D is a subalgebra of $\mathrm{H}^{\bullet}(L_{n,m})$ and $\mathbb{F}X_0^* \bigwedge \left(\bigwedge^{\bullet} \mathcal{I}^*/\mathrm{Im}\ D \right)$ is an ideal of $\mathrm{H}^{\bullet}(L_{n,m})$ with trivial multiplication. From Lemma 3.1, the proof is complete. □

In order to calculate the dimension of $\mathrm{H}^{\bullet}(L_{n,m})$, it suffices to calculate the dimension of Ker D.

Let

$$f_i = \frac{1}{(n-i)!} X_i^*,\ 1 \le i \le n, \quad g_j = \frac{1}{(m-j)!} Y_j^*,\ 1 \le j \le m.$$

Let $x, y, h \in \mathrm{End}_{\mathbb{F}}(\mathcal{I}_0^*)$, such that

$$\begin{aligned} &x = D\mid_{\mathcal{I}_0^*}, \\ &y(f_i) = i f_{i+1},\ 1 \le i \le n-1, \quad y(f_n) = 0, \\ &h(f_i) = (n+1-2i) f_i,\ 1 \le i \le n. \end{aligned}$$

Let $x', y', h' \in \mathrm{End}_{\mathbb{F}}(\mathcal{I}_1^*)$, such that

$$\begin{aligned} &x' = D\mid_{\mathcal{I}_1^*}, \\ &y'(g_j) = j g_{j+1},\ 1 \le j \le m-1, \quad y'(g_m) = 0, \\ &h'(g_j) = (m+1-2j) g_j,\ 1 \le j \le m. \end{aligned}$$

Obviously, $\mathrm{span}_{\mathbb{F}}\{x, y, h\}$, $\mathrm{span}_{\mathbb{F}}\{x', y', h'\}$ are subalgebras of $\mathfrak{gl}(\mathcal{I}_0^*)$ and $\mathfrak{gl}(\mathcal{I}_1^*)$, respectively. Moreover, the following Lie algebra isomorphism holds:

$$\mathrm{span}_{\mathbb{F}}\{x, y, h\} \cong \mathrm{span}_{\mathbb{F}}\{x', y', h'\} \cong \mathfrak{sl}(2).$$

By Weyl's Theorem and representation theory of $\mathfrak{sl}(2)$ (see [17]), $\mathcal{I}_0^*$ and $\mathcal{I}_1^*$ are simple modules of $\mathfrak{sl}(2)$ and, for $k \ge 0$, $\bigwedge^k \mathcal{I}^*$ is a completely reducible module of $\mathfrak{sl}(2)$. Moreover, we can obtain the following theorem.

Lemma 3.2. Suppose $k \ge 0$. Then

$$\dim \mathrm{Ker}\ D_k = \sum_{k_0=0}^{k} \mathfrak{f}_{n,m}(k_0, k-k_0),$$

where $\mathfrak{f}_{n,m}(k_0, k-k_0) = \mathrm{card}\Big\{ (i_1, \ldots, i_{k_0}, j_1, \ldots, j_{k-k_0}) \in \mathbb{Z}^k \mid 1 \le i_1 < \ldots < i_{k_0} \le n, 1 \le j_1 < \ldots < j_{k-k_0} \le m+k-k_0-1, \sum_{a=1}^{k_0} i_a + \sum_{b=1}^{k-k_0} j_b = \left\lfloor \frac{k_0(n+1)+(k-k_0)(m+1)}{2} \right\rfloor + \frac{(k-k_0)(k-k_0-1)}{2} \Big\}$.

Proof. For $k \geq 0$, set

$$\bigwedge^k \mathcal{I}^* = \bigoplus_{i=1}^r V_i,$$

where $V_1, \ldots, V_r$ are simple $\mathfrak{sl}(2)$-modules. Moreover,

$$\text{Ker } D_k = \bigoplus_{i=1}^r \text{Ker } D_k \bigcap V_i.$$

So we obtain that for any $v \in V_i$, $D_k(v) = 0$ if and only if v is a maximal vector of V_i. Moreover, the following conclusions hold:

$$\dim \text{Ker } D_k = r = \dim \big(\bigwedge^k \mathcal{I}^*\big)_0 + \big(\bigwedge^k \mathcal{I}^*\big)_1, \tag{3.1.1}$$

where $\big(\bigwedge^k \mathcal{I}^*\big)_0$, $\big(\bigwedge^k \mathcal{I}^*\big)_1$ are weight spaces of weight 0, 1, respectively.

Let $\mathfrak{sl}(2) = \text{span}_{\mathbb{F}}\{X_+, H, X_-\}$, where $\mathbb{F}H$ is a Cartan subalgebra of $\mathfrak{sl}(2)$, and $f_{i_1} \wedge \ldots \wedge f_{i_{k_0}} \wedge g_{j_1} \wedge \ldots \wedge g_{j_{k-k_0}}$ is a standard basis of $\bigwedge^k \mathcal{I}^*$, where $0 \leq k_0 \leq k$, $i_1 < \ldots < i_{k_0} \leq n$, $1 \leq j_1 \leq \ldots \leq j_{k-k_0} \leq m$. Note that

$$H(f_{i_1} \wedge \ldots \wedge f_{i_{k_0}} \wedge g_{j_1} \wedge \ldots \wedge g_{j_{k-k_0}}) = \Bigg(k_0(n+1) + (k-k_0)(m+1) - 2\Big(\sum_{a=1}^{k_0} i_a + \sum_{b=1}^{k-k_0} j_b\Big)\Bigg)$$
$$f_{i_1} \wedge \ldots \wedge f_{i_{k_0}} \wedge g_{j_1} \wedge \ldots \wedge g_{j_{k-k_0}}.$$

From Eq. (3.1.1), it is sufficient to calculate the dimensions of $\big(\bigwedge^k \mathcal{I}^*\big)_0$ and $\big(\bigwedge^k \mathcal{I}^*\big)_1$. We consider the following cases:

Case 1 : $k_0(n+1) + (k-k_0)(m+1)$ is even. Then

$$\Bigg(k_0(n+1) + (k-k_0)(m+1) - 2\Big(\sum_{a=1}^{k_0} i_a + \sum_{b=1}^{k-k_0} j_b\Big)\Bigg) = 0,$$

if and only if $\sum_{a=1}^{k_0} i_a + \sum_{b=1}^{k-k_0} j_b = \dfrac{k_0(n+1) + (k-k_0)(m+1)}{2}$.

Case 2 : $k_0(n+1) + (k-k_0)(m+1)$ is odd. Then

$$\Bigg(k_0(n+1) + (k-k_0)(m+1) - 2\Big(\sum_{a=1}^{k_0} i_a + \sum_{b=1}^{k-k_0} j_b\Big)\Bigg) = 1,$$

if and only if $\sum_{a=1}^{k_0} i_a + \sum_{b=1}^{k-k_0} j_b = \dfrac{k_0(n+1) + (k-k_0)(m+1) - 1}{2}$.

Thus, the conclusion holds. □

Theorem 2. Suppose $k \geq 0$. Then

$$\dim \mathrm{H}^k(L_{n,m}) = \sum_{k_0=0}^{k} \mathfrak{f}_{n,m}(k_0, k-k_0) + \sum_{k_0=0}^{k-1} \mathfrak{f}_{n,m}(k_0, k-k_0-1).$$

Proof. It follows from Lemmas 3.1 and 3.2. □

In order to characterize the superalgebra structure of $\mathrm{H}^{\bullet}(L_{n,m})$, we make a $\mathbb{Z}$-gradation of $\bigwedge^k \mathcal{I}^*$ for any $k \geq 0$.

For any $0 \leq s \leq k$, let l such that $\alpha_{k,s} \leq l \leq \beta_{k,s}$, where $\alpha_{k,s} = \frac{s(s+1)}{2} + k - s$, $\beta_{k,s} = \frac{s(2n-s+1)}{2} + (k-s)m$.

Let $\left(\bigwedge^k \mathcal{I}^*\right)_s^l$ be the space spanned by

$$X_{i_1}^* \wedge \ldots \wedge X_{i_s}^* \wedge Y_1^{*^{\alpha_1}} \wedge \ldots \wedge Y_m^{*^{\alpha_m}},$$

where $1 \leq i_1 < \ldots < i_s \leq n$, and $\alpha_1, \ldots, \alpha_m \geq 0$, satisfying that

$$\sum_{j=1}^{m} \alpha_j = k - s, \quad \sum_{a=1}^{s} i_a + \sum_{b=1}^{m} \alpha_b b = l.$$

Let

$$D_s^{k,l} : \left(\bigwedge^k \mathcal{I}^*\right)_s^l \longrightarrow \left(\bigwedge^k \mathcal{I}^*\right)_s^{l-1},$$
$$x \longmapsto D_k(x).$$

Obviously,

$$\bigwedge^k \mathcal{I}^* = \bigoplus_{s=0}^{k} \bigoplus_{l=\alpha_{k,s}}^{\beta_{k,s}} \left(\bigwedge^k \mathcal{I}^*\right)_s^l,$$

since the linear mapping D_k is compatible with this graduation, we have:

$$\text{Ker } D_k = \bigoplus_{s=0}^{k} \bigoplus_{l=\alpha_{k,s}}^{\beta_{k,s}} \text{Ker } D_s^{k,l}, \tag{3.1.2}$$

$$\text{Im } D_k = \bigoplus_{s=0}^{k} \bigoplus_{l=\alpha_{k,s}}^{\beta_{k,s}} \text{Im } D_s^{k,l}, \tag{3.1.3}$$

where $\alpha_{k,s} = \frac{s(s+1)}{2} + k - s$, $\beta_{k,s} = \frac{s(2n-s+1)}{2} + (k-s)m$.

We describe the superalgebra structures of the cohomology for some low-dimensional filiform Lie superalgebras by the decomposition in Theorem 1.

Example 1. Then the following $\mathbb{Z}$-graded superalgebra isomorphism holds:

$$\mathrm{H}^{\bullet}(L_{1,2}) \cong \mathcal{U}_{1,2} \ltimes \mathcal{V}_{1,2},$$

where $\mathcal{U}_{1,2}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_{1,i}, \alpha_i \mid i \geq 0\},$$

satisfying that $|\alpha_{1,i}| = |\alpha_i| = \bar{i} \pmod 2$, $\|\alpha_{1,i}\| = i+1$, $\|\alpha_i\| = i$, and the multiplication is given by

$$\alpha_i \alpha_j = \alpha_{i+j}, \quad \alpha_{1,i}\alpha_j = \alpha_{1,i+j}, \ i, j \geq 0.$$

$\mathcal{V}_{1,2}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_{0,i}, \alpha_{0,1,i} \mid i \geq 0\},$$

satisfying that $|\alpha_{0,i}| = |\alpha_{0,1,i}| = \bar{i} \pmod 2$, $\|\alpha_{0,i}\| = i+1$, $\|\alpha_{0,1,i}\| = i+2$, and the trivial multiplication. The multiplication between $\mathcal{U}_{1,2}$ and $\mathcal{V}_{1,2}$ is graded-supercommutative, and the multiplication is given by

$$\alpha_{0,i}\alpha_{1,0} = (-1)^i \alpha_{0,1,i}, \quad \alpha_{0,i}\alpha_0 = \alpha_{0,i}, \quad \alpha_{0,1,i}\alpha_0 = \alpha_{0,1,i}, \ i \geq 0.$$

Proof. Note that

$$\Big(\bigwedge^k \mathcal{I}^*\Big)^l_s = \begin{cases} \mathbb{F}Y_1^{*^{2k-l}} \wedge Y_2^{*^{l-k}}, & s = 0; \\ \mathbb{F}X_1^* \wedge Y_1^{*^{2k-l-1}} \wedge Y_2^{*^{l-k}}, & s = 1; \\ 0, & \text{else.} \end{cases}$$

Moreover, we have

$$\text{Ker } D^{k,l}_s = \begin{cases} \mathbb{F}Y_1^{*^k}, & s = 0, l = k; \\ \mathbb{F}X_1^* \wedge Y_1^{*^{k-1}}, & s = 1, l = k; \\ 0, & \text{else.} \end{cases}$$

$$\text{Im } D^{k,l}_s = \begin{cases} \mathbb{F}Y_1^{*^{2k-l+1}} \wedge Y_2^{*^{l-k-1}}, & s = 0, k+1 \leq l \leq 2k; \\ \mathbb{F}X_1^* \wedge Y_1^{*^{2k-l}} \wedge Y_2^{*^{l-k-1}}, & s = 1, k+1 \leq l \leq 2k-1; \\ 0, & \text{else.} \end{cases}$$

From Theorem 1 and Eqs. (3.1.2), (3.1.3), $\mathrm{H}^\bullet(L_{1,2})$ has a basis:

$$Y_1^{*^i}, \quad X_1^* \wedge Y_1^{*^i}, \quad X_0^* \wedge Y_2^{*^i}, \quad X_0^* \wedge X_1^* \wedge Y_2^{*^i}, \ i \geq 0.$$

The conclusion can be obtained by a direct calculation. □

Example 2. The following $\mathbb{Z}$-graded superalgebra isomorphism holds:

$$\mathrm{H}^\bullet(L_{2,1}) \cong \mathcal{U}_{2,1} \ltimes \mathcal{V}_{2,1},$$

where $\mathcal{U}_{2,1}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_i, \alpha_{1,i}, \alpha_{1,2,i} \mid i \geq 0\},$$

satisfying that $|\alpha_i| = |\alpha_{1,i}| = |\alpha_{1,2,i}| = \bar{i} \pmod 2$, $\|\alpha_i\| = i$, $\|\alpha_{1,i}\| = i+1$, $\|\alpha_{1,2,i}\| = i+2$, and the multiplication is given by

$$\alpha_i\alpha_j = \alpha_{i+j}, \quad \alpha_{1,i}\alpha_j = \alpha_{1,i+j}, \quad \alpha_{1,2,i}\alpha_j = \alpha_{1,2,i+j}, \ i,j \geq 0.$$

$\mathcal{V}_{2,1}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_{0,i}, \alpha_{0,2,i}, \alpha_{0,1,2,i} \mid i \geq 0\},$$

satisfying that $|\alpha_{0,i}| = |\alpha_{0,2,i}| = |\alpha_{0,1,2,i}| = \bar{i} \pmod 2$, $\|\alpha_{0,i}\| = i+1$, $\|\alpha_{0,2,i}\| = i+2$, $\|\alpha_{0,1,2,i}\| = i+3$, and the trivial multiplication. The multiplication between $\mathcal{U}_{2,1}$ and $\mathcal{V}_{2,1}$ is graded-supercommutative, and the multiplication is given by

$$\alpha_{0,i}\alpha_j = \alpha_{0,i+j}, \quad \alpha_{0,2,i}\alpha_j = \alpha_{0,2,i+j}, \quad \alpha_{0,1,2,i}\alpha_j = \alpha_{0,1,2,i+j},$$

$$\alpha_{0,2,i}\alpha_{1,j} = (-1)^{i+1}\alpha_{0,1,2,i+j}, \quad \alpha_{0,i}\alpha_{1,2,j} = \alpha_{0,1,2,i+j}, \ i,j \geq 0.$$

Proof. Note that

$$\big(\bigwedge^k \mathcal{I}^*\big)^l_s = \begin{cases} \mathbb{F}{Y_1^*}^l, & s=0; \\ \mathbb{F}X_1^* \wedge {Y_1^*}^{l-1}, & s=1, l=k; \\ \mathbb{F}X_2^* \wedge {Y_1^*}^{l-2}, & s=1, l=k+1; \\ \mathbb{F}X_1^* \wedge X_2^* \wedge {Y_1^*}^{l-3}, & s=2, l=k+1; \\ 0, & \text{else.} \end{cases}$$

Moreover, we have

$$\text{Ker } D^{k,l}_s = \begin{cases} \mathbb{F}{Y_1^*}^l, & s=0; \\ \mathbb{F}X_1^* \wedge {Y_1^*}^{l-1}, & s=1, l=k; \\ \mathbb{F}X_1^* \wedge X_2^* \wedge {Y_1^*}^{l-3}, & s=2, l=k+1; \\ 0, & \text{else.} \end{cases}$$

$$\text{Im } D^{k,l}_s = \begin{cases} \mathbb{F}X_1^* \wedge {Y_1^*}^{l-2}, & s=1, l=k+1; \\ 0, & \text{else.} \end{cases}$$

From Theorem 1 and Eqs. (3.1.2), (3.1.3), $\mathrm{H}^\bullet(L_{2,1})$ has a basis:

$${Y_1^*}^i, \quad X_1^* \wedge {Y_1^*}^i, \quad X_1^* \wedge X_2^* \wedge {Y_1^*}^i, \quad X_0^* \wedge {Y_1^*}^i, \quad X_0^* \wedge X_2^* \wedge {Y_1^*}^i, \quad X_0^* \wedge X_1^* \wedge X_2^* \wedge {Y_1^*}^i, \ i \geq 0.$$

The conclusion can be obtained by a direct calculation. □

Example 3. The following $\mathbb{Z}$-graded superalgebra isomorphism holds:

$$\mathrm{H}^\bullet(L_{1,3}) \cong \mathcal{U}_{1,3} \ltimes \mathcal{V}_{1,3},$$

where $\mathcal{U}_{1,3}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_{i,j}, \alpha_{1,i,j}, \mid i,j \geq 0\},$$

satisfying that $|\alpha_{i,j}| = |\alpha_{1,i,j}| = \overline{2j+i} \pmod 2$, $\|\alpha_{i,j}\| = 2j+i$, $\|\alpha_{1,i,j}\| = 2j+i+1$, and the multiplication is given by

$$\alpha_{i,j}\alpha_{i',j'} = \alpha_{i+i',j+j'}, \quad \alpha_{1,i,j}\alpha_{i',j'} = \alpha_{1,i+i',j+j'}, \ i,i',j,j' \geq 0.$$

$\mathcal{V}_{1,3}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_{0,i,j}, \alpha_{0,1,i,j} \mid i,j \geq 0\},$$

satisfying that $|\alpha_{0,i,j}| = |\alpha_{0,1,i,j}| = \overline{2i+j} \pmod 2$, $\|\alpha_{0,i,j}\| = 2i+j+1$, $\|\alpha_{0,1,i,j}\| = 2i+j+2$, and the trivial multiplication. The multiplication between $\mathcal{U}_{1,3}$ and $\mathcal{V}_{1,3}$ is graded-supercommutative, and the multiplication is given by

$$\alpha_{0,i,j}\alpha_{i',j'} = \rho_{i,j,i',j'}\alpha_{0,i+i'+j',j-i'},$$

$$\alpha_{0,i,j}\alpha_{1,i',j'} = (-1)^{2i+j}\alpha_{0,1,i,j}\alpha_{i',j'} = (-1)^{2i+j}\rho_{i,j,i',j'}\alpha_{0,i+i'+j',j-i'}, \ i,j \geq 0,$$

where $\rho_{i,j,i',j'} = \prod_{s=0}^{i'-1} \frac{s-j}{2(i+j'+s)+1} + \sum_{t=i'+1}^{i'+j'} \left(\prod_{a=1}^{t-i'} \frac{2(a-1-j')}{a} \prod_{b=0}^{t-1} \frac{b-j-t+i'}{2(i+i'+j'-t+b)+1} \right)$.

Proof. Note that

$$\big(\bigwedge^k \mathcal{I}^*\big)^l_s = \begin{cases} \bigoplus\limits_{i=2k-l}^{\lfloor \frac{3k-l}{2} \rfloor} \mathbb{F}Y_1^i \wedge Y_2^{3k-l-2i} \wedge Y_3^{i+l-2k}, & s=0, k \le l \le 2k; \\ \bigoplus\limits_{i=0}^{\lfloor \frac{3k-l}{2} \rfloor} \mathbb{F}Y_1^i \wedge Y_2^{3k-l-2i} \wedge Y_3^{i+l-2k}, & s=0, 2k+1 \le l \le 3k \\ \bigoplus\limits_{i=2k-l-1}^{\lfloor \frac{3k-l}{2} \rfloor -1} \mathbb{F}X_1 \wedge Y_1^i \wedge Y_2^{3k-l-2i-2} \wedge Y_3^{i+l-2k+1}, & s=1, k \le l \le 2k-1; \\ \bigoplus\limits_{i=0}^{\lfloor \frac{3k-l}{2} \rfloor -1} \mathbb{F}X_1 \wedge Y_1^i \wedge Y_2^{3k-l-2i-2} \wedge Y_3^{i+l-2k+1}, & s=1, 2k \le l \le 3k-2; \\ 0, & \text{else.} \end{cases}$$

From a direct computation, we have the following conclusion:

If $2k+1 \le l \le 3k$, or $k \le l \le 2k$ and $3k-l$ is odd, we have $\mathrm{Ker}\ D_0^{k,l} = 0$.

If $k \le l \le 2k$, and $3k-l$ is even, we have $\dim \mathrm{Ker}\ D_0^{k,l} = 1$ and $\mathrm{Ker}\ D_0^{k,l}$ has a basis:

$$Y_1^{*^{2k-l}} \wedge Y_2^{*^{l-k}} + \sum_{i=2k-l+1}^{\frac{3k-l}{2}} \lambda_i^{k,l} Y_1^{*^i} \wedge Y_2^{*^{3k-l-2i}} \wedge Y_3^{*^{i+l-2k}},$$

where $\lambda_i^{k,l} = \prod\limits_{j=1}^{i-2k+l} \frac{2(j-1)+k-l}{j}$. Moreover, we have

$$\big(\bigwedge^k \mathcal{I}^*\big)^{l-1}_0 / \mathrm{Im}\ D_0^{k,l} = \begin{cases} \mathbb{F}Y_2^{*^{3k-l+1}} \wedge Y_3^{*^{l-2k-1}}, & 2k+1 \le l \le 3k \text{ and } 3k-l \text{ is odd;} \\ 0, & \text{else.} \end{cases}$$

From Theorem 1 and Eqs. (3.1.2), (3.1.3), $\mathrm{H}^\bullet(L_{1,3})$ has a basis:

$$Y_1^{*^i} \wedge Y_2^{*^{2j}} + \sum_{s=i+1}^{i+j} \lambda_s^{2j+i,4j+i} Y_1^{*^s} \wedge Y_2^{*^{2(i+j-s)}} \wedge Y_3^{*^{s-i}},$$

$$X_1^* \wedge Y_1^{*^i} \wedge Y_2^{*^{2j}} + \sum_{s=i+1}^{i+j} \lambda_s^{2j+i,4j+i} X_1^* \wedge Y_1^{*^s} \wedge Y_2^{*^{2(i+j-s)}} \wedge Y_3^{*^{s-i}},$$

$$X_0^* \wedge Y_2^{*^{2i}} \wedge Y_3^{*^j},\ X_0^* \wedge X_1^* \wedge Y_2^{*^{2i}} \wedge Y_3^{*^j},\ i,j \ge 0.$$

The conclusion can be obtained by a direct calculation. □

Example 4. The following $\mathbb{Z}$-graded superalgebra isomorphism holds:

$$\mathrm{H}^\bullet(L_{3,1}) \cong \mathcal{U}_{3,1} \ltimes \mathcal{V}_{3,1},$$

where $\mathcal{U}_{3,1}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_i, \alpha_{1,i}, \alpha_{1,2,i}, \alpha_{1,2,3,i} \mid i \ge 0\},$$

satisfying that $|\alpha_i| = |\alpha_{1,i}| = |\alpha_{1,2,i}| = |\alpha_{1,2,3,i}| = \bar{i}$ (mod 2), $\|\alpha_i\| = i$, $\|\alpha_{1,i}\| = i+1$, $\|\alpha_{1,2,i}\| = i+2$, $\|\alpha_{1,2,3,i}\| = i+3$, and the multiplication is given by

$$\alpha_i\alpha_j = \alpha_{i+j}, \quad \alpha_{1,i}\alpha_j = \alpha_{1,i+j}, \quad \alpha_{1,2,i}\alpha_j = \alpha_{1,2,i+j}, \quad \alpha_{1,2,3,i}\alpha_j = \alpha_{1,2,3,i+j},\ i,j \geq 0.$$

$\mathcal{V}_{3,1}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_{0,i}, \alpha_{0,3,i}, \alpha_{0,2,3,i}, \alpha_{0,1,2,3,i} \mid i \geq 0\},$$

satisfying that $|\alpha_{0,i}| = |\alpha_{0,3,i}| = |\alpha_{0,2,3,i}| = |\alpha_{0,1,2,3,i}| = \bar{i}$ (mod 2), $\|\alpha_{0,i}\| = i+1$, $\|\alpha_{0,3,i}\| = i+2$, $\|\alpha_{0,2,3,i}\| = i+3$, $\|\alpha_{0,1,2,3,i}\| = i+4$, and the trivial multiplication. The multiplication between $\mathcal{U}_{3,1}$ and $\mathcal{V}_{3,1}$ is graded-supercommutative, and the multiplication is given by

$$\alpha_{0,i}\alpha_j = \alpha_{0,i+j}, \quad \alpha_{0,3,i}\alpha_j = \alpha_{0,3,i+j}, \quad \alpha_{0,2,3,i}\alpha_j = \alpha_{0,2,3,i+j}, \quad \alpha_{0,1,2,3,i}\alpha_j = \alpha_{0,1,2,3,i+j},$$

$$\alpha_{0,2,3,i}\alpha_{1,j} = \alpha_{0,i}\alpha_{1,2,3,j} = (-1)^i\alpha_{0,3,i}\alpha_{1,2,j} = (-1)^i\alpha_{0,1,2,3,i+j},\ i,j \geq 0.$$

Proof. Note that

$$\big(\bigwedge^k \mathcal{I}^*\big)^l_s = \begin{cases} \mathbb{F}Y_1^{*^l}, & s=0; \\ \mathbb{F}X_1^* \wedge Y_1^{*^{l-1}}, & s=1, l=k; \\ \mathbb{F}X_2^* \wedge Y_1^{*^{l-2}}, & s=1, l=k+1; \\ \mathbb{F}X_3^* \wedge Y_1^{*^{l-3}}, & s=1, l=k+2; \\ \mathbb{F}X_1^* \wedge X_2^* \wedge Y_1^{*^{l-3}}, & s=2, l=k+1; \\ \mathbb{F}X_1^* \wedge X_3^* \wedge Y_1^{*^{l-4}}, & s=2, l=k+2; \\ \mathbb{F}X_2^* \wedge X_3^* \wedge Y_1^{*^{l-5}}, & s=2, l=k+3; \\ \mathbb{F}X_1^* \wedge X_2^* \wedge X_3^* \wedge Y_1^{*^{l-6}}, & s=3, l=k+3; \\ 0, & \text{else.} \end{cases}$$

Moreover, we have

$$\text{Ker } D_s^{k,l} = \begin{cases} \mathbb{F}Y_1^{*^l}, & s=0; \\ \mathbb{F}X_1^* \wedge Y_1^{*^{l-1}}, & s=1, l=k; \\ \mathbb{F}X_1^* \wedge X_2^* \wedge Y_1^{*^{l-3}}, & s=2, l=k+1; \\ \mathbb{F}X_1^* \wedge X_2^* \wedge X_3^* \wedge Y_1^{*^{l-6}}, & s=3, l=k+3; \\ 0, & \text{else.} \end{cases}.$$

$$\text{Im } D_s^{k,l} = \begin{cases} \mathbb{F}X_1^* \wedge Y_1^{*^{l-2}}, & s=1, l=k+1; \\ \mathbb{F}X_2^* \wedge Y_1^{*^{l-3}}, & s=1, l=k+2; \\ \mathbb{F}X_1^* \wedge X_2^* \wedge Y_1^{*^{l-4}}, & s=2, l=k+2; \\ \mathbb{F}X_1^* \wedge X_3^* \wedge Y_1^{*^{l-5}}, & s=2, l=k+3; \\ 0, & \text{else.} \end{cases}.$$

From Theorem 1 and Eqs. (3.1.2), (3.1.3), $\mathrm{H}^\bullet(L_{3,1})$ has a basis:

$$Y_1^{*^i}, \quad X_1^* \wedge Y_1^{*^i}, \quad X_1^* \wedge X_2^* \wedge Y_1^{*^i}, \quad X_1^* \wedge X_2^* \wedge X_3^* \wedge Y_1^{*^i},$$

$$X_0^* \wedge Y_1^{*^i}, \quad X_0^* \wedge X_3^* \wedge Y_1^{*^i}, \quad X_0^* \wedge X_2^* \wedge X_3^* \wedge Y_1^{*^i}, \quad X_0^* \wedge X_1^* \wedge X_2^* \wedge X_3^* \wedge Y_1^{*^i},\ i \geq 0.$$

The conclusion can be obtained by a direct calculation. □

Example 5. The following $\mathbb{Z}$-graded superalgebra isomorphism holds:

$$\mathrm{H}^\bullet(L_{2,2}) \cong \mathcal{U}_{2,2} \ltimes \mathcal{V}_{2,2},$$

where $\mathcal{U}_{2,2}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_i, \alpha_{1,i}, \alpha_{1,2,i}, \beta_i \mid i \geq 0\},$$

satisfying that $|\alpha_i| = |\alpha_{1,i}| = |\alpha_{1,2,i}| = \bar{i} \pmod 2$, $|\beta_i| = \overline{i+1} \pmod 2$, $\|\alpha_i\| = i$, $\|\alpha_{1,i}\| = i+1$, $\|\alpha_{1,2,i}\| = \|\beta_i\| = i+2$, and the multiplication is given by

$$\alpha_i\alpha_j = \alpha_{i+j}, \quad \alpha_{1,i}\alpha_j = \alpha_{1,i+j}, \quad \alpha_{1,2,i}\alpha_j = \alpha_{1,2,i+j},$$
$$\beta_i\alpha_j = \beta_{i+j}, \quad \alpha_{1,i}\beta_j = (-1)^{i+1}\alpha_{1,2,i+j+1}, \ i,j \geq 0.$$

$\mathcal{V}_{2,2}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_{0,i}, \alpha_{0,1,i}, \alpha_{0,2,i}, \alpha_{0,1,2,i} \mid i \geq 0\},$$

satisfying that $|\alpha_{0,i}| = |\alpha_{0,2,i}| = |\alpha_{0,1,2,i}| = \bar{i} \pmod 2$, $|\alpha_{0,1,i}| = \overline{i+1} \pmod 2$, $\|\alpha_{0,i}\| = i+1$, $\|\alpha_{0,2,i}\| = i+2$, $\|\alpha_{0,1,i}\| = \|\alpha_{0,1,2,i}\| = i+3$, and the trivial multiplication. The multiplication between $\mathcal{U}_{2,2}$ and $\mathcal{V}_{2,2}$ is graded-supercommutative, and the multiplication is given by

$$\alpha_{0,i}\alpha_0 = \alpha_{0,i}, \quad \alpha_{0,1,i}\alpha_0 = \alpha_{0,1,i}, \quad \alpha_{0,2,i}\alpha_0 = \alpha_{0,2,i}, \quad \alpha_{0,1,2,i}\alpha_0 = \alpha_{0,1,2,i},$$
$$\alpha_{0,2,i}\alpha_{1,0} = (-1)^{i+1}\alpha_{0,i}\alpha_{1,2,0} = (-1)^{i+1}\alpha_{0,1,2,i},$$
$$\alpha_{0,2,i}\beta_0 = (-1)^{i+1}\alpha_{0,1,2,i+1}, \quad \alpha_{0,i+1}\alpha_{1,0} = (-1)^{i+1}\alpha_{0,1,i},$$
$$\alpha_{0,i}\beta_0 = (-1)^i\frac{i+2}{i+1}\alpha_{0,1,i}, \quad \alpha_{0,2,i}\alpha_1 = -\frac{1}{i+1}\alpha_{0,1,i}, \ i \geq 0.$$

Proof. Note that

$$(\bigwedge^k \mathcal{I}^*)^l_s = \begin{cases} \mathbb{F}Y_1^{*^{2k-l}} \wedge Y_2^{*^{l-k}}, & s=0; \\ \mathbb{F}X_1^* \wedge Y_1^{*^{2k-l-1}} \wedge Y_2^{*^{l-k}} \bigoplus \mathbb{F}X_2^* \wedge Y_1^{*^{2k-l}} \wedge Y_2^{*^{l-k-1}}, & s=1; \\ \mathbb{F}X_1^* \wedge X_2^* \wedge Y_1^{*^{2k-l-1}} \wedge Y_2^{*^{l-k-1}}, & s=2; \\ 0, & \text{else}. \end{cases}$$

Moreover, we have

$$\mathrm{Ker}\, D_s^{k,l} = \begin{cases} \mathbb{F}Y_1^{*^l}, & s=0, l=k; \\ \mathbb{F}X_1^* \wedge Y_1^{*^{l-1}}, & s=1, l=k; \\ \mathbb{F}(X_1^* \wedge Y_1^{*^{l-3}} \wedge Y_2^* - X_2^* \wedge Y_1^{*^{l-2}}), & s=1, l=k+1; \\ \mathbb{F}X_1^* \wedge X_2^* \wedge Y_1^{*^{l-3}}, & s=2, l=k+1; \\ 0, & \text{else}. \end{cases}.$$

$$\mathrm{Im}\, D_s^{k,l} = \begin{cases} \mathbb{F}Y_1^{*^{2k-l+1}} \wedge Y_2^{*^{l-k-1}}, & s=0, k+1 \leq l \leq 2k; \\ \mathbb{F}X_1^* \wedge Y_1^{*^{l-2}}, & s=1, l=k+1; \\ \mathbb{F}X_1^* \wedge Y_1^{*^{2k-l}} \wedge Y_2^{*^{l-k-1}} \bigoplus \mathbb{F}X_2^* \wedge Y_1^{*^{2k-l+1}} \wedge Y_2^{*^{l-k-2}}, & s=1, k+2 \leq l \leq 2k; \\ \mathbb{F}X_1^* \wedge X_2^* \wedge Y_1^{*^{2k-l}} \wedge Y_2^{*^{l-k-2}}, & s=2, k+2 \leq l \leq 2k-1; \\ 0, & \text{else}. \end{cases}.$$

From Theorem 1 and Eqs. (3.1.2), (3.1.3), $\mathrm{H}^{\bullet}(L_{3,1})$ has a basis:

$$Y_1^{*^i},\quad X_1^*\wedge Y_1^{*^i},\quad X_1^*\wedge X_2^*\wedge Y_1^{*^i},\quad X_1^*\wedge Y_1^{*^i}\wedge Y_2 - X_2^*\wedge Y_1^{*^{i+1}},$$

$$X_0^*\wedge Y_2^{*^i},\quad X_0^*\wedge X_1^*\wedge Y_2^{*^{i+1}},\quad X_0^*\wedge X_2^*\wedge Y_2^{*^i},\quad X_0^*\wedge X_1^*\wedge X_2^*\wedge Y_2^{*^i},\ i\geq 0.$$

The conclusion can be obtained by a direct calculation. □

3.2 Low-dimensional filiform Lie superalgebras

In this section, we describe the cohomology of $\mathcal{F}_{1,2}$, $\mathcal{F}_{2,2}$ by using the Hochschild-Serre spectral sequence.

Lemma 3.3. For $\mathcal{F}_{s,2}^{t_s}$, $s=1,2$, $t_1=1,2,3$, $t_2=1,3,4,5$, let $\mathcal{I}_{s,2}^{t_s}=[\mathcal{F}_{s,2}^{t_s},\mathcal{F}_{s,2}^{t_s}]$. For $k\geq 0$, $0\leq i\leq k$, the following conclusions hold:
(1) $\mathrm{E}_2^{k-i,i}=\bigwedge^{k-i}(\mathcal{F}_{s,2}^{t_s}/\mathcal{I}_{s,2}^{t_s})^*\bigotimes\bigwedge^{i}(\mathcal{I}_{s,2}^{t_s})^*\Longrightarrow \mathrm{H}^k(\mathcal{F}_{s,2}^{t_s})$.
(2) $\mathrm{E}_\infty^{k-i,i}=\mathrm{E}_3^{k-i,i}$.

Proof. (1) Since $\mathcal{I}_{s,2}^{t_s}\subseteq C(\mathcal{F}_{s,2}^{t_s})$, the action of $\mathcal{F}_{s,2}^{t_s}/\mathcal{I}_{s,2}^{t_s}$ on $(\mathcal{I}_{s,2}^{t_s})^*$ is trivial. Moreover, from Eq. (2.0.3), we have

$$\mathrm{E}_2^{k-i,i}=\mathrm{H}^{k-i}(\mathcal{F}_{s,2}^{t_s}/\mathcal{I}_{s,2}^{t_s},\mathrm{H}^i(\mathcal{I}_{s,2}^{t_s}))=\bigwedge^{k-i}(\mathcal{F}_{s,2}^{t_s}/\mathcal{I}_{s,2}^{t_s})^*\bigotimes\bigwedge^{i}(\mathcal{I}_{s,2}^{t_s})^*\Longrightarrow \mathrm{H}^k(\mathcal{F}_{s,2}^{t_s}).$$

(2) From (1), we have

$$\mathrm{E}_r^{k-i,i}=\mathrm{E}_3^{k-i,i},\quad r\geq 3.$$

Moreover, $\mathrm{E}_\infty^{k-i,i}=\mathrm{E}_3^{k-i,i}$. □

Theorem 3. The following $\mathbb{Z}$-graded superalgebra isomorphisms hold:
(1) $\mathrm{H}^{\bullet}(\mathcal{F}_{1,2}^2)\cong\mathcal{U}_{1,2}^2\ltimes\mathcal{V}_{1,2}^2$, where $\mathcal{U}_{1,2}^2$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_i,\alpha_{0,i}\mid i\geq 0\},$$

satisfying that

$$|\alpha_i|=|\alpha_{0,i}|=\bar{i}\ (\mathrm{mod}\ 2);\quad \|\alpha_i\|=i,\ \|\alpha_{0,i}\|=i+1,$$

and the multiplication is given by

$$\alpha_i\alpha_j=\alpha_{i+j},\quad \alpha_{0,i}\alpha_j=\alpha_{0,i+j},\ i,j\geq 0,$$

$\mathcal{V}_{1,2}^2$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_{0,1}^i,\alpha_{0,1,i}\mid i\geq 0\},$$

satisfying that

$$|\alpha_{0,1}^i|=|\alpha_{0,1,i}|=\bar{i}\ (\mathrm{mod}\ 2);\quad \|\alpha_{0,1}^i\|=i+1,\ \|\alpha_{0,1,i}\|=i+2,$$

and the trivial multiplication. The multiplication between $\mathcal{U}_{1,2}^2$ and $\mathcal{V}_{1,2}^2$ is graded-super -commutative, and the multiplication is given by

$$\alpha_{0,1}^i\alpha_0=\alpha_{0,1}^i,\quad \alpha_{0,1,i}\alpha_0=\alpha_{0,1,i},\quad \alpha_{0,0}\alpha_{0,1}^i=\alpha_{0,1,i},\ i\geq 0.$$

In particular,

$$\dim \mathrm{H}^k(\mathcal{F}^2_{1,2}) = \begin{cases} 1, & k=0; \\ 3, & k=1; \\ 4, & k \geq 2. \end{cases}$$

(2) $\mathrm{H}^\bullet(\mathcal{F}^3_{1,2}) \cong \mathcal{U}^3_{1,2}$, where $\mathcal{U}^3_{1,2}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_0, \alpha_1, \alpha_{0,i}, \beta_i \mid i \geq 0\},$$

satisfying that

$$|\alpha_0| = \bar{0} \ (\mathrm{mod}\ 2),\ |\alpha_1| = \bar{1} \ (\mathrm{mod}\ 2),\ |\alpha_{0,i}| = |\beta_i| = \bar{i} \ (\mathrm{mod}\ 2);$$

$$\|\alpha_0\| = 0,\ \|\alpha_1\| = 1,\ \|\alpha_{0,i}\| = i+1,\ \|\beta_i\| = i+2,$$

and the multiplication is given by

$$\alpha_0\alpha_0 = \alpha_0, \quad \alpha_0\alpha_1 = \alpha_1, \quad \alpha_0\alpha_{0,i} = \alpha_{0,i}, \quad \alpha_0\beta_i = \beta_i,\ i \geq 0.$$

In particular,

$$\dim \mathrm{H}^k(\mathcal{F}^3_{1,2}) = \begin{cases} 1, & k=0; \\ 2, & k \geq 1. \end{cases}$$

(3) $\mathrm{H}^\bullet(\mathcal{F}^3_{2,2}) \cong \mathcal{U}^3_{2,2}$, where $\mathcal{U}^3_{2,2}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_0, \alpha_1, \beta_1, \alpha_{1,1}, \alpha_{0,i}, \beta_{0,i}, \alpha_{0,1,i}, \beta_{0,1,i} \mid i \geq 0\},$$

satisfying that

$$|\alpha_0| = |\alpha_1| = \bar{0}\ (\mathrm{mod}\ 2),\ |\beta_1| = |\alpha_{1,1}| = \bar{1}\ (\mathrm{mod}\ 2),\ |\alpha_{0,i}| = |\alpha_{0,1,i}| = |\beta_{0,i}| = |\beta_{0,1,i}| = \bar{i}\ (\mathrm{mod}\ 2);$$

$$\|\alpha_0\| = 0,\ \|\alpha_1\| = \|\beta_1\| = 1,\ \|\alpha_{1,1}\| = 2,\ \|\alpha_{0,i}\| = i+1,\ \|\alpha_{0,1,i}\| = \|\beta_{0,i}\| = i+2,\ \|\beta_{0,1,i}\| = i+3,$$

and the multiplication is given by

$$\alpha_0\alpha_0 = \alpha_0, \quad \alpha_0\alpha_1 = \alpha_1, \quad \alpha_0\beta_1 = \beta_1, \quad \alpha_0\alpha_{1,1} = \alpha_{1,1},$$

$$\alpha_0\alpha_{0,i} = \alpha_{0,i}, \quad \alpha_0\alpha_{0,1,i} = \alpha_{0,1,i}, \quad \alpha_0\beta_{0,i} = \beta_{0,i}, \quad \alpha_0\beta_{0,1,i} = \beta_{0,1,i},$$

$$\alpha_1\beta_1 = \alpha_{1,1}, \quad \alpha_1\alpha_{0,i} = -\alpha_{0,1,i}, \quad \alpha_1\beta_{0,i} = \beta_{0,1,i}, \quad \beta_1\beta_1 = 2\alpha_{0,1,0},$$

$$\beta_1\beta_{0,i} = 2\alpha_{0,1,i+1}, \quad \beta_{0,i}\beta_{0,j} = 2\alpha_{0,1,i+j+2},\ i \geq 0.$$

In particular,

$$\dim \mathrm{H}^k(\mathcal{F}^3_{2,2}) = \begin{cases} 1, & k=0; \\ 3, & k=1; \\ 4, & k \geq 2. \end{cases}$$

(4) $\mathrm{H}^\bullet(\mathcal{F}^4_{2,2}) \cong \mathcal{U}^4_{2,2} \ltimes \mathcal{V}^4_{2,2}$, where $\mathcal{U}^4_{2,2}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_i, \beta_i \mid i \geq 0\},$$

satisfying that

$$|\alpha_i| = \bar{i}\ (\mathrm{mod}\ 2),\ |\beta_i| = \overline{i+1}\ (\mathrm{mod}\ 2); \quad \|\alpha_i\| = i,\ \|\beta_i\| = i+2,$$

and the multiplication is given by

$$\beta_i\alpha_0 = \beta_i, \quad \alpha_i\alpha_j = \alpha_{i+j}, \ i, j \geq 0.$$

$\mathcal{V}^4_{2,2}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha, \beta, \alpha_{0,i}, \alpha_{0,1,i}, \alpha_{0,2,i}, \alpha_{0,1,2,i}, \beta_{1,i}, \beta_{2,i} \mid i \geq 0\},$$

satisfying that

$$|\alpha| = |\beta| = 0 \ (\mathrm{mod}\ 2), \ |\alpha_{0,i}| = |\alpha_{0,2,i}| = |\alpha_{0,1,2,i}| = \bar{i} \ (\mathrm{mod}\ 2),$$

$$|\alpha_{0,1,i}| = |\beta_{1,i}| = |\beta_{2,i}| = \overline{i+1} \ (\mathrm{mod}\ 2);$$

$$\|\alpha\| = 1, \ \|\beta\| = 2, \ \|\alpha_{0,i}\| = i+1, \ \|\alpha_{0,2,i}\| = \|\beta_{1,i}\| = i+2,$$

$$\|\alpha_{0,1,i}\| = \|\alpha_{0,1,2,i}\| = \|\beta_{2,i}\| = i+3,$$

and the multiplication is given by

$$\alpha_{0,0}\beta_{2,i} = \alpha_{0,1,2,i+1}, \quad \alpha_{0,0}\beta = \alpha_{0,1,2,0}, \quad \alpha_{0,0}\beta_{1,0} = \alpha_{0,2,1} - \alpha_{0,1,0},$$

$$\alpha_{0,i}\beta_{1,j+1} = (-1)^i\alpha_{0,2,i+j+2}, \quad \alpha_{0,i+1}\beta_{1,j} = (-1)^{i+1}\alpha_{0,2,i+j+2},$$

$$\alpha_{0,2,0}\alpha = -\alpha_{0,1,2,0}, \quad \alpha\beta_{1,0} = -\alpha_{0,1,0} - \alpha_{0,2,1},$$

$$\alpha_{0,2,0}\beta_{1,0} = \alpha_{0,1,2,1}, \quad \beta_{2,i}\alpha = (-1)^i\alpha_{0,1,2,i+1},$$

$$\beta_{2,i}\beta_{1,0} = (-1)^{i+1}\alpha_{0,1,2,i+2}, \quad \alpha\beta_{1,i+1} = -\alpha_{0,2,i+2} \ i, j \geq 0.$$

The multiplication between $\mathcal{U}^4_{2,2}$ and $\mathcal{V}^4_{2,2}$ is graded-supercommutative, and the multiplication is given by

$$\alpha_{0,1,i}\alpha_0 = \alpha_{0,1,i}, \quad \alpha_{0,1,2,i}\alpha_0 = \alpha_{0,1,2,i}, \quad \beta_{2,i}\alpha_0 = \beta_{2,i}, \quad \alpha\alpha_0 = \alpha, \quad \beta\alpha_0 = \beta,$$

$$\alpha_{0,0}\beta_i = \alpha_{0,1,i}, \quad \alpha_{0,i}\alpha_j = \alpha_{0,i+j}, \quad \alpha_{0,2,i}\alpha_j = \alpha_{0,2,i+j}, \quad \beta_i\alpha = (-1)^{i+1}\alpha_{0,1,i},$$

$$\beta_{2,i}\alpha_1 = -\frac{1}{i+2}\alpha_{0,1,i+1}, \quad \alpha\alpha_{i+1} = -\alpha_{0,i+1}, \quad \alpha_i\beta_{1,j} = (-1)^i\beta_{1,i+j},$$

$$\alpha_{0,2,0}\beta_i = -\alpha_{0,1,2,i+1}, \quad \beta_i\beta = \alpha_{0,1,2,i+1}, \quad \beta_i\beta_{1,0} = (-1)^i\frac{i+3}{i+2}\alpha_{0,1,i+1},$$

$$\beta\alpha_1 = -\alpha_{0,1,0} - \alpha_{0,2,1}, \quad \beta\alpha_{i+2} = -\alpha_{0,2,i+2}, \ i, j \geq 0.$$

In particular,

$$\dim \mathrm{H}^k(\mathcal{F}^4_{2,2}) = \begin{cases} 1, & k = 0; \\ 3, & k = 1; \\ 6, & k = 2; \\ 8, & k \geq 3. \end{cases}$$

(5) $\mathrm{H}^\bullet(\mathcal{F}^5_{2,2}) \cong \mathcal{U}^5_{2,2}$, where $\mathcal{U}^5_{2,2}$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_0, \alpha_1, \alpha_{0,1,i}, \alpha^i_{0,1}, \beta_{1,i}, \beta_{2,i} \mid i \geq 0\},$$

satisfying that

$$|\alpha_0| = |\alpha_1| = \bar{0} \ (\mathrm{mod}\ 2), \ |\alpha_{0,1,i}| = |\alpha^i_{0,1}| = \bar{i} \ (\mathrm{mod}\ 2), \ |\beta_{1,i}| = |\beta_{2,i}| = \overline{i+1} \ (\mathrm{mod}\ 2);$$

$$\|\alpha_0\| = 0,\ \|\alpha_1\| = 1,\ \|\alpha_{0,1}^i\| = \|\beta_{1,i}\| = i+1,\ \|\alpha_{0,1,i}\| = \|\beta_{2,i}\| = i+2,$$

and the multiplication is given by

$$\alpha_0\alpha_0 = \alpha_0, \quad \alpha_0\alpha_1 = \alpha_1, \quad \alpha_0\alpha_{0,1,i} = \alpha_{0,1,i}, \quad \alpha_0\alpha_{0,1}^i = \alpha_{0,1}^i, \quad \alpha_0\beta_{1,i} = \beta_{1,i}, \quad \alpha_0\beta_{2,i} = \beta_{2,i},$$

$$\alpha_1\alpha_{0,1}^i = \alpha_{0,1,i}, \quad \alpha_1\beta_{1,i} = \beta_{2,i}, \quad \beta_{1,i_1}\beta_{1,i_2} = 2\alpha_{0,1,i_1+i_2},\ i, i_1, i_2 \geq 0.$$

In particular,

$$\dim \mathrm{H}^k(\mathcal{F}_{2,2}^5) = \begin{cases} 1, & k = 0; \\ 3, & k = 1; \\ 4, & k \geq 2. \end{cases}$$

Proof. (1) For $k \geq 0$, $0 \leq i \leq k$, consider the mapping

$$\begin{aligned} d_2^{k-i,i} : \bigwedge^{k-i}(\mathcal{F}_{1,2}^2/\mathbb{F}Y_2)^* \bigotimes \mathbb{F}{Y_2^*}^i &\longrightarrow \bigwedge^{k-i+2}(\mathcal{F}_{1,2}^2/\mathbb{F}Y_2)^* \bigotimes \mathbb{F}{Y_2^*}^{i-1}, \\ f \otimes {Y_2^*}^i &\longmapsto (-1)^{\|f\|} i f \wedge d(Y_2^*) \otimes {Y_2^*}^{i-1}, \end{aligned}$$

where

$$d(Y_2^*) = (X_0^* + X_1^*) \wedge Y_1^*.$$

By Lemma 3.3, we have

$$\mathrm{E}_\infty^{k-i,i} = \begin{cases} \dfrac{\bigwedge^k(\mathcal{F}_{1,2}^2/\mathbb{F}Y_2)^*}{\bigwedge^{k-2}(\mathcal{F}_{1,2}^2/\mathbb{F}Y_2)^* \bigwedge \mathbb{F}d(Y_2^*)}, & i = 0; \\ \dfrac{\mathbb{F}(X_0^*+X_1^*)\wedge Y_1^* {Y_1^*}^{k-i-1} \otimes {Y_2^*}^i \bigoplus \mathbb{F}X_0^*\wedge X_1^*\wedge {Y_1^*}^{k-i-2} \otimes {Y_2^*}^i}{\bigwedge^{k-2-i}(\mathcal{F}_{1,2}^2/\mathbb{F}Y_2)^* \bigwedge \mathbb{F}d(Y_2^*) \bigotimes \mathbb{F}{Y_2^*}^i}, & 1 \leq i \leq k-2; \\ \mathbb{F}(X_0^* + X_1^*) \bigotimes {Y_2^*}^{k-1}, & i = k-1; \\ 0, & i = k. \end{cases}$$

From $\mathrm{H}^k(\mathcal{F}_{1,2}^2) = \bigoplus_{i=0}^{k} \mathrm{E}_\infty^{k-i,i}$, we can obtain the conclusion.

The proofs of (2), (3), (4), (5) are similar to (1). □

Lemma 3.4. For $\mathcal{F}_{2,2}^2$, let $I = \mathrm{span}_{\mathbb{F}}\{X_1, X_2, Y_1, Y_2\}$.

(1) For $k \geq 0$, the following conclusion holds:

$$\mathrm{H}^k(F_{2,2}^2) \cong \mathrm{H}^0(\mathbb{F}X_0, \mathrm{H}^k(I)) \bigoplus \mathrm{H}^1(\mathbb{F}X_0, \mathrm{H}^{k-1}(I)).$$

(2) For $k \geq 2$, $\mathrm{H}^k(I)$ has a basis:

$$ {Y_2^*}^k, \quad X_1^* \wedge {Y_2^*}^{k-1} - X_2^* \wedge Y_1^* \wedge {Y_2^*}^{k-2},$$

(3) For $k \geq 2$, $\mathrm{H}^0(\mathbb{F}X_0, \mathrm{H}^k(I))$ has a basis:

$$ {Y_2^*}^k, \quad X_1^* \wedge {Y_2^*}^{k-1} - X_2^* \wedge Y_1^* \wedge {Y_2^*}^{k-2},$$

(4) For $k \geq 3$, $\mathrm{H}^1(\mathbb{F}X_0, \mathrm{H}^{k-1}(I))$ has a basis:

$$X_0^* \wedge {Y_2^*}^{k-1}, \quad X_0^* \wedge X_1^* \wedge {Y_2^*}^{k-2} - X_0^* \wedge X_2^* \wedge Y_1^* \wedge {Y_2^*}^{k-3}.$$

Proof. (1) Note I is an ideal of $\mathcal{F}_{2,2}^2$. From Eq. (2.0.3), we have

$$\mathrm{E}_\infty^{i,k-i} \cong \begin{cases} \mathrm{H}^0(\mathbb{F}X_0, \mathrm{H}^k(I)), & i=0; \\ \mathrm{H}^1(\mathbb{F}X_0, \mathrm{H}^{k-1}(I)), & i=1; \\ 0, & \text{else}. \end{cases}$$

Moreover, we have

$$\mathrm{H}^k(F_{2,2}^2) = \bigoplus_{i=0}^{k} \mathrm{E}_\infty^{i,k-i} \cong \mathrm{H}^0(\mathbb{F}X_0, \mathrm{H}^k(I)) \bigoplus \mathrm{H}^1(\mathbb{F}X_0, \mathrm{H}^{k-1}(I)).$$

(2) We use the Hochschild-Serre spectral sequence relative to the ideal $I_1 = [I, I]$. Note that $I_1 \subseteq C(I)$, we have

$$\mathrm{E}_2^{k-i,i} = \begin{cases} \bigwedge^{k-i}(I/I_1)^* \bigotimes \bigwedge^i I_1^*, & 0 \le i \le 2; \\ 0, & \text{else}. \end{cases}$$

Moreover, we have

$$\mathrm{E}_\infty^{k-i,i} = \mathrm{E}_3^{k-i,i} = \begin{cases} \mathbb{F}Y_2^{*^k}, & i=0; \\ \mathbb{F}(X_1^* \wedge Y_2^{*^{k-1}} - X_2^* \wedge Y_1^* \wedge Y_2^{*^{k-2}}), & i=1; \\ 0, & \text{else}. \end{cases}$$

Form $\mathrm{H}^k(I) = \bigoplus_{i=0}^{k} \mathrm{E}_\infty^{k-i,i}$, we can obtain the conclusion.

(3) By the definitions of the low cohomology (see [10]), we have

$$\mathrm{H}^0(\mathbb{F}X_0, \mathrm{H}^k(I)) = \mathrm{H}^k(I).$$

(4) From (3), we have

$$\mathrm{H}^1(\mathbb{F}X_0, \mathrm{H}^{k-1}(I)) = \mathbb{F}X_0^* \bigwedge \mathrm{H}^{k-1}(I).$$

□

Theorem 4. The following $\mathbb{Z}$-graded superalgebra isomorphism holds:

$$\mathrm{H}^\bullet(F_{2,2}^2) \cong \mathcal{U}_{2,2}^2,$$

where $\mathcal{U}_{2,2}^2$ is an infinite-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_0, \alpha_1, \alpha_{0,i}, \beta_0, \beta_{1,i}, \beta_{2,i}, \beta_{3,i} \mid i \ge 0\},$$

satisfying that

$$|\alpha_0| = \bar{0} \pmod 2,\ |\alpha_1| = |\beta_0| = \bar{1} \pmod 2,$$
$$|\alpha_{0,i}| = |\beta_{1,i}| = |\beta_{3,i}| = \bar{i} \pmod 2,\ |\beta_{2,i}| = \overline{i+1} \pmod 2;$$
$$\|\alpha_0\| = 0,\ \|\alpha_1\| = 1,\ \|\beta_0\| = 2,\ \|\alpha_{0,i}\| = i+1,\ \|\beta_{1,i}\| = i+2,\ \|\beta_{2,i}\| = \|\beta_{3,i}\| = i+3,$$

and the multiplication is given by

$$\alpha_0\alpha_0 = \alpha_0, \quad \alpha_0\alpha_1 = \alpha_1, \quad \alpha_0\alpha_{0,i} = \alpha_{0,i}, \quad \alpha_0\beta_0 = \beta_0, \quad \alpha_0\beta_{1,i} = \beta_{1,i}, \quad \alpha_0\beta_{2,i} = \beta_{2,i},$$

$$\alpha_0\beta_{3,i}=\beta_{3,i},\quad \beta_{1,i}\alpha_1=2\beta_{2,i},\quad \alpha_{0,i}\beta_0=(-1)^i\beta_{2,i},\quad \alpha_{0,i}\beta_{1,j}=\alpha_{0,i+j+2},$$
$$\alpha_{0,i}\beta_{3,j}=(-1)^i\beta_{2,i+j+1},\quad \beta_{1,i}\beta_0=(-1)^i\beta_{3,i+1},\quad \beta_{1,i}\beta_{1,j}=\beta_{1,i+j+2},$$
$$\beta_{2,i}\beta_{1,j}=\beta_{2,i+j+2},\quad \beta_{3,i}\beta_{1,j}=\beta_{3,i+j+2},\ i,j\geq 0.$$

In particular,

$$\dim \mathrm{H}^k(\mathcal{F}^2_{2,2})=\begin{cases}1, & k=0;\\ 2, & k=1;\\ 3, & k=2;\\ 4, & k\geq 3.\end{cases}$$

Proof. For $k\geq 3$, by Lemma 3.4, let

$$\begin{aligned}\Phi:\mathrm{H}^0(\mathbb{F}X_0,\mathrm{H}^k(I))&\longrightarrow \mathrm{H}^k(\mathcal{F}^2_{2,2}),\\ {Y_2^*}^k&\longmapsto {Y_2^*}^k-2kX_0^*\wedge X_2^*\wedge {Y_2^*}^{k-2},\\ X_1^*\wedge {Y_2^*}^{k-1}-X_2^*\wedge Y_1^*\wedge {Y_2^*}^{k-2}&\longmapsto X_1^*\wedge {Y_2^*}^{k-1}-X_2^*\wedge Y_1^*\wedge {Y_2^*}^{k-2}\\ &\qquad +2(k-2)X_0^*\wedge X_1^*\wedge X_2^*\wedge {Y_2^*}^{k-3},\end{aligned}$$

and

$$\begin{aligned}\Psi:\mathrm{H}^1(\mathbb{F}X_0,\mathrm{H}^{k-1}(I))&\longrightarrow \mathrm{H}^k(\mathcal{F}^2_{2,2}),\\ X_0^*\wedge {Y_2^*}^{k-1}&\longmapsto X_0^*\wedge {Y_2^*}^{k-1},\\ X_0^*\wedge X_1^*\wedge {Y_2^*}^{k-2}-X_0^*\wedge X_2^*\wedge Y_1^*\wedge {Y_2^*}^{k-3}&\longmapsto X_0^*\wedge X_1^*\wedge {Y_2^*}^{k-2}-X_0^*\wedge X_2^*\wedge Y_1^*\wedge {Y_2^*}^{k-3}.\end{aligned}$$

Then Φ and Ψ are injective linear mappings, and $\mathrm{Im}\ \Phi\cap \mathrm{Im}\ \Psi=0$. Moreover, we have

$$\mathrm{H}^k(\mathcal{F}^2_{2,2})=\Phi(\mathrm{H}^0(\mathbb{F}X_0,\mathrm{H}^k(I)))\bigoplus\Psi(\mathrm{H}^1(\mathbb{F}X_0,\mathrm{H}^{k-1}(I))).$$

Thus, $\mathrm{H}^\bullet(\mathcal{F}^2_{2,2})$ has a basis:

$$1,\quad Y_1^*,\quad X_0^*\wedge {Y_2^*}^i,\quad X_1^*\wedge Y_2^*-X_2^*\wedge Y_1^*,\quad {Y_2^*}^{i+2}-2(i+2)X_0^*\wedge X_2^*\wedge {Y_2^*}^i,$$
$$X_0^*\wedge X_1^*\wedge {Y_2^*}^{i+1}-X_0^*\wedge X_2^*\wedge Y_1^*\wedge {Y_2^*}^i,$$
$$X_1^*\wedge {Y_2^*}^{i+2}-X_2^*\wedge Y_1^*\wedge {Y_2^*}^{i+1}+2(i+1)X_0^*\wedge X_1^*\wedge X_2^*\wedge {Y_2^*}^i,\ i\geq 0.$$

The conclusion can be obtained by a direct calculation. □

4 Characteristic $p>3$

Throughout this section the ground field $\mathbb{F}$ is an algebraically closed field of characteristic $p>3$.

Lemma 4.1. For $\mathcal{F}^{t_s}_{s,2}$, $s=1,2$, $t_1=1,2,3$, $t_2=1,3,4,5$, let $\mathcal{I}^{t_s}_{s,2}=[\mathcal{F}^{t_s}_{s,2},\mathcal{F}^{t_s}_{s,2}]$. For $k\geq 0$, $0\leq i\leq k$, the following conclusions hold:
(1) $\mathrm{E}_2^{k-i,i}=\mathcal{O}_{k-i}((\mathcal{F}^{t_s}_{s,2}/\mathcal{I}^{t_s}_{s,2})^*;\underline{t})\bigotimes\mathcal{O}_i((\mathcal{I}^{t_s}_{s,2})^*;\underline{t})\Longrightarrow \mathrm{DPH}^k(\mathcal{F}^{t_s}_{s,2})$.
(2) $\mathrm{E}_\infty^{k-i,i}=\mathrm{E}_3^{k-i,i}$.

Proof. The proof is similar to Lemma 3.3. □

4.1 Divided power cohomology of $\mathcal{F}_{1,2}$

Theorem 5. The following $\mathbb{Z}$-graded superalgebra isomorphisms hold:

(1) $\mathrm{DPH}^{\bullet}(\mathcal{F}^1_{1,2}) \cong \mathcal{U}^1_{1,2}(\underline{t})$, where $\mathcal{U}^1_{1,2}(\underline{t})$ is a $p^{t_1-1}(4p^{t_2}+2p-2)$-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_0^{i,j}, \alpha_{1,k}, \alpha_k, \alpha_{0,1}^{i,j}, \alpha_1^{s,t}, \alpha^{s,t} \mid 0 \le i \le p^{t_1-1}-1, 0 \le j \le p^{t_2}-1, 0 \le k \le p^{t_1}-1,$$
$$1 \le s \le p^{t_1-1}, 1 \le t \le p^{t_2}-1\},$$

satisfying that

$$|\alpha_0^{i,j}| = |\alpha_{0,1}^{i,j}| = \overline{i+j} \ (\mathrm{mod}\ 2),\ |\alpha_k| = |\alpha_{1,k}| = \overline{k} \ (\mathrm{mod}\ 2),\ |\alpha^{s,t}| = |\alpha_1^{s,t}| = \overline{s+t-1} \ (\mathrm{mod}\ 2);$$

$$\|\alpha_0^{i,j}\| = ip+j+1,\ \|\alpha_{0,1}^{i,j}\| = ip+j+2,\ \|\alpha_k\| = k,\ \|\alpha_{1,k}\| = k+1,\ \|\alpha^{s,t}\| = sp+t-1,\ \|\alpha_1^{s,t}\| = sp+t,$$

and the multiplication is given by

$$\alpha_0^{i_1,j}\alpha_{1,i_2p} = (-1)^{i_1+j}\binom{(i_1+i_2)p}{i_1p}\alpha_{0,1}^{i_1+i_2,j}, \quad \alpha_0^{i_1,j}\alpha_{i_2p} = \binom{(i_1+i_2)p}{i_1p}\alpha_0^{i_1+i_2,j},$$

$$\alpha_{1,k_1}\alpha_{k_2} = \binom{k_1+k_2}{k_1}\alpha_{1,k_1+k_2}, \quad \alpha_{1,ip}\alpha^{s,t} = \binom{(i+s)p-1}{ip}\alpha_1^{i+s,t},$$

$$\alpha_{k_1}\alpha_{k_2} = \binom{k_1+k_2}{k_1}\alpha_{k_1+k_2}, \quad \alpha_{0,1}^{i_1,j}\alpha_{i_2p} = \binom{(i_1+i_2)p}{i_1p}\alpha_{0,1}^{i_1+i_2,j},$$

$$\alpha_1^{s,t}\alpha_{ip} = \binom{(i+s)p-1}{ip}\alpha_1^{i+s,t}, \quad \alpha_{ip}\alpha^{s,t} = \binom{(i+s)p-1}{ip}\alpha^{i+s,t},$$

where $0 \le i, i_1, i_2 \le p^{t_1-1}-1$, $0 \le j \le p^{t_2}-1$, $0 \le k_1, k_2 \le p^{t_1}-1$, $1 \le s \le p^{t_1-1}$, $1 \le t \le p^{t_2}-1$.

(2) $\mathrm{DPH}^{\bullet}(\mathcal{F}^2_{1,2}) \cong \mathcal{U}^2_{1,2}(\underline{t})$, where $\mathcal{U}^2_{1,2}(\underline{t})$ is a $p^{t_1-1}(4p^{t_2}+2p-2)$-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_0^{i,j}, \alpha_{0,1}^{s,k}, \alpha_{i,j}, \alpha_0^l, \alpha_l, \alpha_{0,1,s,k} \mid 1 \le i \le p^{t_1-1}, 1 \le j \le p^{t_2}-1, 0 \le s \le p^{t_1-1}-1,$$
$$0 \le k \le p^{t_2}-1, 0 \le l \le p^{t_1}-1\},$$

satisfying that

$$|\alpha_0^{i,j}| = |\alpha_{i,j}| = \overline{i+j-1} \ (\mathrm{mod}\ 2),\ |\alpha_{0,1}^{s,k}| = |\alpha_{0,1,s,k}| = \overline{s+k} \ (\mathrm{mod}\ 2),\ |\alpha_0^l| = |\alpha_l| = \overline{l} \ (\mathrm{mod}\ 2);$$

$$\|\alpha_0^{i,j}\| = ip+j,\ \|\alpha_{i,j}\| = ip+j-1,\ \|\alpha_{0,1}^{s,k}\| = sp+k+1,\ \|\alpha_{0,1,s,k}\| = sp+k+2,\ \|\alpha_0^l\| = l+1,\ \|\alpha_l\| = l,$$

and the multiplication is given by

$$\alpha_0^{i,j}\alpha_{sp} = \binom{(i+s)p-1}{sp}\alpha_0^{i+s,j}, \quad \alpha_0^{sp}\alpha_{i,j} = \binom{(i+s)p-1}{sp}\alpha_0^{i+s,j},$$

$$\alpha_{i,j}\alpha_{sp} = \binom{(i+s)p-1}{sp}\alpha_{i+s,j}, \quad \alpha_0^{s_1p}\alpha_{0,1}^{s_2,k} = (-1)^{s_1}\binom{(s_1+s_2)p}{s_1p}\alpha_{0,1,s_1+s_2,k},$$

$$\alpha_{0,1}^{s_1,k}\alpha_{s_2p} = \binom{(s_1+s_2)p}{s_1p}\alpha_{0,1}^{s_1+s_2,k}, \quad \alpha_{0,1,s_1,k}\alpha_{s_2p} = \binom{(s_1+s_2)p}{s_1p}\alpha_{0,1,s_1+s_2,k},$$

$$\alpha_0^{l_1}\alpha_{l_2}=\begin{pmatrix} l_1+l_2 \\ l_1 \end{pmatrix}\alpha_0^{l_1+l_2},\quad \alpha_{l_1}\alpha_{l_2}=\begin{pmatrix} l_1+l_2 \\ l_1 \end{pmatrix}\alpha_{l_1+l_2},$$

where $1\le i\le p^{t_1-1}$, $1\le j\le p^{t_2}-1$, $0\le s,s_1,s_2\le p^{t_1-1}-1$, $0\le k\le p^{t_2}-1$, $0\le l,l_1,l_2\le p^{t_1}-1$.

(3) $\mathrm{DPH}^{\bullet}(\mathcal{F}^3_{1,2})\cong\mathcal{U}^3_{1,2}(\underline{t})$, where $\mathcal{U}^3_{1,2}(\underline{t})$ is a $p^{t_1-1}(4p^{t_2}+1)$-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_{0,1}^{i,j},\alpha_i,\alpha_i^1,\alpha_0^{i,k},\alpha_1^{s,k},\alpha_{1,s},\alpha^{s,t}\mid 0\le i\le p^{t_1-1}-1,0\le j\le p^{t_2}-2,0\le k\le p^{t_2}-1,$$
$$1\le s\le p^{t_1-1},1\le t\le p^{t_2}-1\},$$

satisfying that

$$|\alpha_{0,1}^{i,j}|=\overline{i+j}\ (\mathrm{mod}\ 2),\ |\alpha_i|=\bar{i}\ (\mathrm{mod}\ 2),\ |\alpha_i^1|=\overline{i+1}\ (\mathrm{mod}\ 2),$$
$$|\alpha_0^{i,k}|=\overline{i+k}\ (\mathrm{mod}\ 2),\ |\alpha_1^{s,k}|=\overline{s+k-1}\ (\mathrm{mod} 2),\ |\alpha_{1,s}|=\bar{s}\ (\mathrm{mod}\ 2),\ |\alpha^{s,t}|=\overline{s+t-1}\ (\mathrm{mod}\ 2);$$
$$\|\alpha_{0,1}^{i,j}\|=ip+j+2,\ \|\alpha_i\|=ip,\ \|\alpha_i^1\|=ip+1,\ \|\alpha_0^{i,k}\|=ip+k+1,$$
$$\|\alpha_1^{s,k}\|=sp+k,\ \|\alpha_{1,s}\|=sp-1,\ \|\alpha^{s,t}\|=sp+t-1,$$

and the multiplication is given by

$$\alpha_{0,1}^{i_1,j}\alpha_{i_2}=\begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix}\alpha_{0,1}^{i_1+i_2,j},\quad \alpha_{i_1}\alpha_{i_2}=\begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix}\alpha_{i_1+i_2},$$
$$\alpha_{i_1}\alpha_{i_2}^1=\begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix}\alpha_{i_1+i_1}^1,\quad \alpha_0^{i_1,k}\alpha_{i_2}=\begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix}\alpha_0^{i_1+i_2,k},$$
$$\alpha_1^{s,k}\alpha_i=\begin{pmatrix} (i+s)p-1 \\ ip \end{pmatrix}\alpha_1^{i+s,k},\quad \alpha_{1,s}\alpha_i=\begin{pmatrix} (i+s)p-2 \\ ip \end{pmatrix}\alpha_{1,i+s},$$
$$\alpha^{s,t}\alpha_i=\begin{pmatrix} (i+s)p-1 \\ ip \end{pmatrix}\alpha^{i+s,t},\quad \alpha_{1,s}\alpha_i^1=\begin{pmatrix} (i+s)p-1 \\ ip+1 \end{pmatrix}\alpha_1^{i+s,0},$$
$$\alpha_0^{i,k}\alpha_{1,s}=(-1)^{i+k+1}\begin{pmatrix} (i+s)p-2 \\ ip \end{pmatrix}\frac{1}{2}\alpha^{i+s,k+1},$$

where $0\le i,i_1,i_2\le p^{t_1-1}-1$, $0\le j\le p^{t_2}-2$, $0\le k\le p^{t_2}-1$, $1\le s\le p^{t_1-1}$, $1\le t\le p^{t_2}-1$.

Proof. (1) For $k\ge 0$, $0\le i\le k$, consider the mapping

$$d_2^{k-i,i}:\mathcal{O}_{k-i}((\mathcal{F}^1_{1,2}/\mathbb{F}Y_2)^*;\underline{t})\bigotimes\mathbb{F}Y_2^{*(i)}\longrightarrow\mathcal{O}_{k-i+2}((\mathcal{F}^1_{1,2}/\mathbb{F}Y_2)^*;\underline{t})\bigotimes\mathbb{F}Y_2^{*(i-1)},$$
$$f\otimes Y_2^{*(i)}\longmapsto(-1)^{\|f\|}f\wedge d(Y_2^*)\otimes Y_2^{*(i-1)},$$

where

$$d(Y_2^*)=X_0^*\wedge Y_1^*.$$

By Lemma 4.1, we have

$$\mathrm{E}_\infty^{k-i,i}=\begin{cases}\dfrac{\mathcal{O}_k((\mathcal{F}^1_{1,2}/\mathbb{F}Y_2)^*;\underline{t})}{\mathcal{O}_{k-2}((\mathcal{F}^1_{1,2}/\mathbb{F}Y_2)^*;\underline{t})\bigwedge\mathbb{F}d(Y_2^*)}, & i=0;\\ \dfrac{\mathrm{Ker}\ d_\infty^{k-i,i}}{\mathcal{O}_{k-2-i}((\mathcal{F}^2_{1,2}/\mathbb{F}Y_2)^*;\underline{t})\bigwedge\mathbb{F}d(Y_2^*)\bigotimes\mathbb{F}Y_2^{*(i)}}, & 1\le i\le k-2;\\ \mathbb{F}X_0^*\bigotimes Y_2^{*(k-1)}, & i=k-1;\\ 0, & i=k.\end{cases}$$

where

$$\text{Ker } d_\infty^{k-i,i} = \mathbb{F}X_0^* \wedge Y_1^{*(k-i-1)} \otimes Y_2^{*(i)} \bigoplus \mathbb{F}X_0^* \wedge X_1^* \wedge Y_1^{*(k-i-2)} \otimes Y_2^{*(i)}$$

$$\bigoplus \mathbb{F}\delta_{k-i} X_1^* \wedge Y_1^{*(k-i-1)} \otimes Y_2^{*(i)} \bigoplus \mathbb{F}\delta_{k+1-i} Y_1^{*(k-i)} \otimes Y_2^{*(i)},$$

where $\delta_a = 1$ when $a = 0(\text{mod } p)$ and $\delta_a = 0$ otherwise. From $\mathrm{DPH}^k(\mathcal{F}_{1,2}^1) = \bigoplus_{i=0}^{k} \mathrm{E}_\infty^{k-i,i}$, we can obtain the conclusion.

The proofs of (2), (3) are similar to (1). □

4.2 Divided power cohomology of $\mathcal{F}_{2,2}$

Theorem 6. The following $\mathbb{Z}$-graded superalgebra isomorphisms hold:

(1) $\mathrm{DPH}^\bullet(\mathcal{F}_{2,2}^1) \cong \mathcal{U}_{2,2}^1(\underline{t})$, where $\mathcal{U}_{2,2}^1(\underline{t})$ is a $p^{t_1-1}(8p^{t_2}+4p-7)$-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_i, \alpha_0^{s,j}, \alpha_1^i, \alpha_{0,2}^{s,j}, \alpha_{0,1}^{s,t}, \alpha_{0,1,2}^{s,j}, \alpha_{u,t}, \alpha_1^{u,v}, \alpha_{1,2}^{u,t}, \beta_{u,h}, \alpha_{1,2}^i, \beta_i \mid 0 \le i \le p^{t_1}-1, 0 \le j \le p^{t_2}-1,$$

$$0 \le s \le p^{t_1-1}-1, 1 \le t \le p^{t_2}-1, 1 \le u \le p^{t_1-1}, 2 \le v \le p^{t_2}-1, 1 \le h \le p^{t_2}-2\},$$

satisfying that

$$|\alpha_i| = |\alpha_1^i| = |\alpha_{1,2}^i| = \bar{i} \,(\text{mod } 2),\ |\beta_i| = \overline{i+1} \,(\text{mod } 2),\ |\alpha_0^{s,j}| = |\alpha_{0,2}^{s,j}| = |\alpha_{0,1,2}^{s,j}| = \overline{s+j} \,(\text{mod } 2),$$

$$|\alpha_{0,1}^{s,t}| = \overline{s+t} \ (\text{mod } 2),\ |\alpha_{u,t}| = |\alpha_{1,2}^{u,t}| = \overline{u+t-1} \ (\text{mod } 2),$$

$$|\alpha_1^{u,v}| = \overline{u+v-1} \ (\text{mod} 2),\ |\beta_{u,h}| = \overline{u+h-1} \ (\text{mod } 2);$$

$$\|\alpha_i\| = i,\ \|\alpha_1^i\| = i+1,\ \|\alpha_{1,2}^i\| = \|\beta_i\| = i+2,\ \|\alpha_0^{s,j}\| = sp+j+1,$$

$$\|\alpha_{0,2}^{s,j}\| = sp+j+2,\ \|\alpha_{0,1,2}^{s,j}\| = sp+j+3,\ \|\alpha_{0,1}^{s,t}\| = sp+t+2,$$

$$\|\alpha_{u,t}\| = up+t-1,\ \|\alpha_{1,2}^{u,t}\| = up+t+1,\ \|\alpha_1^{u,v}\| = up+v,\ \|\beta_{u,h}\| = up+h,$$

and the multiplication is given by

$$\alpha_{i_1}\alpha_{i_2} = \binom{i_1+i_2}{i_1}\alpha_{i_1+i_2}, \quad \alpha_1^{i_1}\alpha_{i_2} = \binom{i_1+i_2}{i_1}\alpha_1^{i_1+i_2},$$

$$\alpha_{1,2}^{i_1}\alpha_{i_2} = \binom{i_1+i_2}{i_1}\alpha_{1,2}^{i_1+i_2}, \quad \beta_{i_1}\alpha_{i_2} = \binom{i_1+i_2}{i_1}\beta_{i_1+i_2},$$

$$\alpha_0^{s_1,j}\alpha_{s_2p} = \binom{(s_1+s_2)p}{s_1p}\alpha_0^{s_1+s_2,j}, \quad \alpha_{0,2}^{s_1,j}\alpha_{s_2p} = \binom{(s_1+s_2)p}{s_1p}\alpha_{0,2}^{s_1+s_2,j},$$

$$\alpha_{0,1}^{s_1,t}\alpha_{s_2p} = \binom{(s_1+s_2)p}{s_1p}\alpha_{0,1}^{s_1+s_2,t}, \quad \alpha_{0,1,2}^{s_1,j}\alpha_{s_2p} = \binom{(s_1+s_2)p}{s_1p}\alpha_{0,1,2}^{s_1+s_2,j},$$

$$\alpha_{0,2}^{s_1,j}\alpha_{s_2p+1} = -\binom{(s_1+s_2)p}{s_1p}\alpha_{0,1}^{s_1+s_2,j+1}, \quad \alpha_0^{s_1,j}\alpha_1^{s_2p} = (-1)^{s_1+j}\binom{(s_1+s_2)p}{s_1p}\alpha_{0,1}^{s_1+s_2,j},$$

$$\alpha_0^{s_1,j}\alpha_{1,2}^{s_2p} = \binom{(s_1+s_2)p}{s_1p}\alpha_{0,1,2}^{s_1+s_2,j}, \quad \alpha_1^{s_1p}\alpha_{0,2}^{s_2,j} = -\binom{(s_1+s_2)p}{s_1p}\alpha_{0,1,2}^{s_1+s_2,j},$$

$$\alpha_{sp}\alpha_{u,t} = \binom{(s+u)p-1}{sp}\alpha_{s+u,t}, \quad \alpha_1^{u,v}\alpha_{sp} = \binom{(s+u)p-1}{sp}\alpha_1^{s+u,v},$$

$$\alpha_{1,2}^{u,t}\alpha_{sp} = \binom{(s+u)p-1}{sp}\alpha_{1,2}^{s+u,t}, \quad \beta_{u,h}\alpha_{sp} = \binom{(s+u)p-1}{sp}\beta_{s+u,h},$$

$$\beta_{u,h}\alpha_{sp+1} = -\binom{(s+u)p-1}{sp}\alpha_1^{s+u,h+1}, \quad \alpha_1^{sp}\alpha_{u,t} = \binom{(s+u)p-1}{sp}\alpha_1^{s+u,t},$$

$$\alpha_1^{sp}\beta_{u,h} = (-1)^s\binom{(s+u)p-1}{sp}\alpha_{1,2}^{s+u,h}, \quad \alpha_{1,2}^{sp}\alpha_{u,t} = \binom{(s+u)p-1}{sp}\alpha_{1,2}^{s+u,t},$$

$$\beta_{sp}\alpha_{u,t} = \binom{(s+u)p-1}{sp}(t+1)\alpha_1^{s+u,t+1}, \quad \beta_{u,h}\beta_{sp} = (-1)^{u+h}\binom{(s+u)p-1}{sp}h\alpha_{1,2}^{u+s,h+1},$$

$$\alpha_0^{s_1,j}\beta_{s_2p} = (-1)^{s_1+j}\binom{(s_1+s_2)p}{s_1p}(j+2)\alpha_{0,1}^{s_1+s_2,j+1},$$

$$\alpha_{0,2}^{s_1,j}\beta_{s_2p} = (-1)^{s_1+j+1}\binom{(s_1+s_2)p}{s_1p}(j+1)\alpha_{0,1,2}^{s_1+s_2,j+1},$$

$$\alpha_1^{i_1}\beta_{i_2} = (-1)^{i_1+1}\binom{i_1+i_2+1}{i_1}(i_2+1)\alpha_{1,2}^{i_1+i_2+1}, \quad 0 \le i_1, i_2 \le p^{t_1}-1,$$

where $0 \le s, s_1, s_2 \le p^{t_1-1}-1$, $0 \le j \le p^{t_2}-1$, $1 \le t \le p^{t_2}-1$, $1 \le u \le p^{t_1-1}$, $2 \le v \le p^{t_2}-1$, $1 \le h \le p^{t_2}-2$.

(2) $\mathrm{DPH}^\bullet(\mathcal{F}_{2,2}^3) \cong \mathcal{U}_{2,2}^3(\underline{t})$, where $\mathcal{U}_{2,2}^3(\underline{t})$ is a $p^{t_1-1}(8p^{t_2}+1)$-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_1^i, \alpha_1^{i,1}, \alpha_0^{i,j}, \alpha_{0,1}^{i,j}, \alpha_{1,2}^s, \alpha_{1,2}^{s,j}, \alpha_{1,s,h}, \beta_s, \beta_{i,t}, \beta_{1,i,t}, \beta_{2,s,t}, \gamma_i, \gamma_i^1, \gamma_{s,h} \mid 0 \le i \le p^{t_1-1}-1,$$
$$0 \le j \le p^{t_2}-1, 1 \le s \le p^{t_1-1}, 0 \le t \le p^{t_2}-2, 1 \le h \le p^{t_2}-1\},$$

satisfying that

$$|\alpha_1^i| = |\gamma_i| = \bar{i} \ (\mathrm{mod}\ 2), \ |\alpha_1^{i,1}| = |\gamma_i^1| = \overline{i+1} \ (\mathrm{mod}\ 2), \ |\alpha_{1,2}^s| = |\beta_s| = \bar{s} \ (\mathrm{mod}\ 2),$$

$$|\alpha_0^{i,j}| = |\alpha_{0,1}^{i,j}| = \overline{i+j} \ (\mathrm{mod}\ 2), \ |\beta_{i,t}| = |\beta_{1,i,t}| = \overline{i+t} \ (\mathrm{mod}\ 2), \ |\beta_{2,s,t}| = \overline{s+t-1} \ (\mathrm{mod} 2),$$

$$|\alpha_{1,2}^{s,j}| = \overline{s+j-1} \ (\mathrm{mod}\ 2), \ |\alpha_{1,s,h}| = |\gamma_{s,h}| = \overline{s+h-1} \ (\mathrm{mod}\ 2);$$

$$\|\gamma_i\| = ip, \ \|\alpha_1^i\| = \|\gamma_i^1\| = ip+1, \ \|\alpha_1^{i,1}\| = ip+2, \ \|\beta_s\| = sp-1,$$

$$\|\alpha_{1,2}^s\| = sp, \ \|\alpha_0^{i,j}\| = ip+j+1, \ \|\alpha_{0,1}^{i,j}\| = ip+j+2, \ \|\beta_{i,t}\| = ip+t+2,$$

$$\|\beta_{1,i,t}\| = ip+t+3, \ \|\beta_{2,s,t}\| = sp+t, \ \|\alpha_{1,2}^{s,j}\| = sp+j+1, \ \|\alpha_{1,s,h}\| = sp+h, \ \|\gamma_{s,h}\| = sp+h-1,$$

and the multiplication is given by

$$\alpha_{1,s,h}\gamma_i = \binom{(i+s)p-1}{ip}\alpha_{1,i+s,h}, \quad \alpha_{1,2}^{s,j}\gamma_i = \binom{(i+s)p-1}{ip}\alpha_{1,2}^{i+s,j},$$

$$\gamma_{s,h}\gamma_i = \binom{(i+s)p-1}{ip}\gamma_{i+s,h}, \quad \alpha_{1,2}^s\gamma_i = \binom{(i+s)p-1}{ip}\alpha_{1,2}^{i+s},$$

$$\beta_s\gamma_i = \binom{(i+s)p-1}{ip}\beta_{i+s}, \quad \beta_{2,s,t}\gamma_i^1 = \binom{(i+s)p-1}{ip+1}\alpha_{1,i+s,t+1},$$

$$\alpha_{1,2}^s\gamma_i^1 = \binom{(i+s)p-1}{ip+1}\alpha_{1,2}^{i+s,0}, \quad \beta_s\gamma_i^1 = \binom{(i+s)p-1}{ip+1}2\beta_{2,i+s,0},$$

$$\alpha_1^{i_1}\beta_{i_2,t} = \binom{(i_1+i_2)p}{i_1p}\beta_{1,i_1+i_2,t}, \quad \alpha_0^{i,j}\alpha_{1,2}^s = \binom{(i+s)p-1}{ip}\frac{1}{2}\alpha_{1,i+s,j+1},$$

$$\alpha_{0,1}^{i,j}\beta_s = (-1)^{i+j+1}\binom{(i+s)p-1}{ip}\alpha_{1,i+s,j+1}, \quad \alpha_1^i\gamma_{s,h} = \binom{(i+s)p-1}{ip}\alpha_{1,i+s,h},$$

$$\alpha_1^i\beta_{2,s,t} = (-1)^i\binom{(i+s)p-1}{ip}\alpha_{1,2}^{i+s,t}, \quad \alpha_1^i\beta_s = (-1)^i\binom{(i+s)p-1}{ip}2\alpha_{1,2}^{i+s},$$

$$\alpha_1^{i,1}\beta_s = (-1)^{i+1}\binom{(i+s)p-1}{ip+1}2\alpha_{1,2}^{i+s,0}, \quad \beta_{2,s,t}\gamma_i = \binom{(i+s)p-1}{ip}\beta_{2,i+s,t},$$

$$\alpha_{1,2}^s\beta_{i,t} = \binom{(i+s)p-1}{ip+1}\alpha_{1,2}^{i+s,t+1}, \quad \beta_{i,t}\beta_s = (-1)^{i+t}\binom{(i+s)p-1}{ip+1}2\beta_{2,i+s,t+1},$$

$$\gamma_{i_1}\gamma_{i_2} = \binom{(i_1+i_2)p}{i_1p}\gamma_{i_1+i_2}, \quad \gamma_{i_1}\gamma_{i_2}^1 = \binom{(i_1+i_2)p}{i_1p}\gamma_{i_1+i_2}^1,$$

$$\alpha_0^{i_1,j}\gamma_{i_2} = \binom{(i_1+i_2)p}{i_1p}\alpha_0^{i_1+i_2,j}, \quad \alpha_{0,1}^{i_1,j}\gamma_{i_2} = \binom{(i_1+i_2)p}{i_1p}\alpha_{0,1}^{i_1+i_2,j},$$

$$\alpha_1^{i_1}\gamma_{i_2} = \binom{(i_1+i_2)p}{i_1p}\alpha_1^{i_1+i_2}, \quad \alpha_1^{i_1,1}\gamma_{i_2} = \binom{(i_1+i_2)p}{i_1p}\alpha_1^{i_1+i_2,1},$$

$$\beta_{i_1,t}\gamma_{i_2} = \binom{(i_1+i_2)p}{i_1p}\beta_{i_1+i_2,t}, \quad \beta_{1,i_1,t}\gamma_{i_2} = \binom{(i_1+i_2)p}{i_1p}\beta_{1,i_1+i_2,t},$$

$$\gamma_{i_1}^1\gamma_{i_2}^1 = 2\binom{(i_1+i_2)p}{i_1p}\alpha_{0,1}^{i_1+i_2,0}, \quad \alpha_1^{i_1}\gamma_{i_2}^1 = \binom{(i_1+i_2)p}{i_1p}\alpha_1^{i_1+i_2,1},$$

$$\beta_{i_1,t}\gamma_{i_2}^1 = 2\binom{(i_1+i_2)p}{i_1p}\alpha_{0,1}^{i_1+i_2,t+1}, \quad \alpha_0^{i_1,j}\alpha_1^{i_2} = (-1)^{i_1+j}\binom{(i_1+i_2)p}{i_1p}\alpha_{0,1}^{i_1+i_2,j},$$

$$\alpha_0^{i,j}\beta_s = (-1)^{i+j+1}\binom{(i+s)p-1}{ip}\gamma_{i+s,j+1},$$

$$\beta_s\beta_{1,i,t} = (-1)^{s-1}\binom{(i+s)p-1}{ip+1}2\alpha_{1,2}^{i+s,t+1},$$

$$\beta_{i,t_1}\beta_{2,s,t_2} = (-1)^{i+t_1}\binom{(i+s)p-1}{ip+1}\binom{t_1+t_2+1}{t_1+1}\alpha_{1,i+s,t_1+t_2+2},$$

$$\beta_{i_1,t_1}\beta_{i_2,t_2} = 2\binom{(i_1+i_2)p}{i_1p}\binom{t_1+t_2+2}{t_1+1}\alpha_{0,1}^{i_1+i_2,t_1+t_2+2},$$

where $0 \le i, i_1, i_2 \le p^{t_1-1}-1$, $1 \le s \le p^{t_1-1}$, $1 \le h \le p^{t_2}-1$, $0 \le j \le p^{t_2}-1$, $0 \le t, t_1, t_2 \le p^{t_2}-2$.

(3) $\mathrm{DPH}^{\bullet}(\mathcal{F}^4_{2,2}) \cong \mathcal{U}^4_{2,2}(\underline{t}) \ltimes \mathcal{V}^4_{2,2}(\underline{t})$, where $\mathcal{U}^4_{2,2}(\underline{t})$ is a $p^{t_1-1}(2p+1)$-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_i, \alpha_{0,i}, \alpha_{1,s} \mid 0 \leq i \leq p^{t_1} - 1, 0 \leq s \leq p^{t_1-1} - 1\},$$

satisfying that

$$|\alpha_i| = |\alpha_{0,i}| = \bar{i} \ (\mathrm{mod}\ 2),\ \ |\alpha_{1,s}| = \bar{s} \ (\mathrm{mod}\ 2);$$

$$\|\alpha_i\| = i,\ \ \|\alpha_{0,i}\| = i+1,\ \ \|\alpha_{1,s}\| = sp+1,$$

and the multiplication is given by

$$\alpha_{i_1}\alpha_{i_2} = \begin{pmatrix} i_1+i_2 \\ i_1 \end{pmatrix} \alpha_{i_1+i_2}, \quad \alpha_{0,i_1}\alpha_{i_2} = \begin{pmatrix} i_1+i_2 \\ i_1 \end{pmatrix} \alpha_{0,i_1+i_2},$$

$$\alpha_{1,s_1}\alpha_{s_2p} = \begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix} \alpha_{1,s_1+s_2}, \quad \alpha_{1,s}\alpha_{qp+r} = -\begin{pmatrix} (s+q)p+r \\ sp \end{pmatrix} \alpha_{0,(s+q)p+r},$$

where $0 \leq i_1, i_2 \leq p^{t_1} - 1$, $0 \leq s_1, s_2 \leq p^{t_1-1} - 1$, $0 \leq s \leq p^{t_1-1} - 1$, $0 \leq qp+r \leq p^{t_1} - 1$, $1 \leq r \leq p-1$. $\mathcal{V}^4_{2,2}(\underline{t})$ is a $2p^{t_1-1}(4p^{t_2}+p-3)-1$-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_{0,2,i}, \beta_{0,2,i}, \beta^j_{0,1,s}, \beta^j_{0,1,2,s}, \alpha^j_{0,1,s}, \alpha^k_{0,1,2,s}, \alpha_{1,2,s}, \alpha^j_h, \alpha^j_{0,h}, \alpha^j_{0,2,h}, \beta^\rho_{0,2,h}, \alpha^1_{1,2,l} \mid 0 \leq i \leq p^{t_1} - 1,$$

$$1 \leq j \leq p^{t_2}-1, 0 \leq s \leq p^{t_1-1}-1, 1 \leq h \leq p^{t_1-1}, 0 \leq k \leq p^{t_2}-1, 1 \leq l \leq p^{t_1-1}-1, 1 \leq \rho \leq p^{t_2}-2\},$$

satisfying that

$$|\alpha_{0,2,i}| = \bar{i} \ (\mathrm{mod}\ 2),\ |\beta_{0,2,i}| = \overline{i+1} \ (\mathrm{mod}\ 2),\ |\alpha_{1,2,s}| = \bar{s} \ (\mathrm{mod}\ 2),$$

$$|\alpha^1_{1,2,l}| = \overline{l+1} \ (\mathrm{mod}\ 2),\ |\alpha^k_{0,1,2,s}| = \overline{s+k} \ (\mathrm{mod}\ 2),\ |\beta^\rho_{0,2,h}| = \overline{h+\rho-1} \ (\mathrm{mod}\ 2),$$

$$|\beta^j_{0,1,s}| = |\beta^j_{0,1,2,s}| = |\alpha^j_{0,1,s}| = \overline{s+j} \ (\mathrm{mod}\ 2),\ |\alpha^j_h| = |\alpha^j_{0,h}| = |\alpha^j_{0,2,h}| = \overline{h+j-1} \ (\mathrm{mod} 2);$$

$$\|\alpha_{0,2,i}\| = \|\beta_{0,2,i}\| = i+2,\ \|\alpha_{1,2,s}\| = sp+2,\ \|\alpha^1_{1,2,l}\| = lp+3,\ \|\beta^j_{0,1,s}\| = sp+j+1,$$

$$\|\beta^j_{0,1,2,s}\| = \|\alpha^j_{0,1,s}\| = sp+j+2,\ \|\alpha^k_{0,1,2,s}\| = sp+k+3,\ \|\alpha^j_h\| = hp+j-1,\ \|\alpha^j_{0,h}\| = hp+j,$$

$$\|\alpha^j_{0,2,h}\| = hp+j+1,\ \|\beta^\rho_{0,2,h}\| = hp+\rho,$$

and the multiplication

$$\alpha_{0,2,sp}\alpha^j_h = \begin{pmatrix} (s+h)p-1 \\ sp \end{pmatrix} \alpha^j_{0,2,s+h}, \quad \alpha_{1,2,s}\alpha^j_h = -\begin{pmatrix} (s+h)p-1 \\ sp \end{pmatrix} \alpha^j_{0,2,s+h},$$

$$\beta_{0,2,sp}\alpha^j_h = \begin{pmatrix} (s+h)p-1 \\ sp \end{pmatrix} (j+1)\alpha^{j+1}_{0,s+h}, \quad \beta^j_{0,1,s_1}\alpha_{1,2,s_2} = \begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix} \alpha^j_{0,1,2,s_1+s_2},$$

$$\beta_{0,2,s_1p}\alpha_{1,2,s_2} = \begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix} \alpha^1_{0,1,2,s_1+s_2},$$

$$\alpha_{0,2,s_1p}\beta^j_{0,1,s_2} = (-1)^{s_1+1} \begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix} \alpha^j_{0,1,2,s_1+s_2},$$

$$\beta^{j}_{0,1,s_1}\beta_{0,2,s_2p} = (-1)^{s_1+j+1}\begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}(j+2)\alpha^{j+1}_{0,1,s_1+s_2},$$

$$\beta_{0,2,sp}\beta^{\rho}_{0,2,h} = (-1)^{s+1}\begin{pmatrix} (s+h)p-1 \\ sp+1 \end{pmatrix}\rho\alpha^{\rho+1}_{0,2,s+h},$$

$$\beta_{0,2,s_1p}\beta^{j}_{0,1,2,s_2} = \begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}(j+1)\alpha^{j+1}_{0,1,2,s_1+s_2},$$

where $0 \leq s, s_1, s_2 \leq p^{t_1-1}-1$, $1 \leq h \leq p^{t_1-1}$, $1 \leq j \leq p^{t_2}-1$, $1 \leq \rho \leq p^{t_2}-2$. The multiplication between $\mathcal{U}^4_{2,2}(\underline{t})$ and $\mathcal{V}^4_{2,2}(\underline{t})$ is graded-supercommutative, and the multiplication is given by

$$\alpha_{1,2,0}\alpha_1 = -\alpha^1_{0,1,0} - \alpha_{0,2,1}, \quad \alpha_{1,2,l}\alpha_{sp+1} = \begin{pmatrix} (s+l)p \\ sp \end{pmatrix}\alpha^1_{1,2,s+l},$$

$$\beta_{0,2,0}\alpha_{1,0} = -2\alpha^1_{0,1,0} - \alpha_{0,2,1}, \quad \alpha_{0,s_1p}\beta^{j}_{0,1,s_2} = (-1)^{s_1}\begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}\alpha^{j}_{0,1,s_1+s_2},$$

$$\alpha^1_{1,2,l}\alpha_{hp-1} = \begin{pmatrix} (h+l)p \\ lp+1 \end{pmatrix}\alpha_{1,2,h+l}, \quad \alpha_{0,hp-1}\alpha^1_{1,2,l} = \begin{pmatrix} (h+l)p \\ lp+1 \end{pmatrix}\alpha^0_{0,1,2,h+l},$$

$$\alpha^{j}_{h}\alpha_{sp} = \begin{pmatrix} (s+h)p-1 \\ sp \end{pmatrix}\alpha^{j}_{s+h}, \quad \beta^{j}_{0,1,s_1}\alpha_{s_2p} = \begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}\beta^{j}_{0,1,s_1+s_2},$$

$$\beta^{\rho}_{0,2,h}\alpha_{sp} = \begin{pmatrix} (s+h)p-1 \\ sp \end{pmatrix}\beta^{\rho}_{0,2,s+h}, \quad \beta^{\rho}_{0,2,h}\alpha_{sp+1} = \begin{pmatrix} (s+h)p-1 \\ sp+1 \end{pmatrix}\alpha^{\rho+1}_{0,s+h},$$

$$\alpha^{j}_{0,h}\alpha_{sp} = \begin{pmatrix} (s+h)p-1 \\ sp \end{pmatrix}\alpha^{j}_{0,s+h}, \quad \beta^{j}_{0,1,2,s_1}\alpha_{s_2p} = \begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}\beta^{j}_{0,1,2,s_1+s_2},$$

$$\beta^{j}_{0,1,2,s_1}\alpha_{s_2p+1} = -\begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}\alpha^{j+1}_{0,1,s_1+s_2}, \quad \alpha^{j}_{0,2,h}\alpha_{sp} = \begin{pmatrix} (s+h)p-1 \\ sp \end{pmatrix}\alpha^{j}_{0,2,s+h},$$

$$\alpha^{j}_{0,1,s_1}\alpha_{s_2p} = \begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}\alpha^{j}_{0,1,s_1+s_2}, \quad \alpha^{k}_{0,1,2,s_1}\alpha_{s_2p} = \begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}\alpha^{k}_{0,1,2,s_1+s_2},$$

$$\alpha_{0,2,i_1}\alpha_{i_2} = \begin{pmatrix} i_1+i_2 \\ i_1 \end{pmatrix}\alpha_{0,2,i_1+i_2}, \quad \alpha_{1,2,s_1}\alpha_{s_2p} = \begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}\alpha_{1,2,s_1+s_2},$$

$$\alpha_{1,2,s}\alpha_{lp+1} = \begin{pmatrix} (s+l)p \\ sp \end{pmatrix}\alpha^1_{1,2,s+l}, \quad \alpha_{1,2,s}\alpha_{q_1p+r_1} = -\begin{pmatrix} (s+q_1)p+r_1 \\ sp \end{pmatrix}\alpha_{0,2,(s+q_1)p+r_1},$$

$$\alpha^1_{1,2,l}\alpha_{sp} = \begin{pmatrix} (s+l)p \\ sp \end{pmatrix}\alpha^1_{1,2,s+l}, \quad \alpha^1_{1,2,l}\alpha_{q_2p+r_2} = -\begin{pmatrix} (q_2+l)p+r+1 \\ lp+1 \end{pmatrix}\alpha_{0,2,(q_2+l)p+r_2+1},$$

$$\beta_{0,2,i_1}\alpha_{i_2} = \begin{pmatrix} i_1+i_2 \\ i_1 \end{pmatrix}\beta_{0,2,i_1+i_2}, \quad \alpha_{0,sp}\alpha^{j}_{h} = \begin{pmatrix} (s+h)p-1 \\ sp \end{pmatrix}\alpha^{j}_{0,s+h},$$

$$\alpha_{0,s_1p}\beta^{j}_{0,1,2,s_2} = \begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}\alpha^{j}_{0,1,2,s_1+s_2}, \quad \alpha_{0,s_1p}\alpha_{1,2,s_2} = \begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}\alpha^0_{0,1,2,s_1+s_2},$$

$$\alpha_{1,s}\alpha^{j}_{h} = -\begin{pmatrix} (s+h)p-1 \\ sp \end{pmatrix}\alpha^{j}_{0,s+h}, \quad \alpha_{1,s_1}\alpha_{0,2,s_2p} = -\begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}\alpha^0_{0,1,2,s_1+s_2},$$

$$\beta_{0,2,lp}\alpha_{1,s} = (-1)^{l+1}\begin{pmatrix} (s+l)p \\ sp \end{pmatrix}\alpha^1_{0,1,s+l} + (-1)^l \begin{pmatrix} (s+l)p+1 \\ sp \end{pmatrix}\alpha^1_{1,2,s+l},$$

$$\beta_{0,2,sp}\alpha_{1,l} = (-1)^{s+1}\begin{pmatrix} (s+l)p \\ sp \end{pmatrix}\alpha^1_{0,1,s+l} + (-1)^s \begin{pmatrix} (s+l)p+1 \\ lp \end{pmatrix}\alpha^1_{1,2,s+l},$$

$$\alpha_{0,sp}\beta^{\rho}_{0,2,h} = (-1)^{s+1}\begin{pmatrix} (s+h)p-1 \\ sp \end{pmatrix}\alpha^{\rho}_{0,2,s+h},$$

$$\alpha_{0,i_1}\beta_{0,2,i_2} = (-1)^{i_1}(i_2+1)\begin{pmatrix} i_1+i_2+1 \\ i_1 \end{pmatrix}\alpha_{0,2,i_1+i_2+1},$$

$$\beta^{\rho}_{0,2,h}\alpha_{1,s} = (-1)^{h+\rho}\begin{pmatrix} (s+h)p-1 \\ sp \end{pmatrix}\alpha^{\rho}_{0,2,s+h},$$

$$\beta^{j}_{0,1,s_1}\alpha_{1,s_2} = (-1)^{s_1+j}\begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}\alpha^{j}_{0,1,s_1+s_2},$$

$$\beta^{j}_{0,1,2,s_1}\alpha_{1,s_2} = (-1)^{s_1+j+1}\begin{pmatrix} (s_1+s_2)p \\ s_1p \end{pmatrix}\alpha^{j}_{0,1,2,s_1+s_2},$$

$$\beta_{0,2,q_3p+r_3}\alpha_{1,s} = (-1)^{q_3+r_3+1}(r_3+1)\begin{pmatrix} (s+q_3)p+r_3+1 \\ q_3p+r_3+1 \end{pmatrix}\alpha_{0,2,(s+q_3)p+r_3+1},$$

where $0 \le i_1, i_2, q_1p+r_1, q_2p+r_2, q_3p+r_3 \le p^{t_1}-1$, $0 \le s, s_1, s_2 \le p^{t_1-1}-1$, $1 \le h \le p^{t_1-1}$, $0 \le k \le p^{t_2}-1$, $1 \le l \le p^{t_1-1}-1$, $1 \le j \le p^{t_2}-1$, $1 \le \rho \le p^{t_2}-2$, $2 \le r_1 \le p-1$, $1 \le r_2, r_3 \le p-2$.

(4) $\mathrm{DPH}^{\bullet}(\mathcal{F}^5_{2,2}) \cong \mathcal{U}^5_{2,2}(\underline{t})$, where $\mathcal{U}^5_{2,2}(\underline{t})$ is a $p^{t_1-1}(8p^{t_2}+2)$-dimensional $\mathbb{Z}$-graded superalgebra with a $\mathbb{Z}_2$-homogeneous basis

$$\{\alpha_i, \alpha^i_1, \alpha^i_{0,1}, \alpha^i_0, \alpha^{k,l}_0, \alpha^{i,j}_{0,1}, \alpha^{k,j}_{0,2}, \alpha^k_{0,2}, \alpha_{k,l}, \beta_k, \beta^{i,j}_{0,1}, \beta_{1,i,s}, \beta_{2,i,s}, \beta_{3,k,j} \mid 0 \le i \le p^{t_1-1}-1,$$
$$0 \le j \le p^{t_2}-1, 0 \le s \le p^{t_2}-2, 1 \le k \le p^{t_1-1}, 1 \le l \le p^{t_2}-1\},$$

satisfying that

$$|\alpha_i| = |\alpha^i_0| = \bar{i} \pmod 2,\ |\alpha^i_{0,1}| = |\alpha^i_1| = \overline{i+1} \pmod 2,\ |\alpha^{i,j}_{0,1}| = |\beta^{i,j}_{0,1}| = \overline{i+j} \pmod 2,$$

$$|\beta_{1,i,s}| = |\beta_{2,i,s}| = \overline{i+s} \pmod 2,\ |\alpha^{k,l}_0| = |\alpha_{k,l}| = \overline{k+l-1} \pmod 2,$$
$$|\alpha^{k,j}_{0,2}| = |\beta_{3,k,j}| = \overline{k+j-1} \pmod 2,\ |\alpha^k_{0,2}| = |\beta_k| = \bar{k} \pmod 2;$$

$$\|\alpha_i\| = ip,\ \|\alpha^i_0\| = \|\alpha^i_1\| = ip+1,\ \|\alpha^i_{0,1}\| = ip+2,\ \|\alpha^{i,j}_{0,1}\| = ip+j+2,$$

$$\|\beta^{i,j}_{0,1}\| = ip+j+1,\ \|\beta_{1,i,s}\| = ip+s+2,\ \|\beta_{2,i,s}\| = ip+s+3,\ \|\alpha^{k,l}_0\| = kp+l,$$

$$\|\alpha_{k,l}\| = kp+l-1,\ \|\alpha^{k,j}_{0,2}\| = kp+j+1,\ \|\beta_{3,k,j}\| = kp+j,\ \|\alpha^k_{0,2}\| = kp,\ \|\beta_k\| = kp-1,$$

and the multiplication is given by

$$\alpha^{i_1,j}_{0,1}\alpha_{i_2} = \begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix}\alpha^{i_1+i_2,j}_{0,1},\quad \alpha^{i_1}_0\beta^{i_2,j}_{0,1} = (-1)^{i_1}\begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix}\alpha^{i_1+i_2,j}_{0,1},$$

$$\beta^{i_1,j}_{0,1}\alpha_{i_2} = \begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix}\beta^{i_1+i_2,j}_{0,1},\quad \alpha^{i_1}_0\alpha_{i_2} = \begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix}\alpha^{i_1+i_2}_0,$$

$$
\begin{aligned}
\alpha_0^{i_1}\alpha_1^{i_2} &= \begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix} \alpha_{0,1}^{i_1+i_2}, \quad \alpha_0^{i_1}\beta_{1,i_2,s} = \begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix} \beta_{2,i_1+i_2,s}, \\
\alpha_{0,1}^{i_1}\alpha_{i_2} &= \begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix} \alpha_{0,1}^{i_1+i_2}, \quad \alpha_{i_1}\alpha_{i_2} = \begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix} \alpha_{i_1+i_2}, \\
\alpha_{i_1}\alpha_1^{i_2} &= \begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix} \alpha_1^{i_1+i_2}, \quad \beta_{1,i_1,s}\alpha_{i_2} = \begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix} \beta_{1,i_1+i_2,s}, \\
\beta_{2,i_1,s}\alpha_{i_2} &= \begin{pmatrix} (i_1+i_2)p \\ i_1p \end{pmatrix} \beta_{2,i_1+i_2,s}, \quad \alpha_{0,1}^{i,j}\beta_k = (-1)^{i+j+1} \begin{pmatrix} (i+k)p-2 \\ ip \end{pmatrix} \frac{1}{2}\alpha_0^{i+k,j+1}, \\
\alpha_{0,2}^{k}\alpha_i &= \begin{pmatrix} (i+k)p-2 \\ ip \end{pmatrix} \alpha_{0,2}^{i+k}, \quad \alpha_{0,2}^{k}\beta_{0,1}^{i,j} = (-1)^k \begin{pmatrix} (i+k)p-2 \\ ip \end{pmatrix} \frac{1}{2}\alpha_0^{i+k,j+1}, \\
\beta_k\alpha_i &= \begin{pmatrix} (i+k)p-2 \\ ip \end{pmatrix} \beta_{i+k}, \quad \beta_{0,1}^{i,j}\beta_k = (-1)^{i+j+1} \begin{pmatrix} (i+k)p-2 \\ ip \end{pmatrix} \frac{1}{2}\alpha_{i+k,j+1}, \\
\alpha_0^{i}\alpha_{k,l} &= \begin{pmatrix} (i+k)p-1 \\ ip \end{pmatrix} \alpha_0^{i+k,l}, \quad \alpha_0^{i}\beta_k = (-1)^i \begin{pmatrix} (i+k)p-2 \\ ip \end{pmatrix} \alpha_{0,2}^{i+k}, \\
\alpha_0^{k,l}\alpha_i &= \begin{pmatrix} (i+k)p-1 \\ ip \end{pmatrix} \alpha_0^{i+k,l}, \quad \alpha_{0,2}^{k,j}\alpha_i = \begin{pmatrix} (i+k)p-1 \\ ip \end{pmatrix} \alpha_{0,2}^{i+k,j}, \\
\alpha_i\alpha_{k,l} &= \begin{pmatrix} (i+k)p-1 \\ ip \end{pmatrix} \alpha_{i+k,l}, \quad \beta_{3,k,j}\alpha_i = \begin{pmatrix} (i+k)p-1 \\ ip \end{pmatrix} \beta_{3,i+k,j}, \\
\alpha_0^{i}\beta_{3,k,j} &= (-1)^i \begin{pmatrix} (i+k)p-1 \\ ip \end{pmatrix} \alpha_{0,2}^{i+k,j}, \quad \beta_k\alpha_1^{i} = \begin{pmatrix} (i+k)p-1 \\ ip+1 \end{pmatrix} \beta_{3,i+k,0}, \\
\alpha_{0,1}^{i}\beta_k &= (-1)^{i+1} \begin{pmatrix} (i+k)p-1 \\ ip+1 \end{pmatrix} \alpha_{0,2}^{i+k,0}, \quad \alpha_{0,2}^{k}\alpha_1^{i} = \begin{pmatrix} (i+k)p-1 \\ ip+1 \end{pmatrix} \alpha_{0,2}^{i+k,0}, \\
\alpha_{0,2}^{k}\beta_{1,i,s} &= \begin{pmatrix} (i+k)p-1 \\ ip+1 \end{pmatrix} \alpha_{0,2}^{i+k,s+1}, \quad \beta_k\beta_{1,i,s} = \begin{pmatrix} (i+k)p-1 \\ ip+1 \end{pmatrix} \beta_{3,i+k,s+1}, \\
\beta_{2,i,s}\beta_k &= (-1)^{i+s} \begin{pmatrix} (i+k)p-1 \\ ip+1 \end{pmatrix} \alpha_{0,2}^{i+k,s+1}, \quad \alpha_1^{i_1}\alpha_1^{i_2} = \begin{pmatrix} (i_1+i_2)p+2 \\ i_1p+1 \end{pmatrix} \alpha_{0,1}^{i_1+i_2,0}, \\
\beta_{1,i_1,s}\alpha_1^{i_2} &= \begin{pmatrix} (i_1+i_2)p+2 \\ i_1p+1 \end{pmatrix} \alpha_{0,1}^{i_1+i_2,s+1}, \quad \beta_{3,k,j}\alpha_1^{i} = -\begin{pmatrix} (i+k)p-2 \\ ip+1 \end{pmatrix} \frac{1}{2}\alpha_0^{i+k,j+1}, \\
\beta_{1,i_1,s_1}\beta_{1,i_2,s_2} &= \begin{pmatrix} (i_1+i_2)p+2 \\ i_1p+1 \end{pmatrix} \begin{pmatrix} s_1+s_2+2 \\ s_1+1 \end{pmatrix} \alpha_{0,1}^{i_1+i_2,s_1+s_2+2}, \\
\beta_{3,k,j}\beta_{1,i,s} &= -\begin{pmatrix} (i+k)p-2 \\ ip+1 \end{pmatrix} \begin{pmatrix} j+s+1 \\ j \end{pmatrix} \frac{1}{2}\alpha_0^{i+k,j+s+2},
\end{aligned}
$$

where $0 \le i, i_1, i_2 \le p^{t_1-1}-1$, $0 \le j \le p^{t_2}-1$, $0 \le s, s_1, s_2 \le p^{t_2}-2$, $1 \le k \le p^{t_1-1}$, $1 \le l \le p^{t_2}-1$.

Proof. The proof is similar to Theorem 5. □

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