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A fundamental theorem for submanifolds in semi-Riemannian warped products

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ABSTRACT

In this paper we find necessary and sufficient conditions for a nondegenerate arbitrary signature manifold to be realized as a submanifold in the large class of warped product manifolds. This comprises the case of hypersurfaces immersed in semi-Riemannian warped products proved by M.A. Lawn and M. Ortega in 2015.

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1. Introduction

A fundamental question in the theory of submanifolds is to know when a (pseudo)-Riemannian manifold can be isometrically immersed in a given ambient space. In the case where the ambient space is a Riemannian space form, it is a well-known fact that the Gauss, Codazzi and Ricci equations are necessary and sufficient (see [12]). This fact is also true for pseudo-Riemannian manifolds, and a short proof of it can be found in [9], where the authors there used it to obtain isometric immersion results in product of space forms. In particular for immersions of codimension 1, the Ricci equation is trivial and the Gauss and Codazzi equations are equivalent to the existence of a local isometric immersion into the desired space form.

Versions of the fundamental theorem of submanifold theory were recently achieved by Benoit Daniel in [1,2], Julien Roth in [11] and by Marie-Amélie Lawn and Miguel Ortega in [5]. In [2], Daniel considers hypersurfaces in $\mathbb{M}^n \times \mathbb{R}$, where $\mathbb{M}^n$ is the sphere $\mathbb{S}^n$ or the hyperbolic space $\mathbb{H}^n$ of dimension n. He observed that for a hypersurface M of $\mathbb{M}^n \times \mathbb{R}$ the Gauss and Codazzi equations depend only on the first and second fundamental form, the projection T of the vertical vector ∂t onto the tangent bundle TM and its normal component ν, and proved that these two equations together with additional first order differential equations in T and ν give necessary and sufficient conditions for a manifold M to be immersed locally and isometrically into $\mathbb{M}^n \times \mathbb{R}$. In [11], Roth expanded Daniel's result and proved a fundamental theorem of hypersurfaces for the case of the Lorentzian products $\mathbb{M}^n \times \mathbb{R}_1$. In [5], Lawn and Ortega generalized Roth's result and obtained a fundamental theorem for hypersurfaces in semi-Riemannian warped products. It is interesting to bring to

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attention that none of these three results already mentioned above deal with submanifolds of codimension greater than one. It is also worth to point out that theorems of submanifolds for the case of arbitrary codimension have been proved (see, e.g., [4,6–9]).

Our aim in this paper is to give a proof of a fundamental theorem for submanifolds in semi-Riemannian warped products. It extends the result of Lawn and Ortega. Namely, we consider isometric immersions of arbitrary codimensions. Like the proof in [5] we use moving frames and integrable distributions, the standard method based on differential forms first used by Cartan and after Tenenblat [12] or Daniel [2] for instance.

2. Preliminaries

Let (P^m, g_P) be a semi-Riemannian manifold of dimension $\dim P = m$. All our manifolds will be connected and of class C^∞, unless otherwise stated. We consider a smooth function $a : I \subset \mathbb{R} \to \mathbb{R}^+$, where I is an interval, a (sign) constant $\varepsilon = \pm 1$ and the warped product

$$\bar{P}^{m+1} = \varepsilon I \times_a P^m, \qquad \langle \cdot, \cdot \rangle = \varepsilon dt^2 + a(t)^2 g_P$$

We set our convention for the curvature operator $\mathcal{R}$ of a connection $\mathcal{D}$ as

$$\mathcal{R}(X, Y)Z = \mathcal{D}_X \mathcal{D}_Y Z - \mathcal{D}_Y \mathcal{D}_X Z - \mathcal{D}_{[X,Y]} Z.$$

Let $\bar{R}_P$ and R_P be the curvature operator of $\bar{P}^{m+1}$ and P^m, respectively. Since there is no confusion, we will use the same notation for the associated curvature tensors along the paper. Let $\bar{\nabla}$ and ∇^P be the Levi-Civita connection of $\bar{P}^{m+1}$ and P^m, respectively. We recall the following formulae, which can be checked in [10], bearing in mind the change of sign.

Lemma 1 (*Cf. [10]*). *On the semi-Riemannian manifold $\bar{P}^{m+1}$, the following statements hold, for any V, W lifts of vector fields tangent to P^m:*

1. $\bar{\nabla}_{\partial t} \partial t = 0$, $\bar{\nabla}_V \partial t = \frac{a'}{a} V$,
2. $\operatorname{grad}(a) = \varepsilon a' \partial t$,
3. $\bar{\nabla}_V W = \nabla^P_V W - \frac{\varepsilon a'}{a} \langle V, W \rangle \partial t$,
4. $\bar{R}_P(V, \partial t)\partial t = -\frac{a''}{a} V$, $\bar{R}_P(\partial t, V)W = \frac{\varepsilon a''}{a} \langle V, W \rangle \partial t$, $\bar{R}_P(V, W)\partial t = 0$.

We also need to set the standard Euclidean space of dimension $N + 1 \geq 3$ and index λ, $\mathbb{R}^{N+1}_\lambda$ with its metric

$$g_0 = \sum_{i=0}^{N-\lambda} dx_i^2 - \sum_{i=N-\lambda+1}^{N} dx_i^2.$$

Let $\mathbb{M}^N_\lambda(c)$ be the semi-Riemannian space form of constant sectional curvature $c \in \{-1, 1\}$ and index λ, with metric g. We put

$$\mathbb{E}^{N+1}_\lambda = \begin{cases} \mathbb{R}^{N+1}_\lambda, & \text{if } \mathbb{M}^N_\lambda(c) = \mathbb{S}^N_\lambda,\ c = +1, \\ \mathbb{R}^{N+1}_{\lambda+1}, & \text{if } \mathbb{M}^N_\lambda(c) = \mathbb{H}^N_\lambda,\ c = -1, \end{cases}$$

with its standard pseudo-Riemannian flat metric g_0 and Levi-Civita connection ∇^0. We are considering $\mathbb{S}^N_\lambda$ and $\mathbb{H}^N_\lambda$, respectively, as one of the connected components of

$$\mathbb{S}^N_\lambda = \{p \in \mathbb{E}^{N+1}_\lambda;\ g_0(p, p) = +1\}, \quad \mathbb{H}^N_\lambda = \{p \in \mathbb{E}^{N+1}_\lambda;\ g_0(p, p) = -1\}.$$

Following the previous notations, we consider $\bar{P}^{N+1} = \varepsilon I \times_a \mathbb{M}^N_\lambda(c)$ with metric $\langle \cdot, \cdot \rangle$, Levi-Civita connection $\bar{\nabla}$ and curvature tensor $\bar{R}$. We also consider $\tilde{P}^{N+2} = \varepsilon I \times_a \mathbb{E}^{N+1}_\lambda$ with metric $\langle \cdot, \cdot \rangle_2$, Levi-Civita connection $\tilde{\nabla}$ and curvature tensor $\tilde{R}$. From the usual totally umbilical embedding $\Xi : \mathbb{M}^N_\lambda(c) \to \mathbb{E}^{N+1}_\lambda$, we construct the isometric immersion

$$\tilde{\Xi} : (\varepsilon I \times_a \mathbb{M}^N_\lambda(c), \langle , \rangle) \to (\varepsilon I \times_a \mathbb{E}^{N+1}_\lambda, \langle , \rangle_2), \quad (t, p) \mapsto (t, \Xi(p)).$$

Proposition 1 (*Cf. [10]*). *The curvature tensor $\tilde{R}$ of $\tilde{P}^{N+2}$ is given by*

$$\begin{aligned} \tilde{R}(X, Y, Z, W) = {} & \varepsilon \frac{(a')^2}{a} \Big(\langle X, Z \rangle \langle Y, W \rangle - \langle Y, Z \rangle \langle X, W \rangle \Big) \\ & + \left(\frac{a''}{a} - \frac{(a')^2}{a^2} \right) \Big(\langle X, Z \rangle \langle Y, \partial t \rangle \langle W, \partial t \rangle - \langle Y, Z \rangle \langle X, \partial t \rangle \langle W, \partial t \rangle \\ & - \langle X, W \rangle \langle Y, \partial t \rangle \langle Z, \partial t \rangle + \langle Y, W \rangle \langle X, \partial t \rangle \langle Z, \partial t \rangle \Big), \end{aligned}$$

for any sections $X, Y, Z, W \in \Gamma(T\tilde{P}^{N+2})$.

Using the general Gauss equation for the isometric immersion above, one concludes the following:

Proposition 2 (*Cf. [10]*). *The curvature tensor $\bar{R}$ of $\bar{P}^{N+1}$ is given by*

$$\begin{aligned}\bar{R}(X,Y,Z,W) = &\left(\varepsilon\frac{(a')^2}{a^2} - \frac{c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\Big)\\ &+\left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,\partial t\rangle\langle W,\partial t\rangle - \langle Y,Z\rangle\langle X,\partial t\rangle\langle W,\partial t\rangle\\ &-\langle X,W\rangle\langle Y,\partial t\rangle\langle Z,\partial t\rangle + \langle Y,W\rangle\langle X,\partial t\rangle\langle Z,\partial t\rangle\Big),\end{aligned}$$

for any tangent vector fields $X, Y, Z, W \in T\bar{P}^{N+1}$.

3. Isometric immersions into warped products

Let M^n be a semi-Riemannian manifold isometrically immersed in $\bar{P}^{N+1} = \varepsilon I \times_a \mathbb{M}^N_\lambda(c)$. Let ∇ and $\bar{\nabla}$ be the Levi-Civita connection of M^n and $\bar{P}^{N+1}$, respectively. Denote by R and $R^\perp$ the curvature tensors of the tangent and normal bundles TM and $TM^\perp$, respectively, by $\alpha \in \Gamma(T^*M \otimes T^*M \otimes TM^\perp)$ the second fundamental form of the immersion and by $A_\eta \in \Gamma(T^*M \otimes TM)$ its shape operator in the normal direction η, given by $\langle A_\eta X, Y\rangle = \langle \alpha(X,Y), \eta\rangle$, for all $X, Y \in \Gamma(TM)$. Let $\pi : M \to I$ be the restriction to M of the natural projection $\pi_I : \varepsilon I \times_a \mathbb{M}^N_\lambda(c) \to I$ and $\partial t = T + \xi$, where $T = \varepsilon \cdot \mathrm{grad}(\pi)$. Hence, we have $\xi \in \Gamma(TM^\perp)$, because for any $x_0 \in M$ and $v \in T_{x_0}M$

$$\langle \xi(x_0), v\rangle = \langle \partial t - T(x_0), v\rangle = \langle \partial t, v\rangle - \varepsilon d\pi_{x_0}(v) = 0.$$

In particular, we have the equation

$$\varepsilon = \langle \partial t, \partial t\rangle = \langle T, T\rangle + \langle \xi, \xi\rangle.$$

In the following, we are using these decompositions for the tangent bundle $T\bar{P}^{N+1}$:

$$T_{(t_0,x_0)}\bar{P}^{N+1} = T_{t_0}I \oplus T_{x_0}\mathbb{M}^N_\lambda(c) = T_{(t_0,x_0)}M^n \oplus (T_{(t_0,x_0)}M^n)^\perp, \quad (t_0,x_0) \in \bar{P}^{N+1}.$$

Lemma 2. *Under the previous conditions, the following equations hold for any $X \in TM$:*

1. $\bar{\nabla}_X \partial t = \frac{a'}{a}(X - \varepsilon\langle X, T\rangle \partial t)$.
2. $\nabla_X T = \frac{a'}{a}(X - \varepsilon\langle X, T\rangle T) + A_\xi X$.
3. $\nabla^\perp_X \xi = -\frac{\varepsilon a'}{a}\langle X, T\rangle \xi - \alpha(X,T)$.

Proof. For any $X \in TM$, there exists $X_0 \in T\mathbb{M}^N_\lambda(c)$ such that $X = \varepsilon\langle X, T\rangle \partial t + X_0$. Using Lemma 1, one has

$$\bar{\nabla}_X \partial t = \bar{\nabla}_{X_0}\partial t = \frac{a'}{a}X_0 = \frac{a'}{a}(X - \varepsilon\langle X, T\rangle \partial t).$$

Finally, we obtain

$$\begin{aligned}\nabla_X T + \alpha(X,T) &= \bar{\nabla}_X T\\ &= \bar{\nabla}_X \partial t - \bar{\nabla}_X \xi\\ &= \frac{a'}{a}(X - \varepsilon\langle X, T\rangle \partial t) - (-A_\xi X + \nabla^\perp_X \xi)\\ &= \frac{a'}{a}(X - \varepsilon\langle X, T\rangle T) + A_\xi X - \left(\frac{\varepsilon a'}{a}\langle X, T\rangle \xi + \nabla^\perp_X \xi\right).\end{aligned}$$

Now, the last two equations are just the tangential and the normal part of $\bar{\nabla}_X T$. □

By using Proposition 2 and the classical Gauss equation for submanifolds, which states

$$R(X,Y,Z,W) = \bar{R}(X,Y,Z,W) - \langle \alpha(X,Z), \alpha(Y,W)\rangle + \langle \alpha(X,W), \alpha(Y,Z)\rangle,$$

for any $X, Y, Z, W \in TM$, we obtain:

Proposition 3. *The curvature tensor R of M in $\varepsilon I \times_a \mathbb{M}^N_\lambda(c)$ is*

$$\begin{aligned} R(X,Y,Z,W) = &\left(\varepsilon\frac{(a')^2}{a^2} - \frac{c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\Big) \\ &+ \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,T\rangle\langle W,T\rangle - \langle Y,Z\rangle\langle X,T\rangle\langle W,T\rangle \\ &- \langle X,W\rangle\langle Y,T\rangle\langle Z,T\rangle + \langle Y,W\rangle\langle X,T\rangle\langle Z,T\rangle\Big) \\ &- \langle\alpha(X,Z),\alpha(Y,W)\rangle + \langle\alpha(X,W),\alpha(Y,Z)\rangle\,, \end{aligned}$$

for any $X, Y, Z, W \in TM$.

Proposition 4. *The Codazzi equation of M in $\varepsilon I \times_a \mathbb{M}^N_\lambda(c)$ is*

$$(\bar\nabla_Y\alpha)(X,Z,\eta) - (\bar\nabla_X\alpha)(Y,Z,\eta) = \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\langle\xi,\eta\rangle\Big(\langle T,X\rangle\langle Y,Z\rangle - \langle T,Y\rangle\langle X,Z\rangle\Big)\,,$$

for any $X, Y, Z \in TM$ and $\eta \in TM^\perp$.

Proof. The classical Codazzi equation for submanifolds states

$$(\bar\nabla_Y\alpha)(X,Z,\eta) - (\bar\nabla_X\alpha)(Y,Z,\eta) = -\bar R(X,Y,Z,\eta).$$

Proposition 2 gives us

$$\bar R(X,Y,Z,\eta) = \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\langle\xi,\eta\rangle\Big(\langle T,Y\rangle\langle X,Z\rangle - \langle T,X\rangle\langle Y,Z\rangle\Big)\,,$$

and the proof is complete. □

Proposition 5. *The Ricci equation of M in $\varepsilon I \times_a \mathbb{M}^N_\lambda(c)$ is*

$$R^\perp(X,Y)\eta = \alpha(A_\eta Y, X) - \alpha(A_\eta X, Y),$$

for any $X, Y \in TM$ and $\eta \in TM^\perp$.

Proof. The classical Ricci equation for submanifolds states

$$R^\perp(X,Y)\eta = \bar R(X,Y)\eta + \alpha(A_\eta Y, X) - \alpha(A_\eta X, Y).$$

Proposition 2 gives us $\bar R(X,Y)\eta = 0$, and the proof is complete. □

4. Structure equations and main theorem

In the previous section we found necessary conditions for a nondegenerate arbitrary signature manifold M^n to be realized as a submanifold in the large class of warped product manifolds $\varepsilon I \times_a \mathbb{M}^N_\lambda(c)$, where $\varepsilon = \pm 1$, $a : I \subset \mathbb{R} \to \mathbb{R}^+$ is the scale factor and $\mathbb{M}^N_\lambda(c)$ is the N-dimensional semi-Riemannian space form of index λ and constant curvature $c \in \{-1, 1\}$. Now, we want to prove the converse. Namely, we want to show that if M^n satisfies Gauss, Codazzi and Ricci equations for a submanifold in $\varepsilon I \times_a \mathbb{M}^N_\lambda(c)$, along with some additional conditions that appear in Lemma 2, then M^n can be isometrically immersed into $\varepsilon I \times_a \mathbb{M}^N_\lambda(c)$.

Élie Cartan developed the moving frame technique. Definitions, basic results and some other details can be found in [3]. We will use the following convention on the ranges of indices, unless mentioned otherwise:

$$0 \le \alpha, \beta, \gamma, \ldots \le N+1; \qquad 1 \le i, j, k, \ldots \le n; \qquad n+1 \le u, v, w \cdots \le N+1$$

We consider $\tilde P^{N+2} = \varepsilon I \times_a \mathbb{E}^{N+1}_\lambda$ with a connection $\tilde\nabla$ and a base $(E_0, \ldots, E_N, \partial_t)$, where $(E_0, \ldots, E_N)$ is an orthonormal frame of $\mathbb{E}^{N+1}_\lambda$. We set $\overline{E}_0 = \dfrac{E_0}{ca}, \ldots, \overline{E}_N = \dfrac{E_N}{ca}, \overline{E}_{N+1} = \partial_t$, and, consequently, $\big(\overline{E}_0, \ldots, \overline{E}_{N+1}\big)$ is an orthonormal base of $\varepsilon I \times_a \mathbb{E}^{N+1}_\lambda$. If necessary, we reorder $\big(\overline{E}_1, \ldots, \overline{E}_{N+1}\big)$ so that $\big\langle \overline{E}_\alpha, \overline{E}_\alpha\big\rangle_{\varepsilon I \times_a \mathbb{E}^{N+1}_\lambda} = \varepsilon_\alpha = \pm 1$ and suppose, without lost of generality, that $c = \varepsilon_0$. Also, let $\overline\nabla$ be the Levi-Civita connection of $\bar P^{N+1} = \varepsilon I \times_a \mathbb{M}^N_\lambda(c)$.

From now on, let $(M^n, \langle,\rangle_M)$ be a semi-Riemannian manifold of index p and $(E, \langle,\rangle_E)$ a semi-Riemannian vector bundle of index q and rank $m = N + 1 - n$ over M with compatible connection ∇^E and curvature operator R^E. Let us also give α^E a symmetric section in $\Gamma(T^*M \otimes T^*M \otimes E)$, ξ a section in $\Gamma(E)$, real numbers $c, \varepsilon \in \{-1, 1\}$ and smooth functions $a : I \subset \mathbb{R} \to \mathbb{R}^+$ and $\pi : M \to I$. We define the vector field $T \in TM$ by $T = \varepsilon \cdot \mathrm{grad}(\pi)$ and, for each $\eta \in \Gamma(E)$, we define the section $A_\eta \in \Gamma(T^*M \otimes TM)$ by $\big\langle \alpha^E(X,Y), \eta\big\rangle = \big\langle A_\eta(X), Y\big\rangle$.

Definition 1. Under the previous conditions, we say that M satisfies the **structure equations** if the following conditions hold for any $X, Y, Z, W \in TM$ and $\eta \in \Gamma(E)$:

(A) $\langle T, T\rangle + \langle \xi, \xi\rangle = \varepsilon$.

(B) $\nabla_X T = \dfrac{a'}{a}\left(X - \varepsilon \langle X, T\rangle T\right) + A_\xi X$.

(C) $\nabla^E_X \xi = \dfrac{-\varepsilon a'}{a}\langle X, T\rangle \xi - \alpha^E(T, X)$.

(D) (Gauss)

$$\begin{aligned} R(X,Y,Z,W) = &\left(\varepsilon\frac{(a')^2}{a^2} - \frac{c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\Big)\\ &+ \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,T\rangle\langle W,T\rangle - \langle Y,Z\rangle\langle X,T\rangle\langle W,T\rangle\\ &- \langle X,W\rangle\langle Y,T\rangle\langle Z,T\rangle + \langle Y,W\rangle\langle X,T\rangle\langle Z,T\rangle\Big)\\ &- \langle \alpha(X,Z), \alpha(Y,W)\rangle + \langle \alpha(X,W), \alpha(Y,Z)\rangle\,. \end{aligned}$$

(E) (Codazzi)

$$(\nabla_Y \alpha^E)(X,Z,\eta) - (\nabla_X \alpha^E)(Y,Z,\eta) = \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\langle \xi, \eta\rangle\Big(\langle T, X\rangle\langle Y, Z\rangle - \langle T, Y\rangle\ \langle X, Z\rangle\Big)\,.$$

(F) (Ricci)

$$R^E(X,Y)\eta = \alpha^E(A_\eta Y, X) - \alpha^E(A_\eta X, Y).$$

Remark 1. By abuse of notation, we have written above $a = a \circ \pi$.

Theorem 1.

(i) **Existence:** *Assume that M, under the previous conditions, is simply connected and satisfies the structure equations. Then there exists an isometric immersion $f : M \to \varepsilon I \times_a \mathbb{M}^N_\lambda(c)$ and a vector bundle isometry $\Phi : E \to Tf(M)^\perp$, such that:*

1. $\partial_t = df(T) + \Phi(\xi)$.
2. $\pi = \pi_I \circ f$, *where $\pi_I : \varepsilon I \times_a \mathbb{M}^N_\lambda(c) \to I$ is the projection.*
3. $\alpha_f = \Phi \circ \alpha^E \circ df^{-1}$, *where α_f is the second fundamental form of immersion f.*
4. *(D), (E) e (F) are the Gauss, Codazzi and Ricci equations, respectively, of the immersion founded.*
5. $\nabla^\perp \Phi = \Phi \nabla^E$.

(ii) **Uniqueness:** *Let $f, g : M \to \varepsilon I \times_a \mathbb{M}^N_\lambda(c)$ be isometric immersions of a semi-Riemannian manifold. Assume that there exists a vector bundle isometry $\Phi : T_f M^\perp \to T_g M^\perp$ such that*

$$\Phi\alpha_f = \alpha_g \quad \text{and} \quad \Phi^f\nabla^\perp = {}^g\nabla^\perp\Phi.$$

Then there exists an isometry $\tau : \varepsilon I \times_a \mathbb{M}^N_\lambda(c) \to \varepsilon I \times_a \mathbb{M}^N_\lambda(c)$ such that $\tau \circ f = g$ and $\tau_*|_{T_f M^\perp} = \Phi$.

Regarding the part (*i*), let ∇ be the Levi-Civita connection on TM. Consider the Whitney sum $F = TM \oplus E$ endowed with the orthogonal sum of the metrics in TM and E. Define

$$\begin{aligned} &\nabla^F_X Y = \nabla_X Y + \alpha^E(X,Y), \quad X, Y \in \Gamma(TM),\\ &\nabla^F_X \eta = -A_\eta X + \nabla^E_X \eta, \quad X \in \Gamma(TM), \quad \eta \in \Gamma(E). \end{aligned}$$

It is easy to see that ∇^F is a compatible connection on F. Let us give a point $x \in M$ and consider around it a local orthonormal frame $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}\}$ for F, with their signs $\varepsilon_\alpha = \langle e_\alpha, e_\alpha\rangle = \pm 1$, where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame for TM and $\{e_{n+1}, \ldots, e_{n+m}\}$ is a local orthonormal frame for E. Define in $\Gamma(F^*)$ the 1-forms

$$\omega_\alpha = e^*_\alpha, \text{ se } \alpha \in \{1, \ldots, n+m\}, \text{ e } \omega_0 = 0.$$

Also define $\delta \in \Gamma(F^*)$ by $\delta(X) = \langle X, T + \xi\rangle$, for $X \in F$. With the help of the tensor S in $\Gamma(TM^* \otimes F)$ given by $SX = \dfrac{-1}{ac}\left(X - \varepsilon\langle X, T + \xi\rangle(T + \xi)\right)$, we construct the following 1-forms

$$\omega_{\alpha 0}(X) = -\varepsilon_\alpha \langle e_\alpha, SX\rangle\,, \quad \omega_{ij}(X) = \varepsilon_i\left\langle e_i, \nabla^F_X e_j\right\rangle = \varepsilon_i\left\langle e_i, \nabla_X e_j\right\rangle,$$

$$\omega_{iu}(X) = \varepsilon_i\left\langle e_i, \nabla^F_X e_u\right\rangle = -\varepsilon_i\left\langle e_i, A_{e_u}(X)\right\rangle, \quad \omega_{uv}(X) = \varepsilon_u\left\langle e_u, \nabla^F_X e_v\right\rangle = \varepsilon_u\langle e_u, \nabla_X e_v\rangle\,,$$

$$\omega_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta\omega_{\beta\alpha},$$

for any $X \in TM$, which are known as the connection 1-forms. Now, define the functions $T_\alpha = \delta(e_\alpha)$ for $\alpha \in \{1, \ldots, n+m\}$ and $T_0 = 0$. Next, we consider the matrices $\mathbf{X}$ and Υ, given by

$$\mathbf{X}_{\alpha\beta} = \frac{\varepsilon a'}{a}\left\{T_\beta \omega_\alpha - \varepsilon_\alpha \varepsilon_\beta T_\alpha \omega_\beta\right\}, \quad \Upsilon = \Omega - \mathbf{X},$$

where $\Omega = (\omega_{\alpha\beta})$. A simple computation gives

$$d\Upsilon + \Upsilon \wedge \Upsilon = d\Omega - d\mathbf{X} + \Omega \wedge \Omega - \Omega \wedge \mathbf{X} - \mathbf{X} \wedge \Omega + \mathbf{X} \wedge \mathbf{X}. \tag{1}$$

Our goal now is to prove that the right side of (1) vanishes, and the proof of it will be split in some lemmas.

Lemma 3. *The following equalities hold:*

(1) $\sum_\gamma \varepsilon_\gamma T_\gamma^2 = \varepsilon$;
(2) $\delta = \sum_\gamma T_\gamma \omega_\gamma$;
(3) $dT_\alpha = \sum_\gamma T_\gamma \omega_{\gamma\alpha} + \frac{a'}{a}\varepsilon_\alpha \omega_\alpha - \frac{\varepsilon a'}{a} T_\alpha \delta$;
4. $d\mathcal{W} = -\Omega \wedge \mathcal{W}$, *where* $\mathcal{W} = (\omega_0, \ldots, \omega_{n+m})^\top$.

Proof. Item (1) comes from item (A) of the structure equations, and (2) is immediate from definitions of the δ, T_γ and ω_γ. In order to prove item (3), we see that if $X \in TM$ and $\alpha \geq 1$, then

$$\begin{aligned}
\sum_\gamma T_\gamma \omega_{\gamma\alpha}(X) &= \sum_\gamma \langle e_\gamma, T + \xi\rangle \varepsilon_\gamma \left\langle e_\gamma, \nabla^F_X e_\alpha\right\rangle \\
&= \left\langle T + \xi, \nabla^F_X e_\alpha\right\rangle \\
&= X\left(\langle T + \xi, e_\alpha\rangle\right) - \left\langle \nabla^F_X (T + \xi), e_\alpha\right\rangle \\
&= X(T_\alpha) - \left\langle \frac{a'}{a}\left(X - \varepsilon \langle X, T\rangle (T + \xi)\right), e_\alpha\right\rangle \\
&= dT_\alpha(X) - \frac{a'}{a}\varepsilon \omega_\alpha(X) + \frac{\varepsilon a'}{a} T_\alpha \delta(X),
\end{aligned}$$

where the penultimate equality was obtained from the items (B) and (C) of the structure equations and the definition of the connection ∇^F. If $\alpha = 0$,

$$\begin{aligned}
\sum_\gamma T_\gamma \omega_{\gamma 0}(X) &= -\sum_{\gamma \geq 1} \langle e_\gamma, T + \xi\rangle \varepsilon_\gamma \langle e_\gamma, SX\rangle \\
&= -\langle T + \xi, SX\rangle \\
&= 0.
\end{aligned}$$

Then, $dT_0 = \sum_\gamma T_\gamma \omega_{\gamma 0} + \frac{a'}{a}\varepsilon_0 \omega_0 - \frac{\varepsilon a'}{a} T_0 \delta$, because $dT_0 = T_0 = \omega_0 = 0$.

Finally, to prove item (4), we see that if $\alpha \geq 1$, then

$$\begin{aligned}
d\omega_\alpha(X, Y) &= X(\omega_\alpha(Y)) - Y(\omega_\alpha(X)) - \omega_\alpha([X, Y]) \\
&= X(\varepsilon_\alpha \langle e_\alpha, Y\rangle) - Y(\varepsilon_\alpha \langle e_\alpha, X\rangle) - \varepsilon_\alpha(\langle e_\alpha, [X, Y]\rangle) \\
&= \varepsilon_\alpha\left(\left\langle \nabla^F_X e_\alpha, Y\right\rangle + \left\langle e_\alpha, \nabla^F_X Y\right\rangle\right) - \varepsilon_\alpha\left(\left\langle \nabla^F_Y e_\alpha, X\right\rangle + \left\langle e_\alpha, \nabla^F_Y X\right\rangle\right) - \varepsilon_\alpha(\langle e_\alpha, [X, Y]\rangle) \\
&= \varepsilon_\alpha \left\langle \nabla^F_X e_\alpha, Y\right\rangle - \varepsilon_\alpha \left\langle \nabla^F_Y e_\alpha, X\right\rangle \\
&= \varepsilon_\alpha \left(\sum_{\gamma \geq 1} \omega_{\gamma\alpha}(X)\varepsilon_\gamma \omega_\gamma(Y) - \omega_{\gamma\alpha}(Y)\varepsilon_\gamma \omega_\gamma(X)\right) \\
&= \varepsilon_\alpha \left(\sum_{\gamma \geq 0} \omega_{\gamma\alpha}(X)\varepsilon_\gamma \omega_\gamma(Y) - \omega_{\gamma\alpha}(Y)\varepsilon_\gamma \omega_\gamma(X)\right)
\end{aligned}$$

$$= -\sum_{\gamma\geq 0}\omega_{\alpha\gamma}\wedge\omega_{\gamma}(X,Y)$$
$$= -(\Omega\wedge\mathcal{W})_{\alpha}(X,Y).$$

If $\alpha=0$, then $d\omega_{\alpha}=0$, and

$$(\Omega\wedge\mathcal{W})_{0}(X,Y) = \sum_{\gamma}\omega_{0\gamma}(X)\omega_{\gamma}(Y)-\omega_{0\gamma}(Y)\omega_{\gamma}(X)$$
$$= \frac{\varepsilon}{a}\delta\wedge\left(\sum_{\gamma}T_{\gamma}\omega_{\gamma}\right)(X,Y)$$
$$= \frac{\varepsilon}{a}\delta\wedge\delta(X,Y)=0.$$

This concludes the proof. □

Lemma 4. *The following equalities hold:*

1. $(d\boldsymbol{X})_{\alpha\beta} = \varepsilon\left(\frac{aa''-(a')^2}{aa'}\right)\delta\wedge\boldsymbol{X}_{\alpha\beta}+\frac{\varepsilon a'}{a}\left(dT_{\beta}\wedge\omega_{\alpha}-\varepsilon_{\alpha}\varepsilon_{\beta}dT_{\alpha}\wedge\omega_{\beta}\right)+\frac{\varepsilon a'}{a}\left(T_{\beta}d\omega_{\alpha}-\varepsilon_{\alpha}\varepsilon_{\beta}T_{\alpha}d\omega_{\beta}\right)$;
2. $(\boldsymbol{X}\wedge\boldsymbol{X})_{\alpha\beta} = -\left(\frac{\varepsilon a'}{a}\right)\delta\wedge\boldsymbol{X}_{\alpha\beta}-\left(\frac{a'}{a}\right)^2\varepsilon\varepsilon_{\beta}\omega_{\alpha}\wedge\omega_{\beta}$;
3. $(\Omega\wedge\boldsymbol{X})_{\alpha\beta}+(\boldsymbol{X}\wedge\Omega)_{\alpha\beta} = -\left(\frac{\varepsilon a'}{a}\right)\left(T_{\beta}d\omega_{\alpha}-T_{\alpha}\varepsilon_{\alpha}\varepsilon_{\beta}d\omega_{\beta}\right)-\left(\frac{\varepsilon a'}{a}\right)\left(dT_{\beta}\wedge\omega_{\alpha}-\varepsilon_{\alpha}\varepsilon_{\beta}dT_{\alpha}\wedge\omega_{\beta}\right)$
$-\left(\frac{\varepsilon a'}{a}\right)\delta\wedge\boldsymbol{X}_{\alpha\beta}-2\left(\frac{a'}{a}\right)^2\varepsilon\varepsilon_{\beta}\omega_{\alpha}\wedge\omega_{\beta}$;
4. $(d\Omega)_{\alpha\beta}+(\Omega\wedge\Omega)_{\alpha\beta} = -\left(\frac{a'}{a}\right)^2\varepsilon\varepsilon_{\beta}\omega_{\alpha}\wedge\omega_{\beta}+\varepsilon\left(\frac{aa''-(a')^2}{aa'}\right)\delta\wedge\boldsymbol{X}_{\alpha\beta}$.

Proof. First of all, it is important to remember that $T=\varepsilon\cdot\mathrm{grad}(\pi)$ and that we have been written $a=a\circ\pi$. For (1), we have

$$(d\boldsymbol{X})_{\alpha\beta} = d\left(\frac{\varepsilon a'}{a}\left\{T_{\beta}\omega_{\alpha}-\varepsilon_{\alpha}\varepsilon_{\beta}T_{\alpha}\omega_{\beta}\right\}\right)$$
$$= d\left(\frac{\varepsilon a'}{a}\right)\wedge\left\{T_{\beta}\omega_{\alpha}-\varepsilon_{\alpha}\varepsilon_{\beta}T_{\alpha}\omega_{\beta}\right\}+\frac{\varepsilon a'}{a}d\left(\left\{T_{\beta}\omega_{\alpha}-\varepsilon_{\alpha}\varepsilon_{\beta}T_{\alpha}\omega_{\beta}\right\}\right)$$
$$= \varepsilon\left(\frac{aa''-(a')^2}{a^2}\right)\frac{a}{a'}\delta\wedge\boldsymbol{X}_{\alpha\beta}+\frac{\varepsilon a'}{a}\left(dT_{\beta}\wedge\omega_{\alpha}-\varepsilon_{\alpha}\varepsilon_{\beta}dT_{\alpha}\wedge\omega_{\beta}\right)+\frac{\varepsilon a'}{a}\left(T_{\beta}d\omega_{\alpha}-\varepsilon_{\alpha}\varepsilon_{\beta}T_{\alpha}d\omega_{\beta}\right).$$

For (2), we have

$$(\boldsymbol{X}\wedge\boldsymbol{X})_{\alpha\beta} = \sum_{\gamma}\boldsymbol{X}_{\alpha\gamma}\wedge\boldsymbol{X}_{\gamma\beta}$$
$$= \left(\frac{a'}{a}\right)^2\sum_{\gamma}\left(T_{\gamma}\omega_{\alpha}-\varepsilon_{\alpha}\varepsilon_{\gamma}T_{\alpha}\omega_{\gamma}\right)\wedge\left(T_{\beta}\omega_{\gamma}-\varepsilon_{\gamma}\varepsilon_{\beta}T_{\gamma}\omega_{\beta}\right)$$
$$= \left(\frac{a'}{a}\right)^2\left(T_{\beta}\omega_{\alpha}\wedge\sum_{\gamma}T_{\gamma}\omega_{\gamma}\right)-\left(\frac{a'}{a}\right)^2\varepsilon_{\beta}\left(\sum_{\gamma}\varepsilon_{\gamma}T_{\gamma}^2\right)\omega_{\alpha}\wedge\omega_{\beta}$$
$$+\left(\frac{a'}{a}\right)^2\varepsilon_{\alpha}\varepsilon_{\beta}T_{\alpha}\left(\sum_{\gamma}T_{\gamma}\omega_{\gamma}\right)\wedge\omega_{\beta}$$
$$= \left(\frac{a'}{a}\right)^2\left(T_{\beta}\omega_{\alpha}\wedge\delta\right)-\left(\frac{a'}{a}\right)^2\varepsilon_{\beta}\varepsilon\omega_{\alpha}\wedge\omega_{\beta}+\left(\frac{a'}{a}\right)^2\varepsilon_{\alpha}\varepsilon_{\beta}T_{\alpha}\delta\wedge\omega_{\beta}$$
$$= \left(\frac{\varepsilon a'}{a}\right)\boldsymbol{X}_{\alpha\beta}\wedge\delta-\left(\frac{a'}{a}\right)^2\varepsilon\varepsilon_{\beta}\omega_{\alpha}\wedge\omega_{\beta}.$$

For (3), we initially calculate $(\Omega \wedge \boldsymbol{X})_{\alpha\beta}$. We have

$$\begin{aligned}
(\Omega \wedge \boldsymbol{X})_{\alpha\beta} &= \sum_{\gamma} \omega_{\alpha\gamma} \wedge \boldsymbol{X}_{\gamma\beta} \\
&= \left(\frac{\varepsilon a'}{a}\right) \sum_{\gamma} \omega_{\alpha\gamma} \wedge \left(T_\beta \omega_\gamma - \varepsilon_\gamma \varepsilon_\beta T_\gamma \omega_\beta\right) \\
&= \left(\frac{\varepsilon a'}{a}\right) T_\beta \sum_{\gamma} \omega_{\alpha\gamma} \wedge \omega_\gamma + \left(\frac{\varepsilon a'}{a}\right) \varepsilon_\alpha \varepsilon_\beta \left(\sum_{\gamma} T_\gamma \omega_{\gamma\alpha}\right) \wedge \omega_\beta \\
&= \left(\frac{\varepsilon a'}{a}\right) T_\beta \sum_{\gamma} \omega_{\alpha\gamma} \wedge \omega_\gamma + \left(\frac{\varepsilon a'}{a}\right) \varepsilon_\alpha \varepsilon_\beta \left(dT_\alpha - \frac{a'}{a} \varepsilon_\alpha \omega_\alpha + \frac{\varepsilon a'}{a} T_\alpha \delta\right) \wedge \omega_\beta \\
&= -\left(\frac{\varepsilon a'}{a}\right) T_\beta d\omega_\alpha + \left(\frac{\varepsilon a'}{a}\right) \varepsilon_\alpha \varepsilon_\beta dT_\alpha \wedge \omega_\beta - \left(\frac{a'}{a}\right)^2 \varepsilon \varepsilon_\beta \omega_\alpha \wedge \omega_\beta + \left(\frac{a'}{a}\right)^2 \varepsilon_\alpha \varepsilon_\beta T_\alpha \delta \wedge \omega_\beta.
\end{aligned}$$

Now, we compute $(\boldsymbol{X} \wedge \Omega)_{\alpha\beta}$. We have

$$\begin{aligned}
(\boldsymbol{X} \wedge \Omega)_{\alpha\beta} &= \sum_{\gamma} \boldsymbol{X}_{\alpha\gamma} \wedge \omega_{\gamma\beta} \\
&= \left(\frac{\varepsilon a'}{a}\right) \sum_{\gamma} \left(T_\gamma \omega_\alpha - \varepsilon_\alpha \varepsilon_\gamma T_\alpha \omega_\gamma\right) \wedge \omega_{\gamma\beta} \\
&= \left(\frac{\varepsilon a'}{a}\right) \omega_\alpha \wedge \left(\sum_{\gamma} T_\gamma \omega_{\gamma\beta}\right) - \left(\frac{\varepsilon a'}{a}\right) T_\alpha \varepsilon_\alpha \varepsilon_\beta \left(\sum_{\gamma} \omega_{\beta\gamma} \wedge \omega_\gamma\right) \\
&= \left(\frac{\varepsilon a'}{a}\right) \omega_\alpha \wedge \left(dT_\beta - \frac{a'}{a} \varepsilon_\beta \omega_\beta + \frac{\varepsilon a'}{a} T_\beta \delta\right) - \left(\frac{\varepsilon a'}{a}\right) T_\alpha \varepsilon_\alpha \varepsilon_\beta \left(\sum_{\gamma} \omega_{\beta\gamma} \wedge \omega_\gamma\right) \\
&= -\left(\frac{\varepsilon a'}{a}\right) dT_\beta \wedge \omega_\alpha - \left(\frac{a'}{a}\right)^2 \varepsilon \varepsilon_\beta \omega_\alpha \wedge \omega_\beta - \left(\frac{a'}{a}\right)^2 T_\beta \delta \wedge \omega_\alpha + \left(\frac{\varepsilon a'}{a}\right) T_\alpha \varepsilon_\alpha \varepsilon_\beta d\omega_\beta.
\end{aligned}$$

Hence, $(\Omega \wedge \boldsymbol{X})_{\alpha\beta} + (\boldsymbol{X} \wedge \Omega)_{\alpha\beta} = -\left(\frac{\varepsilon a'}{a}\right)\left(T_\beta d\omega_\alpha - T_\alpha \varepsilon_\alpha \varepsilon_\beta d\omega_\beta\right) - \left(\frac{\varepsilon a'}{a}\right)\left(dT_\beta \wedge \omega_\alpha - \varepsilon_\alpha \varepsilon_\beta dT_\alpha \wedge \omega_\beta\right) - \left(\frac{\varepsilon a'}{a}\right) \delta \wedge \boldsymbol{X}_{\alpha\beta} - 2\left(\frac{a'}{a}\right)^2 \varepsilon \varepsilon_\beta \omega_\alpha \wedge \omega_\beta$.

For (4), if $\alpha, \beta \geq 1$, a straightforward computation gives

$$(d\Omega)_{\alpha\beta} + (\Omega \wedge \Omega)_{\alpha\beta} = \varepsilon_\alpha \left\langle R^F(X, Y) e_\beta, e_\alpha \right\rangle - \omega_{\alpha 0} \wedge \omega_{0\beta}(X, Y).$$

First, we calculate $\left\langle R^F(X, Y) e_\beta, e_\alpha \right\rangle$ in some cases:
Case 1: $\alpha, \beta \in \{1, \ldots, n\}$

$$\begin{aligned}
\left\langle R^F(X, Y) e_\beta, e_\alpha \right\rangle &= \left\langle \nabla^F_X \nabla^F_Y e_\beta - \nabla^F_Y \nabla^F_X e_\beta - \nabla^F_{[X,Y]} e_\beta, e_\alpha \right\rangle \\
&= \left\langle \nabla^F_X \left(\nabla_Y e_\beta + \alpha^E(e_\beta, Y)\right) - \nabla^F_Y \left(\nabla_X e_\beta + \alpha^E(e_\beta, X)\right) - \left(\nabla_{[X,Y]} e_\beta + \alpha^E\left(e_\beta, [X, Y]\right)\right), e_\alpha \right\rangle \\
&= \left\langle \nabla^F_X \nabla_Y e_\beta + \nabla^F_X \alpha^E(e_\beta, Y) - \nabla^F_Y \nabla_X e_\beta - \nabla^F_Y \alpha^E(e_\beta, X) - \nabla_{[X,Y]} e_\beta - \alpha^E\left(e_\beta, [X, Y]\right), e_\alpha \right\rangle \\
&= \left\langle \nabla_X \nabla_Y e_\beta, e_\alpha \right\rangle + \left\langle \alpha^E\left(X, \nabla_Y e_\beta\right), e_\alpha \right\rangle - \left\langle A_{\alpha^E(e_\beta, Y)} X, e_\alpha \right\rangle + \left\langle \nabla^E_X \alpha^E(e_\beta, Y), e_\alpha \right\rangle \\
&\quad - \left\langle \nabla_Y \nabla_X e_\beta, e_\alpha \right\rangle - \left\langle \alpha^E\left(Y, \nabla_X e_\beta\right), e_\alpha \right\rangle + \left\langle A_{\alpha^E(e_\beta, X)} Y, e_\alpha \right\rangle - \left\langle \nabla^E_Y \alpha^E(e_\beta, X), e_\alpha \right\rangle \\
&\quad - \left\langle \nabla_{[X,Y]} e_\beta, e_\alpha \right\rangle - \left\langle \alpha^E\left(e_\beta, [X, Y]\right), e_\alpha \right\rangle \\
&= \left\langle R(X, Y) e_\beta, e_\alpha \right\rangle + \left\langle (\nabla_X \alpha)(e_\beta, Y) - (\nabla_X \alpha)(e_\beta, Y), e_\alpha \right\rangle - \left\langle A_{\alpha^E(e_\beta, Y)} X - A_{\alpha^E(e_\beta, X)} Y, e_\alpha \right\rangle.
\end{aligned}$$

Since $e_\alpha \in TM$,

$$\begin{aligned}
\left\langle R^F(X,Y)e_\beta, e_\alpha\right\rangle &= \left\langle R(X,Y)e_\beta, e_\alpha\right\rangle - \left\langle \alpha^E\left(e_\beta, Y\right), \alpha^E\left(e_\alpha, X\right)\right\rangle + \left\langle \alpha^E\left(e_\beta, X\right), \alpha^E\left(e_\alpha, Y\right)\right\rangle \\
&= \left(\varepsilon\frac{(a')^2}{a^2} - \frac{c}{a^2}\right)\left(\left\langle X, e_\beta\right\rangle\langle Y, e_\alpha\rangle - \left\langle Y, e_\beta\right\rangle\langle X, e_\alpha\rangle\right) \\
&\quad + \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\left(\left\langle X, e_\beta\right\rangle\delta(Y)T_\alpha - \left\langle Y, e_\beta\right\rangle\delta(X)T_\alpha - \langle X, e_\alpha\rangle\,\delta(Y)T_\beta + \langle Y, e_\alpha\rangle\,\delta(X)T_\beta\right) \\
&= -\left(\varepsilon\frac{(a')^2}{a^2} - \frac{c}{a^2}\right)\varepsilon_\alpha\varepsilon_\beta\omega_\alpha\wedge\omega_\beta(X,Y) \\
&\quad - \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\varepsilon_\alpha(T_\beta\omega_\alpha - T_\alpha\varepsilon_\alpha\varepsilon_\beta\omega_\alpha)\wedge\delta(X,Y) \\
&= -\left(\varepsilon\frac{(a')^2}{a^2} - \frac{c}{a^2}\right)\varepsilon_\alpha\varepsilon_\beta\omega_\alpha\wedge\omega_\beta(X,Y) - \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\frac{a}{a'}\varepsilon_\alpha\varepsilon\boldsymbol{X}_{\alpha\beta}\wedge\delta(X,Y).
\end{aligned}$$

But, it is easy to see that

$$\omega_{\alpha 0}\wedge\omega_{0\beta} = \frac{-\varepsilon_0\varepsilon_\beta}{a^2}\omega_\alpha\wedge\omega_\beta + \frac{\varepsilon_0\varepsilon}{aa'}\boldsymbol{X}_{\alpha\beta}\wedge\delta,$$

and, in this case, we conclude that

$$(d\Omega)_{\alpha\beta} + (\Omega\wedge\Omega)_{\alpha\beta} = -\varepsilon\frac{(a')^2}{a^2}\varepsilon_\beta\omega_\alpha\wedge\omega_\beta - \left(\frac{a''}{a} - \frac{(a')^2}{a^2}\right)\frac{\varepsilon a}{a'}\boldsymbol{X}_{\alpha\beta}\wedge\delta.$$

For the other cases, we use a similar computation, noting that $\omega_\alpha(TM) = 0$ if $\alpha \in \{n+1,\dots,n+m\}$. Now, for α or β equal zero, we have, considering $\beta = 0$ for instance,

$$\begin{aligned}
(d\Omega)_{\alpha 0} + (\Omega\wedge\Omega)_{\alpha 0} &= -\varepsilon_\alpha X\langle e_\alpha, SY\rangle + \varepsilon_\alpha Y\langle e_\alpha, SX\rangle + \varepsilon_\alpha\langle e_\alpha, S[X,Y]\rangle \\
&\quad + \sum_{\gamma\geq 1} -\varepsilon_\alpha\varepsilon_\gamma\left\langle e_\alpha, \nabla^F_X e_\gamma\right\rangle\langle e_\gamma, SY\rangle - \sum_{\gamma\geq 1} -\varepsilon_\alpha\varepsilon_\gamma\left\langle e_\alpha, \nabla^F_Y e_\gamma\right\rangle\langle e_\gamma, SX\rangle \\
&= -\varepsilon_\alpha X\langle e_\alpha, SY\rangle + \varepsilon_\alpha Y\langle e_\alpha, SX\rangle + \varepsilon_\alpha\langle e_\alpha, S[X,Y]\rangle \\
&\quad + \varepsilon_\alpha\sum_{\gamma\geq 1}\varepsilon_\gamma\left\langle \nabla^F_X e_\alpha, e_\gamma\right\rangle\langle e_\gamma, SY\rangle - \varepsilon_\alpha\sum_{\gamma\geq 1}\varepsilon_\gamma\left\langle \nabla^F_Y e_\alpha, e_\gamma\right\rangle\langle e_\gamma, SX\rangle \\
&= -\varepsilon_\alpha X\langle e_\alpha, SY\rangle + \varepsilon_\alpha Y\langle e_\alpha, SX\rangle + \varepsilon_\alpha\langle e_\alpha, S[X,Y]\rangle + \varepsilon_\alpha\left\langle \nabla^F_X e_\alpha, SY\right\rangle - \varepsilon_\alpha\left\langle \nabla^F_Y e_\alpha, SX\right\rangle \\
&= -\varepsilon_\alpha\left\langle e_\alpha, \nabla^F_X SY\right\rangle + \varepsilon_\alpha\left\langle e_\alpha, \nabla^F_Y SX\right\rangle + \varepsilon_\alpha\langle e_\alpha, S[X,Y]\rangle \\
&= \varepsilon_\alpha\left\langle e_\alpha, -\nabla^F_X SY + \nabla^F_Y SX + S[X,Y]\right\rangle.
\end{aligned}$$

Now, we need to compute $-\nabla^F_X SY + \nabla^F_Y SX + S[X,Y]$. First, we note that for any $U \in \Gamma(F)$ it holds $\nabla^F_X\left(\frac{-1}{ac}U\right) = \frac{\varepsilon a'}{ca^2}\delta(X)U - \frac{1}{a\varepsilon_0}\nabla^F_X U$ and $\nabla^F_X(T+\xi) = -ca'SX$. Therefore,

$$\begin{aligned}
\nabla^F_X SY &= \nabla^F_X\left(\frac{-1}{ac}\left(Y - \varepsilon\delta(Y)(T+\xi)\right)\right) \\
&= \frac{\varepsilon a'}{ca^2}\delta(X)\left(Y - \varepsilon\delta(Y)(T+\xi)\right) - \frac{1}{ac}\nabla^F_X\left(Y - \varepsilon\delta(Y)(T+\xi)\right) \\
&= \frac{\varepsilon a'}{ca^2}\delta(X)Y - \frac{a'}{ca^2}\delta(X)\delta(Y)(T+\xi) - \frac{1}{ac}\nabla^F_X Y + \frac{\varepsilon}{ac}X(\delta(Y))(T+\xi) - \frac{\varepsilon a'}{a}\delta(Y)SX.
\end{aligned}$$

Analogously,

$$\nabla^F_Y SX = \frac{\varepsilon a'}{ca^2}\delta(Y)X - \frac{a'}{c}\delta(Y)\delta(X)(T+\xi) - \frac{1}{ac}\nabla^F_Y X + \frac{\varepsilon}{ac}Y(\delta(X))(T+\xi) - \frac{\varepsilon a'}{a}\delta(X)SY.$$

Then,

$$\begin{aligned}\nabla^F_Y SX - \nabla^F_X SY + S[X,Y] &= \frac{\varepsilon a'}{ca^2}(\delta(Y)X - \delta(X)Y) + \frac{1}{ac}[X,Y] + \frac{\varepsilon}{ac}Y(\delta(X)) - X(\delta(Y))(T+\xi)\\ &\quad - \frac{\varepsilon a'}{a}(\delta(X)SY - \delta(Y)SX) + S[X,Y]\\ &= \frac{\varepsilon a'}{ca^2}(\delta(Y)X - \delta(X)Y) + \frac{1}{ac}[X,Y] - \frac{\varepsilon}{ac}\delta([X,Y])(T+\xi)\\ &\quad - \frac{\varepsilon a'}{a}(\delta(X)SY - \delta(Y)SX) + S[X,Y]\\ &= \frac{\varepsilon a'}{ca^2}(\delta(Y)X - \delta(X)Y) - \frac{\varepsilon a'}{a}(\delta(X)SY - \delta(Y)SX)\\ &= 0.\end{aligned}$$

Therefore, it follows that $(d\Omega)_{\alpha 0} + (\Omega \wedge \Omega)_{\alpha 0} = 0$. Since

$$-\varepsilon\frac{(a')^2}{a^2}\varepsilon_\beta\omega_\alpha \wedge \omega_\beta - \left(\frac{a''}{a} - \frac{(a')^2}{a^2}\right)\frac{\varepsilon a}{a'}\boldsymbol{X}_{\alpha\beta} \wedge \delta$$

vanishes when $\beta = 0$, the item (4) is proved. This concludes the proof. □

Lemma 5. *$d\Upsilon + \Upsilon \wedge \Upsilon = 0$.*

Proof. Eq. (1) says that

$$d\Upsilon + \Upsilon \wedge \Upsilon = d\Omega - d\boldsymbol{X} + \Omega \wedge \Omega - \Omega \wedge \boldsymbol{X} - \boldsymbol{X} \wedge \Omega + \boldsymbol{X} \wedge \boldsymbol{X},$$

and the items of Lemma 4 say that the right side of this equation vanishes. This finishes the proof. □

Set $N = m + n - 1$, $\lambda = p + q + \dfrac{|c-1|}{2}$ and define $\mathcal{S} = \{Z \in \mathcal{M}_{N+2}(\mathbb{R}); Z^t G Z = G\}$, where $G_{\alpha\beta} = \varepsilon_\alpha\delta_{\alpha\beta}$. Also set the map $s : \mathcal{S} \to \mathbb{S}\left(\mathbb{E}^{N+2}_\lambda\right) = \{X \in \mathbb{E}^{N+2}_\lambda; \langle X, X\rangle = \varepsilon\}$ given by $Z \mapsto \left(Z_{(N+1)0}, \ldots, Z_{N+1N+1}\right)^t$.

Proposition 6. *The map s described above is a submersion.*

Proof. First, notice that $W \in T_Z\mathcal{S}$ if and only if $Z^{-1}W \in T_I\mathcal{S}$. If $\mathbf{e}_k = (0, 0, \ldots, 0, \underbrace{1}_{k+1\text{th}}, 0, \ldots, 0, 0)^t$, then it is clear that $s(U) = U^t\mathbf{e}_{N+1}$. We have to show that $(ds)_Z : T_Z\mathcal{S} \to T_{s(Z)}\mathbb{S}\left(\mathbb{E}^{N+2}_\lambda\right)$ is surjective. In fact, $(ds)_Z(W) = s(W) = W^t\mathbf{e}_{N+1}$. Therefore, for a given $V \in T_{s(Z)}\mathbb{S}\left(\mathbb{E}^{N+2}_\lambda\right)$, we want to find $W \in T_Z\mathcal{S}$ such that $W^t\mathbf{e}_{N+1} = V$, which is equivalent to $(Z^{-1}W)^t(Z^t\mathbf{e}_{N+1}) = V$. So we want to find $H \in T_I\mathcal{S} = \mathfrak{s} = \{H \in M_{N+2}(\mathbb{R}); H^tG + GH = 0\}$ such that $H(Z^t\mathbf{e}_{N+1})$. Since $\{Z^t\mathbf{e}_0 \ldots, Z^t\mathbf{e}_{N+1}\}$ is a base to $\mathbb{E}^{N+2}_k$ and $\langle V, Z^t\mathbf{e}_{N+1}\rangle = \langle V, s(Z)\rangle = 0$, this is possible (to see this, change to base $\{Z^t\mathbf{e}_\alpha\}$, set the last column of H to being V and construct the rest of H using that $H_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta H_{\beta\alpha}$). This finishes the proof. □

Now, we prove the following

Proposition 7. *Let $(M, \langle,\rangle)$ be a semi-Riemannian manifold satisfying the structure equations. Set $\mathcal{Z}(x) = \{Z \in \mathcal{S} | Z_{n+1\beta} = T_\beta(x), \beta = 0, 1, \ldots, N+1\}$. Then for each $x_0 \in M$ and $B_0 \in \mathcal{Z}(x_0)$, there exists a neighborhood $\mathcal{U}$ of x_0 in M and a unique map $B : \mathcal{U} \to \mathcal{S}$, such that*

$$B^{-1}dB = \Omega - \boldsymbol{X}, \quad B(x_0) = B_0,$$

and $B(x) \in \mathcal{Z}(x)$, for all $x \in \mathcal{U}$.

Proof. For an open neighborhood $\mathcal{U}$ of $x_0 \in M$, we define the set

$$\mathcal{F} = \{(x, Z) \in \mathcal{U} \times \mathcal{S} | Z \in \mathcal{Z}(x)\}.$$

Since s is a submersion by Proposition 6, the dimension of the manifold $\mathcal{F}$ is

$$\dim \mathcal{F} = n + \frac{(N+1)(N+2)}{2} - (N+1) = n + \frac{N(N+1)}{2}$$

and its tangent space is

$$T_{(x,Z)}\mathcal{F} = \{(U, V) \in T_x\mathcal{U} \oplus T_Z\mathcal{S} | (dT_\beta)_x(U) = V_{N+1\beta}, \beta = 0, \ldots, N+1\}.$$

Consider on $\mathcal{F}$ the distribution $\mathcal{D}(x,Z) = \ker \Theta_{(x,Z)}$, where $\Theta = \Upsilon - Z^{-1}dZ = \Omega - \boldsymbol{X} - Z^{-1}dZ$. Also consider $\mathcal{H} = \{H \in \mathfrak{s} | ZH \in \ker(ds)_Z\}$. It is easy to see that

$$\dim \mathcal{H} = \dim \ker(ds)_Z = \dim \ker(ds)_{I_{N+2}} = \dim \mathfrak{s} - (N+1) = \frac{(N+1)(N+2)}{2} - (N+1) = \frac{N(N+1)}{2}.$$

Now, we observe the equivalences

$$\begin{aligned}\Theta(U,V) \in \mathcal{H} &\Leftrightarrow \Omega(U) - \boldsymbol{X}(U) - Z^{-1}V \in \mathcal{H}\\ &\Leftrightarrow Z\Omega(U) - Z\boldsymbol{X}(U) - V \in \ker\,(ds)_Z\\ &\Leftrightarrow (Z\Omega(U))_{N+1\beta} - (Z\boldsymbol{X}(U))_{N+1\beta} - V_{N+1\beta} = 0, \text{ for } \beta = 0,\dots,N+1.\end{aligned}$$

But

$$\begin{aligned}(Z\Omega(U))_{N+1\beta} - (Z\boldsymbol{X}(U))_{N+1\beta} - (dT_\beta)_x(U) &= \sum_\gamma Z_{N+1\gamma}\Omega(U)_{\gamma\beta} - \sum_\gamma Z_{N+1\gamma}\boldsymbol{X}(U)_{\gamma\beta} - (dT_\beta)_x(U)\\ &= \sum_\gamma T_\gamma(x)\omega_{\gamma\beta}(U) - \sum_\gamma T_\gamma(x)\boldsymbol{X}(U)_{\gamma\beta} - (dT_{N+1\beta})_x(U)\\ &= (dT_\beta)_x(U) - \frac{a'}{a}\varepsilon_\beta(\omega_\beta)_x(U) + \frac{\varepsilon a'}{a}T_\beta(x)\delta_x(U)\\ &\quad - \frac{\varepsilon a'}{a}\sum_\gamma T_\gamma(x)\left(T_\beta(x)(\omega_\gamma)_x(U) - \varepsilon_\gamma\varepsilon_\beta T_\gamma(x)(\omega_\beta)_x(U)\right)\\ &\quad - (dT_\beta)_x(U)\\ &= 0,\end{aligned}$$

for $\beta = 0,\dots,N+1$. Therefore, Im $\Theta \subset \mathcal{H}$. Now, if $H \in \mathcal{H}$, then $(0,-ZH) \in T_{(x,Z)}\mathcal{F}$, which means that $\Theta(0,-ZH) = H$, and it follows that Θ is surjective, that is, Im $\Theta = \mathcal{H}$. Hence, we conclude that $\dim \mathcal{D}(x,Z) = \dim T_{(x,Z)}\mathcal{F} - \dim \text{Im}\,\Theta = n$. Now, we want to prove that $\mathcal{D}$ is integrable. Since $d\Upsilon + \Upsilon \wedge \Upsilon = 0$, we have

$$d\Theta = d\Upsilon + Z^{-1}dZ \wedge Z^{-1}dZ = d\Upsilon + (\Upsilon - \Theta)\wedge(\Upsilon - \Theta) = \Upsilon\wedge\Theta - \Theta\wedge\Upsilon + \Theta\wedge\Theta.$$

Thus, if $U, V \in \mathcal{D}$, we obtain $d\Theta(U,V) = (\Upsilon\wedge\Theta - \Theta\wedge\Upsilon + \Theta\wedge\Theta)(U,V) = 0$. On the other hand, we have $d\Theta(U,V) = U(\Theta(V)) - V(\Theta(U)) - \Theta([U,V]) = -\Theta([U,V])$, and we conclude that $[U,V] \in \mathcal{D}$.

For what is missing, let L be an integral manifold through (x_0,B_0). For each $(0,V) \in \mathcal{D}_{(x_0,B_0)}$, we have $\Theta_{(x_0,B_0)}(0,V) = B_0^{-1}V = 0$, which gives $V = 0$. This means that L intersects $\{x_0\}\times\mathcal{S}$ transversely in (x_0,B_0). Shrinking $\mathcal{U}$, if necessary, and using the Local Characterization of Graphs, we conclude that L is a graph of a unique function $B : \mathcal{U} \to \mathcal{S}$. Since $L \subset \mathcal{F}$, for each $x \in \mathcal{U}$ we have $B(x) \in Z(x)$. Finally, since $\Theta \equiv 0$ on L, B satisfies the equation $B^{-1}dB = \Omega - \boldsymbol{X}$. □

5. Proof of the main theorem

Let $\mathcal{U}$ be a neighborhood in M of a given point $x_0 \in M$ and $B : \mathcal{U} \to \mathcal{S}$ the map found in Proposition 7. Define $f : \mathcal{U} \to \varepsilon I \times_a \mathbb{E}^{N+1}_\lambda$ by

$$f_0 = cB_{00}, \quad \dots \quad , \quad f_N = \varepsilon_N B_{N0}, \quad f_{N+1} = \pi.$$

A direct computation gives us Im $f \subset \varepsilon I \times_a \mathbb{M}^N_\lambda(c)$. In order to see that f is an isometric immersion, we compute

$$\begin{aligned}df(e_i) &= \sum_\gamma df_\gamma(e_i)E_\gamma\\ &= \sum_{\gamma\le N} df_\gamma(e_i)E_\gamma + d\pi(e_i)E_{N+1}\\ &= \sum_{\gamma\le N}\varepsilon_\gamma dB_{\gamma 0}(e_i)E_\gamma + \varepsilon T_i E_{N+1}\\ &= \sum_{\gamma\le N}\varepsilon_\gamma\left(B\Omega(e_i) - B\mathbf{X}(e_i)\right)_{\gamma 0}E_\gamma + \varepsilon_{N+1}B_{N+1i}E_{N+1}\\ &= \left(\sum_{\substack{\gamma\le N\\ \theta}}\varepsilon_\gamma B_{\gamma\theta}\omega_{\theta 0}(e_i)E_\gamma\right) + \varepsilon_{N+1}B_{N+1i}E_{N+1}\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{\substack{\gamma \leq N \\ \theta}} -\varepsilon_\gamma B_{\gamma\theta} \left(\frac{\varepsilon\varepsilon_\theta T_i T_\theta - \delta_{\theta i}}{ac} \right) E_\gamma \right) + \varepsilon_{N+1} B_{N+1i} E_{N+1} \\
&= \sum_{\gamma \leq N} \frac{-\varepsilon_\gamma}{ac} \left(\varepsilon T_i \left(\sum_\theta \varepsilon_\theta T_\theta B_{\gamma\theta} \right) - B_{\gamma i} \right) E_\gamma + \varepsilon_{N+1} B_{N+1i} E_{N+1} \\
&= \sum_{\gamma \leq N} \frac{-\varepsilon_\gamma}{ac} \left(\varepsilon T_i \left(\sum_\theta \varepsilon_\theta B_{\gamma\theta} B_{N+1\theta} \right) - B_{\gamma i} \right) E_\gamma + \varepsilon_{N+1} B_{N+1i} E_{N+1} \\
&= \sum_{\gamma \leq N} \frac{-\varepsilon_\gamma}{ac} \left(\varepsilon T_i \varepsilon_\gamma \delta_{\gamma N+1} - B_{\gamma i} \right) E_\gamma + \varepsilon_{N+1} B_{N+1i} E_{N+1} \\
&= \sum_{\gamma \leq N} \frac{\varepsilon_\gamma}{ac} B_{\gamma i} E_\gamma + \varepsilon_{N+1} B_{N+1i} E_{N+1} \\
&= \sum_\gamma \varepsilon_\gamma B_{\gamma i} \overline{E}_\gamma .
\end{aligned}
$$

Since $B \in \mathbb{S}$, we have

$$
\left\langle df(e_i), df(e_j) \right\rangle = \left\langle \sum_\alpha \varepsilon_\gamma B_{\alpha i} \overline{E}_\alpha, \sum_\gamma \varepsilon_\gamma B_{\gamma j} \overline{E}_\gamma \right\rangle = \sum_\gamma \varepsilon_\gamma B_{\gamma i} B_{\gamma j} = \varepsilon_i \delta_{ij} = \left\langle e_i, e_j \right\rangle ,
$$

which means that f is an isometric immersion. Now, let $\tilde{E}_\gamma = \sum_\alpha \varepsilon_\alpha B_{\alpha\gamma} \overline{E}_\alpha$. We have that $\tilde{E}_i = df(e_i)$ is tangent to $f(\mathcal{U})$ and $\tilde{E}_u$ is normal to $f(\mathcal{U})$. In fact, for each i, we have

$$
\left\langle \tilde{E}_u, \tilde{E}_i \right\rangle = \left\langle \sum_\alpha \varepsilon_\alpha B_{\alpha u} \overline{E}_\alpha, \sum_\gamma \varepsilon_\gamma B_{\gamma i} \overline{E}_\gamma \right\rangle = \sum_\gamma \varepsilon_\gamma B_{\gamma u} B_{\gamma i} = \varepsilon_i \delta_{ui} = \langle e_u, e_i \rangle = 0,
$$

since $i \neq u$. Then, $Tf(\mathcal{U})^\perp = \text{span}\,\{\tilde{E}_u\}$. Let Φ be an extension of the immersion f defined as follows:

$$
\begin{aligned}
\Phi : T\mathcal{U} \oplus \sigma^{-1}(\mathcal{U}) \subset TM \oplus E &\to Tf(\mathcal{U}) \oplus Tf(\mathcal{U})^\perp \\
\left(p, \sum_\gamma \lambda_\gamma e_\gamma \right) &\to \left(f(p), \sum_\gamma \lambda_\gamma \tilde{E}_\gamma \right),
\end{aligned}
$$

where $\sigma : E \to M$ is the projection. It is easy to see that Φ is a isomorphism between the bundles $T\mathcal{U} \oplus \sigma^{-1}(\mathcal{U})$ and $Tf(\mathcal{U}) \oplus Tf(\mathcal{U})^\perp$. Moreover, the metric induced by Φ also coincides with the given bundle metric since $\Phi(e_\gamma) = \tilde{E}_\gamma$. If $\alpha, \beta, \gamma \geq 1$, then

$$
\begin{aligned}
\left\langle \nabla^\perp_{\Phi(e_i)} \Phi(e_u), \Phi(e_v) \right\rangle &= \left\langle \tilde{\nabla}_{\tilde{E}_i} \tilde{E}_u, \tilde{E}_v \right\rangle \\
&= \left\langle \tilde{\nabla}_{\sum_\rho \varepsilon_\rho B_{\rho i} \overline{E}_\rho} \sum_\theta \varepsilon_\theta B_{\theta u} \overline{E}_\theta, \sum_\lambda \varepsilon_\lambda B_{\lambda v} \overline{E}_\lambda \right\rangle \\
&= \sum_\theta \varepsilon_\theta B_{\theta v} dB_{\theta u}(e_i) + \sum_{\theta,\lambda,\rho} \varepsilon_\theta \varepsilon_\lambda \varepsilon_\rho B_{\rho i} B_{\lambda v} B_{\theta u} \left\langle \tilde{\nabla}_{\overline{E}_\rho} \overline{E}_\theta, \overline{E}_\lambda \right\rangle \\
&= \varepsilon_v (B^{-1} dB(e_i))_{vu} + \sum_{\theta,\lambda,\rho} \varepsilon_\theta \varepsilon_\lambda \varepsilon_\rho B_{\rho i} B_{\lambda v} B_{\theta u} \left\langle \tilde{\nabla}_{\overline{E}_\rho} \overline{E}_\theta, \overline{E}_\lambda \right\rangle \\
&= \varepsilon_v \omega_{vu}(e_i) + \frac{\varepsilon a'}{a} B_{N+1u} \sum_{\theta \leq N} \varepsilon_\theta B_{\theta i} B_{\theta v} - \frac{\varepsilon a'}{a} B_{N+1v} \sum_{\theta \leq N} \varepsilon_\theta B_{\theta i} B_{\theta u} \\
&= \left\langle e_v, \nabla^E_{e_i} e_u \right\rangle = \left\langle \Phi(e_v), \Phi(\nabla^E_{e_i} e_u) \right\rangle .
\end{aligned}
$$

If we call, by abuse of notation, the isomorphism $\Phi_{|_{\sigma^{-1}(\mathcal{U})}} : \sigma^{-1}(\mathcal{U}) \subset E \to Tf(\mathcal{U})^\perp$ by Φ too, then we conclude that

$$
\Phi \nabla^E = \nabla^\perp \Phi .
$$

A similar computation gives $\alpha_f = \Phi \circ \alpha^E \circ df^{-1}$, where α_f is the second fundamental form of f. In fact,

$$\begin{aligned}(df \circ \alpha_f)(e_i, e_j) &= df(\alpha_f(e_i, e_j)) = \nabla^{\perp}_{e_i} df(e_j) = \nabla^{\perp}_{e_i} \tilde{E}_j = \nabla^{\perp}_{e_i} \Phi(e_j) \\ &= \Phi \nabla^E_{e_i} e_j = \Phi(\alpha^E(e_i, e_j)) = (\Phi \circ \alpha^E)(e_i, e_j).\end{aligned}$$

We also notice that

$$\begin{aligned}\partial_t &= \sum_\gamma \varepsilon_\gamma \left\langle \tilde{E}_\gamma, \partial_t \right\rangle \tilde{E}_\gamma \\ &= \sum_\gamma \varepsilon_\gamma \left\langle \sum_\theta \varepsilon_\theta B_{\theta\gamma} \overline{E}_\theta, \partial_t \right\rangle \tilde{E}_\gamma \\ &= \sum_\gamma \varepsilon_\gamma B_{N+1\gamma} \tilde{E}_\gamma \\ &= \sum_\gamma \varepsilon_\gamma T_\gamma \tilde{E}_\gamma \\ &= \sum_\gamma \varepsilon_\gamma \left\langle e_\gamma, T + \xi \right\rangle \tilde{E}_\gamma \\ &= \sum_\gamma \varepsilon_\gamma \left\langle \Phi(e_\gamma), \Phi(T + \xi) \right\rangle \tilde{E}_\gamma \\ &= \sum_\gamma \varepsilon_\gamma \left\langle \tilde{E}_\gamma, \Phi(T + \xi) \right\rangle \tilde{E}_\gamma \\ &= \Phi(T) + \Phi(\xi).\end{aligned}$$

Now, we finish the proof by showing that the local immersion f is unique up to a global isometry of $\varepsilon I \times_a \mathbb{M}^N_\lambda(c)$. For this, let $\tilde{f} : \tilde{\mathcal{U}} \to \varepsilon I \times_a \mathbb{M}^N_\lambda(c)$ be another isometric immersion satisfying all the properties of Theorem 1, with $\tilde{\mathcal{U}}$ a simply connected neighborhood of x_0 included in $\mathcal{U}$ and $\{\tilde{V}_\alpha\}$ the associated frame, with $\tilde{V}_i = d\tilde{f}(e_i)$ and $\tilde{V}_u$ normal to $\tilde{f}(\tilde{U})$. Let $\tilde{B}$ e be the matrix of the coordinates of the vectors $\tilde{V}_\alpha$ in the frame $\{\tilde{E}_\alpha\}$. Obviously, up to a direct isometry of $\varepsilon I \times_a \mathbb{M}^N_\lambda(c)$, we can assume that $f(x_0) = \tilde{f}(x_0)$ and the frame $\{\tilde{E}_\alpha\}$ and $\{\tilde{V}_\alpha\}$ coincide at the point x_0 and hence $B(x_0) = \tilde{B}(x_0)$. Moreover, these two matrices satisfy all the properties of Proposition 7, so by uniqueness of the solution in Proposition 7, we have $B(x) = \tilde{B}(x)$, for all $x \in \tilde{\mathcal{U}}$. Hence, by construction of f and $\tilde{f}$ from B and $\tilde{B}$, we deduce that $f = \tilde{f}$ in $\tilde{\mathcal{U}}$.

Now, we will prove that f can be extended in a unique way to M. For this, we consider x_1 in M and a curve $\Gamma : [0, 1] \to M$ so that $\Gamma(0) = x_0$ and $\Gamma(1) = x_1$. Then, for each point $\Gamma(t)$ there exists a neighborhood of $\Gamma(t)$ such that there exists an isometric immersion of this neighborhood into $\varepsilon I \times_a \mathbb{M}^N_\lambda(c)$ satisfying the properties of the theorem. From this family of neighborhood, we can extract a finite subsequence $(\mathcal{U}_0, \ldots, \mathcal{U}_r)$ covering Γ with $\mathcal{U}_0 = \mathcal{U}$. Hence, by uniqueness, we can extend f to $\mathcal{U}_r$ and define $f(x_1)$. We conclude by noting that since M is simply connected, the value of $f(x_1)$ does not depend on the choice of the curve Γ. Finally, having the global immersion $f : M \to \varepsilon I \times_a \mathbb{M}^N_\lambda(c)$ we obtain the desired global bundle isomorphism $\Phi : E \to Tf(M)^{\perp}$.

This finishes the proof of Theorem 1.

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