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Dolbeault cohomology groups of compact pseudo-Kähler homogeneous manifolds

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ABSTRACT

It is well known that a pseudo-Kähler structure is one of the natural generalizations of a Kähler structure. In this paper, we consider the Dolbeault cohomology groups of compact pseudo-Kähler homogeneous manifolds.

0. Introduction

A *pseudo-Kähler manifold* is a symplectic manifold (M, ω) with a complex structure J that is compatible with ω, i.e., ω induces an indefinite Hermitian metric on M. The manifolds represent an interesting class containing the class of Kähler manifolds in symplectic geometry. For example, the Kodaira–Thurston manifold, which is the first example of compact non-Kähler symplectic manifolds, has a pseudo-Kähler structure. On the other hand, some results (e.g. [6,9]) have shown that pseudo-Kähler manifolds are natural generalizations of Kähler manifolds.

Dorfmeister–Guan [6] proved that a compact homogeneous pseudo-Kähler manifold is biholomorphic to the direct product of a homogeneous rational manifold and a complex torus (cf. also [15]). By a homogeneous pseudo-Kähler manifold we mean a pseudo-Kähler manifold on which the group of holomorphic *isometric* transformations act transitively. Huckleberry [13] investigated pseudo-Kähler manifolds from a Hamiltonian viewpoint, and gave a simpler proof of the result of Dorfmeister–Guan. However, in the non-homogeneous pseudo-Kähler case, the results are still not complete.

Our aim in the present paper is to construct some methods for investigating these manifolds by considering the Dolbeault cohomology groups. This is because there exist many important studies related to Dolbeault cohomology groups of homogeneous complex manifolds (for example, [1,14,17,20,21,25]), and we can investigate pseudo-Kähler homogeneous manifolds by using some of the results of these studies (cf. [28,29], and Corollary 5.4). By a pseudo-Kähler homogeneous manifold we mean a pseudo-Kähler manifold on which the group of holomorphic transformations act transitively.

We mainly prove the following three theorems, one of which was announced in the previous paper [29]:

Theorem 2.2. *For a compact pseudo-Kähler manifold, the following two assertions are equivalent:*

(a) *Any Dolbeault cohomology class contains an almost $\bar{\partial}$-harmonic representative (see Definition 1.9).*

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(b) *The pseudo-Kähler manifold satisfies the hard Lefschetz theorem with respect to the Dolbeault cohomology groups, i.e., every Lefschetz mapping is an isomorphism.*

For a compact pseudo-Kähler manifold (M, ω, J), we can consider a subspace of the Dolbeault cohomology group of type (p, q) such that each cohomology class contains an almost $\bar{\partial}$-harmonic representative. We denote the subspace by $K_{\omega}^{p,q}(M)$. Then we have:

Theorem 3.3. *Let (M, J) be a compact complex manifold, and ω, ω' cohomologous pseudo-Kähler structures. Then $K_{\omega}^{p,q}(M) = K_{\omega'}^{p,q}(M)$ for each p, q.*

Let Γ be a discrete subgroup of a real Lie group G with a compact left coset space G/Γ. We can project the left G-invariant forms on G onto G/Γ. As an application of Theorem 3.3, we see:

Theorem 3.4. *Let $(G/\Gamma, \omega, J)$ be a compact pseudo-Kähler manifold such that J is a left G-invariant complex structure. Assume that each Dolbeault cohomology class contains a left G-invariant representative. Then a left G-invariant form ω_0 that is cohomologous to ω is a pseudo-Kähler structure such that $K_{\omega_0}^{p,q}(G/\Gamma) = K_{\omega}^{p,q}(G/\Gamma)$. In particular, each cohomology class of $K_{\omega_0}^{p,q}(G/\Gamma)$ contains a left G-invariant almost $\bar{\partial}$-harmonic representative.*

Note that condition (a) can be rewritten as follows: (a′) *The subspace $K_{\omega}^{p,q}(M)$ is equal to the Dolbeault cohomology group of type (p, q) for each p, q.* Thus, there exists an important relationship among these theorems. As one of the applications, we can exactly compute the dimensions of $K_{\omega}^{p,q}(G/\Gamma)$ for investigating pseudo-Kähler structures of $(G/\Gamma, \omega, J)$ if it satisfies the above assumptions. Thus, we can also notice the properties of Lefschetz mappings.

Cordero–Fernández–Ugarte [4] investigated condition (b) of Theorem 2.2, and they proved that if a compact pseudo-Kähler nilmanifold satisfies condition (b), then it is a complex torus. On the other hand, there exists a non-toral compact pseudo-Kähler solvmanifold which satisfies condition (b) (see Example 6.1). Cordero–Fernández–Gray–Ugarte [3] proved that if G is nilpotent, and if the complex structure J satisfies the *nilpotent condition*, then each Dolbeault cohomology class contains a left G-invariant representative. For example, the Kodaira–Thurston manifold is a nilmanifold which satisfies the nilpotent condition. Moreover, Cordero–Fernández–Ugarte [5] showed that *the complex structure underlying a pseudo-Kähler structure of a 6-dimensional nilpotent Lie algebra satisfies the nilpotent condition.*

In Section 1, we prepare several notation and propositions to prove our main theorems. In Sections 2 and 3, we prove Theorems 2.2 and 3.4.

A complex manifold M is said to be *complex parallelizable* if the holomorphic tangent bundle of M is holomorphically trivial. Then M can be written as G/Γ, where G is a complex Lie group, and Γ is a lattice of G. In the previous paper [29], we proved that if a compact complex parallelizable solvmanifold M has a pseudo-Kähler structure, then M is the total space of a complex torus bundle over a complex torus. In particular, the derived Lie subgroup of G is abelian. Thus we have the following natural question:

"*When does a compact homogeneous manifold which is the total space of a complex torus bundle over a complex torus have a pseudo-Kähler structure?*"

With respect to this question, in Section 5, we point out a remarkable property if M is complex parallelizable (see Theorem 5.8 and Remark 5.9).

In Section 4, we construct a compact pseudo-Kähler manifold which is a homogeneous space of a *real* solvable Lie group G such that the derived Lie subgroup of G is not abelian. Moreover, we prove that the curvature tensor of the pseudo-Kähler manifold vanishes.

1. The space of almost $\bar{\partial}$-harmonic forms

In this section, we exactly define an almost $\bar{\partial}$-harmonic form, and apply an $\mathfrak{sl}(2)$-representation to the space of all almost $\bar{\partial}$-harmonic forms in order to prove Theorems 2.2 and 3.4.

Let (M^{2m}, ω) be a symplectic manifold and $\mathbf{G}$ the skew-symmetric bivector field dual to ω. Hence, $\mathbf{G}$ induces a mapping $i(\mathbf{G}) : \Omega^k(M) \to \Omega^{k-2}(M)$. We define a star operator

$$*_{\omega} : \Omega^k(M) \to \Omega^{2m-k}(M) \qquad \text{for } k = 0, \ldots, 2m$$

by requiring

$$\alpha \wedge *_{\omega} \beta = (\wedge^k(\mathbf{G}))(\alpha, \beta) v_M \quad \text{for } \alpha, \beta \in \Omega^k(M),$$

where $v_M = \omega^m/m!$. We define $L_{\omega} : \Omega^k(M) \longrightarrow \Omega^{k+2}(M)$ by $L_{\omega}(\alpha) = \alpha \wedge \omega$. If there exist no possibilities of confusion, then we write $*_{\omega} = *$ and $L_{\omega} = L$ for simplicity. We define $d^* : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ by $d^* = (-1)^k * d *$. Brylinski [2] proved the following:

Proposition 1.1. $d^* = [d, i(\mathbf{G})] = d \circ i(\mathbf{G}) - i(\mathbf{G}) \circ d.$

We define an operator L^* by

$$L^* = *L* : \Omega^k(M) \to \Omega^{k-2}(M).$$

We can easily see that L^* is the adjoint operator for L, where we define an inner product $(,)$ on $\Omega^*(M)$ by $(\alpha, \beta) = \int *(\alpha \wedge *\beta)v_M$ for $\alpha, \beta \in \Omega^k(M)$. Moreover, we define an operator A by

$$A = \sum (m-k)\pi_k,$$

where $\pi_k : \Omega^*(M) \to \Omega^k(M)$ is the natural projection.

Proposition 1.2 ([30]). $i(\mathbf{G}) = -L^*$.

These operators satisfy the following relations:

Proposition 1.3 ([30]). $[L^*, L] = A$, $[A, L] = -2L$, $[A, L^*] = 2L^*$.

Let X, H, Y be the standard basis of $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{sl}(2)$. Thus these satisfy the relations $[X, Y] = H$, $[H, X] = 2X$, and $[H, Y] = -2Y$.

Definition 1.4. Let V be an infinite-dimensional vector space. Assume that $\mathfrak{sl}(2)$ acts on V. We say that V is an $\mathfrak{sl}(2)$-*module of finite H-spectrum* if the following two conditions are satisfied:

(a) V can be decomposed as the direct sum of eigenspaces of H.
(b) H has only finitely many distinct eigenvalues.

Let V be an $\mathfrak{sl}(2)$-module of finite H-spectrum, and V_k the eigenspace of H with eigenvalue k. Note that $\{v \in V_k \mid Xv = 0\} = \{v \in V_k \mid Y^{k+1}v = 0\}$. By a basic result on an $\mathfrak{sl}(2)$-representation, we have the following:

Proposition 1.5 ([8]). *For any k, the maps $Y^k : V_k \to V_{-k}$ and $X^k : V_{-k} \to V_k$ are isomorphisms.*

Now we have a representation $\mathfrak{sl}(2)$ on $\Omega^*(M) \otimes \mathbb{C}$ by sending

$$X \longleftrightarrow L^*, \quad Y \longleftrightarrow L, \quad H \longleftrightarrow A.$$

It is easy to check that $\Omega^*(M) \otimes \mathbb{C}$ is an $\mathfrak{sl}(2)$-module of finite H-spectrum, and $\Omega^{m-k}(M) \otimes \mathbb{C} = V_k$.

From now on, we consider the case of pseudo-Kähler manifolds.

Definition 1.6. A non-degenerate real closed $(1, 1)$-form ω on a complex manifold (M, J) is called a *pseudo-Kähler structure*.

Let (M^{2m}, ω, J) be a pseudo-Kähler manifold of $\dim_{\mathbb{R}} M = 2m$. Then we have the following:

Corollary 1.7. *For each $k \le m$, the Lefschetz mapping*

$$L^k : \Omega^{m-k}(M) \otimes \mathbb{C} \longrightarrow \Omega^{m+k}(M) \otimes \mathbb{C}$$

is an isomorphism. In particular, for each $p + q \le m$, the Lefschetz mapping

$$L^k : \Omega^{p,q}(M) \longrightarrow \Omega^{m-q,m-p}(M)$$

is an isomorphism.

Proof. By Proposition 1.5, we have our corollary (note that ω is a $(1, 1)$-form). □

By the definition of $\mathbf{G}$, for $\alpha \in \Omega^{s,t}(M)$ and $\beta \in \Omega^{p,q}(M)$, we have

$$\bigwedge^{p+q}(\mathbf{G})(\alpha, \beta) = 0 \quad \text{if } s \ne q, t \ne p,$$

which implies that $* : \Omega^{p,q}(M) \longrightarrow \Omega^{m-q,m-p}(M)$. Thus we define operators $\partial^* : \Omega^{p,q}(M) \longrightarrow \Omega^{p-1,q}(M)$ and $\bar{\partial}^* : \Omega^{p,q}(M) \longrightarrow \Omega^{p,q-1}(M)$ by $\partial^* = (-1)^{p+q} * \bar{\partial} *$ and $\bar{\partial}^* = (-1)^{p+q} * \partial *$, respectively. Then we have the following lemma.

Lemma 1.8. $\partial^* = [\bar{\partial}, i(\mathbf{G})]$, $\bar{\partial}^* = [\partial, i(\mathbf{G})]$.

Proof. Since $[d, i(\mathbf{G})] = d^*$, $d = \partial + \bar{\partial}$ and $d^* = \partial^* + \bar{\partial}^*$, we have our lemma. □

For a complex manifold (M, J), we denote the space of all $\bar{\partial}$-closed (p, q)-forms on M by $Z^{p,q}_{\bar{\partial}}(M)$, and we write $B^{p,q}_{\bar{\partial}}(M) = \bar{\partial}(\Omega^{p,q-1}(M))$.

Definition 1.9. Let (M, ω, J) be a pseudo-Kähler manifold. A form $\alpha \in \Omega^{p,q}(M)$ is said to be *almost $\bar{\partial}$-harmonic* with respect to ω, if it satisfies $\bar{\partial}\alpha = \partial^*\alpha = 0$. We denote the space of all almost $\bar{\partial}$-harmonic forms of type (p, q) by $\mathcal{K}^{p,q}_{\omega}(M)$, and we set $\mathcal{K}(M) = \sum_r \mathcal{K}^r(M) = \sum_r \sum_{p+q=r} \mathcal{K}^{p,q}(M)$. We define the *almost $\bar{\partial}$-harmonic cohomology group* $K^{p,q}_{\omega}(M)$ by $\mathcal{K}^{p,q}_{\omega}(M)/\mathcal{K}^{p,q}_{\omega}(M) \cap B^{p,q}_{\bar{\partial}}(M)$.

Remark 1.10. (1) It is obvious that $dd^* + d^*d = 0$ by Proposition 1.1, while $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} \neq 0$. However, it is false that if $(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\alpha = 0$, then $\bar{\partial}\alpha = 0$.

(2) Let (M, g, J) be a compact Kähler manifold and $*_g$ the $*$-operator induced naturally. Brylinski proved that $*_\omega = \sqrt{-1}^{p-q} *_g$ on $\Omega^{p,q}(M)$, where ω is the fundamental 2-form induced by g. Thus, in this case, if $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$, then $\bar{\partial}\alpha = \partial^*\alpha = 0$.

Proposition 1.11. *$\mathcal{K}(M)$ is an $\mathfrak{sl}(2)$-submodule of $\Omega^*(M) \otimes \mathbb{C} = \sum_{p,q} \Omega^{p,q}(M)$.*

Proof. Since $[i(\mathbf{G}), d] = d^*$, $[L, d^*] = d$, and $[L, d] = 0$, we see

$$[\bar{\partial}, L^*] = \partial^*, [L^*, \partial^*] = 0, [L, \bar{\partial}] = 0, [L, \partial^*] = \bar{\partial}.$$

Hence, $\mathcal{K}(M)$ is an $\mathfrak{sl}(2)$-submodule. $\square$

Since $\mathcal{K}(M)$ is an $\mathfrak{sl}(2)$-submodule, we have the following:

Corollary 1.12. *For each $k \leq m$, the Lefschetz mapping*

$$L^k : \mathcal{K}^{m-k}(M) \longrightarrow \mathcal{K}^{m+k}(M)$$

is an isomorphism. In particular, for each $p + q \leq m$, the Lefschetz mapping

$$L^{m-p-q} : \mathcal{K}^{p,q}(M) \longrightarrow \mathcal{K}^{m-q,m-p}(M)$$

is an isomorphism.

2. Mathieu's theorem of Dolbeault cohomology groups

In this section, we prove Theorem 2.2, which is a Mathieu's theorem of Dolbeault cohomology groups. The original result due to Mathieu is a theorem related to de Rham cohomology groups of symplectic manifolds (see [30]). The proofs of Theorems 2.2 and 3.4 are essentially the same as that of the case of de Rham cohomology groups (see [30,26]). We use same notation introduced in Section 1.

Proposition 2.1. *Let (M^{2m}, ω, J) be a pseudo-Kähler manifold of $\dim_{\mathbb{R}} M = 2m$. Then for each p, q,*

$$P^{p,q}_\omega(M) \subset K^{p,q}_\omega(M),$$

where $P^{p,q}_\omega(M) = \{[\alpha] \in H^{p,q}_{\bar{\partial}}(M) \mid L^{m-p-q+1}([\alpha]) = 0\}$.

Proof. Let $[\alpha] \in P^{p,q}_\omega(M)$. Hence, $\alpha \wedge \omega^{m-p-q+1} = \bar{\partial}\gamma$, where $\gamma \in \Omega^{m-q+1,m-p}(M)$. Since $L^{m-p-q-1} : \Omega^{p,q-1}(M) \longrightarrow \Omega^{m-q+1,m-p}(M)$ is surjective, there exists $\theta \in \Omega^{p,q-1}(M)$ such that $L^{m-p-q+1}\theta = \gamma$. Then we have $L^{m-p-q+1}(\alpha - \bar{\partial}\theta) = 0$. Therefore, $i(\mathbf{G})(\alpha - \bar{\partial}\theta) = 0$, which implies $\partial^*(\alpha - \bar{\partial}\theta) = 0$ by Lemma 1.8. Hence, $\alpha - \bar{\partial}\theta$ is almost $\bar{\partial}$-harmonic. $\square$

Theorem 2.2. *Let (M^{2m}, ω, J) be a pseudo-Kähler manifold. Then the following two assertions are equivalent:*

(a) *Any Dolbeault cohomology class contains an almost $\bar{\partial}$-harmonic representative.*
(b) *For any $p + q \leq m$, the Lefschetz mapping $L^{m-p-q} : H^{p,q}_{\bar{\partial}}(M) \longrightarrow H^{m-q,m-p}_{\bar{\partial}}(M)$ is surjective.*

Proof. (b) $\Rightarrow$ (a). Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{K}^{p,q}(M) & \xrightarrow{L^{m-p-q}} & \mathcal{K}^{m-q,m-p}(M) \\ \downarrow & & \downarrow \\ H^{p,q}_{\bar{\partial}}(M) & \xrightarrow{L^{m-p-q}} & H^{m-q,m-p}_{\bar{\partial}}(M), \end{array}$$

where the two vertical arrows are surjective by our assumption. It follows from Corollary 1.12 that the second horizontal arrow is also surjective.

(a) $\Rightarrow$ (b). Let $u \in H^{p,q}_{\bar{\partial}}(M)$ and consider $L^{m-p-q+1}(u)$. Since

$$L^{m-p-q} : H^{p,q}_{\bar{\partial}}(M) \longrightarrow H^{m-q,m-p}_{\bar{\partial}}(M)$$

is surjective, there exists $v \in H^{p-1,q-1}_{\bar{\partial}}(M)$ which satisfies

$$L^{m-p-q+1}(u) = L^{m-p-q+2}(v).$$

Hence, $L^{m-p-q+1}(u - L(v)) = 0$. Since $u = (u - L(v)) + L(v)$, we have Theorem 2.2 by Proposition 2.1. $\square$

Condition (b) is sometimes called the *hard Lefschetz property* with respect to Dolbeault cohomology groups.

Corollary 2.3. *Let (M^{2m}, ω, J) be a compact pseudo-Kähler manifold. If any Dolbeault cohomology class contains an almost $\bar\partial$-harmonic representative, then for each p, q,*

$$\dim H^{p,q}_{\bar\partial}(M) = \dim H^{q,p}_{\bar\partial}(M).$$

Proof. By Theorem 2.2, we have

$$\dim H^{p,q}_{\bar\partial}(M) = \dim H^{m-q,m-p}_{\bar\partial}(M).$$

On the other hand, let us consider any Riemannian metric g of M, and consider a Hermitian metric h induced by g and J. By the Hodge decomposition with respect to h, we have

$$\dim H^{q,p}_{\bar\partial}(M) = \dim H^{m-q,m-p}_{\bar\partial}(M).$$

Hence, $\dim H^{p,q}_{\bar\partial}(M) = \dim H^{q,p}_{\bar\partial}(M)$. $\square$

3. Nomizu type theorem on $K^{p,q}_{\omega}(M)$

In this section we prove Theorem 3.4, which is a Nomizu type theorem (cf. [7,18,26]).

Lemma 3.1. *Let (M, J) be a complex manifold of $\dim_{\mathbb{R}} M = 2m$. Assume that ω, ω' are cohomologous pseudo-Kähler structures on M. Then for each p, q,*

$$P^{p,q}_{\omega}(M) = P^{p,q}_{\omega'}(M).$$

Proof. Let $v = [\alpha] \in P^{p,q}_{\omega}(M)$, where $z \in Z^{p,q}_{\bar\partial}(M)$. Since $\omega = \omega' + \bar\partial\gamma$, we have

$$\begin{aligned} L^{k+1}_{\omega}(\alpha) &= \omega^{k+1} \wedge \alpha \\ &= (\omega' + \bar\partial\gamma)^{k+1} \wedge \alpha \\ &= L^{k+1}_{\omega'}(\alpha) + \sum_{r \neq k+1} \binom{k+1}{r} \omega' \wedge (\bar\partial\gamma)^{k-r+1} \wedge \alpha, \end{aligned}$$

where $k = m - p - q$. Therefore,

$$L^{k+1}_{\omega}(v) = [L^{k+1}_{\omega}(\alpha)] = [L^{k+1}_{\omega'}(\alpha)] = L^{k+1}_{\omega'}(v) = 0.$$

Hence, $P^{p,q}_{\omega}(M) = P^{p,q}_{\omega'}(M)$. $\square$

Lemma 3.2. *Let (M, ω, J) be a compact pseudo-Kähler manifold. Then for each p, q,*

$$K^{p,q}_{\omega}(M) = P^{p,q}_{\omega}(M) + L_{\omega}(K^{p-1,q-1}_{\omega}(M)).$$

Proof. Since $\mathcal{K}(M) = \sum_r \sum_{p+q=r} \mathcal{K}^{p,q}(M)$ is an $\mathfrak{sl}(2)$-submodule of $\sum_{p,q} \Omega^{p,q}(M)$, we have Lemma 3.2 in the same manner as in the proof of Theorem 2.2. $\square$

Theorem 3.3. *Let (M, J) be a compact complex manifold of $\dim_{\mathbb{R}} M = 2m$. Assume that ω, ω' are cohomologous pseudo-Kähler structures. Then for each p, q,*

$$K^{p,q}_{\omega}(M) = K^{p,q}_{\omega'}(M).$$

Proof. We prove our theorem by induction. By Lemma 1.8, we have $K^{p,0}_{\omega}(M) = K^{p,0}_{\omega'}(M) = H^{p,0}_{\bar\partial}(M)$ and $K^{0,q}_{\omega}(M) = K^{0,q}_{\omega'}(M) = H^{0,q}_{\bar\partial}(M)$. Assume that if $s + t < p + q < m$, then $K^{s,t}_{\omega}(M) = K^{s,t}_{\omega'}(M)$. Let $[\alpha] \in K^{p-1,q-1}_{\omega}(M)$, where $\alpha \in \mathcal{K}^{p-1,q-1}_{\omega}(M)$. By induction, there exists $[\alpha'] \in K^{p-1,q-1}_{\omega'}(M)$, where $\alpha' \in \mathcal{K}^{p-1,q-1}_{\omega'}(M)$, such that $[\alpha] = [\alpha']$ on $H^{p-1,q-1}_{\bar\partial}(M)$. Then we have $L_{\omega}(K^{p-1,q-1}_{\omega}(M)) = L_{\omega'}(K^{p-1,q-1}_{\omega'}(M))$. Indeed,

$$\begin{aligned} L_{\omega}(K^{p-1,q-1}_{\omega}(M)) \ni [\omega \wedge \alpha] &= [\omega] \wedge [\alpha] \\ &= [\omega] \wedge [\alpha'] \\ &= [\omega \wedge \alpha'] = [(\omega' + \bar\partial\gamma) \wedge \alpha'] \\ &= [\omega' \wedge \alpha'] \in L_{\omega'}(K^{p-1,q-1}_{\omega'}(M)). \end{aligned}$$

Therefore, by Lemma 3.2, we have

$$\begin{aligned} K^{p,q}_{\omega}(M) &= P^{p,q}_{\omega}(M) + L_{\omega}(K^{p-1,q-1}_{\omega}(M)) \\ &= P^{p,q}_{\omega'}(M) + L_{\omega'}(K^{p-1,q-1}_{\omega'}(M)) \\ &= K^{p,q}_{\omega'}(M). \end{aligned}$$

Let $v = [\beta] \in K^{m-q,m-p}_{\omega}(M)$, where $\beta \in \mathcal{K}^{m-q,m-p}_{\omega}(M)$. Since $L^{m-p-q} : \mathcal{K}^{p,q}(M) \to \mathcal{K}^{m-q,m-p}(M)$ is an isomorphism, there exists $\theta \in \mathcal{K}^{p,q}_{\omega}(M)$ such that $v = [\beta] = [L^k_{\omega}(\theta)] = L^k_{\omega}([\theta])$, where $k = m - p - q$. Thus, by the above argument, there exists $\tau \in \mathcal{K}^{p,q}_{\omega'}(M)$ such that $[\theta] = [\tau]$. Since $\omega = \omega' + \bar{\partial}\gamma$, we have $L^k_{\omega}([\theta]) = L^k_{\omega'}([\tau])$. Indeed,

$$\begin{aligned} L^k_{\omega}([\theta]) = L^k_{\omega}([\tau]) &= [\omega^k \wedge \tau] \\ &= [(\omega' + \bar{\partial}\gamma)^k \wedge \tau] \\ &= \left[L^k_{\omega'}(\tau) + \sum_{r\neq k} \binom{k}{r} \omega' \wedge (\bar{\partial}\gamma)^{k-r} \wedge \tau\right] = [L^k_{\omega'}(\tau)] = L^k_{\omega'}([\tau]). \end{aligned}$$

This implies $K^{p,q}_{\omega}(M) = K^{p,q}_{\omega'}(M)$ for $p, q = 0, \ldots, m$. $\square$

Let Γ be a discrete subgroup of a real Lie group G with a compact quotient space $M = G/\Gamma$, J a left G-invariant complex structure on G, and $\bigwedge^k(\mathfrak{g}^{\mathbb{C}})^* = \bigwedge^k \mathfrak{g}^* \otimes \mathbb{C}$ the space of complex valued left G-invariant k-forms on G. By the projection onto G/Γ, we can consider $\bigwedge^k(\mathfrak{g}^{\mathbb{C}})^* \subset \Omega^k(M) \otimes \mathbb{C}$, and J is a complex structure of M. We define $Z^{p,q}_{\bar{\partial}}(\mathfrak{g}^{\mathbb{C}}) = Z^{p,q}_{\bar{\partial}}(M) \cap \bigwedge^*(\mathfrak{g}^{\mathbb{C}})^*$. We also define $B^{p,q}_{\bar{\partial}}(\mathfrak{g}^{\mathbb{C}})$, $H^{p,q}_{\bar{\partial}}(\mathfrak{g}^{\mathbb{C}})$ and $K^{p,q}(\mathfrak{g}^{\mathbb{C}})$ in the same manner.

Theorem 3.4. *Let $(G/\Gamma, \omega, J)$ be a compact pseudo-Kähler manifold such that J is a left G-invariant complex structure. Assume that $H^{p,q}_{\bar{\partial}}(G/\Gamma) = H^{p,q}_{\bar{\partial}}(\mathfrak{g}^{\mathbb{C}})$ for each p, q. Then we have*

$$K^{p,q}_{\omega}(G/\Gamma) = K^{p,q}_{\omega_0}(G/\Gamma) = K^{p,q}_{\omega_0}(\mathfrak{g}^{\mathbb{C}}),$$

where ω_0 is a left G-invariant form on G.

Proof. By Proposition 2.1, we see that $K^{p,q}_{\omega}(G/\Gamma) = K^{p,q}_{\omega_0}(G/\Gamma)$. Since $\bigwedge^* \mathfrak{g}^* \otimes \mathbb{C}$ is an $\mathfrak{sl}(2)$-submodule with respect to ω_0, and since $K^{p,0}_{\omega_0}(G/\Gamma) = K^{p,0}_{\omega_0}(\mathfrak{g}^{\mathbb{C}}) = H^{p,0}_{\bar{\partial}}(\mathfrak{g}^{\mathbb{C}})$ and $K^{0,q}_{\omega_0}(G/\Gamma) = K^{0,q}_{\omega_0}(\mathfrak{g}^{\mathbb{C}}) = H^{0,q}_{\bar{\partial}}(\mathfrak{g}^{\mathbb{C}})$ by our assumption, we have Theorem 3.4 in the same manner as in the proof of Theorem 3.3. $\square$

4. A construction of compact pseudo-Kähler solvmanifolds with flat curvature tensors

In this section, we construct a compact pseudo-Kähler solvmanifold G/Γ such that the derived Lie subgroup of G is not abelian and the curvature tensor vanishes. In the paper [5], Cordero–Fernández–Ugarte considered the curvature tensors of compact invariant pseudo-Kähler nilmanifolds G/Γ, where an invariant pseudo-Kähler structure means a left G-invariant pseudo-Kähler structure on G. We consider the following Lie algebra over $\mathbb{R}$,

$$\mathfrak{g} = \mathfrak{a}_p + \mathfrak{b}_p + \cdots + \mathfrak{b}_1,$$

where $\mathfrak{a}_p$ is abelian subalgebra, and $\mathfrak{b}_i$ is an abelian ideal of $\mathfrak{g} = \mathfrak{a}_p + \mathfrak{b}_p + \cdots + \mathfrak{b}_i$ for each i. Put $a_p = \dim \mathfrak{a}_p$, and $b_i = \dim \mathfrak{b}_i$ for each i. Assume that $\mathfrak{a} = \mathrm{span}_{\mathbb{R}}\{U_1, \ldots, U_{a_p}\}$, and $\mathfrak{b}_i = \mathrm{span}_{\mathbb{R}}\{X^{(i)}_1, \ldots, X^{(i)}_{b_i}\}$ for each i. By our assumption, the basis $\{U_1, \ldots, U_{a_p}, X^{(p)}_1, \ldots, X^{(p)}_{b_p}, \ldots, X^{(1)}_1, \ldots, X^{(1)}_{b_1}\}$ has the following relations for each k, h:

$$[U_k, X^{(i)}_h] = \sum_l {}^iC^l_{kh} X^{(i)}_l, \qquad [X^{(i)}_k, X^{(j)}_h] = \sum_l {}^{ij}C^l_{kh} X^{(j)}_l,$$

where $i > j$, and ${}^iC^l_{kh}, {}^{ij}C^l_{kh} \in \mathbb{R}$.

Consider a Lie subalgebra $\mathfrak{h} = \mathfrak{a}_p + \mathfrak{b}_p + \cdots + \mathfrak{b}_1 + \sqrt{-1}\mathfrak{b}_1$ of ${}_{\mathbb{R}}(\mathfrak{g})^{\mathbb{C}}$, where $\mathfrak{g}^{\mathbb{C}}$ is the complexification of $\mathfrak{g}$. Let us consider an abstract vector space $\mathfrak{k} = \mathrm{span}_{\mathbb{R}}\{V_1, \ldots, V_{a_p}, Y^{(p)}_1, \ldots, Y^{(p)}_{b_p}, \ldots, Y^{(2)}_2, \ldots, Y^{(2)}_{b_2}\}$. Put $\mathfrak{g}^{\mathbb{D}} = \mathfrak{k} + \mathfrak{h}$. We define the product of $\mathfrak{g}^{\mathbb{D}}$ by relations $[U_k, Y^{(i)}_h] = \sum_l {}^iC^l_{kh} Y^{(i)}_l$, $[X^{(i)}_k, Y^{(j)}_h] = \sum_l {}^{ij}C^l_{kh} Y^{(j)}_l$ for $i > j$, and the above relations. Then we can easily see that $\mathfrak{g}^{\mathbb{D}}$ is a Lie algebra. We define a complex structure J on $\mathfrak{g}^{\mathbb{D}}$ by $JU_k = V_k$, $JX^{(i)}_h = Y^{(i)}_h$ for each i, k, h. Let $G^{\mathbb{D}}$ be the simply connected Lie group corresponding to $\mathfrak{g}^{\mathbb{D}}$. Then we see:

Proposition 4.1. *The almost complex structure J is integrable on $G^{\mathbb{D}}$.*

Proof. We prove our proposition by the induction. It is easy to check that the Nijenhuis tensor N_J vanishes on $\mathfrak{a}_p + J\mathfrak{a}_p + \mathfrak{b}_p + J\mathfrak{b}_p$ (see [27]). Assume that N_J vanishes on $\mathfrak{g}^{\mathbb{D}}_{i-1} = \mathfrak{a}_p + J\mathfrak{a}_p + \mathfrak{b}_p + J\mathfrak{b}_p + \cdots + \mathfrak{b}_{i-1} + J\mathfrak{b}_{i-1}$. If $X \in \mathfrak{g}_{i-1} = \mathfrak{a} + \mathfrak{b}_p + \cdots + \mathfrak{b}_{i-1}$ and $Y \in \mathfrak{b}_i + J\mathfrak{b}_i$, then $N_J(X, Y) = J[X, Y] - [X, JY]$, hence $N_J(X, Y) = 0$ by the definition of the brackets. If $X \in J\mathfrak{g}_{i-1}$ and $Y \in \mathfrak{b}_i + J\mathfrak{b}_i$, then $N_J(X, Y) = -[JX, Y] - J[JX, JY]$. Hence, $N_J(X, Y) = 0$ by the definition of the brackets. Thus, N_J vanishes on $\mathfrak{g}^{\mathbb{D}}_i$. $\square$

Let $\{\alpha_1, \ldots, \alpha_{a_p}, \beta_1, \ldots, \beta_{a_p}\}$ be the dual basis of $\{U_1, \ldots, U_{a_p}, V_1, \ldots, V_{a_p}\}$, and $\{\xi_1^{(i)}, \ldots, \xi_{b_i}^{(i)}, \eta_1^{(i)}, \ldots, \eta_{b_i}^{(i)}\}$ be the dual basis of $\{X_1^{(i)}, \ldots, X_{b_i}^{(i)}, Y_1^{(i)}, \ldots, Y_{b_i}^{(i)}\}$ for each i. We put

$$\begin{cases} \lambda_k = \alpha_k + \sqrt{-1}\beta_k, & k = 1, \ldots, a_p, \\ \zeta_{h_i}^{(i)} = \xi_{h_i}^{(i)} + \sqrt{-1}\eta_{h_i}^{(i)}, & h_i = 1, \ldots, b_i, i = 1, \ldots, p. \end{cases}$$

Proposition 4.2. *If $\mathfrak{b}_i$ has a non-degenerate 2-form $\omega_{\mathfrak{b}_i} \in \bigwedge^2 \mathfrak{b}_i^* \hookrightarrow \Omega^2(G)$ such that $d\omega_{\mathfrak{b}_i} = 0$ for each i, then $G^{\mathbb{D}}$ has a pseudo-Kähler structure ω.*

Proof. Let $\omega_{\mathfrak{b}_i} = \sum_{k<h} P_{kh}^i \xi_k^{(i)} \wedge \xi_h^{(i)}$, where $P_{kh}^i \in \mathbb{R}$. Then

$$\omega = \sqrt{-1}\sum \lambda_s \wedge \bar{\lambda}_s + \sum_{k<h} P_{kh}^p(\zeta_k^{(p)} \wedge \bar{\zeta}_h^{(p)} + \bar{\zeta}_k^{(p)} \wedge \zeta_h^{(p)}) + \cdots + \sum_{k<h} P_{kh}^1(\zeta_k^{(1)} \wedge \bar{\zeta}_h^{(1)} + \bar{\zeta}_k^{(1)} \wedge \zeta_h^{(1)})$$

is a pseudo-Kähler structure, because $\sum_{k<h} P_{kh}^i(\zeta_k^{(i)} \wedge \bar{\zeta}_h^{(i)} + \bar{\zeta}_k^{(i)} \wedge \zeta_h^{(i)}) = \sum_{k<h} P_{kh}^i(\xi_k^{(i)} \wedge \xi_h^{(i)} + \eta_k^{(i)} \wedge \eta_h^{(i)})$ (see [27]). □

Remark 4.3. We can prove that J is integrable without the assumption that $\mathfrak{b}_1$ is abelian.

Let ω be the pseudo-Kähler structure on $G^{\mathbb{D}}$ defined in the proof of the above proposition. We define an indefinite inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}^{\mathbb{D}}$ by $\langle X, Y\rangle = \omega(JX, Y)$. We also denote the induced pseudo-Riemann metric on $G^{\mathbb{D}}$ by $\langle\ ,\ \rangle$. Then we have:

Proposition 4.4. *The curvature tensor R corresponding to $\langle\ ,\ \rangle$ vanishes.*

Proof. Let ∇ be the Levi-Civita connection of $\langle\ ,\ \rangle$. Since ω is closed, we have the following relations for each i, j, k, h, s.

$$\nabla_{U_s} = ad(U_s), \qquad \nabla_{V_s} = 0, \qquad \nabla_{X_k^{(i)}} X_h^{(j)} = [X_k^{(i)}, X_h^{(j)}], \qquad \nabla_{Y_k^{(i)}} Y_h^{(j)} = \nabla_{Y_k^{(i)}} X_h^{(j)} = 0,$$

$$\nabla_{X_k^{(i)}} Y_h^{(j)} = \begin{cases} [X_k^{(i)}, Y_h^{(j)}] & \text{for } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $[X_k^{(i)}, Y_h^{(j)}] = 0$ for $i < j$. Indeed, for example, let $X, Y \in \mathfrak{b} + J\mathfrak{b}$, then

$$\begin{aligned} 0 = d\omega(U_s, X, Y) &= -\omega([U_s, X], Y) + \omega([U_s, Y], X) + 0 \\ &= \langle JY, [U_s, X]\rangle + \langle [U_s, JY], X\rangle. \end{aligned}$$

Hence,

$$\begin{aligned} 2\langle \nabla_{U_s} X, JY\rangle &= \langle JY, [U_s, X]\rangle - \langle [U_s, JY], X\rangle \\ &= 2\langle [U_s, X], JY\rangle. \end{aligned}$$

By using the above relations and the Jacobi identity, we see that the curvature tensor vanishes. □

Remark 4.5. Since $\nabla_{U_s} = ad(U_s)$, ∇ is not natural reductive.

Example 4.6. Let us consider a solvable Lie group

$$G = \left\{ \left. \begin{pmatrix} e^{-t} & 0 & e^{-2t}x_1 & 0 & z_1 \\ 0 & e^t & 0 & e^{2t}x_2 & z_2 \\ 0 & 0 & e^{-2t} & 0 & y_1 \\ 0 & 0 & 0 & e^{2t} & y_2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| t, x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{g}$ corresponding to G is $\mathfrak{g} = \mathrm{span}\{\mathbf{t}, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2\}$, where

$$\begin{aligned} &[\mathbf{x}_1, \mathbf{y}_1] = \mathbf{z}_1, \qquad [\mathbf{t}, \mathbf{x}_1] = \mathbf{x}_1, \qquad [\mathbf{x}, \mathbf{y}_1] = -2\mathbf{y}_1, \qquad [\mathbf{t}, \mathbf{z}_1] = -\mathbf{z}_1, \\ &[\mathbf{x}_2, \mathbf{y}_2] = \mathbf{z}_2, \qquad [\mathbf{t}, \mathbf{x}_2] = -\mathbf{x}_2, \qquad [\mathbf{t}, \mathbf{y}_2] = 2\mathbf{y}_2, \qquad [\mathbf{t}, \mathbf{z}_2] = \mathbf{z}_2. \end{aligned}$$

Let $\{\mathbf{t}_1^*, \mathbf{x}_1^*, \mathbf{y}_1^*, \mathbf{z}_1^*, \mathbf{x}_2^*, \mathbf{y}_2^*, \mathbf{z}_2^*\}$ be the dual basis of $\{\mathbf{t}, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2\}$. Consider a decomposition $\mathfrak{a}_2 = \mathrm{span}\{\mathbf{t}\}$, $\mathfrak{b}_2 = \{\mathbf{y}_1, \mathbf{y}_2\}$, $\mathfrak{b}_1 = \{\mathbf{x}_1, \mathbf{z}_1, \mathbf{x}_2, \mathbf{z}_2\}$. Then $\omega_{\mathfrak{b}_1} = \mathbf{y}_1^* \wedge \mathbf{y}_2^*$ and $\omega_{\mathfrak{b}_2} = \mathbf{x}_1^* \wedge \mathbf{z}_1^* + \mathbf{x}_2^* \wedge \mathbf{z}_2^*$ satisfy the condition in the above theorem. Then the modification $G^{\mathbb{D}}$ of complexification of G has a left-invariant pseudo-Kähler structure. Indeed, we see

$$G^{\mathbb{D}} = \left\{ \begin{pmatrix} e^{-\frac{1}{2}(w+\bar{w})} & 0 & e^{-(w+\bar{w})}x_1 & 0 & 0 & 0 & z_1 \\ 0 & e^{\frac{1}{2}(w+\bar{w})} & 0 & e^{(w+\bar{w})}x_2 & 0 & 0 & z_2 \\ 0 & 0 & e^{-(w+\bar{w})} & 0 & 0 & 0 & y_1 + \bar{y}_1 \\ 0 & 0 & 0 & e^{(w+\bar{w})} & 0 & 0 & y_2 + \bar{y}_2 \\ 0 & 0 & 0 & 0 & e^{-(w+\bar{w})} & 0 & y_1 \\ 0 & 0 & 0 & 0 & 0 & e^{(w+\bar{w})} & y_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

where $w, x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{C}$. The derived Lie subgroup of $G^{\mathbb{D}}$ is *not abelian*. The Lie group $G^{\mathbb{D}}$ has a basis $\{\lambda, \mu_1, \mu_2, \nu_1, \nu_2, \zeta_1, \zeta_2\}$ of left-invariant $(1,0)$-forms on $G^{\mathbb{D}}$ which satisfies the following relations:

$$\begin{aligned} d\mu_1 &= -(\lambda+\bar\lambda)\wedge\mu_1, & d\nu_1 &= 2(\lambda+\bar\lambda)\wedge\nu_1, & d\zeta_1 &= (\lambda+\bar\lambda)\wedge\zeta_1 - \mu_1\wedge(\nu_1+\bar\nu_1),\\ d\mu_2 &= (\lambda+\bar\lambda)\wedge\mu_2, & d\nu_2 &= -2(\lambda+\bar\lambda)\wedge\nu_2, & d\zeta_2 &= -(\lambda+\bar\lambda)\wedge\zeta_2 - \mu_2\wedge(\nu_2+\bar\nu_2). \end{aligned}$$

Then we see that

$$\omega = \sqrt{-1}\bar\lambda\wedge\lambda + (\bar\nu_1\wedge\nu_2 + \nu_1\wedge\bar\nu_2) + (\bar\mu_1\wedge\zeta_1 + \mu_1\wedge\bar\zeta_1 + \bar\mu_2\wedge\zeta_2 + \mu_2\wedge\bar\zeta_2)$$

is a left-invariant pseudo-Kähler structure on $G^{\mathbb{D}}$. The Lie group $G^{\mathbb{D}}$ has a lattice Γ (see [22]), hence we have a compact pseudo-Kähler solvmanifold $(G^{\mathbb{D}}/\Gamma, \omega)$ as desired, i.e. the derived Lie subgroup $[G^{\mathbb{D}}, G^{\mathbb{D}}]$ is not abelian.

5. Dolbeault cohomology groups of compact complex parallelizable pseudo-Kähler solvmanifolds

In this section, we consider Dolbeault cohomology groups of compact complex parallelizable pseudo-Kähler manifolds. A complex manifold M is said to be *complex parallelizable* if the holomorphic tangent bundle of M is holomorphically trivial. Wang [24] proved that a complex parallelizable manifold M is of the form G/Γ, where G is a *complex* Lie group and Γ is a lattice of G. Thus in the case where M is complex parallelizable, we can investigate the details of Dolbeault cohomology groups of M.

Let G be a complex Lie group and $G = S\cdot R$ a Levi decomposition, where S is a semi-simple Lie subgroup, and R is the radical. We denote derived Lie subgroups of G, N and R by G', N' and R', respectively. Winkelmann has proven:

Theorem 5.1 (*[25]*). *Let G, Γ, N, S, R, G', N' and R' be as above. Let $A = [S, R]\cdot N'$. Furthermore let W denote the maximal linear subspace of the Lie algebra* $\mathrm{Lie}(R'A/A)$ *of $R'A/A$ such that $\mathrm{Ad}(\gamma)|_W$ is a semi-simple linear endomorphism with only real eigenvalues for each $\gamma\in\Gamma$. Then*

$$\dim H^1(G/\Gamma, \mathcal{O}) \le \dim G/G' + \dim H^1(G/R\Gamma, \mathbb{C}) + \dim W.$$

In the previous paper [28], we have:

Theorem 5.2. *Let G/Γ be an n-dimensional compact complex parallelizable manifold which admits a pseudo-Kähler structure. Then*

$$h^{p,q}(G/\Gamma) \ge \binom{n}{p}\cdot\binom{n}{q}.$$

The following theorem is well known:

Theorem 5.3 (*[16,19]*). *Let G be a complex semi-simple Lie group and Γ a discrete subgroup of G with a compact quotient G/Γ. If G has no simple factors of dimension 3, then $H^1(G/\Gamma, \mathbb{C})$ vanish.*

Thus we have:

Corollary 5.4 (*cf. [9]*). *Let $(G/\Gamma, \omega)$ be an n-dimensional compact complex parallelizable pseudo-Kähler manifold. Let $G = S\cdot R$ be a Levi decomposition. If S has no simple factors of dimension 3, then G is solvable.*

Proof. By Theorems 5.1–5.3, we see

$$\begin{aligned} n \le \dim G/G' + \dim W &\le \dim G - \dim G' + \dim R'A/A\\ &= \dim G - \dim G' + \dim R'/R'\cap A\\ &\le \dim G - \dim G' + \dim R' \le n, \end{aligned}$$

which implies $G = R$. □

Remark 5.5. In the paper [9], Guan proved that if a compact complex parallelizable manifold G/Γ has a pseudo-Kähler structure, then G is a complex solvable Lie group (see also [10]).

Let $F \longrightarrow M \xrightarrow{\pi} B$ be a holomorphic fiber bundle such that M, B, F are connected and F is compact. We denote the higher direct image sheaf of $\mathcal{O}$ by $R^q\pi_*\mathcal{O}_M$. Put $\mathbf{H}^{p,q}(F) = \bigcup_{b\in B} H^{p,q}_{\bar\partial}(F_b)$. Then $\mathbf{H}^{p,q}(F)$ becomes the total space of a *differentiable* vector bundle over B. If every connected component of the structure group of $\pi : M \longrightarrow B$ acts trivially on $\mathbf{H}^{p,q}(F)$, then the vector bundle is a *holomorphic* vector bundle (for details, see [1,11]).

Let G/Γ be a compact complex parallelizable solvmanifold, $\pi : G/\Gamma \longrightarrow B = G/G'\Gamma$ the Albanese mapping and F its fiber [12,25]. Note that $\pi : G/\Gamma \longrightarrow B = G/G'\Gamma$ is a *locally trivial* holomorphic fiber bundle [23]. If the compact complex parallelizable solvmanifold G/Γ has a pseudo-Kähler structure, then F is a complex torus [29]; in particular, we see that $\mathbf{H}^{p,q}(F)$ is a holomorphic vector bundle. Moreover, we see that $\mathbf{H}^{0,q}(F)$ is holomorphically trivial in this case. Indeed, let us consider the Albanese mapping $\pi : G/\Gamma \longrightarrow G/G'\Gamma$. Note that

$$G/\Gamma = G \times_{G'\Gamma} F,$$

where $F = G'/\Gamma_{G'}$, and $\Gamma_{G'} = G' \cap \Gamma$ (cf. also [12], p. 165). Thus by an argument of Lescure ([14], Proposition 3), we see that

$$R^q\pi_*\mathcal{O} = \mathbf{H}^{0,q}(F),$$

where the equality means that the bundle over $G/G'\Gamma$ corresponding to $R^q\pi_*\mathcal{O}$ is $\mathbf{H}^{0,q}(F)$. Assume in addition that F is a complex torus $T^{n-s}_{\mathbb{C}}$, i.e., G' is abelian. If $\mathbf{H}^{0,1}(T^{n-s}_{\mathbb{C}})$ is holomorphically trivial, then $\mathbf{H}^{0,q}(T^{n-s}_{\mathbb{C}})$ is also holomorphically trivial. Indeed, consider $T^{n-s}_{\mathbb{C}}$ as a Kähler manifold. Hence, $H^{0,q}_{\bar\partial}(T^{n-s}_{\mathbb{C}}) = \mathcal{H}^{0,q}(T^{n-s}_{\mathbb{C}})$, where $\mathcal{H}^{0,q}(T^{n-s}_{\mathbb{C}})$ is the space of all harmonic forms of type $(0, q)$. Let $\bar\omega_1, \ldots, \bar\omega_{n-s}$ be a basis of the global sections of $\mathbf{H}^{0,1}(T^{n-s}_{\mathbb{C}})$. Then $\{\bar\omega_{i_1} \wedge \cdots \wedge \bar\omega_{i_q}\}$ becomes a basis of the global sections of $\mathbf{H}^{0,q}(T^{n-s}_{\mathbb{C}})$.

Since $T^{n-s}_{\mathbb{C}}$ is a Kählerian manifold, we see that

$$R^1\pi_*\mathcal{O} = G \times_{G'\Gamma} H^{0,1}_{\bar\partial}(T^{n-s}_{\mathbb{C}}) = G/G' \times_{G'\Gamma/G'} H^{0,1}_{\bar\partial}(T^{n-s}_{\mathbb{C}}).$$

Recall that the following exact sequence

$$0 \longrightarrow H^1(B, \mathcal{O}) \longrightarrow H^1(G/\Gamma, \mathcal{O}) \longrightarrow H^0(B, R^1\pi_*\mathcal{O}).$$

In particular, we see that

$$\dim H^1(G/\Gamma, \mathcal{O}) \le \dim H^1(B, \mathcal{O}) + \dim H^0(B, R^1\pi_*\mathcal{O}).$$

Therefore, since $h^{0,1}(G/\Gamma) = n$, and since $\dim H^0(B, E) = \dim E_0 - \dim B$, where $E = R^1\pi_*\mathcal{O}$, and E_0 is a subbundle of E such that E_0 is holomorphically trivial (see [25], Proposition 6.2.5), we see that E is holomorphically trivial by Theorem 5.2. Thus we have:

Theorem 5.6 (*cf. [29]*). *If a compact complex parallelizable solvmanifold admits a pseudo-Kähler structure, then the bundle over $G/G'\Gamma$ corresponding to $R^q\pi_*\mathcal{O}$ is trivial as a holomorphic vector bundle.*

By the above theorems, we see that if a compact complex parallelizable solvmanifold G/Γ admits a pseudo-Kähler structure, then the lattice Γ satisfies a strong condition:

Proposition 5.7 (*cf. [1], Section 6*). *Let G/Γ be a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure. Then the eigenvalues of $Ad(\gamma)$ are non-zero real for each $\gamma \in \Gamma$.*

Proof. Let $\mathfrak{a}$ be a subspace such that $\mathfrak{g} = \mathfrak{a} + \mathfrak{g}'$, and $a(t) = \exp tA$ for $A \in \mathfrak{a}$. Since $Ad(\gamma)A = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} a_\gamma(\exp tA) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\gamma a(t)\gamma^{-1}a(t)^{-1} + \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}a(t)$, and since $W = \mathfrak{g}'$ by our assumption, where W is the same one in Theorem 5.1, we have our proposition. □

By Theorems 5.1 and 5.2, we have that if a compact complex parallelizable solvmanifold G/Γ has a pseudo-Kähler structure, then $h^{0,q}(G/\Gamma) = \binom{n}{q}$ (see [29] for details). Let (E_r, d_r) be the Leray spectral sequence converging to $H^*(G/\Gamma, \mathcal{O})$ whose second term is given by $E_2^{p,q} = H^p(B, R^q\pi_*\mathcal{O})$ (for details, see [1,14]). We now have:

Theorem 5.8. *Let G/Γ be a compact complex parallelizable pseudo-Kähler solvmanifold, and $\pi : G/\Gamma \longrightarrow G/G'\Gamma$ be as above. Then the spectral sequence (E_r, d_r) satisfies $d_2 = d_3 = \cdots = 0$.*

Proof. Note that in general $\dim E_2^s \ge \dim E_3^s \ge \cdots \ge \dim E_\infty^s$, where $E_r^s = \sum_{p+q=s} E_r^{p,q}$. By our assumption, we see $E_2^{p,q} = H^p(B, R^q\pi_*\mathcal{O}) = H^p(B, \mathcal{O}) \otimes H^q(F, \mathcal{O})$. Hence, $\dim E_2^s = \binom{n}{s} = \dim E_\infty^s$, which implies $d_2 = 0$. □

Remark 5.9. (1) There exists a fiber bundle $\pi : G/\Gamma \longrightarrow B = G/G'\Gamma$ such that the vector bundle corresponding to $R^q\pi_*\mathcal{O}$ is not holomorphically trivial but differentially trivial (cf. [17], case III-(3a) and [25], Proposition 7.9.2).

(2) Let G be the complex 3-dimensional Heisenberg group and Γ a lattice of G. Then the typical fiber of $\pi : G/\Gamma \longrightarrow G/G'\Gamma$ is a complex torus. However, we see that the Leray spectral sequence (E_r, d_r) converging to $H^*(G/\Gamma, \mathcal{O})$ satisfies $d_2 \neq 0$.

(3) Let G/Γ be a compact complex parallelizable manifold. Considering a fiber bundle $G/\Gamma \longrightarrow G/N\Gamma$, the fibers of which are compact complex parallelizable nilmanifold, we can investigate the Dolbeault cohomology group of G/Γ (see [25], Proposition 7.9.2 and [21], Theorem 1).

6. Example

Finally, we give an example which has the hard Lefschetz property.

Example 6.1 ([1,14,17]). Let us consider a representation $\varphi : \mathbb{C} \longrightarrow GL(2, \mathbb{C})$ given by $\varphi(z) = \begin{pmatrix} e^{-z} & 0 \\ 0 & e^{z} \end{pmatrix}$. Let $B \in SL(2, \mathbb{Z})$ be a unimodular matrix with distinct real eigenvalues, say, λ, $1/\lambda$. Take $t_0 = Log\lambda$, i.e., $e^{t_0} = \lambda$. Then there exists a matrix $P \in GL(2, \mathbb{R})$ such that

$$PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Put

$$L_1 = t_0\mathbb{Z} + 2\pi\sqrt{-1}\mathbb{Z}, \qquad L_2 = \left\{ P\begin{pmatrix} \mu \\ \nu \end{pmatrix} \middle| \mu, \nu \in \mathbb{Z}[\sqrt{-1}] \right\}.$$

Let $\varpi_2 : \mathbb{C}^2 \longrightarrow T^2_{\mathbb{C}} = \mathbb{C}^2/L_2$ be the natural projection. Consider $M = \mathbb{C}^1 \times_{L_1} T^2_{\mathbb{C}} = \mathbb{C}^1 \times T^2_{\mathbb{C}}/\sim$, where the equivalence relation is defined by $(z, \varpi_2(w)) \sim (z', \varpi_2(w')) \iff z' = z + \gamma$ and $\varpi_2(w') = \varpi_2(\varphi(-\gamma)(w))$ for some $\gamma \in L_1$. Note that $\mathbb{C}^3 = \{(z, w_1, w_2) | z, w_1, w_2 \in \mathbb{C}\}$ is the universal covering of M. Then M is a compact complex parallelizable solvmanifold, and $\omega = \sqrt{-1}dz \wedge d\bar{z} + dw_1 \wedge d\bar{w}_2 + d\bar{w}_1 \wedge dw_2$ induces a pseudo-Kähler structure on M. Let us consider the Albanese mapping $\pi : M \longrightarrow \mathbb{C}^1/L_1$. Then the bundle corresponding to $R^q\pi_*\mathcal{O}$ is holomorphically trivial. Indeed, $\{e^{-z}d\bar{w}_1, e^{z}d\bar{w}_2\}$ and $\{d\bar{w}_1 \wedge d\bar{w}_2\}$ are bases of global sections of $R^1\pi_*\mathcal{O}$ and $R^2\pi_*\mathcal{O}$ respectively. For each $\gamma = (\gamma_0, \gamma_1, \gamma_2) \in \Gamma = L_1 \ltimes L_2$, we obtain the eigenvalues of $Ad(\gamma)$ as follows.

$$Ad(\gamma)\left(\left.\frac{\partial}{\partial w_1}\right|_{(0,0,0)}\right) = e^{\gamma_0}\left.\frac{\partial}{\partial w_1}\right|_{(0,0,0)}, \qquad Ad(\gamma)\left(\left.\frac{\partial}{\partial w_2}\right|_{(0,0,0)}\right) = e^{-\gamma_0}\left.\frac{\partial}{\partial w_2}\right|_{(0,0,0)},$$
$$Ad(\gamma)\left(\left.\frac{\partial}{\partial z}\right|_{(0,0,0)}\right) = \left.\frac{\partial}{\partial z}\right|_{(0,0,0)} - \gamma_1\left.\frac{\partial}{\partial w_1}\right|_{(0,0,0)} + \gamma_2\left.\frac{\partial}{\partial w_2}\right|_{(0,0,0)}.$$

It is easily to check that (M, ω) has the hard Lefschetz property with respect to Dolbeault cohomology groups (see [29], Corollary 3.5).

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