Journal of Geometry and Physics 59 (2009) 1227–1243

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Hamiltonian structures and their reciprocal transformations for the r-KdV–CH hierarchy

Ming Chen, Si-Qi Liu, Youjin Zhang *

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, PR China

ARTICLE INFO

Article history:
Received 10 February 2009
Received in revised form 19 May 2009
Accepted 7 June 2009
Available online 11 June 2009

MSC:
primary 37K10
secondary 35Q53

Keywords:
Hamiltonian structure
Camassa–Holm hierarchy
Reciprocal transformations
Frobenius manifold
Central invariants

ABSTRACT

The r-KdV–CH hierarchy is a generalization of the Korteweg–de Vries and Camassa–Holm hierarchies parameterized by $r+1$ constants. In this paper we clarify some properties of its multi-Hamiltonian structures including the explicit expressions of the Hamiltonians, the formulae of the central invariants of the associated bihamiltonian structures and the relationship of these bihamiltonian structures with Frobenius manifolds. By introducing a class of generalized Hamiltonian structures, we present in a natural way the transformation formulae of the Hamiltonian structures of the hierarchy under certain reciprocal transformations, and prove the validity of the formulae at the level of dispersionless limit.

1. Introduction

In recent years progress has been made in the study of the problem of classification of bihamiltonian structures of a certain type; the associated bihamiltonian integrable hierarchies include in particular the well-known Korteweg–de Vries (KdV) hierarchy, the Camassa–Holm (CH) hierarchy, the Drinfeld–Sokolov hierarchies and so on [1–4]. For a given bihamiltonian structure defined on the formal loop space of an n-dimensional manifold M, a complete set of its invariants under the so-called Miura type transformations is obtained in [2,4]. It consists of a flat pencil of metrics defined on the manifold M and n functions of one variable; these functions are called the central invariants of the bihamiltonian structure. These invariants enable one to have a better understanding of the bihamiltonian structures and the associated integrable hierarchies. For most of the well-known bihamiltonian integrable hierarchies including the Drinfeld–Sokolov hierarchies associated to untwisted affine Lie algebras, the flat pencil of metrics is given by certain Frobenius manifold structures, and the central invariants are some constants. In particular, for the bihamiltonian structures of the Drinfeld–Sokolov hierarchies associated to the untwisted affine Lie algebras of A-D-E type, the central invariants are all equal to $\frac{1}{24}$ if one chooses the invariant bilinear form of the Lie algebra to be the normalized one [5,3]. This property is one of the most important characteristics of the integrable hierarchies that arise in 2D topological field theory and Gromov–Witten invariants [6,7,5,1,8–10].

On the other hand, to our knowledge the only known bihamiltonian integrable hierarchies with non-constant central invariants are special cases of the so-called r-KdV–CH hierarchy (see its definition given in the next section). This hierarchy is a generalization of the KdV hierarchy parameterized by an ordered set of $r+1$ constants $\mathcal{P}=(a_0,a_1,\dots,a_r)$. Apart from

* Corresponding author.
E-mail addresses: chen-m02@mails.tsinghua.edu.cn (M. Chen), liusq@mail.tsinghua.edu.cn (S.-Q. Liu), youjin@mail.tsinghua.edu.cn (Y. Zhang).

0393-0440/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2009.06.001

the KdV hierarchy, it contains many other important integrable hierarchies as particular examples, such as the CH hierarchy, the AKNS hierarchy and the two-component Camassa–Holm (2-CH) hierarchy. In these cases the corresponding parameters are given by

$$\begin{aligned} &\text{KdV}: r=1, \mathscr{P}=(1,0),\\ &\text{CH}: r=1, \mathscr{P}=(0,1),\\ &\text{AKNS}: r=2, \mathscr{P}=(1,0,0),\\ &\text{2-CH}: r=2, \mathscr{P}=(0,0,1). \end{aligned}$$

The central invariants of the bihamiltonian structures of the KdV hierarchy and AKNS hierarchy are constant, while that of the CH hierarchy and the 2-CH hierarchy are not [11,12,4].

In some cases, bihamiltonian integrable hierarchies with different central invariants are related via certain type of transformations which change, unlike the Miura type transformations, also the independent variables. Such transformations are called reciprocal transformations, they are quite important in studying properties of solutions of the related integrable hierarchies (see [13–19] and references therein). A typical example is given by the relation between the KdV hierarchy and the CH hierarchy [15], the associated reciprocal transformation provides an efficient way to obtain exact solutions of the CH hierarchy by using the known solutions of the KdV hierarchy [15,16].

The main purpose of the present paper is to study, via the example of the r-KdV–CH hierarchy, the transformation rule of Hamiltonian structures under reciprocal transformations. A better understanding of such transformation rules is important in particular for the study of properties of the class of bihamiltonian integrable systems of Camassa–Holm type, and for the study of a generalized classification scheme for integrable hierarchies under reciprocal transformations.

The first step towards the generalization of the KdV hierarchy to the r-KdV–CH hierarchy was made by Martínez Alonso in [20], where he presented the r-KdV–CH hierarchy with the parameters $\mathscr{P}=(1,0,\dots,0)$ and proved its integrability by using the bihamiltonian structure of the hierarchy and the inverse scattering method. Antonowicz and Fordy studied in a series of important papers (see [21–23] and references therein) the spectral problem associated to the r-KdV–CH hierarchy with general parameters, they obtained $r+1$ Hamiltonian operators associated to the spectral problem and pointed out that the compatibility of these Hamiltonian operators can be proved by using Fuchssteiner and Fokas' method of hereditary symmetry [24]. To fix the notations of the present paper, we first give in Section 2 the explicit formulation of the r-KdV–CH hierarchy with general parameters $\mathscr{P}=(a_0,\dots,a_r)$, and describe its $r+1$ Hamiltonian structures including a formula of the Hamiltonians.

Properties of the bihamiltonian structures of the r-KdV–CH hierarchy were considered in [2], where a formula for the central invariants of these bihamiltonian structures was given without a proof. In Section 3 we fill the proof for the formula. We also specify in this section those bihamiltonian structures of the r-KdV–CH hierarchy which are associated to Frobenius manifolds. Recall that the bihamiltonian structures of the r-KdV–CH hierarchy admit hydrodynamic limits, and under certain conditions a bihamiltonian structures of hydrodynamic type is associated to the flat pencil of metrics defined on certain Frobenius manifold [7,25,1]. Here the notion of Frobenius manifold was invented by Dubrovin as a coordinate free formulation of the WDVV equation which arises in 2D topological field theory [26,27,6,7]. The rich geometry structures of Frobenius manifolds reveal in a natural way the close relationship between integrable hierarchies and 2D topological field theory.

Under reciprocal transformations an evolutionary PDE which possesses a local Hamiltonian structure will in general be transformed to a system with nonlocal Hamiltonian structures. For a Hamiltonian system of hydrodynamic type, such transformation properties were studied by Ferapontov and Pavlov in [14] where the nonlocal Hamiltonian structure was obtained from the expressions of the transformed systems. In order to generalize their results to Hamiltonian structures with dispersive terms such as the ones for the r-KdV–CH hierarchy, we introduce in Section 4 a class of generalized Hamiltonian structures which includes in particular the type of nonlocal Hamiltonian structures that we are interested in, and we also give some important properties of such generalized Hamiltonian structures while leaving their proofs to a separate publication [28]. We then apply these results in Section 5 to the study of properties of the Hamiltonian structures of the r-KdV–CH hierarchy under certain reciprocal transformations.

We give some concluding remarks in the last section.

2. The *r*-KdV–CH hierarchy

In this section we recall the definition of the r-KdV–CH hierarchy and their Hamiltonian structures.

Let M be a contractible manifold with local coordinates $w^0,\dots,w^{r-1}$, and φ be a smooth map from the circle $S^1=\mathbb{R}/\mathbb{Z}$ to M

$$\varphi: S^1\to M,\quad x\mapsto (w^0(x),\dots,w^{r-1}(x)).$$

We denote the derivatives $\partial_x w^i(x), \partial_x^2 w^i(x),\dots$ by $w^i_x, w^i_{xx},\dots$, and denote $\partial_x^k w^i(x)=w^{i,k}$ in general. Let $\bar{\mathcal{A}}$ be the polynomial ring

$$\bar{\mathcal{A}}=C^\infty(M)[\epsilon\, w^i_x, \epsilon^2\, w^i_{xx},\dots].$$

There is a natural gradation on $\bar{\mathcal{A}}$ given by

$$\deg f(w) = 0, \qquad \deg\left(\epsilon\, w^i_x\right) = 1, \qquad \deg\left(\epsilon^2\, w^i_{xx}\right) = 2, \ldots.$$

We denote the completion of $\bar{\mathcal{A}}$ w.r.t. this gradation by $\mathcal{A}$.

By definition, elements of $\mathcal{A}$ are all formal power series of ϵ, they converge in the formal power series topology. Evolutionary equations that we consider in this paper have the following form in general:

$$\epsilon\, w^i_t = X^i, \quad i = 0, \ldots, r-1 \tag{2.1}$$

for some $X^i \in \mathcal{A}$. As a typical example of such generalized evolutionary equations, the Camassa–Holm equation [11,12] can be written as

$$\epsilon\, w_t = (1-\epsilon^2\partial_x^2)^{-1}K = \sum_{k\geq 0}\epsilon^{2k}\partial_x^{2k}K \in \mathcal{A},$$

where $K = 2\,\epsilon^3\, w_x\, w_{xx} + \epsilon^3\, w\, w_{xxx} - 3\,\epsilon\, w\, w_x - \kappa\,\epsilon\, w_x \in \bar{\mathcal{A}}$, and κ is a constant parameter.

Given $r \in \mathbb{N}$ and $\mathcal{P} = (a_0, a_1, \ldots, a_r) \in (\mathbb{R}^{r+1})^\times$, Antonowicz and Fordy studied the following system of linear equations in [23]:

$$\epsilon^2\phi_{xx}(x,t) = A(w;\lambda)\phi(x,t), \quad A = \frac{\hat{A}}{\hat{a}}, \tag{2.2}$$

$$\phi_t(x,t) = B(w;\lambda)\phi_x(x,t) - \frac{1}{2}B_x(w;\lambda)\phi(x,t), \tag{2.3}$$

where

$$\hat{A} = w^0 + w^1\lambda + \cdots + w^{r-1}\lambda^{r-1} + \lambda^r, \tag{2.4}$$

$$\hat{a} = a_0 + a_1\lambda + \cdots + a_{r-1}\lambda^{r-1} + a_r\lambda^r, \tag{2.5}$$

and $B(w;\lambda) \in \mathcal{A}[\lambda, \lambda^{-1}]$. The compatibility condition of (2.2) and (2.3) reads

$$A_t = 2\,A\,B_x + A_x\,B - \frac{\epsilon^2}{2}B_{xxx}. \tag{2.6}$$

The isospectral problem (2.2), (2.3) with $\hat{a} = 1$ was first studied by Martínez Alonso [20]. Antonowicz and Fordy generalized it to more general form: when $a_r = 0$, this isospectral problem corresponds to the coupled KdV hierarchy [21]; when $\hat{a} = \alpha^{-1}\lambda^r$, the corresponding hierarchy is the coupled Harry Dym hierarchy [22].

To define the flows, we introduce two generating functions.

$$b = 1 + \sum_{k\geq 1} b_k\,\lambda^{-k} \in \mathcal{A}[\lambda^{-1}], \qquad c = c_0 + \sum_{k\geq 1} c_k\,\lambda^k \in \mathcal{A}[\lambda],$$

they satisfy

$$\hat{A}b^2 - \frac{\epsilon^2}{4}\hat{a}(2\,b\,b_{xx} - b_x^2) = \lambda^r; \tag{2.7}$$

$$\hat{A}c^2 - \frac{\epsilon^2}{4}\hat{a}(2\,c\,c_{xx} - c_x^2) = 1. \tag{2.8}$$

According to the definition of $\mathcal{A}$, Eq. (2.7) determines b uniquely, and Eq. (2.8) determines c up to a sign. Note that c_0 satisfies the following equation

$$c_0^2\, w^0 - \frac{\epsilon^2}{4}a_0\left(2\,c_0\,c_{0,xx} - c_{0,x}^2\right) = 1, \tag{2.9}$$

so by assuming that c_0 has the following Taylor expansion:

$$c_0 = \left(w^0\right)^{-\frac{1}{2}} + \sum_{k\geq 1}\epsilon^k\, C_k, \quad C_k \in \bar{\mathcal{A}}, \deg C_k = k,$$

we can determine c uniquely.

For $n \in \mathbb{Z}$ we define

$$B_n = \begin{cases} (\lambda^n b)_+, & n \geq 0, \\ (\lambda^n c)_-, & n < 0, \end{cases} \tag{2.10}$$

where $(\cdot)_\pm$ denote the positive or negative part of a Laurent series of λ. According to [21,22], the compatibility condition (2.6) with $B = B_n$ yields an evolutionary PDE of $w^0, \ldots, w^{r-1}$ for any $n \in \mathbb{Z}$. They can be represented as

$$\hat{A}_{t_n} = \left(2\,\hat{A}\,\partial_x + \hat{A}_x - \frac{\epsilon^2}{2}\,\hat{a}\,\partial_x^3\right) B_n, \quad n \in \mathbb{Z}. \tag{2.11}$$

When $n > 0$ or $n < 0$, we call ∂_{t_n} the n-th positive flow or the $|n|$-th negative flow respectively, and when $n = 0$ we have $\partial_{t_0} = \partial_x$.

Lemma 2.1. *For all $n \in \mathbb{Z}$, we have*

$$b_{t_n} = B_n\,b_x - b\,B_{n,x}, \tag{2.12}$$

$$c_{t_n} = B_n\,c_x - c\,B_{n,x}. \tag{2.13}$$

Proof. We only prove Eq. (2.12), the proof of (2.13) is similar.

Let $h = b_{t_n} - B_n\,b_x + b\,B_{n,x}$, we need to show that $h = 0$. The derivative of Eq. (2.7) w.r.t. t_n implies

$$A_{t_n}\,b^2 + \left(2\,A\,b - \frac{\epsilon^2}{2}\left(b\,\partial_x^2 - b_x\,\partial_x + b_{xx}\right)\right) b_{t_n} = 0,$$

by using Eqs. (2.6) and (2.7), one can obtain

$$\left(1 - \frac{\epsilon^2}{4}\,\frac{\hat{a}}{\lambda^r}\left(b^2\,\partial_x^2 - b\,b_x\,\partial_x + b_x^2 - b\,b_{xx}\right)\right) h = 0,$$

thus we have $h = 0$. □

The above lemma shows that

$$B_{n,t_m} - B_{m,t_n} + B_n\,B_{m,x} - B_m\,B_{n,x} = 0, \tag{2.14}$$

it implies the commutativity of the flows $\{\partial_{t_n}\}_{n\in\mathbb{Z}}$ defined in (2.11), so they form an integrable hierarchy which we call the r-KdV–CH hierarchy associated to $\mathcal{P} = (a_0, \ldots, a_r)$.

This hierarchy has infinitely many conserved quantities. In fact, by using Eqs. (2.12) and (2.13) we obtain

$$\left(\frac{1}{b}\right)_{t_n} = \left(\frac{B_n}{b}\right)_x, \qquad \left(\frac{1}{c}\right)_{t_n} = \left(\frac{B_n}{c}\right)_x.$$

So we can define local functionals

$$H_n = \begin{cases} \displaystyle\int \left(2\mathrm{res}_{\lambda=0}\frac{\lambda^n}{b}\right) \mathrm{d}x, & n \geq 0, \\ \displaystyle -\int \left(2\mathrm{res}_{\lambda=0}\frac{\lambda^n}{c}\right) \mathrm{d}x, & n < 0, \end{cases} \tag{2.15}$$

they are conserved quantities of the r-KdV–CH hierarchy.

The r-KdV–CH hierarchy possesses $r + 1$ compatible Hamiltonian structures with Hamiltonian operators P_m $(m = 0, 1, \ldots, r)$ [21,22] given by

$$(P_m)^{ij} = f_m^{ij}\,\mathcal{D}_{i+j+1-m}, \quad i, j = 0, 1, \ldots, r-1,$$

where $\mathcal{D}_i = 2\,w^i\,\partial_x + w^i_x - \frac{\epsilon^2}{2}\,a_i\,\partial_x^3$, and

$$f_m^{ij} = 1 - s_{i-m} - s_{j-m}, \quad s_k = \begin{cases} 1, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

Here we assume $w^r = 1$ and $w^i = a_i = 0$ when $i < 0$ or $i > r$. We have the following theorem:

Theorem 2.2. *For any $n \in \mathbb{Z}$ and $m = 0, \ldots, r$, we have*

$$w^i_{t_n} = \begin{cases} P^{ij}_{r-m}\,\dfrac{\delta H_{n+m}}{\delta w^j}, & n \geq 0, \\ P^{ij}_m\,\dfrac{\delta H_{n-m}}{\delta w^j}, & n < 0. \end{cases}$$

In the description of the multi-Hamiltonian structures of the r-KdV–CH hierarchy given by Antonowicz and Fordy, the Hamiltonians of the hierarchy are constructed in an implicit way using multi-Hamiltonian recursion relations. We give briefly a proof that the explicitly defined local functionals (2.15) serve as the Hamiltonians in the multi-Hamiltonian structures of the hierarchy, as stated in the above theorem. To this end, we need the following lemma:

Lemma 2.3. *Let $H = \int f\,\mathrm{d}x$ be a local functional, $f_0, \dots, f_{r-1} \in \mathcal{A}$, then*

$$\frac{\delta H}{\delta w^i} = f_i$$

if and only if for any $X^0, \dots, X^{r-1} \in \mathcal{A}$ there exists $\sigma \in \mathcal{A}$ such that

$$\partial_t(f) = \sum_{i=0}^{r-1} X^i f_i + \partial_x(\sigma),$$

where ∂_t is given by Eqs. (2.1).

The proof of the lemma is technical, we omit it here.

Proof of Theorem 2.2. Let us first show that the variational derivatives of the functionals H_n w.r.t. w^i are given by the formulae

$$\frac{\delta H_n}{\delta w^i} = \begin{cases} b_{n+1-(r-i)}, & n \ge 0, \\ -c_{-n-1-i}, & n < 0, \end{cases} \quad n \in \mathbb{Z}, i = 0, 1, \dots, r-1. \tag{2.16}$$

Here we assume $b_0 = 1$ and $b_k = c_k = 0$ when $k < 0$. The above formulae with $n \ge 0$ can be encoded into the following identity:

$$\frac{\delta}{\delta w^i} \int \frac{2}{b}\,\mathrm{d}x = \lambda^{i-r} b. \tag{2.17}$$

To prove this identity, we only need to prove, due to the above lemma, that for any $X^i \in \mathcal{A}$ there exists $\sigma \in \mathcal{A}$ such that

$$\left(\frac{1}{b}\right)_t = \frac{\hat{A}_t\, b}{2\,\lambda^r} + \sigma_x,$$

where ∂_t is given by (2.1) and $\hat{A}$ is defined in (2.4). In fact, by the action of ∂_t on Eq. (2.7) one can obtain

$$\left(\frac{1}{b}\right)_t = \left(1 - \frac{\epsilon^2}{4}\hat{a}\,\partial_x b\,\partial_x b\right)^{-1} \left(\frac{\hat{A}_t\, b}{2\lambda^r}\right) = \frac{\hat{A}_t\, b}{2\lambda^r} + \sigma_x,$$

where

$$\sigma = \frac{\epsilon^2}{4}\hat{a}\, b\,\partial_x b \left(\sum_{k=0}^{\infty} \left(\frac{\epsilon^2}{4}\hat{a}\,\partial_x b\,\partial_x b\right)^k\right) \left(\frac{\hat{A}_t\, b}{2\lambda^r}\right).$$

So we proved the formulae (2.16) with $n \ge 0$. The proof for the $n < 0$ case is similar.

Now the theorem follows from the formulae (2.16) and the results of Antonowicz and Fordy representing the r-KdV–CH flows in terms of the differential polynomials b_k, c_k [21,22]. □

3. Properties of the bihamiltonian structures

We now consider properties of the bihamiltonian structures given by the $r+1$ Hamiltonian structures of the r-KdV–CH hierarchy. Note that the r-KdV–CH hierarchy is a deformation of certain integrable hierarchy of hydrodynamic type, and the associated bihamiltonian structures are also deformations of certain semisimple bihamiltonian structures of hydrodynamic type. We are mainly interested in the following two aspects of these bihamiltonian structures:

1. Are the hydrodynamic limits of the bihamiltonian structures associated to some Frobenius manifolds? As we know from [7], on the formal loop space of any Frobenius manifold there is defined a natural bihamiltonian structure of hydrodynamic type.
2. To give a proof of the formulae of the central invariants of these bihamiltonian structures given in [2].

Let us first recall some notations. We say that a Hamiltonian structure Q defined on the formal loop space of a manifold M^r is of hydrodynamic type if its components have the form

$$Q^{ij} = g^{ij}(w)\,\partial_x + \Gamma^{ij}_k(w)\,w^k_x, \quad i,j = 0,\dots,r-1,$$

where $(g^{ij}(w))$ is a symmetric nondegenerate bilinear form on T^*M which we will call a metric on M. A bihamiltonian structure (Q_1, Q_2) of hydrodynamic type

$$Q^{ij}_a = g^{ij}_a(w)\,\partial_x + \Gamma^{ij}_{k,a}(w)\,w^k_x, \quad a = 1,2 \tag{3.1}$$

is called semisimple if the roots of the characteristic equation

$$\det(g_2(w) - \lambda\,g_1(w)) = 0 \tag{3.2}$$

are not constant and pairwise distinct.

We are to use the following theorem of Ferapontov:

Theorem 3.1 (*[29]*). *Let* (Q_1, Q_2) *be a semisimple bihamiltonian structure of hydrodynamic type defined on the loop space of a manifold* M^r*, then the roots of the characteristic equation (3.2) form a local coordinate system near every point of* M*, and the metrics* g_1, g_2 *are diagonal in this coordinate system. These coordinates are called the canonical coordinates of the bihamiltonian structures.*

Any two Hamiltonian structures of the r-KdV–CH hierarchy form a bihamiltonian structure $B_{k,l} = (P_k, P_l)$, $0 \le k \ne l \le r$, it has hydrodynamic limit

$$Q^{ij}_a = P^{ij}_a|_{\epsilon=0} = f^{ij}_a\left(2w^{i+j+1-a}\partial_x + w^{i+j+1-a}_x\right), \quad a = k,l. \tag{3.3}$$

We will denote the metric associated to Q_a by g_a as we did in (3.1).

To describe the canonical coordinates of the bihamiltonian structures, we denote by $\lambda_1,\dots,\lambda_r$ the roots of the polynomial

$$P(\lambda) = \lambda^r + w^{r-1}\lambda^{r-1} + \dots + w^1\lambda + w^0.$$

Assume that $\lambda_1,\dots,\lambda_r$ are pairwise distinct on M, then $P'(\lambda_i) \ne 0$, $i = 1,2,\dots,r$, and $\lambda_1,\dots,\lambda_r$ can be used as a system of local coordinates. In [30], Ferapontov and Pavlov showed that in this coordinate system the metrics $g_0,\dots,g_r$ are diagonal and the diagonal components have the expressions

$$g^{ii}_m(\lambda) = -\frac{2\lambda_i^m}{P'(\lambda_i)}.$$

From this it follows the following proposition.

Proposition 3.2. *For any* $k, l = 0,1,\dots,r$ *and* $k \ne l$*, the bihamiltonian structure* (Q_k, Q_l) *is semisimple and has the canonical coordinates*

$$u^i = (\lambda_i)^{l-k}, \quad i = 1,\dots,r. \tag{3.4}$$

In the canonical coordinates the associated metrics g_k, g_l *have the nonzero components*

$$g^{ii}_k = -2(l-k)^2\frac{\lambda_i^{2l-k-2}}{P'(\lambda_i)}, \qquad g^{ii}_l = -2(l-k)^2\frac{\lambda_i^{3l-2k-2}}{P'(\lambda_i)}, \quad i = 1,\dots,r.$$

We are now prepared to consider the relation of the above bihamiltonian structures with Frobenius manifolds. The notion of Frobenius manifold was introduced by Dubrovin in [6,7], it gives a geometric description of the WDVV equation of associativity [27,31] that arises in 2D topological field theory. For a given r-dimensional Frobenius manifold M, let $v^1,\dots,v^r$ be its flat coordinates near a point $v_0 \in M$ which is so chosen that $\frac{\partial}{\partial v^1}$ is the unit vector field. Then the potential $F = F(v^1,\dots,v^r)$ as a function of the flat coordinates satisfies the WDVV equation of associativity

$$\frac{\partial^3 F}{\partial v^1\partial v^i\partial v^j} = \eta_{ij} = \text{constant}, \quad \det(\eta_{ij}) \ne 0,$$
$$\frac{\partial^3 F}{\partial v^i\partial v^j\partial v^k}\eta^{kl}\frac{\partial^3 F}{\partial v^l\partial v^{i'}\partial v^{j'}} = \frac{\partial^3 F}{\partial v^{i'}\partial v^j\partial v^k}\eta^{kl}\frac{\partial^3 F}{\partial v^l\partial v^i\partial v^{j'}}, \quad \text{for any fixed indices } i,j,i',j', \text{ here } (\eta^{ij}) = (\eta_{ij})^{-1},$$
$$\partial_E F = (3-d)F + \text{quadratic terms in } v.$$

Here $E = \sum_{i=1}^r (d_i v^i + r_i)\frac{\partial}{\partial v^i}$ is the Euler vector field, d_i, r_i are some constants, and the constant d is called the charge of the Frobenius manifold.

On the formal loop space of a Frobenius manifold there is a bihamiltonian structure of hydrodynamic type (P_{fm}, Q_{fm}), its components are given by

$$P_{fm}^{ij} = \eta^{ij}\partial_x,$$
$$Q_{fm}^{ij} = g^{ij}(v)\partial_x + \Gamma_k^{ij}(v)\, v_x^k,$$

where

$$g^{ij}(v) = \sum_{k,l,m=1}^{r} (d_m v^m + r_m)\eta^{ik}\eta^{jl}\frac{\partial^3 F}{\partial v^k \partial v^l \partial v^m}$$

are the components of the intersection form of the Frobenius manifold, and

$$\Gamma_k^{ij} = -g^{il}\,\Gamma_{lk}^j$$

is given by the Christoffel symbols of the Levi-Civita connection of the metric $(g_{ij}) = (g^{ij})^{-1}$.

If the Frobenius manifold is semisimple, the associated bihamiltonian structure is also semisimple. The following proposition gives a relation between the above defined bihamiltonian structures of the r-KdV–CH hierarchy with semisimple Frobenius manifolds.

Proposition 3.3. *The semisimple bihamiltonian structures (Q_k, Q_l) associated to the r-KdV–CH hierarchy is given by the bihamiltonian structure (P_{fm}, Q_{fm}) of a semisimple Frobenius manifold if and only if $(k, l) = (0, 1)$ and $r = 1, 2$.*

We need the following lemma of Dubrovin to prove the proposition.

Lemma 3.4 ([7]). *Let (P_{fm}, Q_{fm}) be the bihamiltonian structure associated to a semisimple Frobenius manifold M^r. Denote by*

$$g_1^{ii} = \frac{1}{\eta_{ii}}, \quad i = 1, \dots, r$$

the diagonal components of the flat metric $(\eta^{\alpha\beta})$ in the canonical coordinates $u^1, \dots, u^r$. Then we must have

$$\frac{\partial \eta_{ii}}{\partial u^j} = \frac{\partial \eta_{jj}}{\partial u^i}, \quad i, j = 1, \dots, r. \tag{3.5}$$

Proof of Proposition 3.3. For the bihamiltonian structure $B_{k,l}$, the functions η_{ii} read

$$\eta_{ii} = -\frac{1}{2(l-k)^2}\frac{P'(\lambda_i)}{\lambda_i^{2l-k-2}}.$$

For $i \neq j$ we have

$$\frac{\partial \eta_{ii}}{\partial u^j} = \frac{\partial \lambda^j}{\partial u^j}\frac{\partial \eta_{ii}}{\partial \lambda^j} = -\frac{\lambda_j^{1-l+k}\lambda_i^{2-2l+k}}{2(l-k)^3}\frac{\partial P'(\lambda_i)}{\partial \lambda_j}.$$

Note that

$$P'(\lambda_i) = \prod_{k\neq i}(\lambda_i - \lambda_k),$$

so we obtain

$$\frac{\partial P'(\lambda_i)}{\partial \lambda_j} = -\prod_{k\neq i,j}(\lambda_i - \lambda_k).$$

Thus the derivative $\frac{\partial \eta_{ii}}{\partial u^j}$ reads

$$\frac{\partial \eta_{ii}}{\partial u^j} = \frac{\lambda_j^{1-l+k}\lambda_i^{2-2l+k}}{2(l-k)^3}\prod_{k\neq i,j}(\lambda_i - \lambda_k).$$

If the bihamiltonian structure (Q_k, Q_l) comes from a Frobenius manifold, the equations in (3.5) must hold true, they yield the conditions

$$\lambda_i^{1-l}\prod_{k\neq i,j}(\lambda_i - \lambda_k) = \lambda_j^{1-l}\prod_{k\neq i,j}(\lambda_j - \lambda_k), \quad 1 \leq i \neq j \leq r. \tag{3.6}$$

When $r \geq 3$, Eq. (3.6) cannot be true, since the left hand side depends on λ_i while the right hand side does not. When $r = 2$, Eq. (3.6) becomes

$$\lambda_i^{1-l} = \lambda_j^{1-l},$$

so we have $l = 1$. Thus we only have the following four possible cases:

$$(Q_0, Q_1)(r = 1), \qquad (Q_1, Q_0)(r = 1), \qquad (Q_0, Q_1)(r = 2), \qquad (Q_2, Q_1)(r = 2).$$

It is easy to verify, by using the explicit form of the potentials of the one-dimensional and two-dimensional Frobenius manifolds given in [7], that only the first and the third ones come from Frobenius manifolds. The first case corresponds to the Frobenius manifold with potential

$$F = \frac{(v^1)^3}{12}, \quad v^1 = w^0.$$

The third one corresponds to the Frobenius manifold with potential

$$F = -\frac{(v^1)^2 v^2}{4} + \frac{1}{4}(v^2)^2 \log v^2,$$

where v^1, v^2 are flat coordinates of the metric g_0 given by

$$v^1 = w^1, \qquad v^2 = w^0 - \frac{1}{4}(w^1)^2.$$

The proposition is proved. $\square$

Let us proceed to consider the central invariants of the bihamiltonian structures of r-KdV–CH hierarchy. The notion of central invariants were introduced in [2,4] for deformations of semisimple bihamiltonian structures of hydrodynamic type. Together with the leading terms, they provide a complete set of invariants of the deformations under Miura type transformations of the form

$$w^i \mapsto \tilde{w}^i = F^i \in \mathcal{A}, \quad i = 0, \ldots, r-1,$$

where $\det\left(\frac{\partial (F^i|_{\epsilon=0})}{\partial w^j}\right) \neq 0$.

To explain the meaning of central invariants, assume we have a semisimple bihamiltonian structure (3.1) of hydrodynamic type. A deformation of (Q_1, Q_2) is a bihamiltonian structure (P_1, P_2) of the following form

$$P_a^{ij} = Q_a^{ij} + \epsilon\left[A_a^{ij}(w)\partial_x^2 + B_{a,k}^{ij}(w)w_x^k\partial_x + C_{a,k}^{ij}(w)w_{xx}^k + D_{a,kl}^{ij}(w)w_x^k w_x^l\right]$$
$$+\epsilon^2\left[E_a^{ij}(w)\partial_x^3 + F_{a,k}^{ij}(w)w_x^k\partial_x^2 + \cdots\right] + \cdots, \quad a = 1, 2.$$

A lemma in [2] shows that one can always find a Miura type transformation eliminating the ϵ-term of (P_1, P_2), but one cannot find Miura type transformations to eliminate the ϵ^2-term in general. The central invariants are introduced to measure the failure to find such transformations. Suppose the ϵ-term of (P_1, P_2) has been eliminated by a Miura type transformation, we denote by f^i, E_1^i, E_2^i the diagonal components[1] under the canonical coordinates $u^1, \ldots, u^r$ of the tensors $g_1^{ij}, E_1^{ij}, E_2^{ij}$ in (P_1, P_2). The i-th central invariants c_i is defined as

$$c_i(u) = \frac{E_2^i - u^i E_1^i}{3\left(f^i\right)^2},$$

it is in fact a function of u^i only, i.e. we have $\frac{\partial c_i}{\partial u^j} = 0, i \neq j$. The main result of [4,2] shows that two deformations of (Q_1, Q_2) are equivalent via a Miura type transformation if and only if they possess same central invariants. In particular, if all c_i's vanish, then one can find a Miura type transformation eliminating all the deformation terms in (P_1, P_2), not only the ϵ^2-term. In this sense, we say the central invariants provide a *complete* set of invariants of the deformations of semisimple bihamiltonian structures of hydrodynamic type under Miura type transformations.

The central invariants of the bihamiltonian structures $B_{k,l}$ were considered in [2], where explicit formulae (see (3.7) below) to compute these invariants were given. We now give a proof of the formulae.

[1] The tensors E_1^{ij}, E_2^{ij} may not be diagonal under the canonical coordinates in general. In the definition of central invariants, we only use their diagonal components.

Theorem 3.5. *Let $\mathcal{P} = (a_0, a_1, \ldots, a_r) \in (\mathbb{R}^{r+1})^\times$. Define a polynomial*

$$p(\lambda) = a_0 + a_1\lambda + \cdots + a_r\lambda^r.$$

Let $B_{k,l}$ be a bihamiltonian structure of the r-KdV–CH hierarchy associated to $\mathcal{P}$, then the central invariants of $B_{k,l}$ are given by the following formulae:

$$c_i(u_i) = \frac{p(\lambda_i)}{24(k-l)\lambda_i^{l-1}}, \quad i = 1, \ldots, r. \tag{3.7}$$

Here u_i, λ_i are defined in Proposition 3.2.

Proof. We write the Hamiltonian structure P_m as

$$P_m = Q_m + \epsilon^2(E_m\partial_x^3 + \cdots),$$

where the dots represent terms with lower order in ∂_x. The components of the tensor E_m in the coordinates $w^0, \ldots, w^{r-1}$ read

$$E_m^{ij}(w) = -\frac{1}{2}f_m^{ij}\, a_{i+j+1-m}.$$

The diagonal components of the tensor E_m in the coordinates $\lambda_1, \ldots, \lambda_r$ have the expressions

$$\begin{aligned} E_m^{ii}(\lambda) &= -\frac{1}{2}\sum_{k,l=0}^{r-1}\frac{\partial\lambda_i}{\partial w^k}f_m^{kl}a_{k+l+1-m}\frac{\partial\lambda_i}{\partial w^l} = -\frac{1}{2\,(P'(\lambda_i))^2}\sum_{k,l=0}^{r-1}f_m^{kl}a_{k+l+1-m}\lambda_i^{k+l} \\ &= \frac{\lambda_i^m p'(\lambda_i) - m\lambda_i^{m-1}p(\lambda_i)}{2\,(P'(\lambda_i))^2}, \end{aligned}$$

transform them into canonical coordinates $u^i = \lambda_i^{l-k}$ we get

$$E_m^{ii}(u) = \left(\frac{\partial u^i}{\partial\lambda^i}\right)^2 E_m^{ii}(\lambda).$$

Then we can obtain the formula (3.7) by a simple computation. □

According to the main result of [2,4], the above theorem shows that the bihamiltonian structures $B_{k,l}$ associated to different $\mathcal{P}$'s are not equivalent under Miura type transformations. In particular, there do not exist Miura type transformations that convert the CH hierarchy to the KdV hierarchy.

4. Quasi-local Hamiltonian structures and their reciprocal transformations

Reciprocal transformations are certain type of change of the independent variables of a system of PDEs, they are important in the study of the classification problem of integrable systems. For example, the bihamiltonian structure of the KdV hierarchy and that of the CH hierarchy are different deformations of a semisimple bihamiltonian structure of hydrodynamic type, in the sense that they have different central invariants, so there is no Miura type transformation that converts one to another. However it is well known that there is a reciprocal transformation which relates these two hierarchies. We consider in this section the transformation rule of Hamiltonian structures under certain reciprocal transformations.

In [14], Ferapontov and Pavlov investigated the reciprocal transformations of local or weakly nonlocal Hamiltonian operators of hydrodynamic type (cf. [42]). The general form of the Hamiltonian operators they considered are

$$J^{ij} = g^{ij}(w)\partial_x + \Gamma_k^{ij}(w)w_x^k + \sum_{\alpha,\beta}\eta^{\alpha\beta}X_\alpha^i\partial_x^{-1}X_\beta^j, \tag{4.1}$$

where X_α^i $(\alpha = 1, \ldots, m)$ is a family of evolutionary vector fields

$$X_\alpha^i = V_{(\alpha)k}^i(w)w_x^k,$$

and $\eta^{\alpha\beta}$ is a constant nondegenerate symmetric matrix. The reciprocal transformations they considered are in general form, here we only consider ones with the following form

$$\mathrm{d}y = \rho(w)\mathrm{d}x + \sigma(w)\mathrm{d}t, \quad \mathrm{d}s = \mathrm{d}t. \tag{4.2}$$

Note that a general reciprocal transformation can be represented as compositions of linear reciprocal transformations and the ones having the form (4.2). One of the main results of [14] shows that a reciprocal transformation of the above form transforms a local Hamiltonian operator

$$J^{ij} = g^{ij}(w)\partial_x + \Gamma_k^{ij}(w)w_x^k$$

to a weakly nonlocal one

$$\tilde{J}^{ij} = \tilde{g}^{ij}(w)\partial_y + \tilde{\Gamma}^{ij}_k(w)w^k_y + \tilde{V}^i_k(w)w^k_y\partial_y^{-1}w^j_y + w^i_y\partial_y^{-1}\tilde{V}^j_k(w)w^k_y, \tag{4.3}$$

where $\tilde{g}^{ij}$ and $\tilde{V}^i_k$ are given by g^{ij} and ρ via the formulae

$$\tilde{g}^{ij} = \rho^2 g^{ij}, \qquad \tilde{V}^i_k = \rho\nabla^i\nabla_k(\rho) - \frac{1}{2}\nabla^j(\rho)\nabla_j(\rho)\delta^i_k.$$

Here ∇ is the Levi-Civita connection of the metric $(g_{ij}) = (g^{ij})^{-1}$. This result shows that the space of local Hamiltonian operators is not a good space to investigate reciprocal transformations.

We observed that the above formula of Ferapontov and Pavlov can be generalized to reciprocal transformations of general weakly nonlocal Hamiltonian operators (4.1). Moreover, if the Hamiltonian operator J takes the form (4.3), then the transformed Hamiltonian operator $\tilde{J}$ has the same form. So the space of weakly nonlocal Hamiltonian operators of the form (4.3) is a good one to study reciprocal transformations.

Furthermore, we observe that the Hamiltonian operators of the form (4.3) are in fact local in the following sense. Let J be a Hamiltonian operator of the following form

$$P^{ij} = \alpha^{ij}_s\partial^s_x + X^i\partial_x^{-1}w^j_x + w^i_x\partial_x^{-1}X^j, \tag{4.4}$$

and F, G be two local functionals, then it is easy to obtain the following identities:

$$\begin{aligned}\{F, G\}_P &= \int \frac{\delta F}{\delta w^i}P^{ij}\frac{\delta G}{\delta w^j}\,\mathrm{d}x \\ &= \int \left(\frac{\delta F}{\delta w^i}\left(\alpha^{ij}_s\partial^s_x\right)\frac{\delta G}{\delta w^j} + X^i\left(E(F)\frac{\delta G}{\delta w^i} - \frac{\delta F}{\delta w^i}E(G)\right)\right)\mathrm{d}x,\end{aligned} \tag{4.5}$$

where E is the energy operator

$$E = \sum_{s\geq 0}\sum_{t\geq 1}(-1)^s w^{i,t}\partial^s_x\frac{\partial}{\partial w^{i,s+t}} - 1, \tag{4.6}$$

it satisfies the following identities

$$E\,\partial_x = 0, \qquad \partial_x E = -w^i_x\frac{\delta}{\delta w^i}.$$

Note that E is a local operator (i.e. the value of $E(F)$ only depends on the germ of a chosen density f of the local functional F), so $\{\ ,\ \}_P$ is also a local operation. In the expression (4.5) of the Poisson bracket we do not need to use the inverse of ∂_x, which is a major difficulty in the definition of nonlocal Hamiltonian structures.

We see from the above observations that the Hamiltonian operators of the form (4.4) are interesting objects. They are similar to local Hamiltonian operators, the only difference is the appearance of an operator E in addition of the operation of variational derivatives in the definition of a Poisson bracket, and their forms are preserved by reciprocal transformations. The local Hamiltonian operators and general weakly nonlocal Hamiltonian operators do not have these properties. Due to the above reason we give the name *quasi-local Hamiltonian operators* to the Hamiltonian operators of the form (4.4).

In [32,33], Getzler, Kersten, Krasil'shchik and Verbovetsky developed a formulism of local Hamiltonian structures with the help of super-variables, which is very convenient to compute the Jacobi identities. We find that one can generalize their construction to quasi-local case, and their reciprocal transformation can be presented by transformations of the super-variables. We will give the details of these results in a separate publication [28], here in this section we just introduce the notations and list the main results of [28].

We denote by $\Lambda_x = \mathcal{A}/\partial_x\mathcal{A}$ the space of local functionals and denote the canonical projection $\mathcal{A} \to \Lambda_x$ by $f \mapsto \int f\,\mathrm{d}x$, the function f is called a density of the local functional. Define $\mathcal{V}^p_x = \mathrm{Alt}^p(\Lambda_x, \Lambda_x)$ to be the linear space of p-linear alternating maps from Λ_x to Λ_x. In particular, we denote $\mathcal{V}^0_x = \Lambda_x$, and $\mathcal{V}_x = \bigoplus_{p\geq 0}\mathcal{V}^p_x$. It is easy to see that every local Hamiltonian structure $P = P^{ij}_s\,\partial^s_x$ corresponds to an element of $\mathcal{V}^2_x$ via the formula

$$P(F_1, F_2) = \int \frac{\delta F_1}{\delta w^i}P^{ij}_s\partial^s_x\frac{\delta F_2}{\delta w^j}\mathrm{d}x, \quad F_1, F_2 \in \Lambda_x.$$

It can be shown [28] that there exists a unique bilinear map

$$[\ ,\ ] : \mathcal{V}^p_x \times \mathcal{V}^q_x \to \mathcal{V}^{p+q-1}_x$$

satisfying the following conditions:

$$[P, Q] = (-1)^{pq}[Q, P], \tag{4.7}$$

$$(-1)^{pk}[[P, Q], R] + (-1)^{qp}[[Q, R], P] + (-1)^{kq}[[R, P], Q] = 0, \tag{4.8}$$

$$[P, F_1](F_2, \ldots, F_p) = P(F_1, \ldots, F_p), \tag{4.9}$$

for any $P \in \mathcal{V}_x^p, Q \in \mathcal{V}_x^q, R \in \mathcal{V}_x^k, F_1, F_2, \dots, F_p \in \Lambda_x$. This bilinear map is introduced in [34,35] for any vector space, it is called the Nijenhuis–Richardson bracket (note that there is a sign difference between the definition of the map given here and the one given in [34,35]).

Let $P \in \mathcal{V}_x^p$ be a p-linear alternative map, we say that P is a quasi-local p-vector if the action of P on $F_1, \dots, F_p \in \Lambda_x$ takes the following form

$$P(F_1, \dots, F_p) = \int \Bigg(Q^{i_1\cdots i_p}_{s_1\cdots s_p} \cdot \partial^{s_1}\delta_{i_1}(F_1)\cdots\partial^{s_p}\delta_{i_p}(F_p) + R^{j_1\cdots j_{p-1}}_{t_1\cdots t_{p-1}} \cdot \sum_{k=1}^{p}(-1)^{k-1}E(F_k)\,\partial^{t_1}\delta_{j_1}(F_1)\cdots\hat{F}_k\cdots\partial^{t_{p-1}}\delta_{j_{p-1}}(F_p)\Bigg)\,\mathrm{d}x,$$

where $Q^{i_1\cdots i_p}_{s_1\cdots s_p}, R^{j_1\cdots j_{p-1}}_{t_1\cdots t_{p-1}} \in \mathcal{A}$. If the second term in the above expression vanishes, then we call P a local p-vector.

All quasi-local multi-vectors form a subspace of $\mathcal{V}_x$, and the Nijenhuis–Richardson bracket can be restricted on this subspace. We need some preparations to describe this procedure.

First we introduce a family of super-variables θ_i^s, ζ, where $i = 0, 1, \dots, r-1, s = 0, 1, 2, \dots$, and define

$$\hat{\mathcal{A}}_x = \mathcal{A} \otimes \wedge^*(V), \quad \text{where } V = \bigoplus_{i,s}\left(\mathbb{R}\theta_i^s\right) \oplus \mathbb{R}\zeta.$$

We extend the derivation ∂_x on $\mathcal{A}$ to a derivation on $\hat{\mathcal{A}}_x$ by the formula

$$\partial_x = \sum_{s\ge 0}\left(w^{i,s+1}\frac{\partial}{\partial w^{i,s}} + \theta_i^{s+1}\frac{\partial}{\partial\theta_i^s}\right) - \left(w_x^i\theta_i\right)\frac{\partial}{\partial\zeta} : \quad \hat{\mathcal{A}}_x \to \hat{\mathcal{A}}_x,$$

and we define

$$\hat{\Lambda}_x = \hat{\mathcal{A}}_x / \partial_x\hat{\mathcal{A}}_x. \tag{4.10}$$

The space $\hat{\Lambda}_x$ (as well as $\hat{\mathcal{A}}_x$) possesses a natural gradation $\hat{\Lambda}_x = \bigoplus_{p\ge 0}\hat{\Lambda}_x^p$ which is induced from the gradation of the exterior algebra $\wedge^*(V)$. A coset $P \in \hat{\Lambda}_x$ is called local, if it contains an element that does not depend on ζ. The space of all local elements of $\hat{\Lambda}_x$ is denoted by $\hat{\Lambda}_x^{loc}$, which is a subspace of $\hat{\Lambda}_x$.

It can be shown [28] that there is an embedding[2] $\jmath : \hat{\Lambda}_x \to \mathcal{V}_x$ defined by

$$\jmath(P)(F_1, \dots, F_p) = \sum_{s_k\ge 0}\int\Bigg(\partial^{i_p}_{s_p}\cdots\partial^{i_1}_{s_1}(\alpha)\cdot\delta^{s_1}_{i_1}(F_1)\cdots\delta^{s_p}_{i_p}(F_p) + \partial^{i_{p-1}}_{s_{p-1}}\cdots\partial^{i_1}_{s_1}\partial_\zeta(\alpha)\cdot\sum_{k=1}^{p}(-1)^{k-1}E(F_k)\,\delta^{s_1}_{i_1}(F_1)\cdots\hat{F}_k\cdots\delta^{s_{p-1}}_{i_{p-1}}(F_p)\Bigg)\,\mathrm{d}x, \tag{4.11}$$

here $\partial_\zeta = \frac{\partial}{\partial\zeta}$, $\partial^i_s = \frac{\partial}{\partial\theta_i^s}$, $\delta_i^s = \partial_x^s\frac{\delta}{\delta w^i}$, and $\alpha \in \hat{\mathcal{A}}_x^p$ is a representative of P, i.e. $P = \int\alpha\,\mathrm{d}x$, and $F_1, \dots, F_p \in \Lambda_x$. It is easy to see that the image of $\jmath$ is just the space of quasi-local multi-vectors, so we also call elements of $\hat{\Lambda}_x$ quasi-local multi-vectors. From now on, we will identify the space $\hat{\Lambda}_x$ and the image of $\jmath$, and omit the symbol $\jmath$.

The restriction of the Nijenhuis–Richardson bracket to $\hat{\Lambda}_x \subset \mathcal{V}_x$ can be represented by the following formula [28]:

$$[P, Q] = \int\left(\frac{\delta\alpha}{\delta\theta_i}\frac{\delta\beta}{\delta w^i} + (-1)^p\frac{\delta\alpha}{\delta w^i}\frac{\delta\beta}{\delta\theta_i} + \frac{\partial\alpha}{\partial\zeta}\hat{E}(\beta) + (-1)^p\hat{E}(\alpha)\frac{\partial\beta}{\partial\zeta}\right)\mathrm{d}x, \tag{4.12}$$

where $P = \int\alpha\,\mathrm{d}x \in \hat{\Lambda}_x^p$, $Q = \int\beta\,\mathrm{d}x \in \hat{\Lambda}_x^q$ with $\alpha \in \hat{\mathcal{A}}_x^p$, $\beta \in \hat{\mathcal{A}}_x^q$, and the operator $\hat{E}$ is given by

$$\hat{E} = \sum_{s\ge 0}\sum_{t\ge 1}(-1)^s\left(w^{i,t}\partial_x^s\frac{\partial}{\partial w^{i,s+t}} + \theta_i^t\partial_x^s\frac{\partial}{\partial\theta_i^{s+t}}\right) + \theta_i\frac{\delta}{\delta\theta_i} - 1.$$

If we restrict the bracket to $\hat{\Lambda}_x^{loc}$ further, the above formula coincides with Getzler's [32] and Kersten, Krasil'shchik, Verbovetsky's [33] Schouten–Nijenhuis brackets for local multi-vectors. So we also call the bracket (4.12) defined on the space of quasi-local multi-vectors the Schouten–Nijenhuis bracket.

2 Note that if we do not introduce the super-variable ζ, then the analogue of the map $\jmath$ no longer gives an embedding, it has a one-dimensional kernel generated by $w_x^i\theta_i$.

Definition 4.1. A bivector $P \in \hat{\Lambda}_x^2$ is called a Poisson bivector if $[P, P] = 0$. The Poisson bracket

$$\{F_1, F_2\}_P = P(F_1, F_2), \quad F_1, F_2 \in \Lambda_x \tag{4.13}$$

defined by a Poisson bivector via the formula (4.11) is called a quasi-local Hamiltonian structure.

Example 4.2. Let $P_0 = \frac{1}{2}\int \alpha_s^{ij}\theta_i\theta_j^s\,dx$ be a local bivector, so it corresponds to a skew-symmetric matrix differential operator $(\alpha^{ij}) = (\alpha_s^{ij}\partial_x^s)$. Let $X^i \in \mathcal{A}$ $(i = 0, \ldots, r-1)$, consider a Poisson bivector of the form $P = P_0 + \int X^i\zeta\theta_i\,dx$. By definition it gives the following Poisson bracket:

$$\begin{aligned}\{F, G\}_P &= P(F, G) = \int \left(\frac{\delta F}{\delta w^i}\alpha^{ij}\frac{\delta G}{\delta w^j} + X^i\left(E(F)\frac{\delta G}{\delta w^i} - \frac{\delta F}{\delta w^i}E(G)\right)\right) dx \\ &= \int \frac{\delta F}{\delta w^i}\left(\alpha^{ij} + X^i\partial_x^{-1}w_x^j + w_x^i\partial_x^{-1}X^j\right)\frac{\delta G}{\delta w^j}\,dx,\end{aligned}$$

so we obtain a usual weakly nonlocal Hamiltonian structure. In particular, if the operators α^{ij} of the local bivector P_0 and X^i have the form

$$\alpha^{ij} = g^{ij}(w)\partial_x + \Gamma_k^{ij}(w)w_x^k, \qquad X^i = V_j^i(w)w_x^j,$$

then one can easily obtain the conditions satisfied by g^{ij}, Γ_k^{ij}, X^i such that P is a Hamiltonian structure by using the formula (4.12). These conditions coincide with Ferapontov's results on the Hamiltonian operators associated with conformally flat metrics [36].

If a linear space V_1 is isomorphic to another linear space V_2, then there is a canonical isomorphism between $\mathrm{Alt}^*(V_1, V_1)$ and $\mathrm{Alt}^*(V_2, V_2)$. We will show in the rest of this section that the reciprocal transformations of Hamiltonian structures like the one between the KdV hierarchy and the CH hierarchy is just the restriction of an isomorphism of this type.

Let $\rho \in \mathcal{A}$ be an invertible element, it defines a derivation $\partial_y = \rho^{-1}\partial_x$ on $\mathcal{A}$. Denote $\Lambda_y = \mathcal{A}/\partial_y\mathcal{A}$ and $\mathcal{V}_y = \mathrm{Alt}^*(\Lambda_y, \Lambda_y)$. There is an isomorphism between the space of local functionals Λ_x and Λ_y

$$\Phi_0 : \Lambda_x \to \Lambda_y, \qquad \int f\,dx \mapsto \int \rho^{-1}f\,dy.$$

This isomorphism induces an isomorphism $\Phi : \mathcal{V}_x \to \mathcal{V}_y$ as follows:

$$\Phi(P)(F_1, F_2, \ldots, F_p) = \Phi_0\left(P\left(\Phi_0^{-1}F_1, \Phi_0^{-1}F_2, \ldots, \Phi_0^{-1}F_p\right)\right),$$

where $P \in \mathcal{V}_x^p$ and $F_1, F_2, \ldots, F_p \in \Lambda_y$. It is easy to see that the isomorphism Φ preserves the Nijenhuis–Richardson bracket, we call it the reciprocal transformation w.r.t. ρ.

To derive the transformation formula of quasi-local multi-vectors under reciprocal transformations, we need to use a new coordinate system on $\mathcal{A}$ defined by

$$\tilde{w}^i = w^i, \qquad \tilde{w}^{i,s} = \partial_y^s w^i, \quad s = 1, 2, \ldots. \tag{4.14}$$

The map $w^{i,s} \mapsto \tilde{w}^{i,s}$ is invertible since ρ is invertible, thus elements of $\mathcal{A}$ (as well as Λ_y) can be written in terms of differential polynomials in $\tilde{w}^{i,s}$. One can also define the operators $\frac{\delta}{\delta\tilde{w}^i}$ and $\tilde{E}$ on $\mathcal{A}$ by replacing w and ∂_x by $\tilde{w}$ and ∂_y in the original definitions of the operators $\frac{\delta}{\delta w^i}$ and E.

Let $f \in \mathcal{A}$ and $\tilde{f} = \rho^{-1}f$, then the following formulae can be proved [28]:

$$\frac{\delta f}{\delta w^i} = \rho\,\frac{\delta\tilde{f}}{\delta\tilde{w}^i} - \sum_{s\ge 0}(-\partial_x)^s\left(\frac{\partial\rho}{\partial w^{i,s}}\tilde{E}(\tilde{f})\right), \tag{4.15}$$

$$E(f) = \rho\,\tilde{E}(\tilde{f}) - \sum_{t\ge 0}\sum_{q\ge 1} w^{i,q}(-\partial_x)^t\left(\frac{\partial\rho}{\partial w^{i,t+q}}\tilde{E}(\tilde{f})\right). \tag{4.16}$$

By using the above formulae, we can write down an explicit formula for the reciprocal transformations of quasi-local multi-vectors. To this end, we introduce a new family of super-variables $\tilde{\theta}_i^s$, $\tilde{\zeta}$, where $i = 0, 1, \ldots, r-1$, $s = 0, 1, 2, \ldots$, and define

$$\hat{\mathcal{A}}_y = \mathcal{A} \otimes \wedge^*(\tilde{V}), \quad \text{where } \tilde{V} = \bigoplus_{i,s}\left(\mathbb{R}\tilde{\theta}_i^s\right) \oplus \mathbb{R}\tilde{\zeta}.$$

We also extend the derivation ∂_y on $\mathcal{A}$ to $\hat{\mathcal{A}}_y$ by

$$\partial_y = \sum_{s\ge 0}\left(\tilde{w}^{i,s+1}\frac{\partial}{\partial\tilde{w}^{i,s}} + \tilde{\theta}_i^{s+1}\frac{\partial}{\partial\tilde{\theta}_i^s}\right) - \left(\tilde{w}_x^i\tilde{\theta}_i\right)\frac{\partial}{\partial\tilde{\zeta}} : \quad \hat{\mathcal{A}}_y \to \hat{\mathcal{A}}_y,$$

and define $\hat{\Lambda}_y = \hat{\mathcal{A}}_y/\partial_y\hat{\mathcal{A}}_y$ which can also be regarded as a subspace of $\mathcal{V}_y$.

Theorem 4.3 ([28]). *The reciprocal transformation Φ is an isomorphism between the graded Lie algebras $\hat{\Lambda}_x$ and $\hat{\Lambda}_y$. More precisely, for $\alpha \in \hat{\mathcal{A}}_x$, we have*

$$\Phi\left(\int \alpha\, dx\right) = \int \rho^{-1}\hat{\Phi}(\alpha)\, dy,$$

where $\hat{\Phi} : \hat{\mathcal{A}}_x \to \hat{\mathcal{A}}_y$ is an isomorphism of super-commutative algebras which is defined by

$$\hat{\Phi}(f) = f, \quad \text{for } f \in \mathcal{A},$$
$$\hat{\Phi}(\theta_i^s) = \left(\rho\partial_y\right)^s \left(\rho\,\tilde{\theta}_i - \sum_{s\geq 0}(-\rho\partial_y)^s \left(\frac{\partial\rho}{\partial w^{i,s}}\tilde{\zeta}\right)\right),$$
$$\hat{\Phi}(\zeta) = \rho\,\tilde{\zeta} - \sum_{t\geq 0}\sum_{q\geq 1} w^{i,q}(-\rho\partial_y)^t \left(\frac{\partial\rho}{\partial w^{i,t+q}}\tilde{\zeta}\right).$$

Example 4.4. We consider an evolutionary PDE: $\epsilon\, w_t^i = X^i$, where $X^i \in \mathcal{A}$. It corresponds to a local vector $X = \int X^i\theta_i\, dx$. Let $\rho \in \mathcal{A}$ be an invertible element, we can define the reciprocal transformation Φ w.r.t. ρ. By Theorem 4.3, we have

$$\Phi(X) = \int \left(X^i\tilde{\theta}_i - \rho^{-1}\partial_t(\rho)\tilde{\zeta}\right) dy.$$

If ρ is a conserved density of ∂_t, i.e. if there exists $\sigma \in \mathcal{A}$ such that $\partial_t(\rho) = \partial_x(\sigma)$, then

$$\Phi(X) = \int \left(X^i - \sigma\, w_y^i\right)\tilde{\theta}_i\, dy,$$

so we obtain another evolutionary PDE: $w_s^i = X^i - \sigma\, w_y^i$, it coincides with the result of the following reciprocal transformation

$$dy = \rho\, dx + \sigma\, dt, \qquad ds = dt.$$

Example 4.5. Let $\alpha^{ij} = g^{ij}\partial_x + \Gamma_k^{ij} w_x^k$ be a local Hamiltonian structure, it corresponds to a local bivector $P = \frac{1}{2}\int \theta_i\alpha^{ij}\theta_j\, dx$ satisfying $[P, P] = 0$. Let ρ be a nowhere zero smooth function on M, so we can define the reciprocal transformation Φ w.r.t. ρ. Theorem 4.3 implies

$$\Phi(P) = \frac{1}{2}\int \left(\rho\tilde{\theta}_i - \frac{\partial\rho}{\partial w^i}\tilde{\zeta}\right)\left(g^{ij}\partial_y + \Gamma_k^{ij}\tilde{w}_y^k\right)\left(\rho\tilde{\theta}_j - \frac{\partial\rho}{\partial w^j}\tilde{\zeta}\right) dy, \tag{4.17}$$

which is equivalent to Ferapontov and Pavlov's transformation formula [36,37,14]. Note that Φ preserves the Schouten–Nijenhuis bracket

$$[\Phi(P), \Phi(P)] = \Phi([P, P]) = 0,$$

so $\Phi(P)$ is an quasi-local Hamiltonian structure automatically.

Remark 4.6. One can generalize our approach to include more general weakly nonlocal Hamiltonian structures. In [33], Kersten, Krasil'shchik, Verbovetsky gave such generalization for some particular examples. However, the problem of defining Schouten–Nijenhuis bracket among this kind of nonlocal multi-vectors is still open.

5. Reciprocal transformation of the *r*-KdV–CH hierarchy

The r-KdV–CH hierarchy possesses an interesting reciprocal transformation which was first introduced by Antonowicz and Fordy in [22] for the particular case of $\mathcal{P} = (0, 0, \ldots, 0, \alpha^{-1})$. In this section we first generalize their reciprocal transformation to the general case.

Lemma 5.1. *Let $w^0, w^1, \ldots, w^{r-1}$ be a solution to the r-KdV–CH hierarchy, then the following 1-form*

$$\omega = \omega_0\, dx + \sum_{n\neq 0}\omega_n\, dt_n = \frac{1}{c_0}\left(dx + \sum_{n>0} b_n\, dt_n - \sum_{n<0} c_{-n} dt_n\right) \tag{5.1}$$

is closed. Here b_n, c_n are defined in (2.7), (2.8).

Proof. This is a direct consequence of Lemma 2.1. □

This lemma shows that for any given solution $w^0, w^1, \ldots, w^{r-1}$ of the r-KdV–CH hierarchy we can define a set of new coordinates $(y, \{s_n\}_{n\in\mathbb{Z}})$ by

$$\mathrm{d}y = \mathrm{d}s_0 = \omega = \omega_0\,\mathrm{d}x + \sum_{n\neq 0}\omega_n\,\mathrm{d}t_n, \tag{5.2}$$

$$\mathrm{d}s_n = \mathrm{d}t_{-n}, \quad n \neq 0, n \in \mathbb{Z}. \tag{5.3}$$

It is called a reciprocal transformation of the r-KdV–CH hierarchy.

Proposition 5.2. *Let $w^0, w^1, \ldots, w^{r-1}$ be a solution to the r-KdV–CH hierarchy and ϕ be a solution to the linear system of the associated Lax pair*

$$\epsilon^2\phi_{xx} = A\phi, \qquad \phi_{t_n} = B_n\phi_x - \frac{1}{2}B_{n,x}\phi, \quad n \in \mathbb{Z}.$$

We define

$$\tilde{\phi} = \frac{\phi}{\sqrt{c_0}}, \qquad \tilde{A} = c_0^2 A - \frac{\epsilon^2}{4}\left(2\,c_0\,c_{0,xx} - c_{0,x}^2\right),$$

$$\tilde{B}_n = \begin{cases} \dfrac{B_{-n}}{c_0} - \omega_{-n}, & n \neq 0, \\ 1, & n = 0, \end{cases}$$

then $\tilde{\phi}, \tilde{A}, \tilde{B}$ satisfy

$$\epsilon^2\tilde{\phi}_{yy} = \tilde{A}\tilde{\phi}, \qquad \tilde{\phi}_{s_n} = \tilde{B}_n\tilde{\phi}_y - \frac{1}{2}\tilde{B}_{n,y}\tilde{\phi}, \quad n \in \mathbb{Z}. \tag{5.4}$$

Proof. The reciprocal transformation (5.2), (5.3) implies

$$\partial_y = \partial_{s_0} = c_0\,\partial_x, \qquad \partial_{s_n} = \partial_{t_{-n}} - c_0\,\omega_{-n}\,\partial_x, \quad n \neq 0.$$

The proposition is proved by a straightforward computation. □

The above reciprocal transformation is invertible, i.e. we can represent x as a function of $(y, \{s_n\}_{n\in\mathbb{Z}})$ from the total differential equation

$$\mathrm{d}x = c_0\,\mathrm{d}y + \sum_{k>0} c_k\,\mathrm{d}s_k - \sum_{k<0} b_{-k}\,\mathrm{d}s_k, \qquad \mathrm{d}t_n = \mathrm{d}s_{-n}, \quad n \neq 0, n \in \mathbb{Z}.$$

Corollary 5.3. *Let $w^i(x, t)$ $(i = 0, \ldots, r-1)$ be a solution to the r-KdV–CH hierarchy associated to $\mathcal{P} = (a_0, a_1, \ldots, a_r)$. We define*

$$v^{r-i}(y, s) = \left[c_0^2\,w^i - \frac{\epsilon^2}{4}a_i\left(2\,c_0\,c_{0,xx} - c_{0,x}^2\right)\right]_{x\mapsto x(y,s)}, \quad \tilde{a}_{r-i} = a_i,$$

then the functions $v^0(y, s), \ldots, v^{r-1}(y, s)$ give a solution to the r-KdV–CH hierarchy associated to $\mathcal{P}' = (\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_r)$.

The above proposition and its corollary show that the reciprocal transformation converts the n-th positive flow of the r-KdV–CH hierarchy associated to $\mathcal{P} = (a_0, a_1, \ldots, a_{r-1}, a_r)$ to the n-th negative flow of the r-KdV–CH hierarchy associated to $\mathcal{P}' = (a_r, a_{r-1}, \ldots, a_1, a_0)$.

Now let us consider the transformation rule of the Hamiltonian structures of the r-KdV–CH hierarchy under the reciprocal transformation. Besides the $r+1$ local Hamiltonian structures, the r-KdV–CH hierarchy also has infinitely many nonlocal Hamiltonian structures [21,22], two of which are quasi-local Hamiltonian structures. Let $R = P_r \cdot P_{r-1}^{-1}$, and define $P_{r+k} = R^k P_r$, $k = 1, 2$. The components of P_{r+1}, P_{r+2} read

$$P_{r+1}^{ij} = \mathcal{D}_{i+j-r} - \mathcal{D}_i\mathcal{D}_r^{-1}\mathcal{D}_j,$$

$$P_{r+2}^{ij} = \mathcal{D}_{i+j-r-1} - \mathcal{D}_{i-1}\mathcal{D}_r^{-1}\mathcal{D}_j - \mathcal{D}_i\mathcal{D}_r^{-1}\mathcal{D}_{j-1} + \mathcal{D}_i\mathcal{D}_r^{-1}\mathcal{D}_{r-1}\mathcal{D}_r^{-1}\mathcal{D}_j.$$

We need to show that $P_{r+1}, P_{r+2} \in \hat{\Lambda}_x^2$. Note that

$$\mathcal{D}_r^{-1} = \left(2\partial_x - \frac{\epsilon^2}{2}a_r\partial_x^3\right)^{-1} = \frac{1}{2}\partial_x^{-1} + O(\partial_x),$$

where $O(\partial_x)$ stands for a differential operator, so we have

$$-\mathcal{D}_i\mathcal{D}_r^{-1}\mathcal{D}_j = -\frac{1}{2}w_x^i\partial_x^{-1}w_x^j + O(\partial_x),$$

thus $P_{r+1} \in \hat{\Lambda}_x^2$ (see Example 4.2). Similarly,

$$\mathcal{D}_i\mathcal{D}_r^{-1}\mathcal{D}_{r-1}\mathcal{D}_r^{-1}\mathcal{D}_j = w_x^i\partial_x^{-1}X^j + X^i\partial_x^{-1}w_x^j + O(\partial_x),$$

where X^i are differential polynomials, so we also have $P_{r+2} \in \hat{\Lambda}_x^2$.

The reciprocal transformation (5.2), (5.3) converts the r-KdV–CH hierarchy associated to $\mathcal{P} = (a_0, a_1, \ldots, a_{r-1}, a_r)$ to the r-KdV–CH hierarchy associated to $\mathcal{P}' = (a_r, a_{r-1}, \ldots, a_1, a_0)$. We denote the flows, the Hamiltonians and the Hamiltonian structures of the r-KdV–CH hierarchy associated to $\mathcal{P}$ and $\mathcal{P}'$ by $\epsilon w_{t_n} = X_n, H_n, P_m$ and $\epsilon \tilde{w}_{s_n} = \tilde{X}_n, \tilde{H}_n, \tilde{P}_m$ respectively. Then by using the notations introduced in Section 4, we have

$$\Phi(X_n) = \tilde{X}_{-n}, \quad n \neq 0,$$

and

$$\Phi(H_n) = -\tilde{H}_{-n-2}, \quad n \neq -1.$$

Here Φ is the reciprocal transformation of the space of quasi-local multi-vectors defined by the function $\rho = \omega_0$ in (5.1). More precisely, for $H = \int h(w, w', \ldots)\,dx \in \Lambda_x$,

$$\Phi(H) = \int \rho^{-1}h(w, w', \ldots)\,dy,$$

and for an evolutionary PDE $\epsilon\, w_t^i = X^i$, $\Phi(X)$ is given in Example 4.4.

Now let us consider the following identity which is proved in Theorem 2.2:

$$X_n = -[P_m, H_{n-m}], \quad m = 0, \ldots, r, n < 0,$$

here [,] is the Schouten–Nijenhuis bracket defined by (4.12), and we view the Hamiltonian operator P_m as the associated Poisson bivector, see Example 4.2. Since the reciprocal transformation preserves the Schouten–Nijenhuis bracket, we have

$$\tilde{X}_n = -[\Phi(P_m), -\tilde{H}_{n+m-2}], \quad n > 0.$$

On the other hand, we know by Theorem 2.2 that

$$\tilde{X}_n = -[-\tilde{P}_{r-m+2}, -\tilde{H}_{n+m-2}], \quad m = 2, \ldots, r+2, n > 0.$$

So from the above two expressions of $\tilde{X}_n$ we can formulate the following conjecture:

Conjecture 5.4. *For $m = 0, 1, \ldots, r+1, r+2$, we have*

$$\Phi(P_m) = -\tilde{P}_{r+2-m}.$$

It seems to us that the trouble in proving the Conjecture 5.4, if it holds true, lies on the fact that the reciprocal transformation is defined in terms of c_0 which does not have an explicit expression. In this paper, we only prove the coincidence of the leading terms of $\Phi(P_m)$ with that of $-\tilde{P}_{r+2-m}$.

The leading terms $P_m^{[0]}$ of P_m ($m = 0, 1, \ldots, r+1, r+2$) read

$$(P_m^{[0]})^{ij} = f_m^{ij}\left(2w^{i+j+1-m}\partial_x + w_x^{i+j+1-m}\right), \quad m = 0, 1, \ldots, r-1, r,$$

$$(P_{r+1}^{[0]})^{ij} = 2\left(w^{i+j-r} - w^iw^j\right)\partial_x + \left(w^{i+j-r} - w^iw^j\right)_x + \frac{1}{2}w_x^i\partial_x^{-1}w_x^j,$$

$$(P_{r+2}^{[0]})^{ij} = 2\,U^{ij}\partial_x + U_x^{ij} - V^i\partial_x^{-1}w_x^j - w_x^i\partial_x^{-1}V^j,$$

where

$$U^{ij} = w^{i+j-r-1} - w^{i-1}w^j - w^iw^{j-1} + w^{r-1}w^iw^j,$$

$$V^i = \frac{1}{2}w_x^{i-1} + \frac{1}{2}w^iw_x^{r-1} + \frac{1}{4}w^{r-1}w_x^i.$$

Proposition 5.5. *For $m = 0, 1, \ldots, r+1, r+2$, we have*

$$\Phi(P_m^{[0]}) = -\tilde{P}_{r+2-m}^{[0]}.$$

Proof. In the dispersionless case, the reciprocal transformation (5.1) takes the form

$$dy = \omega_0^{[0]}\,dx + \sum_{n\neq 0}\omega_n^{[0]}\,dt_n,$$

where $\omega_n^{[0]} = \omega_n|_{\epsilon=0}$. In particular, $\omega_0^{[0]} = \sqrt{w^0}$. By applying the formula (4.17), the proposition is proved by a straightforward computation. □

6. Conclusion

In this paper, we investigate the reciprocal transformations of the multi-Hamiltonian structures of the r-KdV–CH hierarchy. We first give an explicit expression of the Hamiltonians for the multi-Hamiltonian structures of the r-KdV–CH hierarchy, specify those bihamiltonian structures of the hierarchy which are associated to certain Frobenius manifolds and prove the formulae for the central invariants of the bihamiltonian structures. Then we introduce the space of quasi-local multi-vectors, define the Schouten–Nijenhuis bracket and reciprocal transformations on this space. By using these techniques, we give in a natural way the transformation formulae of the Hamiltonian structures of the r-KdV–CH hierarchy under the reciprocal transformation of the hierarchy, and prove the formulae at the level of its dispersionless limit.

The problem of how to find solutions to the r-KdV–CH hierarchy is still open for a general parameter set $\mathcal{P}$. When $\mathcal{P} = (1, 0)$ and $\mathcal{P} = (1, 0, 0)$, the associated hierarchies are the well-known KdV and AKNS hierarchy respectively, the integration schemes of these hierarchies of evolutionary PDEs are well studied. Note that these two hierarchies are also the only particular cases among the general r-KdV–CH hierarchies that possess bihamiltonian structures of topological type.[3] So we may imagine that the rich properties of these two integrable hierarchies are due to their close relations to topological field theory. This fact may also give an explanation on why some well-known integration schemes suitable for these two hierarchies cannot be applied to the r-KdV–CH hierarchy associated to a general parameter set. When $\mathcal{P} = (0, 1)$ and $\mathcal{P} = (0, 0, 1)$, we obtain the CH and 2-CH hierarchy respectively. We can find solutions of these two hierarchies via certain reciprocal transformations, since they are transformed to the KdV and the AKNS hierarchy after such transformations [13,15,16]. However, we do not know how to find exact solutions for the general r-KdV–CH hierarchy.

We can regard the r-KdV–CH hierarchy as an *energy dependent generalization* of the KdV hierarchy. It is natural to ask: *Whether one can perform similar generalizations to other integrable hierarchies that possess Lax pair representations*? For example, the x-part of the Lax pair for the Gelfand–Dickey hierarchy reads

$$\left(\partial_x^{n+1} + u_{n-1}\partial_x^{n-1} + \cdots + u_1\partial_x + (u_0 - \lambda)\right)\phi = 0,$$

if we replace it by

$$\left(\left(\sum_{i=0}^{r} a_i\,\lambda^i\right)\partial_x^{n+1} + \sum_{k=0}^{n-1}\left(\sum_{i=0}^{r} w_{i,k}\lambda^i\right)\partial_x^k\right)\phi = 0, \tag{6.1}$$

where a_i's are some constants, can we obtain some interesting integrable systems? In [38], Antonowicz, Fordy and Liu considered the $n = 2$ case. Their results show that the $n = 2$ case is quite different from the $n = 1$ case, strong restriction needs to be imposed on the form of the operator (6.1) in order to obtain integrable hierarchies.

Moreover, note that the Gelfand–Dickey hierarchy is the Drinfeld–Sokolov hierarchy associated to the affine Lie algebra $A_n^{(1)}$ and the fixed vertex c_0 of the Dynkin diagram, so can we formulate a similar question for the Drinfeld–Sokolov hierarchy associated to general $(\mathfrak{g}, c_k)$?

We have the following two examples of such a generalization.

Recently the Degasperis–Procesi equation draws much attentions [39], it has the form

$$m_t + 3\,m\,u_x + m_x\,u = 0, \quad m = u - u_{xx}.$$

Its Lax pair reads

$$\lambda\,\phi_{xxx} - \lambda\,\phi_x + m\,\phi = 0,$$
$$\phi_t + \lambda\,\phi_{xx} + u\,\phi_x - \left(u_x + \frac{2\lambda}{3}\right)\phi = 0.$$

The x-part has exactly the form of (6.1). It is well known that after a reciprocal transformation the Degasperis–Procesi equation is transformed to the first negative flow of the Kaup–Kupershmidt hierarchy, which is the Drinfeld–Sokolov hierarchy associated to $(A_2^{(2)}, c_1)$. So we can regard the Degasperis–Procesi equation as a particular case of an energy dependent generalization of the Drinfeld–Sokolov hierarchy associated to the affine Lie algebra $A_2^{(2)}$ and its vertex c_1, just like the relationship between the Camassa–Holm equation and the r-KdV–CH hierarchies.

The second example is the Sawada–Kotera hierarchy [40], which is the Drinfeld–Sokolov hierarchy associated to $(A_2^{(2)}, c_0)$. There is also a reciprocal transformation that transforms the first negative flow of this hierarchy to the Novikov equation [41]

$$m_t + 3\,m\,u\,u_x + m_x\,u^2 = 0, \quad m = u - u_{xx}.$$

The Lax pair of the Novikov equation given in [41] is of matrix form, which is equivalent to a scalar form similar to (6.1). So the Novikov equation can also be viewed as a particular case of an energy dependent generalization of the Drinfeld–Sokolov hierarchy associated to the affine Lie algebra $A_2^{(2)}$ and the vertex c_0 of its associated Dynkin diagram.

These examples support a positive answer to the above questions; we hope to return to them in subsequent publications.

[3] We say a semisimple bihamiltonian structure is of topological type, if its leading term is associated to a Frobenius manifold, and its central invariants are equal to a single constant.

Acknowledgments

The authors would like to express their sincere thanks to the referee for his helpful comments and suggestions to improve the presentation of the paper. They would also like to thank Boris Dubrovin for his interest in this work and for his comments. The work is partially supported by the National Basic Research Program of China (973 Program) No. 2007CB814800, and the NSFC, No. 10631050 and No. 10801084.

References

[1] B. Dubrovin, Y. Zhang, Normal forms of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants, 2001. arXiv:math.DG/0108160.
[2] B. Dubrovin, S.-Q. Liu, Y. Zhang, On Hamiltonian perturbations of hyperbolic systems of conserved laws, I: Quasi-triviality of bi-Hamiltonian perturbations, Comm. Pure Appl. Math. 59 (2006) 559–615.
[3] B. Dubrovin, S.-Q. Liu, Y. Zhang, Frobenius manifolds and central invariants for the Drinfeld–Sokolov bi-Hamiltonian structures, Adv. Math. 219 (2008) 780–837.
[4] S.-Q. Liu, Y. Zhang, Deformations of semisimple bihamiltonian structures of hydrodynamic type, J. Geom. Phys. 54 (2005) 427–453.
[5] B. Dubrovin, Y. Zhang, Bihamiltonian hierarchies in 2D topological field theory at one-loop approximation, Comm. Math. Phys. 198 (1998) 311–361.
[6] B. Dubrovin, Integrable systems and classification of 2-dimensional topological field theories, in: O. Babelon, P. Cartier, Y. Kosmann-Schwarzbach (Eds.), Integrable Systems (The Verdier Memorial Conference), in: Progress in Mathematics, vol. 115, Birkhäuser, 1993, pp. 313–359.
[7] B. Dubrovin, Geometry of 2D topological field theories, in: Integrable Systems and Quantum Groups (Montecatini Terme, 1993), in: Lecture Notes in Math., vol. 1620, Springer, Berlin, 1996, pp. 120–348.
[8] T. Eguchi, Y. Yamada, S. Yang, On the genus expansion in the topological string theory, Rev. Math. Phys. 7 (1995) 279–309.
[9] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Comm. Math. Phys. 147 (1992) 1–23.
[10] E. Witten, Two-dimensional gravity and intersection theory on moduli space, Surv. Differ. Geom. 1 (1991) 243–310.
[11] R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993) 1661–1664.
[12] R. Camassa, D.D. Holm, J.M. Hyman, A new integrable shallow water equation, Adv. Appl. Mech. 31 (1994) 1–33.
[13] M. Chen, S.-Q. Liu, Y. Zhang, A two-component generalization of the Camassa–Holm equation and its solutions, Lett. Math. Phys. 75 (2006) 1–15.
[14] E.V. Ferapontov, M.V. Pavlov, Reciprocal transformations of Hamiltonian operators of hydrodynamic type: Nonlocal Hamiltonian formalism for linearly degenerate systems, J. Math. Phys. 44 (2003) 1150–1172.
[15] B. Fuchssteiner, Some tricks from the symmetry-toolbox for nonlinear equations: Generalizations of the Camassa–Holm equation, Physica D 95 (1996) 229–243.
[16] Y.S. Li, J.E. Zhang, The multiple-soliton solution of the Camassa–Holm equation, R. Soc. London Proc. Ser. A Math. Phys. Eng. Sci. 460 (2004) 2617–2627.
[17] M.V. Pavlov, Conservation of the forms of the Hamiltonian structures upon linear substitution for independent variables, Math. Notes 57 (1995) 489–495.
[18] C. Rogers, W.F. Shadwick, Bäcklund transformations and their applications, in: Mathematics in Science and Engineering, vol. 161, Academic Press, Inc., New York, London, 1982.
[19] T. Xue, Y. Zhang, Bihamiltonian systems of hydrodynamic type and reciprocal transformations, Lett. Math. Phys. 75 (2006) 79–92.
[20] L. Martínez Alonso, Schrödinger spectral problems with energy-dependent potentials as sources of nonlinear Hamiltonian evolution equations, J. Math. Phys. 21 (1980) 2342–2349.
[21] M. Antonowicz, A.P. Fordy, Coupled KdV equations with multi-Hamiltonian structures, Physica D 28 (1987) 345–357.
[22] M. Antonowicz, A.P. Fordy, Coupled Harry Dym equations with multi-Hamiltonian structures, J. Phys. A 21 (1988) 269–275.
[23] M. Antonowicz, A.P. Fordy, Factorisation of energy dependent Schrödinger operators: Miura maps and modified systems, Comm. Math. Phys. 124 (1989) 465–486.
[24] B. Fuchssteiner, A.S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, Physica D 4 (1981/82) 47–66.
[25] B. Dubrovin, Flat pencils of metrics and Frobenius manifolds, in: M.-H. Saito, Y. Shimizu, K. Ueno (Eds.), Proceedings of 1997 Taniguchi Symposium Integrable Systems and Algebraic Geometry, World Scientific, 1998, pp. 47–72.
[26] R. Dijkgraaf, E. Witten, Mean field theory, topological field theory, and multi-matrix models, Nuclear Phys. B 342 (1990) 486–522.
[27] R. Dijkgraaf, H. Verlinde, E. Verlinde, Loop equations and Virasoro constraints in nonperturbative two-dimensional quantum gravity, Nuclear Phys. B 348 (1991) 435–456.
[28] S.-Q. Liu, Y. Zhang, Jacobi structures of evolutionary PDEs and their reciprocal transformations (in preparation).
[29] E.V. Ferapontov, Compatible Poisson brackets of hydrodynamic type, in: Kowalevski Workshop on Mathematical Methods of Regular Dynamics (Leeds, 2000), J. Phys. A 34 (2001) 2377–2388.
[30] E.V. Ferapontov, M.V. Pavlov, Quasiclassical limit of coupled KdV equations, Riemann invariants and multi-Hamiltonian structure, Physica D 52 (1991) 211–219.
[31] E. Witten, On the structure of the topological phase of two-dimensional gravity, Nuclear Phys. B 340 (1990) 281–332.
[32] E. Getzler, A Darboux theorem for Hamiltonian operators in the formal calculus of variations, Duke Math. J. 111 (2002) 535–560.
[33] P. Kersten, I. Krasil'shchik, A. Verbovetsky, Hamiltonian operators and l^*-coverings, J. Geom. Phys. 50 (2004) 273–302.
[34] A. Nijenhuis, R. Richardson, Cohomology and deformations in graded Lie algebras, Bull. Amer. Math. Soc. 72 (1966) 1–29.
[35] A. Nijenhuis, R. Richardson, Deformation of Lie algebra structures, J. Math. Mech. 17 (1967) 89–105.
[36] E.V. Ferapontov, Conformally flat metrics, systems of hydrodynamic type and nonlocal Hamiltonian operators, Russian Math. Surveys 50 (1995) 811–813.
[37] E.V. Ferapontov, Hamiltonian systems of hydrodynamic type and their realization on hypersurfaces of a pseudo-Euclidean space, J. Math. Sci. 55 (1991) 1970–1995.
[38] M. Antonowicz, A.P. Fordy, Q.P. Liu, Energy-dependent third-order Lax operators, Nonlinearity 4 (3) (1991) 669–684.
[39] A. Degasperis, D.D. Holm, A.N.W. Hone, A new integrable equation with peakon solutions, Theoret. and Math. Phys. 133 (2002) 1463–1474.
[40] K. Sawada, T. Kotera, A method for finding N-soliton solutions of the K.d.V. equation and K.d.V.-like equation, Progr. Theoret. Phys. 51 (1974) 1355–1367.
[41] A.N.W. Hone, J.P. Wang, Integrable peakon equations with cubic nonlinearity, 2008. arXiv:0805.4310.
[42] A.Ya. Maltsev, S.P. Novikov, On the local systems Hamiltonian in the weakly non-local Poisson brackets, Physica D 156 (2001) 53–80.