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Chern–Simons theory on $L(p, q)$ lens spaces and Gopakumar–Vafa duality☆

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ABSTRACT

We consider aspects of Chern–Simons theory on $L(p, q)$ lens spaces and its relation with matrix models and topological string theory on Calabi–Yau threefolds, searching for possible new large N dualities via geometric transition for non-$SU(2)$ cyclic quotients of the conifold. To this aim we find, on one hand, a useful matrix integral representation of the $SU(N)$ CS partition function in a generic flat background for the whole $L(p, q)$ family and provide a solution for its large N dynamics; on the other hand, we perform in full detail the construction of a family of would-be dual closed string backgrounds through conifold geometric transition from $T^*L(p, q)$. We can then explicitly prove the claim in [5] that Gopakumar–Vafa duality in a fixed vacuum fails in the case $q > 1$, and briefly discuss how it could be restored in a non-perturbative setting.

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1. Overview

After the seminal work of Witten [1], Chern–Simons (CS) theory has been deeply studied both in Mathematics and Physics. A most attractive property of this topological field theory is its large N duality with the A-model topological string as discovered by Gopakumar and Vafa in [2]. From the Physics viewpoint, this is a concrete realization of 't Hooft's intuition [3] that the large N Feynman expansion of a gauge theory with $U(N)$ structure group can be recast as a perturbative expansion of closed oriented strings in a suitable background; mathematically, such duality is a precise (and amazing!) correspondence between two seemingly very unrelated mathematical objects, namely knot invariants and (relative) Gromov–Witten

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invariants. This duality is realized through a particular kind of *geometric transition*, called *conifold transition*, which plays a relevant rôle in the study of the moduli space of Calabi–Yau threefolds ($CY3$) (see *e.g.* [4–6] for reviews).
Let us recall the basic features of Gopakumar–Vafa (GV) duality. We have the following

Proposition 1 (*Witten, [7]*). *Let M be a closed smooth 3-manifold such that T^*M is a Calabi–Yau threefold. Then, the open topological A-model on T^*M, with N Lagrangian branes wrapping M, is equivalent to $U(N)$ Chern–Simons theory on M.*

Proposition 2 (*Gopakumar–Vafa, [2]*). *The topological open A-model on T^*S^3 with N A-branes wrapping the base S^3 is equivalent at large N to the closed topological A-model on $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The closed string coupling g_s is identified with t/N, where t is Kähler volume of the base $\mathbb{P}^1$.*

The closed string target space $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and the open string one T^*S^3 are related by topological surgery — a birational contraction plus a complex deformation of a nodal singularity. This is what goes under the name of "*(conifold) geometric transition*". As it stands, the content of Proposition 2 is striking and somewhat mysterious: a topological invariant of a 3-manifold — the CS partition function, or Reshetikin–Turaev–Witten invariant of S^3 — is also a generating function of symplectic invariants of a non-compact Kähler manifold — its Gromov–Witten potential. It would be then extremely interesting to see if one could find new examples of such a duality along the same lines, replacing S^3 with a generic 3-manifold and engineering the geometric transition in such a case. Indeed, one expects on general grounds the duality to hold for Chern–Simons theories on rational homology spheres, but actually very few examples beyond S^3 are known:

Open problem. *Find when the Reshetikin–Turaev–Witten invariant of a compact smooth 3-manifold M is equal, in suitable coordinates, to the all genus Gromov–Witten potential of an algebraic threefold X_M obtained by geometric transition from T^*M: that is, X_M is given by a complex deformation of T^*M to a normal variety, followed by a birational resolution.*

This issue has been addressed in the case of $\mathbb{Z}_p \subset SU(2)$ cyclic quotients of S^3:

Proposition 3 (*[8]*). *Let $M = L(p, 1)$. The Hori–Vafa mirror curve and differential of the family of $CY3$ obtained by geometric transition from T^*M coincide with the spectral curve and resolvent of the $L(p, 1) = M$ Chern–Simons matrix model* [9].

This has been achieved through a detailed study of the random matrix representation for the Chern–Simons path integral, originally obtained by Mariño in [9]. An early confirmation of the above assertion was provided in [10] for $p = 2$, by matching the 't Hooft expansion of the Chern–Simons 2-matrix model with the solutions of the Picard–Fuchs system at the orbifold point of $K_{\mathbb{P}^1 \times \mathbb{P}^1}$. Notice that, assuming[1] the validity of local mirror symmetry for toric $CY3$, Proposition 3 implies GV duality for $L(p, 1)$ lens spaces at least in genus zero; actually, recent progress [12] strongly suggests that the spectral data contain the full structure of the B-model on toric $CY3$ at all genera, in which case the work [8] would become automatically an all-genus proof.

However, it has been suspected that one could hardly go further along this direction for generic M. Actually, the most exhaustive and detailed monograph on the subject contains the following.

Claim 1 (*[5, Section 7.3]*). *Proposition 3 is false when $M = L(p, 1)$ is replaced by the generic lens space $M = L(p, q)$ for $q > 1$.*

In this note, we will perform a detailed analysis of the whole family $M = L(p, q)$, thus including cyclic subgroups not contained in $SU(2)$, along the lines of [8] and assuming mirror symmetry. The case $q > 1$, in which the lens space is not a $U(1)$ bundle on S^2, appears to be much harder as most of the features of the $q = 1$ case, like the mirror realization of [10] become either unclear or simply are not there. To this aim, in Section 2, we will work out explicitly the conifold transition for the case at hand and obtain a class of would-be large N duals $\widehat{\mathcal{X}_{p,q}}$ as well as their Hori–Vafa mirror curves, correcting *en passant* a few claims in the literature about the impossibility of performing such a geometric transition in the $L(p, q)$ case preserving at the same time the CY condition; in Section 3, we will present a matrix integral representation for the Chern–Simons $U(N)$ partition function on $L(p, q)$ in a fixed flat connection background and consider its large N expansion as governed by a spectral curve and a resolvent. We will find disagreement for $q > 1$ with the B-model spectral data obtained after geometric transition from $T^*L(p, q)$; under the assumption of validity of local mirror symmetry for toric $CY3$, this proves the claim above.

Even though our proof of Claim 1 might appear to be an obstruction to the program of extending the duality of [2] to more general backgrounds, let us outline two possible avenues of further investigation which we think might lead to the solution of the puzzle for the case under scrutiny.

One possible way out is to regard GV duality as an identity between the full CS partition on a 3-manifold M and some suitable *non-perturbative* definition of the $A-$model on the $CY3$ obtained through conifold geometric transition from T^*M. Indeed, as first advocated in[2] [16], a proper non-perturbative definition of the $A-$model on toric target spaces with a dual matrix integral description should be given in terms of a filling fraction independent sum over multi-instanton sectors. This would be dual to the proper definition of the Reshetikin–Turaev–Witten invariant as a sum over flat connections.

[1] There is presently a strikingly large body of evidence of the validity of mirror symmetry for toric $CY3$; see also [11] for rigorous mathematical proofs.

[2] See also [13,14] for related work on background independence and [15, Section 6.3], [14, Section 5.2] for a discussion precisely about the case of topological strings with a $L(p, 1)$ Chern–Simons matrix model representation.

A second possibility, hinted at by the geometric picture arising in the discussion of Section 2, might consist in a refinement of the notion of "orbifold of the *GV* duality for S^3" in order to properly encompass the case of the generic lens space. Indeed, for $1 < q < p - 1$ the cyclic group does no longer act fiberwise on the resolved conifold, giving rise to an orbibundle over a rational curve with marked points (see Remark 1). This new feature with respect to the $q = 1$ case definitely begs for further understanding, in order to clarify the correct formulation of *GV* duality in this case as well as its possible relation with a (suitably twisted) Gromov–Witten theory of orbicurves.

We hope to address both these issues in the future work.

2. Closed string side: Conifold transition for $T^*L(p, q)$

2.1. Geometric transition

According to Propositions 1 and 2, the *GV* duality for the case of the generic $L(p, q)$ lens space should be realized in two steps:

(1) a complex deformation of $\widehat{\mathcal{X}_{p,q}} \equiv T^*L(p, q)$ to a normal variety $\mathcal{X}_{p,q}$ (a suitable $\mathbb{Z}_p$ quotient of the singular conifold);
(2) a complete crepant resolution $\widehat{\mathcal{X}_{p,q}}$ of the latter.

The first step is realized as follows: let us recall that, from [4, Theorem 1.6], the cotangent bundle to the 3-sphere T^*S^3 is diffeomorphic to a smooth hypersurface in $\mathbb{A}^4$

$$xy - zt = \mu \tag{1}$$

which is a complex structure deformation of a conifold singularity. The base S^3 is the real locus $y = \bar{x}$, $t = -\bar{z}$

$$|x|^2 + |z|^2 = \mu. \tag{2}$$

Now, consider the $\mathbb{Z}_p$ action

$$\begin{array}{ccccc} \mathbb{Z}_p & \times & \mathbb{C}^4 & \to & \mathbb{C}^4 \\ \omega & & (x, y, z, t) & \to & (\omega x, \omega^{-1} y, \omega^q z, \omega^{-q} t) \end{array} \tag{3}$$

where $\omega = \mathrm{e}^{2\pi \mathrm{i}/p}$, $1 \leq q < p$, $(p, q) = 1$; the orbit manifold restricted to (2) is a $L(p, q)$ lens space. At first sight, using the same coordinatization as [4], the cyclic group acts both on the fibers and on the base of T^*S^3, thus yielding something a priori different from an $\mathbb{R}^3$-bundle over $L(p, q)$. However, we have the following simple

Lemma 1. *The orbit space of (3) restricted to (1) is smoothly diffeomorphic to $T^*L(p, q)$.*

Proof. Introduce the new set of variables $w_i = q_i + \mathrm{i}p_i$

$$\begin{array}{ll} w_1 = (x + y)/2 & w_2 = \mathrm{i}(-x + y)/2 \\ w_3 = (-z + t)/2 & w_4 = \mathrm{i}(z + t)/2. \end{array} \tag{4}$$

In these coordinates, (1) is the locus in $\mathbb{R}^8$ described by

$$\sum_{j=1}^{4} q_j^2 - p_j^2 = \mu, \qquad \sum_{j=1}^{4} q_j p_j = 0. \tag{5}$$

Consider then the change of variables

$$\begin{array}{lll} \tilde{p}_1 = q_1 p_1 + q_2 p_2, & \tilde{p}_2 = q_1 p_2 - q_2 p_1 \\ \tilde{p}_3 = q_3 p_3 + q_4 p_4, & \tilde{p}_4 = q_3 p_4 - q_4 p_3 \end{array} \tag{6}$$

and $\tilde{q}_i = q_i$, $i = 1, \ldots, 4$. The change of variables for $\mu > 0$ is non-singular everywhere in the set defined by (5), which is then rewritten as

$$\sum_{i=1}^{4} \tilde{q}_i^2 = \mu, \qquad \tilde{p}_1 + \tilde{p}_3 = 0. \tag{7}$$

The $\mathbb{Z}_p$ action is now represented on the tilded $\mathbb{R}^8$ in the form:

$$\begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \\ \tilde{q}_4 \end{pmatrix} \to \begin{pmatrix} \cos 2\pi/p & \sin 2\pi/p & 0 & 0 \\ -\sin 2\pi/p & \cos 2\pi/p & 0 & 0 \\ 0 & 0 & \cos 2\pi q/p & \sin 2\pi q/p \\ 0 & 0 & -\sin 2\pi q/p & \cos 2\pi q/p \end{pmatrix} \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \\ \tilde{q}_4 \end{pmatrix} \tag{8}$$

$$\begin{pmatrix}\tilde{p}_1\\ \tilde{p}_2\\ \tilde{p}_3\\ \tilde{p}_4\end{pmatrix} \to \begin{pmatrix}\tilde{p}_1\\ \tilde{p}_2\\ \tilde{p}_3\\ \tilde{p}_4\end{pmatrix} \tag{9}$$

realizing, therefore, the $\mathbb{Z}_p$ quotient (3) of the deformed conifold as a trivial $\mathbb{R}^3$-bundle over $L(p, q)$. By Stiefel's theorem [17], the latter being an orientable three-manifold, there exists a (strong) C^∞ bundle isomorphism mapping $\mathbb{R}^3 \times L(p, q)$ to $T^*L(p, q) =: \widehat{\mathcal{X}_{p,q}}$. □

With the algebraic realization (1), (3) of $\widehat{\mathcal{X}_{p,q}}$ at hand it is straightforward to perform the second step of the transition. As in the S^3 case, the μ parameter measures the size of the lens space and sending μ to zero amounts to deforming $\widehat{\mathcal{X}_{p,q}}$ to the singular variety $\mathcal{X}_{p,q}$, where the Lagrangian null section $L(p, q)$ has shrunk to zero size. We have the following.

Theorem 4. *The singular variety $\mathcal{X}_{p,q}$, obtained as the orbit space of (3) inside (1) with $\mu = 0$, is a toric variety with trivial canonical sheaf, $K_{\mathcal{X}_{p,q}} \simeq \mathcal{O}_{\mathcal{X}_{p,q}}$.*

Proof. For $q = 1$ the theorem was proven in [8], where the authors exploited the fact that $\mathcal{X}_{p,1}$ is obtained from the resolved conifold (a rank 2 bundle over S^2 as in Proposition 2) by quotienting a fiberwise-acting $\mathbb{Z}_p$ group and "blowing-down" the base S^2. For $q > 1$, though, the $\mathbb{Z}_p$ group does no longer act fiberwise and we have to deal with it in a different way.

By definition,[3] we have to prove that $\mathcal{X}_{p,q}$ contains an algebraic three-torus as an open subset effectively acting through an extension of its obvious action on itself. This is identified as follows: the singular conifold $\mathcal{X}$, as an affine variety

$$\mathcal{X} := \mathrm{Spec}\frac{\mathbb{C}[x, y, z, t]}{\{xy - zt\}}$$

is toric with torus action given by

$$\begin{aligned}&(\mathbb{C}^*)^3 \overset{j}{\hookrightarrow} \mathcal{X}\\ &(t_1, t_2, t_3) \to (t_1, t_2, t_3, t_1t_2t_3^{-1}).\end{aligned} \tag{10}$$

This action descends to an action on the orbifolded conifold $\mathcal{X}_{p,q}$ by (3)

$$(\mathbb{C}^*)^3/\mathbb{Z}_p \overset{\tilde{j}}{\hookrightarrow} \mathcal{X}_{p,q}. \tag{11}$$

Proving that $\mathcal{X}_{p,q}$ is toric, therefore, amounts to find explicitly an isomorphism $\pi : (\mathbb{C}^*)^3/\mathbb{Z}_p \mapsto (\mathbb{C}^*)^3$

$$0 \to \mathbb{Z}_p \overset{i}{\to} (\mathbb{C}^*)^3 \overset{\pi}{\to} (\mathbb{C}^*)^3 \to 0 \tag{12}$$

where the injection i is dictated by (3) to be

$$\begin{array}{ccccc} i: & \mathbb{Z}_p & \hookrightarrow & (\mathbb{C}^*)^3\\ & \omega & \mapsto & (\omega, \omega^{-1}, \omega^q)\end{array} \tag{13}$$

and by (12) we can write for π

$$\begin{array}{ccccc} \pi: & (\mathbb{C}^*)^3 & \hookrightarrow & (\mathbb{C}^*)^3\\ & (t_1, t_2, t_3) & \mapsto & (t_1^p, t_1t_2, t_1^qt_3^{-1}).\end{array} \tag{14}$$

The three-torus inside the quotient of the conifold by the action (3) is then identified by

$$\begin{aligned}&(\mathbb{C}^*)^3 \overset{\tilde{j}\circ\pi^{-1}}{\hookrightarrow} \mathcal{X}_{p,q}\\ &(t_1, t_2, t_3) \to (t_1^{1/p}, t_1^{-1/p}t_2, t_1^{q/p}t_3^{-1}, t_1^{-q/p}t_2t_3).\end{aligned} \tag{15}$$

From (15), we can read off the dual cone as the real tetrahedron spanned by

$$a_1 = \begin{pmatrix}1/p\\ 0\\ 0\end{pmatrix} \qquad a_2 = \begin{pmatrix}-1/p\\ 1\\ 0\end{pmatrix} \qquad a_3 = \begin{pmatrix}q/p\\ 0\\ -1\end{pmatrix} \qquad a_4 = \begin{pmatrix}-q/p\\ 1\\ 1\end{pmatrix}. \tag{16}$$

The fan is then obtained by taking the inward pointing normal to each facet, normalized in such a way to hit the first point on the $\mathbb{Z}^3$ lattice. Modulo an automorphism of the lattice, we thus get that the rays of the fan of $\mathcal{X}_{p,q}$ are given by

[3] See [18] for an introduction to toric geometry.

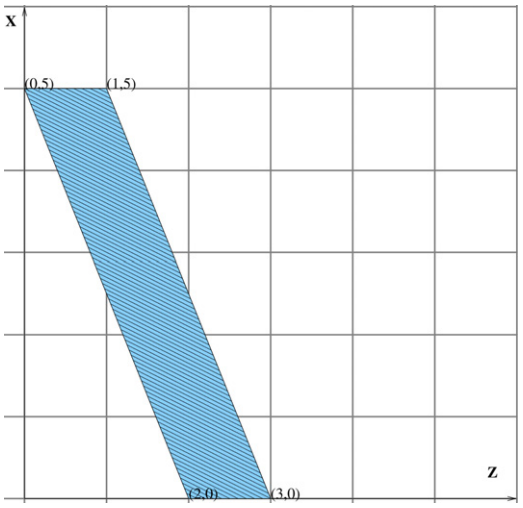


Fig. 1. Fan of $\mathcal{X}_{p,q}$ for $p = 5, q = 2$.

$$b_1 = \begin{pmatrix} 0 \\ 1 \\ q \end{pmatrix} \qquad b_2 = \begin{pmatrix} 0 \\ 1 \\ q+1 \end{pmatrix} \qquad b_3 = \begin{pmatrix} p \\ 1 \\ 1 \end{pmatrix} \qquad b_4 = \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix} \tag{17}$$

and the fan consists of a single cone generated by b_i, $i = 1 \ldots 4$. Notice that the tip of the rays all lie in an affine hyperplane, namely $y = 1$, thus implying triviality of the canonical class [18]. The theorem is proved. □

Remark 1. It is instructive to point out an interesting new geometrical fact in the $1 < q < p - 1$ case. Let us consider the orbifold of the resolved conifold geometry $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ by the $\mathbb{Z}_p$ action (3), which corresponds to a partial resolution of $\mathcal{X}_{p,q}$. This can be described as an orbibundle fibration of a Hirzebruch–Jung singularity over a $\mathbb{P}^1$ with two marked points with $\mathbb{Z}_p vi$-monodromy. One way to do that is to realize the projectivization of the resolved conifold as a subspace $\{[z_0, z_1, z_2], [z_3, z_4, z_5], [r, s] \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1 | z_1 r = z_2 s, z_3 r = z_4 s\}$. The $\mathbb{Z}_p$ action on the above variables is inferred from (3) through the identification $(z_1, z_2, z_3, z_4) = (x, -z, t, -y)$ and imposing invariance of the relations, which gives $(r, s) \to (\omega^{q-1} r, s)$. The fiber over the north pole $r = 0, s = 1$ is parameterized by (z_1, z_3), which describes precisely a Hirzebruch–Jung singularity. The analogue of the above is valid for the fiber over the south pole.

All we are left to do to complete step 2 is to take a complete resolution $\overline{\mathcal{X}_{p,q}}$ of $\mathcal{X}_{p,q}$

$$\overline{\mathcal{X}_{p,q}} \overset{r}{\dashrightarrow} \mathcal{X}_{p,q}.$$

Since $\mathcal{X}_{p,q}$ is Gorenstein, the birational morphism r can be taken to preserve the condition of $\overline{\mathcal{X}_{p,q}}$ being both toric and Calabi–Yau, i.e. to be a crepant toric resolution. We can realize this diagrammatically [18] by adding all the interior lattice vectors inside the tetrahedron spanned by b_i, and declaring that the (top-dimensional part of the) fan of $\overline{\mathcal{X}_{p,q}}$ is made by the cones constructed above the 2-simplices which triangulate the projection of the fan onto the $y = 1$ plane. Notice that the latter is a parallelogram with shorter sides of length 1 (see Fig. 1). This means that the intersection of the parallelogram with the horizontal lines $x = 1, \ldots, p$ contains either one point of the lattice in the interior or two points on the diagonal edges, the latter possibility being excluded by the coprimality condition $(p, q) = 1$. Thus, the number of points of the lattice (apart from the 4 external vertices) in the interior of the parallelogram is precisely $p - 1$. As is clear from the picture, these points have the form (on the plane) $([q + 1 - jq/p], j) = (q - [jq/p], j)$, where square brackets denote the integer part of the argument.

It is straightforward to get a complete crepant resolution of the orbifold by taking a triangulation of the $p + 3$ points

$$v_1 \equiv \begin{pmatrix} q+1 \\ 0 \end{pmatrix} \qquad v_2 \equiv \begin{pmatrix} q \\ 0 \end{pmatrix} \qquad v_{p+3} \equiv \begin{pmatrix} 1 \\ p \end{pmatrix}$$

$$v_{j+2} \equiv \begin{pmatrix} q - [jq/p] \\ j \end{pmatrix}, \quad j = 1, \ldots, p. \tag{18}$$

Definition 1. We will call $\overline{\mathcal{X}_{p,q}}$ the toric variety defined (modulo flops) by a fan supported by the rays

$$b_i \equiv \begin{pmatrix} v_i \\ 1 \end{pmatrix} \tag{19}$$

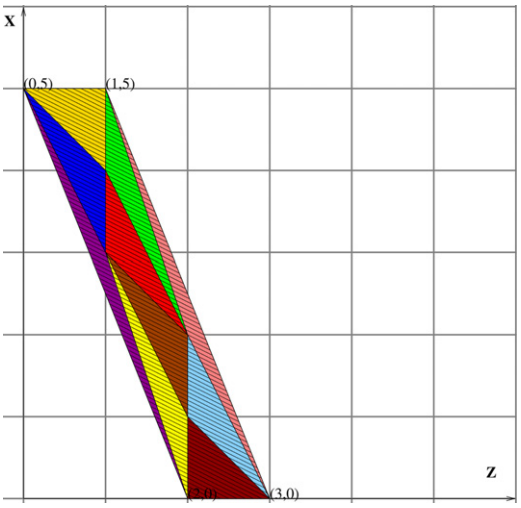


Fig. 2. Fan of $\overline{\mathcal{X}}_{p,q}$ for $p = 5, q = 2$.

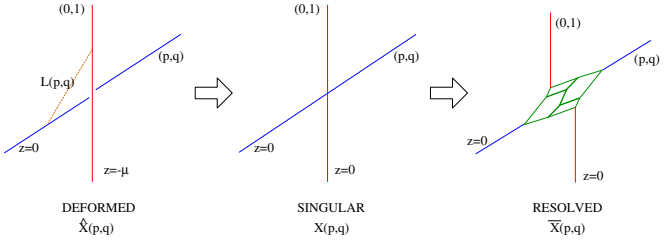


Fig. 3. A pictorial representation of the geometric transition for Lens spaces $L(p, q)$ as $\mathbb{T}^2$ fibrations.

and whose 3-dimensional cones are defined by having their intersection with the $z = 1$ hyperlane coincide with the simplices of a complete triangulation of the convex hull of (19).

By construction $\overline{\mathcal{X}_{p,q}}$ is a simplicial, smooth[4] toric *CY* threefold which is birationally isomorphic to $\mathcal{X}_{p,q}$. Step 2 is completed. □

The toric data (19) allow us to extract some useful information on the geometry of $\overline{\mathcal{X}_{p,q}}$. First of all, since internal vertices are in one-to-one correspondence with linear equivalence classes of (compact) divisors of $\overline{\mathcal{X}_{p,q}}$, we have that the fourth Betti number is

$$b_4(\overline{\mathcal{X}_{p,q}}) = p - 1$$

for every q. Moreover, given that the Euler characteristic $\chi(\overline{\mathcal{X}_{p,q}})$ is simply given by twice the area of the base of the tetrahedron and that odd Betti numbers vanish, we can easily compute the dimension of the second cohomology group as

$$b_2(\overline{\mathcal{X}_{p,q}}) = \chi(\overline{\mathcal{X}_{p,q}}) - b_0(\overline{\mathcal{X}_{p,q}}) - b_4(\overline{\mathcal{X}_{p,q}}) = 2p - 1 - (p - 1) = p \tag{20}$$

for every q. This is expected: the dimension of the Kähler moduli space of $\overline{\mathcal{X}_{p,q}}$ should match the number of inequivalent flat connections of the *CSSU*(N) theory on $L(p, q)$, which is $\pi_1(L(p, q)) = p$.

It is however remarkable, as it is also apparent from Figs. 2–4, that the intersection structure of $\overline{\mathcal{X}_{p,q}}$ for $q > 1$ is significantly more complicated than the simple case $q = 1$. Instead of the simple ladder diagrams describing the pq-webs of the A_{p-1} geometries, which were built out of a single tower of (nef) Hirzebruch surfaces, the compact divisors here are generic toric Fano surfaces and intersect in a wildly more intricate way, due to the fact that the vertices of the rays of the fan are no longer tetravalent and vertically aligned.

[4] A triangulation of the $p + 3$ points (18) realizes the projection of our cone onto the plane $y = 1$ as the disjoint union of precisely $2p$ triangles. The fact that the number of triangles is $2p$ is a consequence of Euler's formula: denoting with m the number of triangles in a triangulation of (18), since each triangle has 3 edges and the convex hull has 4, the number of edges is $(3m + 4)/2$, due to the fact that each edge is incident to exactly two faces. Plugging all the ingredients (number of points and edges) into Euler's formula, it follows that the number of triangles is exactly $2p$. Now, each triangle has half-integer area since each vertex is a site of the lattice, and the area is then given by half the determinant of an integer matrix. But p is the area of the whole parallelogram, so having $2p$ triangles implies that each triangle must have area $1/2$. The cones which project onto those triangles are then simplicial and smooth (i.e. each triple of vectors spanning a cone in the fan of the resolution is an integer basis of the lattice), which is precisely the non-singularity condition for a toric variety.

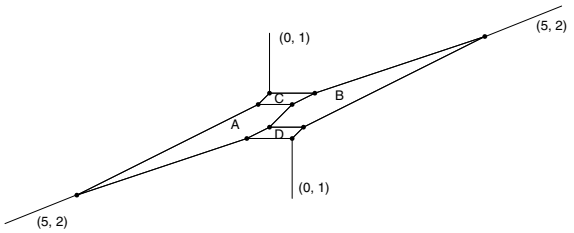


Fig. 4. pq-web diagram for the resolution of the orbifold $p = 5, q = 2$. A, B and C, D represent two dP_2 and two $\mathbb{F}_1$ surfaces, respectively.

2.2. Mirror symmetry

From the toric data (19), we can straightforwardly write down the Hori–Vafa mirror curve to $\overline{\mathcal{X}_{p,q}}$. We recall the following

Definition 2 (*Hori–Vafa mirror, [19]*). The B-model target space mirror to a toric CY threefold X is the hypersurface in $\mathbb{C}^2(x_1, x_2) \times (\mathbb{C}^*)^2(U, V)$

$$x_1 x_2 = P_X(U, V)$$

where $P_X(U, V)$ is the Newton polynomial associated to the fan of X. Periods of the holomorphic 3-form reduce to periods of the differential

$$\mathrm{d}\lambda_X = \log U \, \mathrm{d} \, \log V$$

on the *mirror curve* Σ_X (a projective curve with n punctures) given by the zero locus $P_X(u, v) = 0$. Denoting with $\mathcal{F}_X$ the projection of the fan onto the hyperplane containing the tip of the 1-dimensional cones, the number n of punctures and the genus of its n-point compactification are determined as

$$n = \#\text{ of external points of } \mathcal{F}_X, \qquad g = \#\text{ of internal points of } \mathcal{F}_X \tag{21}$$

While $(\Sigma_X, \mathrm{d}\lambda_X)$ depends on a choice of a representation for the dual lattice of X, i.e. on a $GL(3, \mathbb{Z})$ equivalence class, the periods are automatically invariant. In our case we have from (18), modulo a shift and a reflection on the x direction, that in logarithmic coordinates $u = \log U$, $v = \log V$

$$P_{p,q}(u, v) = \left(\mathrm{e}^{pu+qv} - 1\right)(\mathrm{e}^{v} - 1) + d_p + \sum_{j=1}^{p-1} d_j \mathrm{e}^{ju+(q-[(p-j)q/p])v} \tag{22}$$

where $d_j \in \mathbb{C}$ are complex moduli of the mirror geometry. When $d_j = 0$, corresponding to the singular $\mathcal{X}_{p,q}$, this form for the mirror curve had already been suggested by [10]. Notice, from (18), that there is no $GL(3, \mathbb{Z})$ transformation sending the points in $\mathcal{F}_{\overline{\mathcal{X}_{p,q}}}$ into a strip of horizontal width less than 3 for $1 < q < p - 1$. Moreover, by (3), the fan of $\overline{\mathcal{X}_{p,q}}$ and $\overline{\mathcal{X}_{p,p-q}}$ are related by an automorphism of the lattice, thus yielding isomorphic toric varieties. Collecting it all together, we have proven the following

Proposition 5. *The Hori–Vafa mirror curve and differential are given by*

$$\Sigma^{HV}_{p,q} : P_{p,q}(\mathrm{e}^{u}, \mathrm{e}^{v}) = 0, \qquad \mathrm{d}\lambda_{p,q} = u\mathrm{d}v$$

where $P_{p,q}$ is given by (22) and $u, v \in \mathbb{R} \times S^1$. The curve has 4 punctures and genus $p - 1$ for all q, and its periods have a symmetry given by $q \to p - q$. For $1 < q < p - 1$, $\Sigma^{HV}_{p,q}$ is not hyperelliptic. □

3. Open string side: *CS* theory on $L(p, q)$ and matrix models

The goal of this section is to provide a suitable matrix model representation of the partition function of Chern–Simons theory on a $L(p, q)$ lens space in a given vacuum. This case has been already considered in [9], where a general matrix integral representation for the partition function of Chern–Simons theory on Seifert homology spheres has been derived (see also [20,21]). Here, we find a slightly different, but equivalent, representation more useful for our purposes.

More precisely, let us consider $U(N)$ Chern–Simons theory at level $k \in \mathbb{Z}$ on a $L(p, q)$ lens space ($1 \leq q < p$, p and q coprime) and index with $\mathbf{m} \in \mathbb{Z}^N_p$ the set of $U(N)$-flat connections on $L(p, q)$. We will denote the corresponding partition function, or Reshetikin–Turaev–Witten invariant, as $Z^{L(p,q)}_{U(N)}\left(k, \mathbf{m}\right)$. We have the following

Theorem 6 (*Hansen–Takata, [22]*). *The Chern–Simons partition function for a $L(p, q)$ lens space, gauge group $U(N)$, level k and a fixed choice of flat connection* $\mathbf{m}$ *is given by*

$$Z^{L(p,q)}_{U(N)}\left(k, \mathbf{m}\right) = C_N(p, q; g_s)\, \mathrm{e}^{-\frac{4\pi^2 q}{g_s^2 p}\mathbf{m}^2} \sum_{\tilde{\omega},\omega\in S_N} \varepsilon(\omega)\, \mathrm{e}^{\frac{g_s^2}{2p}\,\omega(\boldsymbol{\rho})\cdot\boldsymbol{\rho}}\, \mathrm{e}^{\frac{2\pi \mathrm{i}}{p}\,\tilde{\omega}(\mathbf{m})\cdot(q\,\boldsymbol{\rho}+\omega(\boldsymbol{\rho}))} \tag{23}$$

where $g_s^2 = \frac{4\pi \mathrm{i}}{k+N} = \frac{4\pi \mathrm{i}}{\hat{k}}$, $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is the Weyl vector of $SU(N)$, S_N is the permutation group of N elements and $C_N(p, q; g_s)$ is a fixed overall factor, not depending on the particular flat connection (the exact expression of $C_N(p, q; g_s)$ is given in [22] and does not play any role here).

To obtain a matrix model representation it is useful to observe that this expression, up to an overall normalization factor, can be also written as

$$Z^{L(p,q)}_{U(N)}\left(k, \mathbf{m}\right) = \sum_{\tilde{\omega},\omega\in S_N} \varepsilon(\omega)\, \mathrm{e}^{\frac{1}{4g_s^2 p}\left(g_s^2(\omega(\boldsymbol{\rho})+\boldsymbol{\rho})+4\mathrm{i}\pi\,\tilde{\omega}(\mathbf{m})\right)^2 + 2\pi\mathrm{i}\frac{(q-1)}{p}\tilde{\omega}(\mathbf{m})\cdot\rho}. \tag{24}$$

By exploiting a trivial integral representation of the Gaussian function, we can rewrite the above partition function as an integral

$$\begin{aligned} Z^{L(p,q)}_{U(N)}\left(k, \mathbf{m}\right) &= \int_{-\infty}^{\infty} \mathrm{d}^N x \sum_{\tilde{\omega},\omega\in S_N} \varepsilon(\omega)\, \mathrm{e}^{\mathrm{i}\hat{k}p\pi\,(x\cdot x)+2\pi\left(\omega(\boldsymbol{\rho})+\boldsymbol{\rho}+\hat{k}\,\tilde{\omega}(\mathbf{m})\right)\cdot x+2\pi\mathrm{i}\frac{(q-1)}{p}\tilde{\omega}(m)\cdot\rho} \\ &= \int_{-\infty}^{\infty} \mathrm{d}^N x \sum_{\tilde{\omega},\omega\in S_N} \varepsilon(\omega)\, \mathrm{e}^{\mathrm{i}\hat{k}p\pi\,(x\cdot x)+2\pi\left(\tilde{\omega}^{-1}(\omega(\boldsymbol{\rho}))+\tilde{\omega}^{-1}(\boldsymbol{\rho})+\hat{k}\,\mathbf{m}\right)\cdot\tilde{\omega}^{-1}(x)+2\pi\mathrm{i}\frac{(q-1)}{p}\mathbf{m}\cdot\tilde{\omega}^{-1}(\rho)}. \end{aligned} \tag{25}$$

Since the measure of integration and $(x\cdot x)$ are symmetric under permutations, the partition function can be rearranged as

$$Z^{L(p,q)}_{U(N)}\left(k, \mathbf{m}\right) = \int_{-\infty}^{\infty} \mathrm{d}^N x \sum_{\tilde{\omega},\omega\in S_N} \varepsilon(\omega)\varepsilon(\omega')\, \mathrm{e}^{\mathrm{i}\hat{k}p\pi\,(x\cdot x)+2\pi\left(\omega(\boldsymbol{\rho})+\tilde{\omega}'(\boldsymbol{\rho})+\hat{k}\,\mathbf{m}\right)\cdot x+2\pi\mathrm{i}\frac{(q-1)}{p}\mathbf{m}\cdot\tilde{\omega}'(\rho)}. \tag{26}$$

To perform the sum over ω and ω', it is sufficient to recall the Weyl formula

$$\sum_{\omega\in S_N} \varepsilon(\omega)\mathrm{e}^{\mathrm{i}(\phi\cdot\omega(\rho))} = \prod_{\alpha>0} 2\sin\left(\frac{\alpha\cdot\phi}{2}\right), \tag{27}$$

and we thus get, restoring $g_s = \frac{4\pi\mathrm{i}}{\hat{k}}$,

$$\begin{aligned} Z^{L(p,q)}_{U(N)}\left(k, \mathbf{m}\right) &= \int_{-\infty}^{\infty} \mathrm{d}^N x\ \mathrm{e}^{\mathrm{i}\hat{k}p\pi\,(x\cdot x)+2\pi\hat{k}\,\mathbf{m}\cdot x} \prod_{\alpha>0} \sinh\left(\pi\alpha\cdot x\right) \sinh\left(\pi\alpha\cdot\left(x+\mathrm{i}\frac{(q-1)}{p}\mathbf{m}\right)\right) \\ &= \int_{-\infty}^{\infty} \mathrm{d}^N x\ \mathrm{e}^{-g_s p(x\cdot x)+4\pi\mathrm{i}\,\mathbf{m}\cdot x} \prod_{i<j} \sinh\left(\Delta_{ij}+\frac{i\pi\,(q-1)}{p}(\mathbf{m}_i-\mathbf{m}_j)\right)\sinh\left(\Delta_{ij}\right), \end{aligned} \tag{28}$$

where $\Delta_{ij} \equiv \frac{g_s}{2}(x_i - x_j)$. All the equalities hold up to irrelevant multiplicative constant factors and we have

Theorem 7. *The partition function of Chern–Simons theory on a $L(p, q)$ lens space for a choice* $\mathbf{m}$ *of flat connection can be written as a multi-eigenvalue integral as*

$$\begin{aligned} Z^{L(p,q)}_{U(N)}\left(k, \mathbf{m}\right) &= \int \prod_{I=1}^{p} \mathrm{d}^{N_I} u_k^{(I)}\, \mathrm{e}^{-\sum_{j=1}^{N} u_j^2 \frac{p}{2g_s}} \prod_{i<j} \sinh\left(\hat{\Delta}_{ij}^{(I)}\right)\sinh\left(\hat{\Delta}_{ij}^{(I)}\right) \\ &\quad\times \prod_{I<J}\prod_{i<j} \sinh\left(\hat{\Delta}_{ij}^{(IJ)}+\frac{\pi\,\mathrm{i}(I-J)}{p}\right)\sinh\left(\hat{\Delta}_{ij}^{(IJ)}+q\frac{\pi\,\mathrm{i}(I-J)}{p}\right) \end{aligned} \tag{29}$$

where $u_i^I \in \mathbb{R}$, $I = 1, \ldots, p$, $i = 1, \ldots, N_I$ and we have defined $\hat{\Delta}_{ij}^{(I)} \equiv \frac{1}{2}\left(u_i^{(I)} - u_j^{(I)}\right)$, $\hat{\Delta}_{ij}^{(IJ)} \equiv \frac{1}{2}\left(u_i^{(I)} - u_j^{(J)}\right)$.

In (29), we have eventually rescaled g_s by a factor of two in order to make contact with the notation of [10,8], to which it reduces in the case $q = 1$ and discarded a constant in front of the final matrix integral. This representation is of course equivalent to the one found in [9], up to an overall multiplicative constant.

Remark 2. At this stage, we can already spot a few signals of the fact that *GV* duality could break down for $q > 1$. Indeed, two $L(p, q)$ and $L(p', q')$ lens spaces are homeomorphic if and only if $p = p'$ and $q = \pm q'$ (mod p) or $qq' = \pm 1$ (mod p): the related partition functions are topological invariants and should thus be equal. This can be verified explicitly when the sum over the flat connections is performed and the Chern–Simons level is correctly quantized [22], but the same property does not seem to show up for the partition function in the background of a fixed flat connection: as one can easily check by explicit examples, different flat connection sectors are mixed under the relevant transformations. On the other hand, as pointed out in Proposition 5, $q = \pm q'$ (mod p) is instead a symmetry of the closed string background described in the previous section. Therefore, it is expected that the spectral data (22) will be different from what we will extract from the large N analysis of (29).

3.1. Large N limit of the CS matrix model

We now would like to prove that, as in the case of hermitian matrix models, the eigenvalue integral (29) is governed by a pair $(\Sigma^{CS}_{p,q}, dR_{p,q})$ made up of a spectral curve $\Sigma^{CS}_{p,q}$ and a resolvent $\mathrm{d}R_{p,q}$, out of which the genus zero free energy is extracted by the usual relations of special geometry. Proving spectral equivalence as in Proposition 3 amounts then to finding an isomorphism of curves ϕ such that

$$\begin{array}{ccccc} \phi: & \Sigma^{HV}_{p,q} & \mapsto & \Sigma^{CS}_{p,q} & \text{isomorphism} \\ & (\phi^{-1})^{*}\mathrm{d}\lambda_{p,q} & = & \mathrm{d}R_{p,q}. & \end{array} \tag{30}$$

Let us first introduce some basic objects in the discussion of the large N limit.

Definition 3. Let $N \in \mathbb{N}_0$, $I = 1, \ldots, p$ and $x \in \mathbb{R}$. For every I, the sequence of tempered distributions $\rho^{(I)}_N \in S'(\mathbb{R})$

$$\rho^{(I)}_N(x) := \frac{1}{N}\sum_{i=1}^{N}\delta(x - u^{(I)}_i)$$

will be called Ith *eigenvalue density at rank N*. Their integral on the real line gives the relative fraction of eigenvalues (**filling fraction**) in the Ith group

$$\int_{\mathbb{R}} \rho_N(x)\mathrm{d}x = \frac{N_I}{N}. \tag{31}$$

We then make the following basic.

Assumption. We *assume* that the $N \to \infty$ distributional limit

$$\rho^{(I)}(x) := \lim_{N\to\infty} \rho^{(I)}_N(x)$$

is a compactly supported continuous function on the real line, $\rho^{(I)} \in \mathcal{C}^0_c(\mathbb{R})$, $\forall I$.

This assumption is motivated by the analogous situation for hermitian matrix ensembles as well as for the *CS* matrix models of [8], and we will prove that it is self-consistent. It will be useful in the following to denote with $t \equiv g_s N$ the total 't Hooft coupling and with S_I the large N limit of the filling fractions

$$S_I := t \lim_{N\to\infty} \frac{N_I}{N}$$

normalized so that $\sum_I S_I = t$.

We will now construct explicitly the spectral curve $\Sigma^{CS}_{p,q}$ and differential $dR_{p,q}$ emerging from the large N study of (29). As usual in random matrix theory, this will be accomplished by finding an implicit algebraic expression $P_{p,q}(u, v)$ for the force $v(u)$ on a probe eigenvalue u at large N, in terms of which we will define

$$\begin{aligned} &\Sigma^{CS}_{p,q} := \{(u, v) \in (\mathbb{R}\times S^1)\times(\mathbb{R}\times S^1) | P_{p,q}(u, v) = 0\} \\ &\mathrm{d}R_{p,q} := v(u)\mathrm{d}u. \end{aligned} \tag{32}$$

At large N, $v(u)$ will have cuts in the complex plane whose discontinuity yields the individual eigenvalue densities ρ^I; its regularized integral from infinity to the Ith cut will instead measure, by construction, the variation of the (leading order) free energy with respect to the Ith filling fraction. This is summarized by the *special geometry* relations

$$\oint_{A_I} \mathrm{d}R_{p,q} = S_I \qquad \oint_{B_I} \mathrm{d}R_{p,q} = \frac{\partial \mathcal{F}}{\partial S_I}. \tag{33}$$

We will now show, from the explicit form of $(\Sigma^{CS}_{p,q}, dR_{p,q})$, that no such a ϕ as in (30) does in fact exist for $1 < q < p - 1$.

Proof of Claim 1. As is customary for *CS* multi-matrix models, the steepest descent (saddle point) method to evaluate (29) at large N yields a singular integral equation for $\rho^{(I)}$ with q-dependent hyperbolic kernels. From (29), we can straightforwardly write the saddle point equation as

$$p\lambda_I = t\fint \coth\left(\frac{\lambda_I-\lambda_I'}{2}\right)\rho_I(\lambda_I')\mathrm{d}\lambda_I' + \frac{t}{2}\sum_{J\neq I}\int_{\mathbb{R}}\left[\coth\left(\frac{\lambda_I-\lambda_J}{2}+d_{IJ}\right)+\coth\left(\frac{\lambda_I-\lambda_J}{2}+qd_{IJ}\right)\right]\rho_J(\lambda_J)\mathrm{d}\lambda_J \tag{34}$$

where $d_{IJ} := \mathrm{i}\pi(I-J)/p$ and the slashed integral indicates the Cauchy principal value ("improper") integral.

Now, let us define the following set of *resolvents*

$$\omega_I(z) \equiv t\int_{\mathbb{R}}\coth\left(\frac{z-\lambda_I}{2}\right)\rho_I(\lambda_I)\mathrm{d}\lambda_I, \tag{35}$$

$$\omega(z) = \frac{1}{2}\sum_{I=1}^{p}\left[\omega_I\left(z-2\pi\mathrm{i}\frac{I}{p}\right)+\omega_I\left(z-2\pi\mathrm{i}\frac{qI}{p}\right)\right]. \tag{36}$$

We will need the following easy generalization of the Sokhotski–Plemelij lemma

Lemma 2. *Define the following limits in* $S'(\mathbb{R})$

$$\coth_\pm(z) := \lim_{\epsilon\to 0}\coth(z\pm\mathrm{i}\epsilon).$$

Then, the following identities in $S'(\mathbb{R})$ *hold true:*

$$\coth_+ + \coth_- = 2pv(\coth), \tag{37}$$

$$\coth_+ - \coth_- = -2\pi\mathrm{i}\delta. \tag{38}$$

For notational purposes, we define accordingly $\omega_\pm(z) \equiv \lim_{\epsilon\to 0}\omega(z+\mathrm{i}\epsilon)$.

Given that the eigenvalue densities are supported on the real axis, we conclude immediately from (35) and (38) that the individual resolvents $\omega_I(z)$ have branch cuts which coincide with[5] $\mathrm{supp}(\rho_i) = [-a_I, a_I]$ for some $a_I \in \mathbb{R}$ (see Fig. 5). This implies that $\omega(z)$, as a function from the cylinder $0 \le \Im mz < 2\pi$ to the Riemann sphere, has p cuts centered at $z = 2\pi\mathrm{i}I/p$, whose width as usual depends on a choice of filling fractions S_I. Explicitly, from (36) we have for $J = 0,\dots,p-1$

$$2\omega\left(z+\frac{2\pi\mathrm{i}J}{p}\right) = \omega_J(z)+\omega_{\hat{J}}(z)+\sum_{I\neq J}\omega_I\left(z-\frac{2\pi\mathrm{i}(I-J)}{p}\right)+\sum_{I\neq\hat{J}}\omega_I\left(z-\frac{2\pi\mathrm{i}(qI-J)}{p}\right) \tag{39}$$

where $\hat{J}$ is defined by $q\hat{J} = J \bmod p$. When $I = 0$ we have $\hat{I} = 0$, and for $x \in [-a_0, a_0]$ we get that

$$\begin{aligned}\frac{\omega_+(x)+\omega_-(x)}{2} &= t\fint\coth\left(\frac{x-\lambda_I'}{2}\right)\rho_0(x')\mathrm{d}x' \\ &\quad+\frac{t}{2}\sum_{J\neq 0}\int_{\mathbb{R}}\left[\coth\left(\frac{x-x'}{2}+d_{pJ}\right)+\coth\left(\frac{x-x'}{2}+qd_{pJ}\right)\right]\rho_J(x')\mathrm{d}x' \\ &= px\end{aligned} \tag{40}$$

due to (34) and Lemma 2. However, a quick inspection shows that for no other $0 < I < p$ it is possible to find a closed expression for the average of the resolvent on the Ith cut. Indeed, since $I \neq \hat{I}$ for $0 < I < p$, different individual resolvents become singular at $x + 2\pi\mathrm{i}I/p$ inside the total sum (36), namely ω_I and $\omega_{\hat{I}} \neq \omega_I$, and it appears to be very intricate to infer the structure of $\omega(z)$ from (34). However, let us restrict ourselves for the moment to the special case in which

$$\rho_I = \rho_1 \quad 1 < I < p. \tag{41}$$

This corresponds to a particular symmetric choice of filling fractions, that is one in which a fraction of S_0 eigenvalues have been put on the cut on the real axis, corresponding to the trivial Chern–Simons connection, and $S_I = (t-S_0)/(p-1)$ for $I > 0$ are democratically distributed between the non-trivial flat connections (see Fig. 6). This would amount to explore a peculiar codimension $p-2$ subspace in the space of 't Hooft parameters, for which the large N data can be described in complete detail. In particular, this would give a 2-parameter closed subset of our sought-for p-dimensional family $(\Sigma^{CS}_{p,q}, dR_{p,q})$.

Under the constraints (41) we now have that, for $x \in [-a_I, a_I]$,

$$\frac{\omega_+\left(x+\frac{2\pi\mathrm{i}l}{p}\right)+\omega_-\left(x+\frac{2\pi\mathrm{i}l}{p}\right)}{2} = px \quad \forall I = 0,\dots,p-1. \tag{42}$$

[5] The eigenvalue integral is parity invariant, which therefore implies a $\mathbb{Z}_2$ symmetry in the location of the branch points.

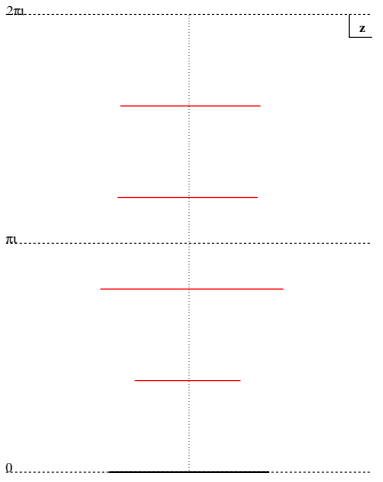


Fig. 5. Cuts of the resolvent for $p = 5$. Cuts relative to non-trivial flat connections are drawn in red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

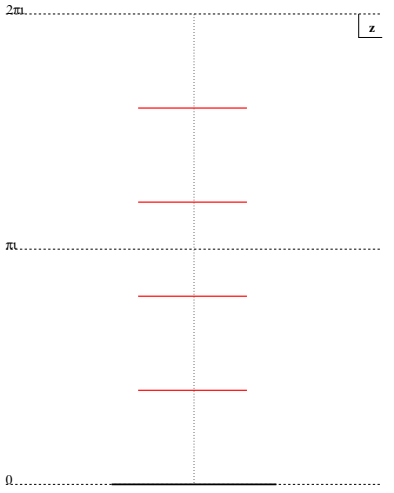


Fig. 6. Cuts of the resolvent for $p = 5$, imposing the constraint (41). Cuts relative to non-trivial flat connections are drawn in red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Now, let us map conformally the cylinder of width 2π to the punctured complex plane through

$$\begin{array}{rccc} Z: & \mathbb{R}\times S^1 & \mapsto & \mathbb{C}^* \\ & z & \mapsto & \mathrm{e}^z. \end{array} \tag{43}$$

Exponentiating (42) yields

$$Z^p = \mathrm{e}^{\omega_+/2}\mathrm{e}^{\omega_-/2}. \tag{44}$$

Introduce now

$$g(z) = \mathrm{e}^{\omega/2} + Z^p\mathrm{e}^{-\omega/2}. \tag{45}$$

This is a function which is single-valued on the whole strip $0 \leq \Im mz < 2\pi$, because for all $x \in [-a_I, a_I]$ we have that

$$g_+(x + 2\pi\mathrm{i}I/p) = \mathrm{e}^{\omega_+/2} + \mathrm{e}^{p(x+2\pi\mathrm{i}I/p)}\mathrm{e}^{-\omega_+/2} = \mathrm{e}^{\omega_+/2} + \mathrm{e}^{px}\mathrm{e}^{-\omega_+/2} = \mathrm{e}^{px}\mathrm{e}^{-\omega_-/2} + \mathrm{e}^{\omega_-/2} = g_-(x + 2\pi\mathrm{i}I/p). \tag{46}$$

and it is regular everywhere, except perhaps at infinity. This implies that $g(Z)$ is an entire analytic function in Z with algebraic growth

$$g(Z) = \sum_{n=0}^{p} d_n Z^n \tag{47}$$

where the d_n's are (still unknown) functions of the two filling fractions $S_0, S_I = S_1 = (t - S_0)/(p-1)$. The resolvent $\omega(z)$ is then determined by solving the quadratic equation (45) with the appropriate boundary condition at infinity, which yields

$$\omega(z) = \log\left[\frac{1}{2}\left(g(z) - \sqrt{g^2(z) - 4e^{pz}}\right)\right]. \tag{48}$$

Defining $u \equiv z$, $v \equiv (t - 2\omega)$ we arrive at the following form for $\Sigma^{CS}_{p,q}$ under the constraint[6] (41)

$$\begin{aligned}\Sigma^{CS}_{p,q}: \quad & e^{t-2v} - e^{t/2-v}e^{-t/2}\left(e^{pu} + \sum_{n=1}^{p-1} d_n e^{nu} + 1\right) + e^{pu} = 0 \\ \Rightarrow\ & e^t - e^v\left(e^{pu} + \sum_{n=1}^{p-1} d_n e^{nu} + 1\right) + e^{pu+2v} = (e^v - 1)(e^{pu+v} - 1) + e^t - 1 + e^v \sum_{n=1}^{p-1} d_n e^{nu} = 0,\end{aligned} \tag{49}$$

which coincides with the Hori–Vafa mirror curve (22) for $q = 1$ and a proper identification of the complex structure parameters. We have then proven the following.

Proposition 8. *Let $\Sigma^{CS}_{p,q}$ and $dR_{p,q} \equiv v\mathrm{d}u$ be the 2-parameter family (49) of large N spectral curves and differentials of the $L(p, q)$ Chern–Simons matrix model (29) under the constraint (41). Then, they are q-independent and they make up a closed subset of the family of Hori–Vafa mirror curves (22) with $q = 1$.*

This concludes the proof of Claim 1 for the following reason. Notice that this restricted class of $L(p, q)$ large N curves consists of hyperelliptic Riemann surfaces, since they coincide with the large N curves of the $q = 1$ case. Moreover, they are generically smooth and have topological genus $p - 1$. But for $1 < q < p - 1$, there is *no* such a subfamily inside the Hori–Vafa family of mirror curves (22), as follows from the discussion preceding Proposition 5.[7] □

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Appendix. CSMM as a unitary matrix model

An alternative matrix model realization can be given, in the trivial vacuum $m = 0$, as an integral over unitary matrices. Starting from (23) and reasoning along the lines of [9], we have that the *CS* partition function can be written as

$$Z^{L(p,q)}_{U(N)}\left(k, \mathbf{0}\right) = \int \mathrm{d}^N x e^{-\frac{p(x\cdot x)}{g_s q}} \prod_{i<j} \sinh\left(\frac{x_i - x_j}{2q}\right) \sinh\left(\frac{x_i - x_j}{2}\right), \tag{50}$$

and by means of the identity

$$\prod_{i<j} \sinh\left(a(x_i - x_j)\right) = \frac{e^{-a(N-1)\sum_i x_i}}{2^{N(N-1)/2}} \Delta(e^{2ax_i}), \tag{51}$$

we can write (50) as follows

$$Z_{CS} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^N x}{2^{N(N-1)/2}} \exp\left[\frac{-p(x\cdot x)}{g_s q} - \frac{(N-1)(q+1)}{2q}\sum_i x^i\right] \Delta(x_i)^2 \frac{\Delta(e^{x_i/q})\Delta(e^{x_i})}{\Delta(x_i)^2}. \tag{52}$$

[6] We also set $d_1 = d_p = 1$ with a redefinition u and dividing by an overall factor.

[7] Indeed, the very definition of the Hori–Vafa map and the fact that the toric diagram is not contained in a strip of width at most two for $1 < q < p - 1$ imply that hyperellipticity can be obtained only imposing a vanishing condition on a coefficient multiplying a monomial associated to an external point in the toric diagram. This amounts to discard a 1-dimensional ray in the fan and all the 3-dimensional cones in which it is contained, as is familiar from the degenerate limit in which local surfaces reduce to local curves. But this fact will automatically lower the number of internal points and thus the genus of the mirror curve. Hence there can be no hyperelliptic *and* genus $p - 1$ subfamily of curves inside (22).

Now, with the help of the Itzykson–Zuber formula [23]

$$\frac{\det(e^{ix_i/q})}{\Delta(i^j)\Delta(x_i^j)} = \frac{1}{\prod_{p=0}^{N-1} p!}\left(\frac{1}{q}\right)^{\frac{N(N-1)}{2}} \int dU_1 e^{\frac{1}{q}\mathrm{Tr}\,(U_1 A_D U_1^\dagger X_d)} \tag{53}$$

with $A_D = \mathrm{diag}\,(1,\ldots,N)$ and using that

$$\int_{\mathfrak{u}(N)} f(X)dX = \Omega_N \int_{\mathbb{R}^N} d^N x \Delta^2(x) f(\mathrm{diag}(x_i)) \tag{54}$$

for any Ad-invariant $f : \mathfrak{u}(N) \to \mathbb{C}$, where $\Omega_N = (2\pi)^{N(N-1)/2}/\prod_{j=1}^N j!$, we can turn (28) for $\mathbf{m} = 0$ into a HUU 3-matrix integral

$$Z_{CS} = \left(\frac{N!}{(4\pi q)^{N(N-1)/2}}\right) \int dX e^{\frac{-p}{gsq}\mathrm{Tr}\,X^2 - \frac{(N-1)(q+1)}{2q}\mathrm{Tr}\,X} \int dU_1 dU_2 e^{\frac{1}{q}\mathrm{Tr}\,(U_1 A_D U_1^\dagger X) + \mathrm{Tr}(U_2 A_D U_2^\dagger X)}. \tag{55}$$

Defining $\hat{X} \equiv U_1^\dagger X U_1$, $U \equiv U_1^\dagger U_2$ and exploiting the translation invariance of the Haar measure on $U(N)$

$$Z_{CS} = \left(\frac{N!}{(4\pi q)^{N(N-1)/2}}\right) \int d\hat{X} e^{\frac{-p}{gsq}\mathrm{Tr}\,\hat{X}^2 - \frac{(N-1)(q+1)}{2q}\mathrm{Tr}\,\hat{X}} \int dU e^{\frac{1}{q}\mathrm{Tr}\,(A_D \hat{X}) + \mathrm{Tr}(U A_D U^\dagger \hat{X})}. \tag{56}$$

The Gaussian integral over $\hat{X}$ gives

$$\begin{aligned} Z_{CS} = & \left(\frac{N!}{(4\pi q)^{N(N-1)/2}}\right) \int dU e^{\frac{g_s}{2p}\mathrm{Tr} UAU^\dagger A} \\ & \times \exp\left\{\frac{g_s}{8pq}\left[\left(\frac{N-1}{4}\right)^2 N(q+1)^2 + (q^2+1)\mathrm{Tr}A^2 - (N-1)(q+1)^2 \mathrm{Tr}A\right]\right\}. \end{aligned} \tag{57}$$

Notice that this last integral can be explicitly evaluated by means of the Itzykson–Zuber formula [23]. The result is in agreement with [21], where the same expression was computed by exploiting the biorthogonal polynomials.

References

[1] E. Witten, Quantum field theory and the Jones polynomial, Comm. Math. Phys. 121 (1989) 351.
[2] R. Gopakumar, C. Vafa, On the gauge theory/geometry correspondence, Adv. Theor. Math. Phys. 3 (1999) 1415 [arXiv:hep-th/9811131].
[3] G. 't Hooft, A planar diagram theory for strong interactions, Nuclear Phys. B 72 (1974) 461.
[4] A. Grassi, M. Rossi, Large N dualities and transitions in geometry, arXiv:math/0209044.
[5] D. Auckly, S. Koshkin, Introduction to the Gopakumar-Vafa large N duality, Geom. Topol. Monog. 8 (2006) 195–456 [arXiv:math/0701568].
[6] M. Rossi, Geometric transitions, J. Geom. Phys. 56 (2006) 1940 [arXiv:math/0412514].
[7] E. Witten, Chern–Simons gauge theory as a string theory, Progr. Math. 133 (1995) 637 [arXiv:hep-th/9207094].
[8] N. Halmagyi, T. Okuda, V. Yasnov, Large N duality, lens spaces and the Chern–Simons matrix model, J. High Energy Phys. 0404 (2004) 014 [arXiv:hep-th/0312145];
N. Halmagyi, V. Yasnov, The spectral curve of the lens space matrix model arXiv:hep-th/0311117.
[9] M. Marino, Chern–Simons theory, matrix integrals, and perturbative three-manifold invariants, Comm. Math. Phys. 253 (2004) 25 [arXiv:hep-th/0207096].
[10] M. Aganagic, A. Klemm, M. Marino, C. Vafa, Matrix model as a mirror of Chern–Simons theory, J. High Energy Phys. 0402 (010) (2004) [arXiv:hep-th/0211098].
[11] A.B. Givental, A mirror theorem for toric complete intersections, in: Topological Field Theory, Primitive Forms and Related Topics (Kyoto, 1996), in: Progr. Math., vol. 160, Birkhäuser Boston, Boston, MA, 1998, pp. 141–175;
B.H. Lian, K. Liu, S.T. Yau, Mirror principle I, Asian J. Math. 1 (1997) 729–763;
A. Elezi, Mirror symmetry for concavex vector bundles on projective spaces, Int. J. Math. Math. Sci. (2003) 159–197.
[12] V. Bouchard, A. Klemm, M. Marino, S. Pasquetti, Remodeling the B-model, arXiv:0709.1453 [hep-th].
[13] B. Eynard, Large N expansion of convergent matrix integrals, holomorphic anomalies, and background independence, arXiv:0802.1788 [math-ph].
[14] B. Eynard, M. Marino, A holomorphic and background independent partition function for matrix models and topological strings, arXiv:0810.4273 [hep-th].
[15] M. Marino, Nonperturbative effects and nonperturbative definitions in matrix models and topological strings, arXiv:0805.3033 [hep-th].
[16] M. Marino, R. Schiappa, M. Weiss, Nonperturbative effects and the large-order behavior of matrix models and topological strings, arXiv:0711.1954 [hep-th].
[17] E. Stiefel, Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten, Comment. Math. Helv. 8 (1) (1935) 305–353.
[18] W. Fulton, Introduction to toric varieties, in: The William H. Roever Lectures in Geometry, in: Annals of Mathematics Studies, vol. 131, Princeton University Press, 1993.
[19] K. Hori, C. Vafa, Mirror symmetry, arXiv:hep-th/0002222;
B. Feng, Y.H. He, K.D. Kennaway, C. Vafa, Dimer models from mirror symmetry and quivering amoebae, arXiv:hep-th/0511287.
[20] S. Garoufalidis, M. Marino, On Chern–Simons matrix models, arXiv:math/0601390.
[21] Y. Dolivet, M. Tierz, Chern–Simons matrix models and Stieltjes-Wigert polynomials, J. Math. Phys. 48 (2007) 023507 [arXiv:hep-th/0609167].
[22] S.K. Hansen, T. Takata, Reshetikhin–Turaev invariants of Seifert 3-manifolds for classical simple Lie algebras, J. Knot Theory Ramifications 13 (2004) [arXiv:math/0209403];
L. Griguolo, D. Seminara, R.J. Szabo, A. Tanzini, Nuclear Phys. B 772 (2007) 1 [arXiv:hep-th/0610155].
[23] C. Itzykson, J.B. Zuber, The planar approximation. 2, J. Math. Phys. 21 (1980) 411.