



Ricci magnetic geodesic motion of vortices and lumps

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ABSTRACT

Ricci magnetic geodesic (RMG) motion in a Kähler manifold is the analogue of geodesic motion in the presence of a magnetic field proportional to the Ricci form. It has been conjectured to model low-energy dynamics of vortex solitons in the presence of a Chern–Simons term, the Kähler manifold in question being the n -vortex moduli space. This paper presents a detailed study of RMG motion in soliton moduli spaces, focusing on the cases of hyperbolic vortices and spherical \mathbb{CP}^1 lumps. It is shown that RMG flow localizes on fixed point sets of groups of holomorphic isometries, but that the flow on such submanifolds does not, in general, coincide with their intrinsic RMG flow. For planar vortices, it is shown that RMG flow differs from an earlier reduced dynamics proposed by Kim and Lee, and that the latter flow is ill-defined on the vortex coincidence set. An explicit formula for the metric on the whole moduli space of hyperbolic two-vortices is computed (extending an old result of Strachan's), and RMG motion of centred two-vortices is studied in detail. Turning to lumps, the moduli space of static n -lumps is Rat_n , the space of degree n rational maps, which is known to be Kähler and geodesically incomplete. It is proved that Rat_1 is, somewhat surprisingly, RMG complete (meaning that the initial value problem for RMG motion has a global solution for all initial data). It is also proved that the submanifold of rotationally equivariant n -lumps, Rat_n^{eq} , a topologically cylindrical surface of revolution, is intrinsically RMG incomplete for $n = 2$ and all $n \geq 5$, but that the extrinsic RMG flow on Rat_2^{eq} (defined by the inclusion $\text{Rat}_2^{\text{eq}} \hookrightarrow \text{Rat}_2$) is complete.

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1. Introduction

Let (M, g, J) be a Kähler manifold with Ricci form ρ , that is, $\rho(X, Y) = \text{Ric}(JX, Y)$ where Ric denotes the Ricci tensor defined by g . A smooth curve $\alpha : I \rightarrow M$ is *Ricci magnetic geodesic* if

$$\nabla_{d/dt}^\alpha \dot{\alpha} = \lambda \sharp \iota_{\dot{\alpha}} \rho, \quad (1.1)$$

where ∇^α is the pullback of the Levi-Civita connexion on M to $\alpha^{-1}TM$, $\sharp : T^*M \rightarrow TM$ denotes the metric isomorphism, ι denotes interior product, and $\lambda \in \mathbb{R}$ is a constant parameter. We shall call such a curve RMG, or RMG_λ if we wish to emphasize the role of the parameter λ . This is an example of magnetic geodesic flow, that is, motion of a charged particle, of electric charge λ , under the influence of a magnetic field, in this case, the two-form ρ . Note that the flow reduces to conventional geodesic motion if $\lambda = 0$, and that, in all cases, RMG curves have constant speed, since

$$\frac{d}{dt} \|\dot{\alpha}(t)\|^2 = 2g(\dot{\alpha}(t), \lambda \sharp \iota_{\dot{\alpha}(t)} \rho) = 2\lambda \rho(\dot{\alpha}(t), \dot{\alpha}(t)) = 0. \quad (1.2)$$

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Unlike geodesics, RMG curves depend on the length, and not just the direction, of their initial velocity. Clearly, $\alpha(t)$ is RMG_λ if and only if $\tilde{\alpha}(t) = \alpha(\lambda_* t)$ is $\text{RMG}_{\lambda_* \lambda}$, so we may, without loss of generality, scale λ to any convenient value, or leave λ general and consider only RMG curves of unit speed.

RMG flow was first proposed by Collie and Tong [1] as a model of the low-energy dynamics of vortex solitons in a certain Chern–Simons variant [2] of the abelian Higgs model on \mathbb{R}^2 . In this setting, $M = M_n \equiv \mathbb{C}^n$, the moduli space of static n -vortex solutions of the (usual) abelian Higgs model, and g is its L^2 metric. In the limit $\lambda \rightarrow 0$, one recovers geodesic motion on M_n , a well-studied problem [3] which is rigorously known to approximate low-energy vortex dynamics in the absence of a Chern–Simons term [4]. RMG flow may thus be regarded as a geometrically natural perturbation of the geodesic approximation of Manton [5], arising from the inclusion of a Chern–Simons term. Low-energy vortex dynamics in this system was previously studied by Kim and Lee [6], who, by a direct perturbative calculation, derived a structurally similar magnetic geodesic flow on M_n . Indeed, Collie and Tong assert [1] that the Kim–Lee flow actually is RMG flow, and that their own contribution is to generalize and give it both a geometric interpretation, and an alternative (rather indirect) derivation. In fact, we will see that the Kim–Lee flow on M_n is *not* RMG flow, as claimed in [1], and, further, is not a well-defined flow on M_n at all, since it is singular on the vortex coincidence set.

In this paper we present a detailed study of RMG flow on the moduli spaces of abelian Higgs vortices and \mathbb{CP}^1 lumps. For vortices on \mathbb{R}^2 , we show that the Kim–Lee flow is ill-defined on the subset of M_n where two or more vortices coincide, and hence that this flow cannot coincide with RMG flow which is, perforce, globally well-defined. We then consider the model on the hyperbolic plane of critical curvature, where the vortex equations are integrable [7], and exact n -vortex solutions can be written down. By a careful analysis of the isometric action of $SL(2, \mathbb{R})$ on M_2 , we find an exact formula for its L^2 metric, generalizing results of Strachan [8], who computed the induced metric on two different two-dimensional submanifolds of M_2 . We then study RMG flow on the submanifold of centred two-vortices M_2^0 in detail, showing that, contrary to a claim of one of us in [9], this does *not* coincide with the intrinsic RMG flow on M_2^0 —in fact, the two flows exhibit qualitative differences.

We go on to study RMG flow on Rat_n , the space of degree n holomorphic maps $S^2 \rightarrow S^2$ (or, equivalently, the moduli space of n \mathbb{CP}^1 lumps on S^2) equipped with its L^2 metric. This geometry arises as the infinite electric charge limit of a certain semi-local vortex model [10,11], so the RMG flow may be relevant to the low energy dynamics of such vortices in the presence of a Chern–Simons term. However, our main interest in it concerns the question of completeness.

Since RMG flow proceeds with constant speed, it is immediate that RMG flow on any geodesically (or, equivalently, metrically) complete Kähler manifold is *complete*, that is, given any initial data $x \in M$, $v \in T_x M$, there is a corresponding RMG curve $\alpha : \mathbb{R} \rightarrow M$ (well-defined for all times $t \in \mathbb{R}$) with $\alpha(0) = x$, $\dot{\alpha}(0) = v$. The converse question is nontrivial, however. If a Kähler manifold is RMG complete, does it follow that it is geodesically complete? The time-scaling properties of RMG flow noted above led one of us to conjecture, in [9], that the answer is yes: if M is RMG complete, then all RMG_λ curves exist for all time and all λ , and RMG flow tends to geodesic flow as $\lambda \rightarrow 0$ (or, equivalently, as speed tends to infinity), so it seems plausible that geodesics should likewise exist for all time. In fact, this conjecture is false, and Rat_1 provides a counterexample: it is known [12] to be Kähler and geodesically incomplete but, as will be shown, is RMG complete. The point is that the Ricci curvature of Rat_1 grows unbounded as one approaches its boundary at infinity so, even though this boundary lies at finite distance, an unbounded “magnetic field” deflects any “charged” particle from hitting it in finite time. We conjecture that Rat_n is RMG complete for all $n \geq 2$ also, despite being geodesically incomplete [13], and present some evidence in favour of this conjecture.

The rest of this paper is structured as follows. In Section 2 we present some generalities on RMG flow on Kähler manifolds, including a useful symmetry reduction lemma. In Section 3 RMG flow on vortex moduli spaces is studied, first for vortices on \mathbb{R}^2 , then on the hyperbolic plane. In Section 4 RMG flow on Rat_n is studied, focusing on Rat_1 . Finally, Section 5 presents some concluding remarks.

2. RMG flow

We have already noted that RMG flow (like any magnetic geodesic flow) conserves speed. Since the Ricci form of a Kähler manifold is closed, one can locally express ρ as $d\mathcal{A}$, for some locally defined one-form \mathcal{A} on M . Then RMG flow has a local Lagrangian formulation,

$$L = \frac{1}{2} \|\dot{\alpha}(t)\|^2 - \lambda \mathcal{A}(\dot{\alpha}(t)), \quad (2.1)$$

that is, $\alpha : [a, b] \rightarrow M$ is RMG if and only if it locally extremizes $S = \int_a^b L dt$ among all paths with fixed endpoints. If $H^2(M) = 0$, as in all cases of interest in this paper, this formulation is actually global. We shall use this fact repeatedly.

Unlike geodesics, RMG curves are not invariant under time reversal, and local isometries do not necessarily map RMG curves to RMG curves. However, *holomorphic* local isometries do preserve RMG curves:

Proposition 1. *Let $\varphi : M \rightarrow N$ be a holomorphic local isometry between two Kähler manifolds M and N and $\alpha : I \rightarrow M$ be an RMG curve on M . Then, $\varphi \circ \alpha$ is an RMG curve on N .*

Proof. Let ∇ and $\bar{\nabla}$ be the Levi-Civita connexions with respect to the Kähler metrics g_M and g_N on M and N , respectively. Similarly, denote by ρ_M, ρ_N and J_M, J_N the Ricci forms and almost complex structures on M, N . Let $\alpha : I \rightarrow M$ be an RMG curve and $\tilde{\alpha} = \varphi \circ \alpha : I \rightarrow N$. Since $\varphi : M \rightarrow N$ is an isometry, then [14]

$$d\varphi(\nabla_{d/dt}^\alpha \dot{\alpha}) = \bar{\nabla}_{d/dt}^{\tilde{\alpha}} \dot{\tilde{\alpha}}. \quad (2.2)$$

Hence, for any $X \in \Gamma(TM)$,

$$\begin{aligned} g_N(\bar{\nabla}_{d/dt}^{\tilde{\alpha}} \dot{\tilde{\alpha}}, d\varphi X) &= g_N(d\varphi(\nabla_{d/dt}^\alpha \dot{\alpha}), d\varphi X) = g_M(\nabla_{d/dt}^\alpha \dot{\alpha}, X) \\ &= g_M(\lambda \sharp_M \iota_{\dot{\alpha}} \rho_M, X) \quad \text{since } \alpha \text{ is RMG} \\ &= \lambda \rho_M(\dot{\alpha}, X) = \lambda \text{Ric}_M(J_M \dot{\alpha}, X) \\ &= \lambda \text{Ric}_N(d\varphi J_M \dot{\alpha}, d\varphi X) \quad \text{since } \varphi \text{ is an isometry} \\ &= \lambda \text{Ric}_N(J_N d\varphi \dot{\alpha}, d\varphi X) \quad \text{since } \varphi \text{ is holomorphic} \\ &= \lambda \rho_N(\tilde{\alpha}, d\varphi X) = g_N(\lambda \sharp_N \iota_{\dot{\tilde{\alpha}}} \rho_N, d\varphi X). \end{aligned} \quad (2.3)$$

But g_N is nondegenerate and $d\varphi$ surjective, so $\tilde{\alpha}$ is RMG. \square

Corollary 2. Let M be a connected component of a fixed point set of a group of holomorphic isometries of a Kähler manifold \bar{M} . Then, any RMG curve α on \bar{M} with initial data $\dot{\alpha}(0) \in T_{\alpha(0)}M$ remains on M .

Proof. Let G be a group of holomorphic isometries from \bar{M} to itself and let M be a connected component of the fixed point set of G . Let also

$$V_p = \{u \in T_p \bar{M} : d\varphi_p u = u, \quad \forall \varphi \in G\}, \quad \forall p \in M. \quad (2.4)$$

We know that M is a totally geodesic submanifold of \bar{M} and $T_p M = V_p$ for all $p \in M$ [15]. Now, let $\alpha : I \rightarrow M$ be an RMG curve on \bar{M} with initial data

$$\alpha(0) = p \in M, \quad \dot{\alpha}(0) = v \in T_p M. \quad (2.5)$$

By Proposition 1, for all $\varphi \in G$, the curve $(\varphi \circ \alpha)(t)$ is RMG on \bar{M} . But its initial data are

$$(\varphi \circ \alpha)(0) = \varphi(p) = p \in M, \quad (d\varphi \dot{\alpha})(0) = d\varphi_p(v) = v \in T_p M. \quad (2.6)$$

Thus, both $\alpha(t)$ and $(\varphi \circ \alpha)(t)$ satisfy the RMG equation on \bar{M} with the same initial data, and so by standard existence and uniqueness theory for ODEs,

$$(\varphi \circ \alpha)(t) = \alpha(t), \quad \forall \varphi \in G. \quad (2.7)$$

Hence, $\alpha(t) \in M$ for all time. \square

Remark 3. One can see that the connected component M of the fixed point set of a group G of holomorphic isometries on a Kähler manifold \bar{M} is a complex submanifold of \bar{M} , and so is Kähler. This follows since for all $u \in T_p M = V_p$ and all $\varphi \in G$,

$$J_p u = J_p(d\varphi_p u) = d\varphi_p(J_p u), \quad (2.8)$$

so, $J_p u \in V_p = T_p M$ for all $u \in T_p M$. It follows that there are two different RMG flows on M : the original RMG flow on $\bar{M} \supset M$, which preserves M , and the RMG flow on M defined by its own Ricci form ρ_M . We shall call these the *extrinsic* and *intrinsic* RMG flows on M respectively. Since $\iota^* \rho_{\bar{M}} \neq \rho_M$ in general (where $\iota : M \rightarrow \bar{M}$ denotes inclusion), these two flows on M do not coincide in general.

Remark 4. For two-dimensional Kähler manifolds, the RMG equation (1.1) simplifies to

$$\nabla_{d/dt}^\alpha \dot{\alpha} = \lambda \frac{S}{2} J \dot{\alpha}, \quad (2.9)$$

where S denotes the scalar curvature of M . Choosing $\|\dot{\alpha}\| = 1$ for convenience, one sees that RMG_λ curves are precisely those curves whose geodesic curvature is λ times the Gauss curvature of M .

3. RMG motion of vortices

The field theory of interest is defined on spacetime \mathbb{R}^3 given a Lorentzian metric $\eta = dt^2 - \Omega(x, y)^2(dx^2 + dy^2)$. The conformal factor Ω will later be chosen so that the spacelike slice $t = 0$ is either the Euclidean plane or the hyperbolic plane, but it is convenient to leave it arbitrary at first. The theory has, like the abelian Higgs model, a complex scalar field ϕ

minimally coupled to a $U(1)$ gauge connexion $A = A_\mu dx^\mu$. It has, in addition, a neutral (real) scalar field N . Its Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \left(D_\mu \phi \overline{D^\mu \phi} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \kappa \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \partial_\mu N \partial^\mu N - \frac{1}{4} (|\phi|^2 - 1 - 2\kappa N)^2 + |\phi|^2 N^2 \right) \quad (3.1)$$

where $D\phi = d\phi - iA\phi$, $F = dA$ and κ is a real parameter (the Chern–Simons constant) which, at the cost of the redefinitions $t \mapsto -t$, $N \mapsto -N$ if necessary, we may assume is non-negative.

In order to have finite total energy

$$E = \frac{1}{2} \int \left(\Omega^2 |D_0 \phi|^2 + |D_1 \phi|^2 + |D_2 \phi|^2 + \Omega^{-2} F_{12}^2 + F_{0i} F_{0i} + \Omega^2 \partial_0 N^2 + \partial_i N \partial_i N + \frac{1}{4} \Omega^2 (|\phi|^2 - 1 - 2\kappa N)^2 - \Omega^2 |\phi|^2 N^2 \right) dx dy \quad (3.2)$$

the fields ϕ , N must have boundary behaviour $|\phi| \rightarrow 1$, $N \rightarrow 0$, or $\phi \rightarrow 0$, $N \rightarrow -(2\kappa)^{-1}$ as $r = \sqrt{x^2 + y^2} \rightarrow \infty$. We choose the first possibility, as this allows vortex solutions. Then, as usual [5], the Higgs field at spatial infinity winds some integer n times around the unit circle in \mathbb{C} , and the total magnetic flux of the field is quantized

$$\int B dx dy = 2\pi n \quad (3.3)$$

where the magnetic field is $B = F_{12}$. There is a Bogomol'nyi argument [2] which shows that among all stationary fields (meaning $\partial_0 N = 0$, $\partial_0 \phi = 0$) of winding n ,

$$E \geq \pi n \quad (3.4)$$

with equality if and only if

$$(D_1 \pm iD_2)\phi = 0 \quad B \pm \frac{\Omega^2}{2} (|\phi|^2 - 1 - 2\kappa N) = 0 \quad (3.5)$$

$$A_0 \mp N = 0 \quad \partial_i \partial_i A_0 - \Omega^2 |\phi|^2 A_0 - \kappa B = 0, \quad (3.6)$$

where the upper (lower) signs apply if n is positive (negative). A formal index calculation indicates the space of (gauge equivalence classes of) winding n solutions of (3.5), (3.6) has real dimension $2n$ [16].

The top two equations (3.5) reduce to the usual Bogomol'nyi equations for vortices when $\kappa = 0$, and in this case the bottom two (3.6) are trivially satisfied by $A_0 = N = 0$. It follows that, when $\kappa = 0$, the moduli space of winding n solutions of (3.5), (3.6) is precisely M_n , the space of abelian Higgs n -vortices [17]. Recall that such vortices are in one-to-one correspondence with unordered n -tuples of points in $\mathbb{R}^2 = \mathbb{C}$, the zeros, with multiplicity, of the field ϕ , and that their low-energy dynamics is governed by geodesic motion in M_n with respect to γ_{L^2} , the L^2 metric [8,3]. There is a useful semi-explicit formula for this metric due to Strachan [8] (on the hyperbolic plane) and Samols [3] (on the Euclidean plane). Let Δ_n denote the subset of M_n on which two or more vortex positions (zeros of ϕ) coincide. Then on $M_n \setminus \Delta_n$ we may use the zeros of ϕ , $(z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ as local complex coordinates for M_n . For a fixed set of vortex positions, we can expand $\log |\phi(z)|^2$ in a neighbourhood of each z_r , $r = 1, \dots, n$,

$$\log |\phi(z)|^2 = \log |z - z_r|^2 + a_r + \frac{1}{2} \{ b_r(z - z_r) + \bar{b}_r(\bar{z} - \bar{z}_r) \} + \dots \quad (3.7)$$

where b_r are n unknown complex functions of (z_1, \dots, z_n) , and a_r are, similarly, unknown real functions. Then the metric on $M_n \setminus \Delta_n$ is

$$\gamma_{L^2} = \pi \sum_{r,s=1}^n \left(\Omega^2 \delta_{rs} + 2 \frac{\partial b_s}{\partial z_r} \right) dz_r d\bar{z}_s. \quad (3.8)$$

Following Kim and Lee [6] and Collie and Tong [1] we assume that, for $\kappa > 0$ but small, n -vortex solutions of (3.5), (3.6) remain in bijective correspondence with unordered n -tuples of points in \mathbb{C} , and hence with points in M_n , and that their low energy dynamics is described by some perturbed geodesic motion in (M_n, γ_{L^2}) . Collie and Tong propose RMG_λ flow on M_n with $\lambda = 2\pi\kappa$. Before examining this flow in detail, we consider Kim and Lee's earlier proposal.

3.1. Kim–Lee flow on M_2

Motivated by a direct perturbative calculation, Kim and Lee [6] propose that low-energy vortex dynamics on the Euclidean plane, for small κ , is described by motion on M_n governed by the Lagrangian

$$L = \frac{1}{2} \gamma_{L^2}(\dot{\alpha}, \dot{\alpha}) + \mathcal{A}_1(\dot{\alpha}) + \mathcal{A}_2(\dot{\alpha}) \quad (3.9)$$

where $\mathcal{A}_1, \mathcal{A}_2$ are two one-forms on M_n , proportional to κ . This, then, is magnetic geodesic motion on M_n in the effective magnetic field $\mathcal{B} = d\mathcal{A}_1 + d\mathcal{A}_2$. On $M_n \setminus \Delta_n$, the one forms $\mathcal{A}_1, \mathcal{A}_2$ are, in terms of the (unknown) functions b_r ,

$$\mathcal{A}_1 = i\frac{\pi\kappa}{2} \left\{ \sum_r (b_r dz_r - \bar{b}_r d\bar{z}_r) - 2 \sum_{r,s \neq r} \left(\frac{dz_r}{z_r - z_s} - \frac{d\bar{z}_r}{\bar{z}_r - \bar{z}_s} \right) \right\} \quad (3.10)$$

$$\mathcal{A}_2 = i\frac{\pi\kappa}{8} \sum_r (H_r dz_r - \bar{H}_r d\bar{z}_r) \quad (3.11)$$

where

$$H_r = -b_r + \sum_{s \neq r} \left\{ (z_r - z_s) \frac{\partial b_r}{\partial z_s} + (\bar{z}_r - \bar{z}_s) \frac{\partial b_r}{\partial \bar{z}_s} \right\}. \quad (3.12)$$

These formulae simplify considerably in the case $n = 2$ (two-vortex dynamics). On $M_2 \setminus \Delta_2$ we define the centre of mass and relative coordinates

$$Z = \frac{1}{2}(z_1 + z_2), \quad \zeta = \sigma e^{i\theta} = \frac{1}{2}(z_1 - z_2)/2 \quad (3.13)$$

respectively. It is known [3] that b_1, b_2 are functions of ζ only, and that

$$b_1(\zeta) = b(\sigma)e^{-i\theta} = -b_2(\zeta) \quad (3.14)$$

where $b(\sigma)$ is some smooth real function on $(0, \infty)$ with the asymptotic behaviour

$$b(\sigma) = \frac{1}{\sigma} - \frac{1}{2}\sigma + O(\sigma^2) \quad (3.15)$$

as $\sigma \rightarrow 0$. Substituting (3.14) into (3.10), (3.11) one sees that

$$\begin{aligned} \mathcal{A}_1 &= 2\pi\kappa [1 - \sigma b(\sigma)] d\theta, \\ \mathcal{A}_2 &= \frac{\pi}{2} \kappa \sigma^2 b'(\sigma) d\theta. \end{aligned} \quad (3.16)$$

It follows that the effective magnetic field is

$$\mathcal{B} = d(\mathcal{A}_1 + \mathcal{A}_2) = f(\sigma) d\sigma \wedge \sigma d\theta \quad (3.17)$$

where

$$f(\sigma) = \frac{\pi\kappa}{\sigma} \frac{d}{d\sigma} \left(-2\sigma b(\sigma) + \frac{1}{2}\sigma^2 b'(\sigma) \right). \quad (3.18)$$

This formula defines the magnetic field, and hence the flow, on $M_2 \setminus \Delta_2$. In order that the flow be well defined on the whole of M_2 , the two-form \mathcal{B} should extend (at least) continuously to Δ_2 . We now show that \mathcal{B} does *not* so extend.

Note that, by virtue of (3.15), $f(\sigma) = \frac{3}{2}\pi\kappa + O(\sigma^2)$ as $\sigma \rightarrow 0$, that is, as the point in M_2 approaches Δ_2 . Recall [3] that ζ is not a globally well-defined coordinate on M_2 because (Z, ζ) and $(Z, -\zeta)$ correspond to exactly the same point in M_2 . To extend any geometric object on M_2 over the coincidence set Δ_2 , we must use the global complex coordinates $Z, w = \zeta^2$. But then

$$\mathcal{B} = f(|w|^{1/2}) \frac{i}{8|w|} dw \wedge d\bar{w} = \frac{3}{2}\pi\kappa \left(\frac{1}{|w|} + O(1) \right) \frac{i}{8} dw \wedge d\bar{w} \quad (3.19)$$

as $|w| \rightarrow 0$. Hence \mathcal{B} blows up on Δ_2 , which calls into question the self-consistency of Kim and Lee's perturbative calculation [6].

Since γ_{12} extends smoothly over Δ_n to give a global Kähler metric on M_n , RMG flow is globally well-defined on M_n . It follows that the Kim–Lee flow cannot, as claimed in [1], coincide with RMG flow.

3.2. The metric on $M_2(\mathbb{H}^2)$

If we wish to study RMG motion of two-vortices on the Euclidean plane, we need the coefficient function $b(\sigma)$ introduced in (3.14), for which no explicit formula is known (although a conjectural large σ asymptotic formula is known [18]). One must resort to numerics even to construct the metric on M_2 , therefore [3]. Matters improve considerably if we consider vortices moving instead on the *hyperbolic* plane with scalar curvature -1 , since the Bogomol'nyi equations (for $\kappa = 0$) are then integrable [7], and the semi-explicit formula for γ_{12} (3.8) can, in some nontrivial cases, be made fully explicit [8]. In this section we will derive an explicit formula for the metric on M_2 . Rather than appealing to (3.8) directly, we will analyse the class of Kähler metrics on M_2 with the same isometries as γ_{12} . This space of metrics is infinite dimensional, but the L^2 metric

is uniquely determined by its restriction to a certain pair of two-dimensional submanifolds of M_2 , and these restrictions are already known [8].

Let \tilde{M}_2 denote the double cover of $M_2 \setminus \Delta_2$, that is, $\tilde{M}_2 = \mathbb{H}^2 \times \mathbb{H}^2 \setminus \{(z, z)z \in \mathbb{H}^2\}$, and $\tilde{\gamma}$ be the pullback of γ_{l^2} to \tilde{M}_2 by the covering map. It is convenient to use the upper half plane model for \mathbb{H}^2 , that is, $\mathbb{H}^2 = \{x + iy \in \mathbb{C} : y > 0\}$ with the Riemannian metric

$$g = \frac{2}{y^2} (dx^2 + dy^2). \quad (3.20)$$

Then there is an isometric action of the projective real linear group $PL(2, \mathbb{R})$ on \mathbb{H}^2 , defined by

$$z \rightarrow \frac{az + b}{cz + d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot z =: M \odot z, \quad (3.21)$$

where $[M] \in PL(2, \mathbb{R})$. This induces an isometric action on $(\tilde{M}_2, \tilde{\gamma})$,

$$(z_1, z_2) \rightarrow (M \odot z_1, M \odot z_2). \quad (3.22)$$

For a generic element of \tilde{M}_2 , the isotropy group of $PL(2, \mathbb{R})$ is trivial. By the Orbit-Stabilizer Theorem [19], it follows that each generic orbit is diffeomorphic to $PL(2, \mathbb{R})$ itself. Hence, the isometric action of $PL(2, \mathbb{R})$ on \tilde{M}_2 has cohomogeneity 1, that is, all generic orbits are submanifolds of \tilde{M}_2 with real codimension 1. Let s denote the distance between two vortices in \mathbb{H}^2 . Then each orbit has a unique element $w_s = (ie^{s/2}, ie^{-s/2}) \in \tilde{M}_2$. Thus, this action decomposes \tilde{M}_2 into a one parameter family of orbits parameterized by $s > 0$, that is, $\tilde{M}_2 \cong (0, \infty) \times PL(2, \mathbb{R})$.

Consider the coframe $\{ds, \sigma_k : k = 1, 2, 3\}$ on \tilde{M}_2 where σ_k are the left-invariant 1-forms dual to the basis $\{e_k : k = 1, 2, 3\}$ of $\mathfrak{p} := T_{[\mathbb{I}_2]}PL(2, \mathbb{R})$ given by

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.23)$$

Any $PL(2, \mathbb{R})$ -invariant metric $\tilde{\gamma}$ on \tilde{M}_2 is determined by a one-parameter family of symmetric bilinear forms $\tilde{\gamma}_s : V_s \times V_s \rightarrow \mathbb{R}$ where $V_s := \partial/\partial s \oplus \mathfrak{p}$ is the tangent space to \tilde{M}_2 at the element w_s .

In terms of the complex coordinate system (z_1, z_2) on \tilde{M}_2 , where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, one can obtain that

$$\begin{aligned} e_1 &= (1 + e^s) \frac{\partial}{\partial x_1} + (1 + e^{-s}) \frac{\partial}{\partial x_2}, & e_2 &= (1 - e^s) \frac{\partial}{\partial x_1} + (1 - e^{-s}) \frac{\partial}{\partial x_2}, \\ e_3 &= 2e^{s/2} \frac{\partial}{\partial y_1} + 2e^{-s/2} \frac{\partial}{\partial y_2}, & \frac{\partial}{\partial s} &= \frac{1}{2}e^{s/2} \frac{\partial}{\partial y_1} - \frac{1}{2}e^{-s/2} \frac{\partial}{\partial y_2}. \end{aligned} \quad (3.24)$$

Hence, the almost complex structure J on \tilde{M}_2 acts as

$$\begin{aligned} Je_1 &= \cosh(s/2) e_3, & Je_2 &= -4 \sinh(s/2) \frac{\partial}{\partial s}, \\ Je_3 &= -\frac{1}{\cosh(s/2)} e_1, & J \frac{\partial}{\partial s} &= \frac{1}{4 \sinh(s/2)} e_2. \end{aligned} \quad (3.25)$$

In addition to the $PL(2, \mathbb{R})$ isometric action on \tilde{M}_2 , there is a discrete isometry on \tilde{M}_2 defined as $P : (z_1, z_2) \rightarrow (z_2, z_1)$. Hence, the group $G := PL(2, \mathbb{R}) \times \{\text{Id}, P\}$, where Id is the identity map, acts isometrically on \tilde{M}_2 .

Proposition 5. Let $\tilde{\gamma}$ be a G -invariant Kähler metric on \tilde{M}_2 . Then, there exists a smooth function $A : (0, \infty) \rightarrow \mathbb{R}$ such that

$$\tilde{\gamma} = A_1(s) ds^2 + A_2(s) \sigma_1^2 + A_3(s) \sigma_2^2 + A_4(s) \sigma_3^2, \quad (3.26)$$

where $A_1(s), \dots, A_4(s)$ are related to $A(s)$ by

$$\begin{aligned} A_1 &= \frac{1}{8 \sinh(s/2)} \frac{d}{ds} \left(\frac{A(s)}{\cosh(s/2)} \right), & A_2 &= A(s), \\ A_3 &= 2 \sinh(s/2) \frac{d}{ds} \left(\frac{A(s)}{\cosh(s/2)} \right), & A_4 &= \frac{A(s)}{\cosh^2(s/2)}. \end{aligned} \quad (3.27)$$

Proof. With respect to the coframe $\{ds, \sigma_k : k = 1, 2, 3\}$ on \tilde{M}_2 , any $PL(2, \mathbb{R})$ -invariant symmetric $(0, 2)$ tensor has the form

$$\begin{aligned} \tilde{\gamma} &= A_1 ds^2 + A_2 \sigma_1^2 + A_3 \sigma_2^2 + A_4 \sigma_3^2 + 2A_5 ds \sigma_1 + 2A_6 ds \sigma_2 + 2A_7 ds \sigma_3 \\ &\quad + 2A_8 \sigma_1 \sigma_2 + 2A_9 \sigma_1 \sigma_3 + 2A_{10} \sigma_2 \sigma_3, \end{aligned} \quad (3.28)$$

where A_1, \dots, A_{10} are smooth functions of s only. The metric is also invariant under the isometry P , which swaps the two points in \mathbb{H}^2 . For a given non-coincident pair of points in \mathbb{H}^2 , this transposition can be accomplished by acting with an isometry in $PL(2, \mathbb{R})$. For the point w_s , we must act with

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3.29)$$

and hence, to swap the pair of points $g \cdot w_s$, where $g \in PL(2, \mathbb{R})$, we must act with gQg^{-1} . Hence, in terms of the coordinates $s \in (0, \infty)$, $g \in PL(2, \mathbb{R})$ the isometry P is

$$P : (s, g) \mapsto (s, gQg^{-1}g) = (s, gQ), \quad (3.30)$$

that is, right multiplication on $PL(2, \mathbb{R})$ by Q . The induced action of P on \mathfrak{p} , identified with the space of left-invariant vector fields on $PL(2, \mathbb{R})$, is $X \mapsto dP(X) = Q^{-1}XQ$, so $e_1 \mapsto -e_1$, $e_2 \mapsto e_2$ and $e_3 \mapsto -e_3$. Hence

$$P^*\sigma_1 = -\sigma_1, \quad P^*\sigma_2 = \sigma_2, \quad P^*\sigma_3 = -\sigma_3, \quad \text{and} \quad P^*ds = ds. \quad (3.31)$$

Now $P^*\tilde{\gamma} = \tilde{\gamma}$, so we conclude that $A_5 = A_7 = A_8 = A_{10} = 0$.

Now, since $\tilde{\gamma}$ is Hermitian, $\tilde{\gamma}_s(u, v) = \tilde{\gamma}_s(Ju, Jv)$ for all $u, v \in V_s$, whence, using (3.24),

$$A_3 \equiv 16 \sinh^2(s/2)A_1, \quad A_4 \equiv \frac{A_2}{\cosh^2(s/2)}, \quad A_9 \equiv A_6 \equiv 0. \quad (3.32)$$

The Kähler form $\omega(\cdot, \cdot) = \tilde{\gamma}(J\cdot, \cdot)$ on \tilde{M}_2 is, therefore,

$$\omega = 4 \sinh(s/2)A_1 ds \wedge \sigma_2 + \frac{A_2}{\cosh(s/2)} \sigma_1 \wedge \sigma_3. \quad (3.33)$$

Hence,

$$d\omega = -\left[8 \sinh(s/2)A_1 - \frac{d}{ds}\left(\frac{A_2}{\cosh(s/2)}\right)\right] ds \wedge \sigma_2 \wedge \sigma_3, \quad (3.34)$$

where we have used the fact that, for our chosen frame/coframe for $PL(2, \mathbb{R})$,

$$d\sigma_i(e_j, e_k) = e_j[\sigma_i(e_k)] - e_k[\sigma_i(e_j)] - \sigma_i([e_j, e_k]) = 0 - 0 - \sigma_i([e_j, e_k]|_{\mathfrak{p}}), \quad (3.35)$$

whence

$$d\sigma_1 = 2 \sigma_2 \wedge \sigma_3, \quad d\sigma_2 = 2 \sigma_1 \wedge \sigma_3, \quad d\sigma_3 = 2 \sigma_1 \wedge \sigma_2. \quad (3.36)$$

Since $\tilde{\gamma}$ is Kähler, $d\omega = 0$, so

$$\frac{d}{ds}\left(\frac{A_2}{\cosh(s/2)}\right) - 8 \sinh(s/2)A_1 = 0. \quad (3.37)$$

So A_1, A_3, A_4 are uniquely determined by the single function $A(\lambda) = A_2(\lambda)$, by (3.37), (3.32) as claimed. \square

Remark 6. We can equally well think of (3.26) as a formula for a general $PL(2, \mathbb{R})$ invariant Kähler metric on M_2 . This space decomposes into a one-parameter family of orbits, parameterized by $s \in [0, \infty)$. Generic orbits are diffeomorphic to $PL(2, \mathbb{R})/K$ where K denotes the subgroup $\{\mathbb{I}_2, Q\}$, and there is an exceptional orbit at $s = 0$ diffeomorphic to \mathbb{H}^2 (the submanifold of coincident vortices). In this picture, one should interpret $\sigma_1^2, \sigma_2^2, \sigma_3^2$ as $Ad(K)$ -invariant symmetric bilinear forms on $\mathfrak{p} = T_{[\mathbb{I}_2]}(PL(2, \mathbb{R})/K)$ (note that σ_1 and σ_3 are not $Ad(K)$ -invariant, so are not well-defined one-forms on $PL(2, \mathbb{R})/K$).

Now, consider the holomorphic isometry of $(\tilde{M}_2, \tilde{\gamma})$ defined by

$$\tilde{K} : (z_1, z_2) \rightarrow \left(-\frac{1}{z_2}, -\frac{1}{z_1}\right). \quad (3.38)$$

The fixed point set in \tilde{M}_2 of this isometry is

$$\tilde{M}_2^0 = \left\{\left(\xi, -\frac{1}{\xi}\right) : \xi \in \mathbb{H}^2\right\} \subset \tilde{M}_2. \quad (3.39)$$

Clearly, \tilde{M}_2^0 is a non-compact 1-dimensional complex submanifold of \tilde{M}_2 . This is the (double cover of the) 2-vortex relative moduli space. The induced metric on \tilde{M}_2^0 is

$$\tilde{\gamma}^0 = A_1(s) ds^2 + A_3(s) \sigma_2^2 = A_1(s) (ds^2 + 16 \sinh^2(s/2) \sigma_2^2). \quad (3.40)$$

Corollary 7. The L^2 metric on the moduli space M_2 is

$$\tilde{\gamma}_{12} = A_1(s) ds^2 + A_2(s) \sigma_1^2 + A_3(s) \sigma_2^2 + A_4(s) \sigma_3^2, \quad (3.41)$$

where A_1, \dots, A_4 are functions of s only determined as in (3.27) by

$$A_{12}(s) = 8\pi \left(1 + \cosh^2(s/2) + 2\sqrt{\cosh^2(s/2) + \sinh^4(s/2)} \right). \quad (3.42)$$

Proof. The lifted L^2 metric on \tilde{M}_2 is a G -invariant Kähler metric, and so is covered by Proposition 5. Hence, we only need to determine the function $A(s)$ of the L^2 metric.

An explicit formula for the induced L^2 metric on the relative moduli space \tilde{M}_2^0 has been determined by Strachan in [8], and rederived and generalized in [9], which uses the same conventions for the abelian Higgs model that we are using. Comparing the formula in [9] with (3.40), we deduce that

$$A_1(s) = 2\pi \frac{\tanh^2(s/4)}{(1 + \tanh^2(s/4))^2} \left[1 + \frac{4(1 + \tanh^4(s/4))}{\sqrt{1 + \tanh^8(s/4) + 14 \tanh^4(s/4)}} \right], \quad (3.43)$$

$$= \frac{\pi}{2} \tanh^2(s/2) \left[1 + \frac{2(\cosh^2(s/2) + 1)/\sinh^2(s/2)}{\sqrt{[\cosh(s/2)/\sinh^2(s/2)]^2 + 1}} \right], \quad (3.44)$$

where we have used the hyperbolic double-angle formulae. Eq. (3.37) now gives a differential equation for $A(\lambda) = A_2$, whose general solution is

$$A(s) = 8\pi \left((\cosh^2(s/2) + 1) + 2 \sinh^2(s/2) \sqrt{[\cosh(s/2)/\sinh^2(s/2)]^2 + 1} \right) + c \cosh(s/2), \quad (3.45)$$

where c is an integration constant.

Strachan also gave an explicit formula for the induced L^2 metric on M_n^{co} , the two-dimensional submanifold of M_n consisting of entirely coincident vortices. By symmetry, this metric must be homothetic to the (physical) metric g on \mathbb{H}^2 (3.20). In fact [8]

$$\gamma_{co} = \frac{1}{2} \pi n(n+2)g \quad (3.46)$$

in our conventions. Consider X , the killing vector field on M_2 generated by $e_1 \in \mathfrak{p}$. Its squared length, with respect to γ_{12} , at the point (i, i) (that is, the coincident two-vortex positioned at $i \in \mathbb{H}^2$) is, by definition, $A_2(0)$. $X((i, i))$ is clearly tangent to M_2^{co} and moves the coincident two-vortex through $x + iy = i$ with initial velocity vector $2\partial/\partial y$, and hence with squared speed 8 (with respect to the metric g). Hence, by (3.46),

$$\gamma_{co}(X, X) = 32\pi. \quad (3.47)$$

Comparing with (3.45), we deduce that $c = 0$, which completes the proof. \square

Proposition 8. Let $\tilde{\gamma}$ be a G -invariant Kähler metric on \tilde{M}_2 , determined as in Proposition 5 by a function $A(s)$. Then, its Ricci curvature tensor is

$$\text{Ric} = C_1(s) ds^2 + C_2(s) \sigma_1^2 + C_3(s) \sigma_2^2 + C_4(s) \sigma_3^2, \quad (3.48)$$

where C_1, \dots, C_4 are smooth functions of s only, defined as in (3.27), by a single function $C(s)$ given by

$$C(s) = -4 \sinh(s/2) \cosh(s/2) \frac{d}{ds} \log \left(\frac{A_1 A_2}{\cosh^2(s/2)} \right) - 8 \cosh^2(s/2). \quad (3.49)$$

Proof. The Ricci curvature tensor with respect to $\tilde{\gamma}$ is a G -invariant symmetric $(0, 2)$ tensor on \tilde{M}_2 which is Hermitian and whose associated 2-form $\rho(\cdot, \cdot) = \text{Ric}(J\cdot, \cdot)$, the Ricci form, is closed. Thus, it is covered by Proposition 5, that is, Ric has the same structure as $\tilde{\gamma}$ and its coefficients C_1, \dots, C_4 are related, as in (3.27), to the function $C(s) := C_2(s) = \text{Ric}_s(e_1, e_1)$. Introducing an orthonormal basis $\{E_k : k = 0, 1, 2, 3\}$ of (V_s, γ_s) as

$$E_0 = \frac{1}{\sqrt{A_1}} \frac{\partial}{\partial s}, \quad E_1 = \frac{1}{\sqrt{A_2}} e_1, \quad E_2 = \frac{1}{\sqrt{A_3}} e_2, \quad E_3 = \frac{1}{\sqrt{A_4}} e_3, \quad (3.50)$$

then, by the definition of the Ricci curvature tensor, we obtain that

$$\begin{aligned}\operatorname{Ric}_s(e_1, e_1) &= \sum_{i=0}^3 \tilde{\gamma}_s(R(E_i, e_1)e_1, E_i), \\ &= -4 \sinh(s/2) \cosh(s/2) \frac{d}{ds} \log\left(\frac{A_1 A_2}{\cosh^2(s/2)}\right) - 8 \cosh^2(s/2),\end{aligned}\quad (3.51)$$

where R is the Riemannian curvature tensor associated with $\tilde{\gamma}$. Hence, the claim is proved. \square

The Ricci form ρ on M_2 has the same structure as the Kähler form ω , that is,

$$\rho = 4 \sinh(s/2) C_1 ds \wedge \sigma_2 + \frac{C_2}{\cosh(s/2)} \sigma_1 \wedge \sigma_3. \quad (3.52)$$

Since M_2 has trivial second cohomology, this (closed) form must be exact. Rewriting C_1 in terms of C one sees that

$$\rho = d\left(\frac{C(s)}{2 \cosh(s/2)} \sigma_2\right). \quad (3.53)$$

3.3. RMG motion on the reduced moduli space

The 2-vortex relative moduli space M_2^0 is the fixed point set in \tilde{M}_2 of the holomorphic isometry \tilde{K} , defined in (3.38). Hence, by Corollary 2, RMG curves with initial data in TM_2^0 remain on M_2^0 for all time. So, RMG flow localizes to M_2^0 . However, the restriction of the Ricci form ρ on M_2 to M_2^0 does not coincide with ρ^0 , the Ricci form on M_2^0 defined by its induced metric γ^0 . Hence, the RMG flow on M_2^0 , thought of as a submanifold of M_2 , does not coincide with the RMG flow on M_2^0 , thought of as a Kähler manifold in its own right. Here, we will compare these flows on M_2^0 , which we call the extrinsic and intrinsic RMG flows, respectively.

It follows from Corollary 7 and Proposition 8 that the restricted and intrinsic Ricci forms on M_2^0 are

$$\rho| = F(s) ds \wedge \sigma_2, \quad \rho^0 = F^0(s) ds \wedge \sigma_2 \quad (3.54)$$

where

$$F(s) = F^0(s) - \frac{d}{ds} \left[2 \sinh(s/2) \frac{d}{ds} \log\left(\frac{A(s)}{\cosh^2(s/2)}\right) \right] - \sinh(s/2) \quad (3.55)$$

$$F^0(s) = -\frac{d}{ds} \left[2 \sinh(s/2) \frac{d}{ds} \log\left(\frac{d}{ds} \left(\frac{A(s)}{\cosh(s/2)}\right)\right) \right] - \frac{1}{2} \sinh(s/2). \quad (3.56)$$

In the case of the L^2 metric, these functions behave asymptotically like

$$F| \sim -\frac{1}{5}s^3, \quad F^0 \sim \frac{7}{40}s^3, \quad \text{as } s \rightarrow 0, \quad (3.57)$$

$$F| \sim -e^{s/2}, \quad F^0 \sim -\frac{1}{2}e^{s/2}, \quad \text{as } s \rightarrow \infty. \quad (3.58)$$

From (3.58), one can see that even as $s \rightarrow \infty$, the restricted and intrinsic Ricci forms do not coincide. Comparing $F|$ with F^0 in (3.58), one expects that the extrinsic and intrinsic RMG flows coincide for large s if the RMG parameters in each are related by

$$\lambda_{\text{extrinsic}} = \frac{1}{2} \lambda_{\text{intrinsic}}. \quad (3.59)$$

Henceforth, when comparing the two flows we choose their parameters to be related in this fashion. In this case, one expects the RMG flows in the core region of M_2^0 (i.e. for s small) to be qualitatively quite different, since $F|$ is uniformly negative, while F^0 is positive for s small and negative for s large. Fig. 1 compares $F|$ and $2F^0$.

Magnetic geodesic flow on M_2^0 with respect to a rotationally invariant effective magnetic field $\mathcal{B} = F(s) ds \wedge \sigma_2$ is governed by the ODE system

$$\begin{aligned}\ddot{s} &= -\frac{1}{2A_1(s)} [A'_1(s)\dot{s}^2 - A'_3(s)\dot{\psi}^2 + 2F(s)\dot{\psi}], \\ \ddot{\psi} &= -\frac{1}{A_3(s)} [A'_3(s)\dot{s}\dot{\psi} - F(s)\dot{s}],\end{aligned}\quad (3.60)$$

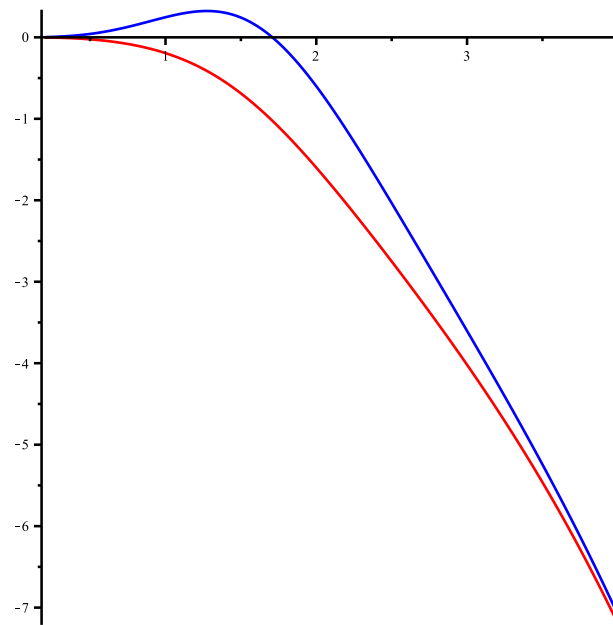


Fig. 1. Comparison of the restricted and intrinsic Ricci forms on the space of centred hyperbolic two vortices: plots of $F(s)$ (red) and $2F^0(s)$ (blue). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

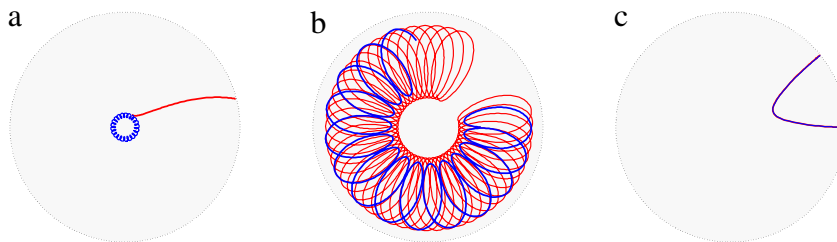


Fig. 2. Plots of vortex trajectories under the extrinsic RMG flow on M_2^0 (red) and the intrinsic RMG flow (blue) with $\lambda_{\text{intrinsic}} = 2\lambda_{\text{extrinsic}}$ and various initial values. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

where ψ is an angular coordinate chosen so that $d\psi = \sigma_2$. Choosing $F(s)$ to be $F(s)$ or $2F^0(s)$ we obtain the extrinsic and intrinsic RMG flows, respectively, normalized so as to coincide asymptotically at large s . We have solved these ODE systems numerically for various initial values. The corresponding RMG trajectories of one of the vortices on the Poincaré disk are depicted in Fig. 2. As expected, RMG trajectories which reach the core region of M_2^0 exhibit marked differences in the two flows.

A related observation concerns orbiting vortex pairs. It is immediate from (3.60) that a circle of constant s is a closed magnetic geodesic if and only if it is traversed at constant angular velocity

$$\dot{\psi} = v(s) = 2 \frac{F(s)}{A'_3(s)}. \quad (3.61)$$

This, then, gives the frequency-separation relation for a pair of vortices orbiting one another at fixed separation. Note that $A'_3(s) > 0$ for all $s > 0$. From Fig. 1, one sees that (for $\lambda > 0$) vortex pairs obeying the intrinsic RMG flow orbit one another anticlockwise for $s < s_0 \approx 1.7$ and clockwise for $s > s_0$, whereas orbiting vortex pairs always circulate clockwise in the extrinsic flow.

Unlike geodesics, the features of RMG flow on a fixed point set of a holomorphic isometry, such as M_2^0 , cannot be deduced by knowing only the metric on the fixed point set. This difference makes RMG flow significantly harder to study than geodesic flow and means that studies of intrinsic RMG flow on low-dimensional submanifolds, such as those presented in [9], are of limited value in trying to understand the true (extrinsic) RMG flow.

4. RMG motion of \mathbb{CP}^1 lumps

As observed in Section 1, the question of *completeness* of RMG flow on a noncompact Kähler manifold is interesting and nontrivial. Certainly, if the manifold is complete (as a metric space or, equivalently, its geodesic flow is complete), then it is

RMG complete since RMG flow conserves speed (so an RMG curve which escaped every compact set in bounded time would define a divergent Cauchy sequence). Since RMG flow converges (pointwise) to geodesic flow in the limit of large speed, it has been conjectured that the converse holds also: if a Kähler manifold is RMG complete, then it is geodesically complete [9]. In fact this is false, and in this section we provide a counterexample of independent interest: the moduli space of unit charge \mathbb{CP}^1 lumps on S^2 , or, equivalently, the space Rat_1 of degree one rational maps $S^2 \rightarrow S^2$, given its L^2 metric.

Recall that Rat_n is the space of degree n holomorphic maps $S^2 \rightarrow S^2$. If one chooses stereographic coordinates z, W on the domain and codomain, such maps take the form

$$W(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_n z^n} \quad (4.1)$$

where $a_0, \dots, a_n, b_0, \dots, b_n \in \mathbb{C}$ are constant. There is a natural inclusion $\text{Rat}_n \hookrightarrow \mathbb{CP}^{2n+1}$ defined by $W(z) \mapsto [a_0, \dots, a_n, b_0, \dots, b_n]$, which equips Rat_n with a complex structure, and a natural metric on Rat_n defined by restricting the L^2 norm on $W^{-1}TS^2$ to $T\text{Rat}_n$. It is known [12] that Rat_n is Kähler with respect to this metric and complex structure. Further, Rat_1 is diffeomorphic to $SO(3) \times \mathbb{R}^3$. One may regard $\mathcal{O} \in SO(3)$ as parametrizing the internal orientation of the lump and $\lambda \in \mathbb{R}^3$ as parametrizing both its sharpness, $\lambda = |\lambda|$, and its position in physical space $-\lambda/|\lambda| \in S^2$. Lumps with $\lambda = \mathbf{0}$ have sharpness 0, that is, constant energy density, so do not have a well-defined position. Explicitly, the point $(\mathbb{I}_3, (0, 0, \lambda))$ corresponds to the rational map

$$W(z) = \mu(\lambda)z, \quad \text{where } \mu = \frac{\Lambda + \lambda}{\Lambda - \lambda} \quad \text{and} \quad \Lambda = \sqrt{1 + \lambda^2}, \quad (4.2)$$

and every other point in Rat_1 can be reached from a point such as this by acting with some isometry: $G = SO(3) \times SO(3)$ acts isometrically on Rat_1 via

$$(\mathcal{L}, \mathcal{R}) : (\mathcal{O}, \lambda) \mapsto (\mathcal{L}\mathcal{O}\mathcal{R}^{-1}, \mathcal{R}\lambda). \quad (4.3)$$

This is just the restriction to Rat_1 of the natural action of G on all smooth maps $S^2 \rightarrow S^2$, namely, $(\mathcal{L}, \mathcal{R}) : \phi \mapsto \mathcal{L} \circ \phi \circ \mathcal{R}^{-1}$.

G -invariance and the Kähler property almost completely determine γ_{12} . By an argument similar to that used to prove Proposition 5, one finds [12] that

$$\gamma_{12} = A_1 d\lambda \cdot d\lambda + A_2 (\lambda \cdot d\lambda)^2 + A_3 \sigma \cdot \sigma + A_4 (\lambda \cdot \sigma)^2 + A_5 \lambda \cdot (\sigma \times d\lambda), \quad (4.4)$$

where A_1, \dots, A_5 are smooth functions of λ only defined by the single function

$$A(\lambda) = 2\pi\mu \frac{[\mu^4 - 4\mu^2 \log \mu - 1]}{(\mu^2 - 1)^3}, \quad (4.5)$$

as follows

$$\begin{aligned} A_1 &= A(\lambda), & A_2 &= \frac{A(\lambda)}{\Lambda^2} + \frac{A'(\lambda)}{\lambda}, & A_3 &= \left(\frac{1 + 2\lambda^2}{4} \right) A(\lambda), \\ A_4 &= \left(\frac{\Lambda^2}{4\lambda} \right) A'(\lambda), & A_5 &= A(\lambda). \end{aligned} \quad (4.6)$$

In (4.4) $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the triple of left invariant one forms on $SO(3)$ dual to the left invariant vector fields $\theta_1, \theta_2, \theta_3$ which, at the identity \mathbb{I}_3 , coincide with the usual basis for $\mathfrak{so}(3)$, that is

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.7)$$

Also, \times denotes vector product in \mathbb{R}^3 , and juxtaposition of one-forms denotes symmetrized tensor product. Note that, although we are using similar notation to that of Section 3.2, the functions A_i and one-forms σ_i in (4.4) are unrelated to the analogous quantities defined there. It follows immediately from (4.4) that Rat_1 is geodesically incomplete (for example, the curve $(\mathbb{I}_3, (0, 0, s))$ with $s \in \mathbb{R}$ has finite length).

Since Rat_1 has trivial second cohomology, its Ricci form ρ is necessarily exact. An explicit formula for ρ was derived in [12], from which it quickly follows that

$$\rho = d\mathcal{A}, \quad \mathcal{A} = \frac{\Lambda}{2} \bar{A}(\lambda) (\lambda \cdot \sigma), \quad (4.8)$$

where

$$\bar{A}(\lambda) = -\frac{1}{2\lambda} \frac{d}{d\lambda} \log(A^2(\lambda) B(\lambda)), \quad (4.9)$$

and

$$B(\lambda) := A_3(\lambda) + \lambda^2 A_4(\lambda) = \frac{1 + 2\lambda^2}{4} A(\lambda) + \frac{\lambda \Lambda^2}{4} A'(\lambda) = \frac{\Lambda}{4} \frac{d}{d\lambda} (\lambda \Lambda A(\lambda)). \quad (4.10)$$

Hence RMG flow on Rat_1 is governed by the Lagrangian

$$L = \frac{1}{2} [A_1(\dot{\lambda} \cdot \dot{\lambda}) + A_2(\lambda \cdot \dot{\lambda})^2 + A_3(\Omega \cdot \Omega) + A_4(\lambda \cdot \Omega)^2 + A_5 \lambda \cdot (\Omega \times \dot{\lambda}) - \Lambda \bar{A}(\lambda \cdot \Omega)] \quad (4.11)$$

for a curve $\chi(t) = (\mathcal{O}(t), \lambda(t))$ on Rat_1 whose angular velocity $\Omega \in \mathbb{R}^3$ is defined such that $\mathcal{O}(t)^{-1} \dot{\mathcal{O}}(t) = \Omega \cdot \mathbf{E} \in \mathfrak{so}(3)$. Note that, to avoid confusion with the radial coordinate $\lambda = \|\lambda\|$, we have used the scaling property of RMG flow to set the effective electric charge (denoted λ in Section 1) to unity. This flow conserves total energy

$$E = \frac{1}{2} [A_1(\dot{\lambda} \cdot \dot{\lambda}) + A_2(\lambda \cdot \dot{\lambda})^2 + A_3(\Omega \cdot \Omega) + A_4(\lambda \cdot \Omega)^2 + A_5 \lambda \cdot (\Omega \times \dot{\lambda})]. \quad (4.12)$$

Furthermore, we have

Proposition 9. RMG flow on Rat_1 conserves the angular momenta $\{P_k, Q_k : k = 1, 2, 3\}$ given by

$$P_k = \sum_{j=1,2,3} \mathcal{O}_{jk} \left[A_3 \Omega_j + A_4 (\Omega \cdot \lambda) \lambda_j + \frac{1}{2} A_1 (\dot{\lambda} \times \lambda)_j - \frac{1}{2} \Lambda \bar{A} \lambda_j \right], \quad (4.13)$$

$$Q_k = \left(A_3 - \frac{1}{2} \lambda^2 A_1 \right) \Omega_k + \left(A_4 + \frac{1}{2} A_1 \right) (\Omega \cdot \lambda) \lambda_k - \frac{1}{2} A_1 (\dot{\lambda} \times \lambda)_k - \frac{1}{2} \Lambda \bar{A} \lambda_k. \quad (4.14)$$

Proof. The RMG Lagrangian, given in (4.11), has $G = SO(3) \times SO(3)$ symmetry. Hence, there is a set of six conserved angular momenta, one for each generator of G . Given $\bar{Y} \in \mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, denote by Y the killing vector field on Rat_1 which it induces. Then the conserved Noether charge associated with the infinitesimal symmetry \bar{Y} is [5]

$$J_Y = \gamma_{L^2}(Y, \dot{\chi}) - \mathcal{A}(Y) + \alpha_Y, \quad (4.15)$$

where α_Y is a real function on Rat_1 such that $d\alpha_Y = \mathcal{L}_Y \mathcal{A}$. Since \mathcal{A} is G -invariant, $\mathcal{L}_Y \mathcal{A} = 0$ for all \bar{Y} , so we may take $\alpha_Y = 0$ for all \bar{Y} .

The six killing vector fields on $\text{Rat}_1 = SO(3) \times \mathbb{R}^3$ generated by the usual basis for \mathfrak{g} are [20]

$$\xi_k, \quad Z_k = \theta_k + \sum_{i,j} \epsilon_{kij} \lambda_i \frac{\partial}{\partial \lambda_j}, \quad k = 1, 2, 3 \quad (4.16)$$

where θ_i are, as before, the left invariant vector fields on $SO(3)$ dual to σ_i , and ξ_i are the *right* invariant vector fields on $SO(3)$ with $\xi_i(\mathbb{I}_3) = \theta_i(\mathbb{I}_3)$, explicitly,

$$\xi_k = \sum_j \mathcal{O}_{jk} \theta_j. \quad (4.17)$$

Setting $Y = \xi_k$ in (4.15) yields the conserved charge $J_Y = P_k$ claimed, and similarly setting $Y = Z_k$ yields the charge Q_k . \square

It is convenient to collect the angular momenta $\{P_k : k = 1, 2, 3\}$ and $\{Q_k : k = 1, 2, 3\}$ into a pair of 3-vectors

$$\mathbf{P} = \mathcal{O}^T \left[A_3 \Omega + A_4 (\Omega \cdot \lambda) \lambda + \frac{1}{2} A_1 (\dot{\lambda} \times \lambda) - \frac{1}{2} \Lambda \bar{A} \lambda \right], \quad (4.18)$$

$$\mathbf{Q} = \left(A_3 - \frac{1}{2} \lambda^2 A_1 \right) \Omega + \left(A_4 + \frac{1}{2} A_1 \right) (\Omega \cdot \lambda) \lambda - \frac{1}{2} A_1 (\dot{\lambda} \times \lambda) - \frac{1}{2} \Lambda \bar{A} \lambda. \quad (4.19)$$

Having determined the conserved quantities E, \mathbf{P} and \mathbf{Q} associated with the RMG flow on Rat_1 , one can eliminate Ω from E to obtain

$$E = \frac{1}{2} \left(A_1 \|\dot{\lambda}\|^2 + A_2 (\lambda \cdot \dot{\lambda})^2 + \frac{1}{A_3} \|\mathbf{P}\|^2 - \frac{A_1^2}{4A_3} \|\dot{\lambda} \times \lambda\|^2 + \frac{\Lambda \bar{A}}{A_3} (\mathbf{Q} \cdot \lambda) - \frac{A_4}{A_3 B} \left[(\mathbf{Q} \cdot \lambda) + \frac{1}{2} \lambda^2 \Lambda \bar{A} \right]^2 + \frac{\lambda^2 \Lambda^2 \bar{A}^2}{4A_3} \right). \quad (4.20)$$

Consider the mapping $q : T\text{Rat}_1 \rightarrow \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ which assigns to each tangent vector the triple $(E, \mathbf{P}, \mathbf{Q})$. By Proposition 9, every RMG curve in Rat_1 is confined to some level set of q . That RMG flow is complete will follow quickly from the following:

Theorem 10. Every level set of $q : T\text{Rat}_1 \rightarrow \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ is compact.

Proof. Choose and fix $(E, \mathbf{P}, \mathbf{Q}) \in \mathbb{R}^7$ and let $X = q^{-1}(E, \mathbf{P}, \mathbf{Q}) \subset T\text{Rat}_1$. Now $T\text{Rat}_1 \equiv TSO(3) \times T\mathbb{R}^3$, and $TSO(3) \equiv SO(3) \times \mathbb{R}^3$ via the identification $\mathcal{O} \mapsto \mathbf{Q}$. We may realize $SO(3)$ as a submanifold of \mathbb{R}^9 by mapping \mathcal{O} to its list of matrix elements. In this way, we may regard $T\text{Rat}_1$ as a 12-dimensional submanifold of $\mathbb{R}^9 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$. The mapping q is smooth, hence certainly continuous, so X is closed. Hence, by Heine–Borel, it suffices to show that $X \subset \mathbb{R}^{18}$ is bounded in Euclidean norm.

Assume, towards a contradiction, that there is some sequence $x_n = (\mathcal{O}_n, \mathbf{Q}_n, \lambda_n, \dot{\lambda}_n) \in X$ which is unbounded in Euclidean norm. By the definition of $SO(3)$, $\|\mathcal{O}_n\|_{\mathbb{R}^9} = \sqrt{3}$ for all n , so (at least) one of $\lambda_n, \dot{\lambda}_n, \mathbf{Q}_n$ must be unbounded. We will now eliminate these possibilities in turn.

Assume λ_n is unbounded. Then it has a subsequence, which we still denote λ_n , with $\|\lambda_n\| \rightarrow \infty$. For all $\lambda \neq \mathbf{0}$ let $\hat{\lambda} = \lambda/\|\lambda\|$ and

$$H(\lambda, \dot{\lambda}) := A_1 \|\dot{\lambda}\|^2 + \lambda^2 A_2 (\hat{\lambda} \cdot \dot{\lambda})^2 - \frac{\lambda^2 A_1^2}{4A_3} \|\hat{\lambda} \times \dot{\lambda}\|^2, \quad (4.21)$$

$$G(\lambda, \mathbf{Q}) := F_1(\lambda) (\mathbf{Q} \cdot \hat{\lambda})^2 + F_2(\lambda) (\mathbf{Q} \cdot \hat{\lambda}) + F_3(\lambda), \quad (4.22)$$

where

$$F_1(\lambda) = -\frac{\lambda^2 A_4}{A_3 B}, \quad F_2(\lambda) = \frac{\lambda A \bar{A}}{A_3} \left(1 - \frac{\lambda^2 A_4}{B}\right), \quad F_3(\lambda) = \frac{\lambda^2 A^2 \bar{A}^2}{4B}. \quad (4.23)$$

Then, the conserved energy E , given in (4.20), can be written as

$$E = \frac{1}{2} \left(H(\lambda, \dot{\lambda}) + \frac{1}{A_3} \|\mathbf{P}\|^2 + G(\lambda, \mathbf{Q}) \right). \quad (4.24)$$

Since the cross and dot products on \mathbb{R}^3 are related by

$$\|\hat{\lambda} \times \dot{\lambda}\|^2 = \|\dot{\lambda}\|^2 - (\hat{\lambda} \cdot \dot{\lambda})^2, \quad (4.25)$$

then,

$$H(\lambda, \dot{\lambda}) = \frac{A^2 A}{1 + 2\lambda^2} \|\dot{\lambda}\|^2 + \lambda^2 \left(\frac{2 + 3\lambda^2}{(1 + 2\lambda^2) A^2} A + \frac{A'(\lambda)}{\lambda} \right) (\hat{\lambda} \cdot \dot{\lambda})^2. \quad (4.26)$$

Here, we have used the definition of A_1, A_2 and A_3 , as in (4.6). Since $B(\lambda)$, given in (4.10), is positive,

$$\frac{A'(\lambda)}{\lambda} > -\frac{1 + 2\lambda^2}{\lambda^2 A^2} A, \quad (4.27)$$

from which it follows that

$$H(\lambda, \dot{\lambda}) \geq \frac{A^2 A}{1 + 2\lambda^2} \|\hat{\lambda} \times \dot{\lambda}\|^2 \geq 0. \quad (4.28)$$

Since both $H(\lambda, \dot{\lambda})$ and A_3 are non-negative, it follows from (4.24) that

$$2E \geq G(\lambda_n, \mathbf{Q}). \quad (4.29)$$

From (4.5), one obtains the following limit

$$\lim_{\lambda \rightarrow \infty} \frac{\log \lambda}{\lambda^4} G(\lambda, \mathbf{Q}) = \frac{4}{\pi} [(\mathbf{Q} \cdot \hat{\lambda}) + 2]^2. \quad (4.30)$$

But $\|\lambda_n\| \rightarrow \infty$, so (4.30) contradicts (4.29) unless $\|\mathbf{Q}\| = 2$.

Hence $\|\mathbf{Q}\| = 2$. Let θ be the angle between λ and \mathbf{Q} , namely,

$$\mathbf{Q} \cdot \hat{\lambda} = \|\mathbf{Q}\| \cos \theta = 2 \cos \theta. \quad (4.31)$$

Then, it follows from (4.22) that

$$G(\lambda, \mathbf{Q}) = 4F_1(\lambda) \cos^2 \theta + 2F_2(\lambda) \cos \theta + F_3(\lambda) =: Z(\lambda, \theta). \quad (4.32)$$

We shall appeal to the following technical lemma whose proof we postpone to [Appendix](#).

Lemma 11. On $(\text{Rat}_1, \gamma_{L^2})$, there exist $c_0, \lambda_0 > 0$ such that for all $\lambda \geq \lambda_0, Z(\lambda, \theta)$, given in (4.32), satisfies

$$Z(\lambda, \theta) > \frac{c_0 \lambda^4}{(\log \lambda)^3}, \quad \forall \theta \in \mathbb{R}. \quad (4.33)$$

Using the above lemma, it follows from (4.29), (4.32) and (4.33) that, for all n sufficiently large,

$$2E \geq G(\lambda_n, \mathbf{Q}) > \frac{c_0 \|\lambda_n\|^4}{(\log \|\lambda_n\|)^3}, \quad (4.34)$$

a contradiction. Hence λ_n is bounded.

Assume now that $\dot{\lambda}_n$ is unbounded. We have shown that $\|\lambda_n\|$ is confined to a closed bounded interval, so $A_i(\|\lambda_n\|)$, $B(\|\lambda_n\|)$ are positive, bounded and bounded away from zero, and $A(\|\lambda_n\|)$ is bounded, by continuity. Hence, from (4.20) we see that

$$2E > A_1(\|\lambda_n\|) \left(1 - \frac{\|\lambda_n\|^2 A_1(\|\lambda_n\|)}{4A_3(\|\lambda_n\|)} \right) \|\dot{\lambda}_n\|^2 + c = A_1(\|\lambda_n\|) \frac{2\|\lambda_n\|^2}{1 + 2\|\lambda_n\|^2} \|\dot{\lambda}_n\|^2 + c \quad (4.35)$$

for some constant $c \in \mathbb{R}$. But this contradicts unboundedness of $\dot{\lambda}_n$.

Finally, assume that $\|\Omega_n\|$ is unbounded. We have already shown that $\|\lambda_n\|$ and $\|\dot{\lambda}_n\|$ are bounded, and by continuity, $A_i(\|\lambda_n\|)$ are positive, bounded and bounded away from 0. But this immediately contradicts (4.12). \square

Corollary 12. $(\text{Rat}_1, \gamma_{L^2})$ is RMG complete.

Proof. For each $K > 0$, let $X_K = \{(\mathcal{O}, \Omega, \lambda, \dot{\lambda}) \in T\text{Rat}_1 : \|\Omega\| + \|\lambda\| + \|\dot{\lambda}\| \leq K\}$. By a standard application of Picaud's method, there exists $T_K > 0$, depending only on K , such that, for all $x_0 \in X_K$ there exists a unique RMG curve $x : [-T_K, T_K] \rightarrow X_{2K}$ with $x(0) = x_0$. Now choose and fix $x_0 \in T\text{Rat}_1$, and let $X = q^{-1}(q(x_0))$, the level set of q containing x_0 . By Theorem 10, there exists $K > 0$ such that $X \subset X_K$. Hence there is a unique RMG curve $x : [-T_K, T_K] \rightarrow X_{2K}$ with $x(0) = x_0$. But, by Proposition 9, $x(\pm T_K) \in X \subset X_K$, so this solution can be extended, both forward and backward in time, to $[-3T_K, 3T_K]$, and $x(\pm 3T_K) \in X \subset X_K$ also. Proceeding inductively, the RMG curve has an extension $x : \mathbb{R} \rightarrow X_K$. Since x_0 was arbitrary, it follows that RMG flow is complete. \square

Remark 13. Theorem 10 is strictly stronger than Corollary 12, since it implies that every RMG curve in $(\text{Rat}_1, \gamma_{L^2})$ is confined to a compact subset of Rat_1 and hence is bounded away from the boundary of Rat_1 at infinity. This is not true of complete RMG flows in general (consider for example the trivial RMG flow on \mathbb{C}^n).

Remark 14. Since $(\text{Rat}_1, \gamma_{L^2})$ is known to be geodesically incomplete, it is a counterexample to the conjecture [9] that every RMG complete Kähler manifold is geodesically complete. Simpler counterexamples can be constructed. For example the surface of revolution \mathbb{C} given the metric $g = \text{sech } |z| dz d\bar{z}$ is manifestly geodesically incomplete and can be shown, by an energy/angular momentum conservation argument analogous to the one presented here for Rat_1 , to be RMG complete [21].

The L^2 geometry of Rat_n , for $n \geq 2$, is comparatively poorly understood. It is known to be G -invariant, Kähler and geodesically incomplete [12], and is conjectured to have finite total volume [10]. Inside Rat_n there is a topologically cylindrical submanifold, Rat_n^{eq} , the fixed point set of the circle group of isometries $W(z) \mapsto e^{i\alpha} W(e^{-i\alpha} z)$. This consists of rotationally equivariant rational maps, of the form $W(z) = cz^n$, where $c \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, and is preserved by (extrinsic) RMG flow on Rat_n by Corollary 2. The induced L^2 metric on Rat_n^{eq} was studied in detail in [22]. It is interesting to compare the intrinsic RMG flow on Rat_n^{eq} with the extrinsic RMG flow, defined by its inclusion in Rat_n .

Denote by $\pi : \mathbb{C}^\times \rightarrow \text{Rat}_n^{\text{eq}}$ the n -fold covering map $a \mapsto [W : z \mapsto (az)^n]$. Then the lifted L^2 metric on \mathbb{C}^\times is

$$\pi^* \gamma_{L^2}^{\text{eq}} = F(|a|) da d\bar{a}, \quad F(\rho) = \pi n^2 \int_0^\infty \frac{s^n}{(1+s^n)^2} \frac{ds}{(\rho^2+s)^2}. \quad (4.36)$$

Now $c \mapsto c^{-1}$ is an isometry of Rat_n^{eq} , whence it follows that $a \mapsto a^{-1}$ is an isometry of the lifted metric. Furthermore, $\lim_{\rho \rightarrow \infty} F(\rho)$ exists for all $n \geq 2$, so $\pi^* \gamma_{L^2}^{\text{eq}}$ has a C^0 extension to $S^2 = \mathbb{C}^\times \cup \{0, \infty\}$ for all $n \geq 2$, which we denote as $\bar{\gamma}_n$. For n sufficiently large, we can obtain useful information about RMG flow on $(\text{Rat}_n^{\text{eq}}, \gamma_{L^2}^{\text{eq}})$ by considering its lift to $(S^2, \bar{\gamma}_n)$. This requires us to establish enhanced regularity of $\bar{\gamma}_n$, as follows.

Proposition 15. For all $n \geq 5$, the extended lifted metric $\bar{\gamma}_n$ on S^2 is C^3 .

Proof. It is known [22] that $\bar{\gamma}_n$ is C^2 for all $n \geq 4$. Further, $\bar{\gamma}_n$ is manifestly smooth on $S^2 \setminus \{0, \infty\}$ so, in light of the isometry $a \mapsto 1/a$, which interchanges 0 and ∞ , it suffices to prove that $f_{xxx}, f_{xxy}, f_{xyy}$ and f_{yyy} exist at $(0, 0)$, where

$f(x, y) = F(\sqrt{x^2 + y^2})$. By computing in polar coordinates, $(x, y) = \rho(\cos \theta, \sin \theta)$, one sees that all these third derivatives exist (and vanish) if and only if

$$\begin{aligned} \lim_{\rho \rightarrow 0} \frac{1}{\rho} \left(F''(\rho) - \frac{F'(\rho)}{\rho} \right) &= 0 \\ \lim_{\rho \rightarrow 0} \left(F'''(\rho) - \frac{3}{\rho} \left(F''(\rho) - \frac{F'(\rho)}{\rho} \right) \right) &= 0. \end{aligned} \quad (4.37)$$

For each pair of integers $n \geq 2$ and $k \geq 0$, define the function $\eta_{n,k} : (0, \infty) \rightarrow \mathbb{R}$,

$$\eta_{n,k}(\rho) = \int_0^\infty \frac{s^n}{(1+s^n)^2} \frac{ds}{(\rho^2+s)^k}. \quad (4.38)$$

For $0 \leq k \leq n$, its integrand is bounded above by the integrable function $s^{n-k}/(1+s^n)^2$, so, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{\rho \rightarrow 0} \eta_{n,k}(\rho) = \int_0^\infty \frac{s^{n-k}}{(1+s^n)^2} ds < \infty. \quad (4.39)$$

It follows from the definition of F that

$$\frac{1}{\rho} \left(F''(\rho) - \frac{F'(\rho)}{\rho} \right) = 24\pi n^2 \rho \eta_{n,4}(\rho) \quad (4.40)$$

$$F'''(\rho) - \frac{3}{\rho} \left(F''(\rho) - \frac{F'(\rho)}{\rho} \right) = -129\pi n^2 \rho^3 \eta_{n,5}(\rho). \quad (4.41)$$

Hence the required limits (4.37) follow from (4.39) provided $n \geq 5$. \square

Corollary 16. For all $n \geq 5$, the intrinsic RMG flow on $(\text{Rat}_n^{\text{eq}}, \gamma_{L^2}^{\text{eq}})$ is incomplete.

Proof. Assume $n \geq 5$. Then $\bar{\gamma}_n$ is C^3 , so its Ricci form is C^1 . Hence, by standard existence and uniqueness theory of ODEs, the RMG flow is globally well-defined on $(S^2, \bar{\gamma}_n)$. In particular, there is an RMG curve $a : (-\varepsilon, \varepsilon) \rightarrow S^2$ with $a(0) = 0$ and $\dot{a}(0) = 1$. Consider the image of $a : (-\varepsilon, 0) \rightarrow S^2$ under the projection $\pi : \mathbb{C}^\times \rightarrow \text{Rat}_n^{\text{eq}}$. By definition, π is a holomorphic isometry, so $\pi \circ a$ is an RMG curve in Rat_n^{eq} , which reaches the singular point $a = 0$ in finite time. Hence intrinsic RMG flow in Rat_n^{eq} is incomplete. \square

Remark 17. By resorting to a case-by-case analysis of the flow on Rat_n^{eq} itself, one can extend the conclusion of Corollary 16 to all $n \geq 2$ [21]. The case $n = 2$ is considered below, see Proposition 19.

As we have remarked, the extrinsic RMG flow on a totally geodesic complex submanifold of a Kähler manifold does not, in general, coincide with its intrinsic RMG flow, so we cannot conclude from Corollary 16 that $(\text{Rat}_n, \gamma_{L^2})$ is RMG incomplete for $n \geq 5$: this would follow if the extrinsic RMG flow on Rat_n^{eq} were incomplete. Remarkably, although we have little information about the L^2 metric on Rat_2 , we have enough to prove that the extrinsic RMG flow on Rat_2^{eq} is complete. This follows from the following formula for the restriction of the Ricci form to Rat_n^{eq} .

Proposition 18. Let $\rho|$ be the restriction of the Ricci form ρ of $(\text{Rat}_n, \gamma_{L^2})$ to Rat_n^{eq} (that is, $\rho| = \iota^* \rho$ where $\iota : \text{Rat}_n^{\text{eq}} \rightarrow \text{Rat}_n$ denotes inclusion) and ρ^{eq} be the intrinsic Ricci form on $(\text{Rat}_n^{\text{eq}}, \gamma_{L^2}^{\text{eq}})$. Then $\rho| = d\mathcal{A}|$ and $\rho^{\text{eq}} = d\mathcal{A}^{\text{eq}}$ where

$$\begin{aligned} \mathcal{A}| &= - \left(\sum_{j=0}^{2n} \frac{|\chi| F'_j(\chi)}{2F_j(\chi)} \right) d\psi, \\ \mathcal{A}^{\text{eq}} &= - \frac{\chi F'_n(\chi)}{2F_n(\chi)} d\psi, \end{aligned}$$

and

$$F_j(\chi) = 16\pi \int_0^\infty \frac{s^j}{(1+\chi^2 s^n)^2} \frac{ds}{(1+s)^2}.$$

The coordinate $\chi e^{i\psi}$ on $\text{Rat}_n^{\text{eq}} \cong \mathbb{C}^\times$ corresponds to the rational map $W(z) = \chi e^{i\psi} z^n$.

Proof. Rat_n^{eq} lies entirely within the coordinate chart on Rat_n on which

$$W(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{1 + a_{n+1} z + \cdots + a_{2n} z^n}. \quad (4.42)$$

It is the surface $a_0 = \cdots = a_{n-1} = a_{n+1} = \cdots = a_{2n} = 0$, $a_n = \chi e^{i\psi} \in \mathbb{C}^\times$. Let $G(a_0, \dots, a_{2n}) = \log \det \gamma$, where γ denotes the Hermitian matrix of metric coefficients of γ with respect to the local complex coordinates a_j . Then [23, p. 82],

$$\rho = -i\partial\bar{\partial}G, \quad (4.43)$$

so

$$\rho| = -i\iota^* \partial\bar{\partial}G = -i\partial\bar{\partial}(G \circ \iota) \quad (4.44)$$

since the inclusion is holomorphic. Now

$$\gamma_{\bar{k}} = 16 \int_{\mathbb{C}} \frac{1}{(1 + |W(z)|^2)^2} \frac{\partial W}{\partial a_j} \overline{\left(\frac{\partial W}{\partial a_k} \right)} \frac{dz d\bar{z}}{(1 + |z|^2)^2} \quad (4.45)$$

and

$$\left. \frac{\partial W}{\partial a_j} \right| = \begin{cases} z^j & 0 \leq j \leq n \\ -\chi e^{i\psi} z^j & n+1 \leq j \leq 2n \end{cases} \quad (4.46)$$

where the vertical stroke denotes evaluation at the rational map $\chi e^{i\psi} z^n$. It follows that $\gamma_{\bar{j}k} = 0$ if $j \neq k$, and that

$$\gamma_{\bar{j}j} = \begin{cases} F_j(\chi) & 0 \leq j \leq n \\ \chi^2 F_j(\chi) & n+1 \leq j \leq 2n. \end{cases} \quad (4.47)$$

Hence

$$G \circ \iota = n \log \chi^2 + \sum_{j=0}^{2n} \log F_j(\chi), \quad (4.48)$$

and the formula for $\rho|$ immediately follows. To obtain the formula for ρ^{eq} we note that the induced metric on Rat_n^{eq} is

$$\gamma_{l^2}^{eq} = \gamma_{n\bar{n}} |da_n d\bar{a}_n = F_n(\chi) dc d\bar{c}, \quad (4.49)$$

where $c = \chi e^{i\psi}$, and use (4.43). \square

Since the integrand in F_j is rational, one can, in principle, evaluate each of these functions as an explicit function of χ . The expressions involved become very complicated as n grows large, however.

Both the extrinsic and intrinsic RMG flows on Rat_n^{eq} are governed by a Lagrangian of the form

$$L = \frac{1}{2} F_n(\chi) (\dot{\chi}^2 + \chi^2 \dot{\psi}^2) - a(\chi) \dot{\psi} \quad (4.50)$$

where $a(\chi) = \mathcal{A}^{\text{eq}}(\partial/\partial\psi)$ or $a(\chi) = \mathcal{A}(\partial/\partial\psi)$ respectively. In each case, both the momentum conjugate to ψ ,

$$P = \chi^2 F_n(\chi) \dot{\psi} - a(\chi) \quad (4.51)$$

and the kinetic energy

$$E = \frac{1}{2} F_n(\chi) (\dot{\chi}^2 + \chi^2 \dot{\psi}^2) = \frac{1}{2} F_n(\chi) \dot{\chi}^2 + \frac{(P + a(\chi))^2}{2\chi^2 F_n(\chi)} \quad (4.52)$$

are conserved. This is equivalent to motion on $(0, \infty)$ with the metric $F_n(\chi) d\chi^2$ in the effective potential

$$V_P(\chi) = \frac{(P + a(\chi))^2}{2\chi^2 F_n(\chi)}. \quad (4.53)$$

Since $(0, \infty)$ has finite total length with respect to this metric [13], the flow is complete if and only if, for each $P \in \mathbb{R}$, the effective potential is unbounded above as $\chi \rightarrow 0$ and $\chi \rightarrow \infty$. Both the intrinsic and extrinsic RMG flows are symmetric under $c = \chi e^{i\psi} \mapsto 1/c$, so in fact it suffices to consider $V_P(\chi)$ in a neighbourhood of 0.

Proposition 19. Rat_2^{eq} is extrinsically RMG complete with respect to the L^2 metric, but intrinsically RMG incomplete.

Proof. As argued above, we must show that the effective potential V_P is unbounded above as $\chi \rightarrow 0$ for all P , in the case of extrinsic flow, and is bounded as $\chi \rightarrow 0$ for at least one choice of P in the case of intrinsic flow. Let $G_j(\chi) = -\frac{1}{2}\chi F_j'(\chi)/F_j(\chi)$, and $G(\chi) = \sum_{j=0}^4 G_j(\chi)$. Then the effective potentials governing the extrinsic and intrinsic RMG flows are

$$V_P^{\text{ext}}(\chi) = \frac{(P + G(\chi))^2}{2\chi^2 F_2(\chi)}, \quad V_P^{\text{int}}(\chi) = \frac{(P + G_2(\chi))^2}{2\chi^2 F_2(\chi)}, \quad (4.54)$$

respectively. With the aid of Maple, for example, one can obtain the following limits:

$$\lim_{\chi \rightarrow 0} \chi F_2(\chi) = 4\pi, \quad \lim_{\chi \rightarrow 0} \frac{G_2(\chi) - \frac{1}{2}}{\chi \log \chi} = -\frac{4}{\pi}, \quad \lim_{\chi \rightarrow 0} (G(\chi) - 3) \log \chi = -\frac{1}{2}. \quad (4.55)$$

It follows that, for all $P \neq -3$,

$$\lim_{\chi \rightarrow 0} \chi V_P^{\text{ext}}(\chi) = \frac{(P + 3)^2}{8\pi} \quad (4.56)$$

and

$$\lim_{\chi \rightarrow 0} \chi (\log \chi)^2 V_{-3}^{\text{ext}}(\chi) = \frac{1}{32\pi}. \quad (4.57)$$

Hence, for all P , V_P^{ext} is unbounded above as $\chi \rightarrow 0$. But

$$\lim_{\chi \rightarrow 0} V_{1/2}^{\text{int}}(\chi) = 0 \quad (4.58)$$

so $V_{1/2}^{\text{int}}$ is bounded. \square

Numerical analysis of the functions $F_j(\chi)$ suggests that Rat_n^{eq} is likely to be extrinsically RMG complete for all $n \geq 2$, but we have been unable to prove this so far. Since the process of a single isolated lump collapsing to a singular spike during RMG flow is prohibited by curvature effects in Rat_1 , and the same is true for a pair of equivariant coincident lumps in Rat_2 , it is plausible that Rat_n should be RMG complete for all $n \geq 2$, despite being geodesically incomplete.

5. Concluding remarks

In this paper we have studied Ricci magnetic geodesic motion on the moduli spaces of abelian Higgs vortices and \mathbb{CP}^1 lumps. In so doing we have established that two assertions and one conjecture about this kind of soliton dynamics in the current literature are false. First, contrary to a claim of Collie and Tong [1], RMG motion on the vortex moduli space does not coincide with the magnetic geodesic flow proposed earlier by Kim and Lee [6] (and, furthermore, we have shown that the Kim–Lee flow is globally ill-defined). Second, we have shown that, while RMG flow localizes to fixed point sets of groups of holomorphic isometries, the flow does not, as claimed by one of us and Krusch [9], coincide with the intrinsic RMG flow on the fixed point set. We have seen that on both the submanifold of centred hyperbolic two-vortices and the space of rotationally equivariant two-lumps, the intrinsic and extrinsic RMG flows are qualitatively different from one another. This aspect of RMG flow is conceptually troubling: since it arises by restricting an infinite dimensional dynamical system (a field theory) to a finite dimensional submanifold, it is somewhat strange that further symmetry reduction is not self-consistent. Third, we have shown that, contrary to a conjecture in [9], there exist Kähler manifolds which are geodesically incomplete but RMG complete: in fact $(\text{Rat}_1, \gamma_{L^2})$ is one such manifold.

Several interesting open questions remain. Can one, by adapting the methods of Stuart for example [4], prove rigorously Collie and Tong's conjecture that Chern–Simons vortex dynamics is controlled by RMG motion in M_n , in the small κ (and small energy) limit? Or can one rigorously derive some alternative magnetic geodesic flow on M_n ? Can one develop a point-vortex formalism for well-separated Chern–Simons vortices, analogous to the one for standard vortices [24,18]? This would provide formal evidence for, or against, Collie and Tong's conjecture. Treating RMG flow as an interesting dynamical system on Kähler manifolds, can one establish geometric criteria which ensure that RMG completeness implies geodesic completeness? Can one find examples of RMG complete but geodesically incomplete manifolds with bounded scalar curvature? Or bounded Ricci curvature? Note that Rat_1 and the surface of revolution described in Remark 14 both have unbounded scalar curvature.

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Appendix. Proof of Lemma 11

One can obtain using, for example, Maple the following asymptotic formulae for $F_i(\lambda)$, given in (4.23), with respect to the L^2 metric on Rat_1 as $\lambda \rightarrow \infty$:

$$\begin{aligned} F_1(\lambda) &= \frac{\lambda^4}{\log \lambda} \left[a_1 + \frac{a_2}{\log \lambda} + \frac{a_3}{(\log \lambda)^2} + O\left(\frac{1}{(\log \lambda)^3}\right) \right], \\ F_2(\lambda) &= \frac{\lambda^4}{\log \lambda} \left[b_1 + \frac{b_2}{\log \lambda} + \frac{b_3}{(\log \lambda)^2} + O\left(\frac{1}{(\log \lambda)^3}\right) \right], \\ F_3(\lambda) &= \frac{\lambda^4}{\log \lambda} \left[c_1 + \frac{c_2}{\log \lambda} + \frac{c_3}{(\log \lambda)^2} + O\left(\frac{1}{(\log \lambda)^3}\right) \right], \end{aligned} \quad (\text{A.1})$$

where

$$\begin{aligned} a_1 &= \frac{4}{\pi}, & a_2 &= \frac{2}{\pi}[1 - 2 \log 2], & a_3 &= \frac{1}{\pi}[1 - 4 \log 2 + 4(\log 2)^2], \\ b_1 &= \frac{16}{\pi}, & b_2 &= \frac{2}{\pi}[3 - 8 \log 2], & b_3 &= \frac{1}{\pi}[2 - 12 \log 2 + 16(\log 2)^2], \\ c_1 &= \frac{16}{\pi}, & c_2 &= \frac{4}{\pi}[1 - 4 \log 2], & c_3 &= \frac{1}{\pi}[1 - 32 \log 2 + 64(\log 2)^2]. \end{aligned} \quad (\text{A.2})$$

It follows from (4.32) and (A.1) that

$$Z(\lambda, \theta) = Z_0(\lambda, \theta) + Z_{\text{error}}(\lambda, \theta), \quad (\text{A.3})$$

where

$$\begin{aligned} Z_0(\lambda, \theta) &= \frac{\lambda^4}{\log \lambda} \left[(4a_1 \cos^2 \theta + 2b_1 \cos \theta + c_1) + \frac{1}{\log \lambda} (4a_2 \cos^2 \theta + 2b_2 \cos \theta + c_2) \right. \\ &\quad \left. + \frac{1}{(\log \lambda)^2} (4a_3 \cos^2 \theta + 2b_3 \cos \theta + c_3) \right], \end{aligned} \quad (\text{A.4})$$

and $Z_{\text{error}}(\lambda, \theta)$ satisfies the following estimate: there exist $c_*, \lambda_* > 0$ such that for all $\lambda \geq \lambda_*$,

$$|Z_{\text{error}}(\lambda, \theta)| < \frac{c_* \lambda^4}{(\log \lambda)^4}, \quad \forall \theta \in \mathbb{R}. \quad (\text{A.5})$$

Hence, it suffices to prove that $Z_0(\lambda, \theta)$ satisfies an estimate of the form (4.33).

Defining $\tau = 1 + \cos \theta$ and $x = 1/\log \lambda$. Then

$$\frac{\log \lambda}{\lambda^4} Z_0(\lambda, \theta) = P_x(\tau), \quad (\text{A.6})$$

where

$$P_x(\tau) = \alpha_1(x)\tau^2 + \alpha_2(x)\tau + \alpha_3(x), \quad (\text{A.7})$$

and the coefficients α_1, α_2 and α_3 are given by

$$\begin{aligned} \alpha_1(x) &= 4(a_1 + a_2 x + a_3 x^2), \\ \alpha_2(x) &= 2(b_2 - 4a_2)x + 2(b_3 - 4a_3)x^2, \\ \alpha_3(x) &= (4a_3 - 2b_3 + c_3)x^2. \end{aligned} \quad (\text{A.8})$$

Since $\alpha_1(0) > 0$, then there exists $x_* > 0$ such that for all $x \in (-x_*, x_*)$, $P_x(\tau)$ has a minimum, occurs at $\tau = \tau_*$, where $dP_x(\tau)/d\tau|_{\tau=\tau_*} = 0$, that is,

$$\tau_*(x) = -\frac{1}{2} \frac{\alpha_2(x)}{\alpha_1(x)}. \quad (\text{A.9})$$

So, for all $x \in (-x_*, x_*)$, the minimum value of $P_x(\tau)$ is

$$P_x(\tau_*(x)) = -\frac{1}{4\alpha_1(x)}[\alpha_2(x)^2 - 4\alpha_1(x)\alpha_3(x)]. \quad (\text{A.10})$$

Note that $P_x(\tau_*(x))$ is a rational function of x , and hence is analytic. Using (A.8), one finds that

$$P_0(\tau_*(0)) = 0, \quad \left. \frac{d}{dx} P_x(\tau_*(x)) \right|_{x=0} = 0, \quad (\text{A.11})$$

and

$$\left. \frac{d^2}{dx^2} P_x(\tau_*(x)) \right|_{x=0} = -\frac{1}{8a_1} [(b_2 - 4a_2)^2 - 16a_1(4a_3 - 2b_3 + c_3)] > 0. \quad (\text{A.12})$$

Thus, there exist $\varepsilon > 0$ and $0 < x_0 < x_*$ such that for all $x \in (-x_0, x_0)$,

$$P_x(\tau_*(x)) \geq \varepsilon x^2. \quad (\text{A.13})$$

Hence, for all $x \in (0, x_0)$,

$$P_x(\tau) \geq \varepsilon x^2, \quad \forall \tau \in \mathbb{R}. \quad (\text{A.14})$$

Hence, it follows from (A.6) that for all $\lambda > e^{1/x_0}$,

$$\frac{\log \lambda}{\lambda^4} Z_0(\lambda, \theta) = P_x(\tau) \geq \varepsilon x^2 = \frac{\varepsilon}{(\log \lambda)^2}, \quad \forall \theta \in \mathbb{R}, \quad (\text{A.15})$$

which implies that $Z_0(\lambda, \theta)$ satisfies the estimate (4.33). \square

References

- [1] B. Collie, D. Tong, Dynamics of Chern–Simons vortices, *Phys. Rev. D* 78 (2008) 065013.
- [2] C. Lee, K. Lee, H. Min, Self-dual Maxwell Chern–Simons solitons, *Phys. Lett. B* 252 (1990) 79–83.
- [3] T.M. Samols, Vortex scattering, *Comm. Math. Phys.* 145 (1992) 149–179.
- [4] D. Stuart, Dynamics of abelian Higgs vortices in the near Bogomolny regime, *Comm. Math. Phys.* 159 (1994) 51–91.
- [5] N.S. Manton, P.M. Sutcliffe, *Topological Solitons*, Cambridge University Press, Cambridge U.K. 2004.
- [6] Y. Kim, K. Lee, First and second order vortex dynamics, *Phys. Rev. D* 66 (2002) 045016.
- [7] E. Witten, Some exact multipseudoparticle solutions of classical Yang–Mills theory, *Phys. Rev. Lett.* 38 (1977) 121–124.
- [8] I.A.B. Strachan, Low-velocity scattering of vortices in a modified abelian Higgs model, *J. Math. Phys.* 33 (1992) 102–110.
- [9] S. Krusch, J.M. Speight, Exact moduli space metrics for hyperbolic vortex polygons, *J. Math. Phys.* 51 (2010) 022304. 13.
- [10] J.M. Baptista, On the L^2 -metric of vortex moduli spaces, *Nuclear Phys. B* 844 (2011) 308–333.
- [11] C.-C. Liu, Dynamics of Abelian vortices without common zeros in the adiabatic limit, *Comm. Math. Phys.* (2014).
- [12] J.M. Speight, The L^2 geometry of spaces of harmonic maps $S^2 \rightarrow S^2$ and $\mathbb{R}P^2 \rightarrow \mathbb{R}P^2$, *J. Geom. Phys.* 47 (2003) 343–368.
- [13] L.A. Sadun, J.M. Speight, Geodesic incompleteness in the CP^1 model on a compact Riemann surface, *Lett. Math. Phys.* 43 (1998) 329–334.
- [14] B. O’Neill, *Semi-Riemannian Geometry*, in: *Pure and Applied Mathematics*, vol. 103, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983, with applications to relativity.
- [15] J. Berndt, S. Console, C. Olmos, *Submanifolds and Holonomy*, in: *Chapman & Hall/CRC Research Notes in Mathematics*, vol. 434, Chapman & Hall/CRC, Boca Raton, FL, 2003.
- [16] C. Lee, H. Min, C. Rim, Zero modes of the self-dual Maxwell Chern–Simons solitons, *Phys. Rev. D* 43 (1991) 4100–4110.
- [17] C.H. Taubes, Arbitrary N -vortex solutions to the first order Ginzburg–Landau equations, *Comm. Math. Phys.* 72 (1980) 277–292.
- [18] N.S. Manton, J.M. Speight, Asymptotic interactions of critically coupled vortices, *Comm. Math. Phys.* 236 (2003) 535–555.
- [19] M.A. Armstrong, *Groups and Symmetry*, in: *Undergraduate Texts in Mathematics*, Springer-Verlag, New York, 1988.
- [20] S. Krusch, J.M. Speight, Quantum lump dynamics on the two-sphere, *Comm. Math. Phys.* 322 (2013) 95–126.
- [21] L.S.M. Alqahtani, *Geometric flows on soliton moduli spaces*, (Ph.D. thesis), University of Leeds, 2013.
- [22] J.A. McGlade, J.M. Speight, Slow equivariant lump dynamics on the two sphere, *Nonlinearity* 19 (2006) 441–452.
- [23] A.L. Besse, *Einstein Manifolds*, in: *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, vol. 10, Springer-Verlag, Berlin, 1987.
- [24] J.M. Speight, Static intervortex forces, *Phys. Rev. D* 55 (1997) 3830–3835.