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Vector bundles for "Matrix algebras converge to the sphere"

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ABSTRACT

In the high-energy quantum-physics literature, one finds statements such as "matrix algebras converge to the sphere". Earlier I provided a general precise setting for understanding such statements, in which the matrix algebras are viewed as quantum metric spaces, and convergence is with respect to a quantum Gromov–Hausdorff-type distance.

But physicists want even more to treat structures on spheres (and other spaces), such as vector bundles, Yang–Mills functionals, Dirac operators, etc., and they want to approximate these by corresponding structures on matrix algebras. In the present paper we treat this idea for vector bundles. We develop a general precise way for understanding how, for two compact quantum metric spaces that are close together, to a given vector bundle on one of them there can correspond in a natural way a unique vector bundle on the other. We then show explicitly how this works for the case of matrix algebras converging to the 2-sphere.

1. Introduction

In several earlier papers [1–4] I showed how to give a precise meaning to statements in the literature of high-energy physics and string theory of the kind "matrix algebras converge to the sphere". (See the references to the quantum physics literature given in [1,5–10].) I did this by introducing and developing a concept of "compact quantum metric spaces", and a corresponding quantum Gromov–Hausdorff-type distance between them. The compact quantum spaces are unital C*-algebras, and the metric data is given by equipping the algebras with suitable seminorms that play the role of the usual Lipschitz seminorms on the algebras of continuous functions on ordinary compact metric spaces. The natural setting for "matrix algebras converge to the sphere" is that of coadjoint orbits of compact semi-simple Lie groups, as shown in [1,3,4].

But physicists need much more than just the algebras. They need vector bundles, gauge fields, Dirac operators, etc. In the present paper I provide a general method for giving precise quantitative meaning to statements in the physics literature of the kind "here are the vector bundles over the matrix algebras that correspond to the monopole bundles on the sphere" [11–20,6]. I then apply this method to the case of the 2-sphere, with full proofs of convergence. Many of the considerations in this paper apply directly to the general case of coadjoint orbits. But some of the detailed estimates needed to prove convergence require fairly complicated considerations (see Section 11) concerning highest weights for representations of compact semi-simple Lie groups. It appears to me that it would be quite challenging to carry out those details for the

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general case, though I expect that some restricted cases, such as matrix-algebra approximations for complex projective spaces [21,14,22–24], are quite feasible to deal with.

In [5] I studied the convergence of ordinary vector bundles on ordinary compact metric spaces for the ordinary Gromov–Hausdorff distance. The approach that worked for me was to use the correspondence between vector bundles and projective modules (Swan's theorem [25]), and by this means represent vector bundles by corresponding projections in matrix algebras over the algebras of continuous functions on the compact metric spaces; and then to prove appropriate convergence of the projections. In the present paper we follow that same approach, in which now we also consider projective modules over the matrix algebras that converge to the 2-sphere, and thus also projections in matrix algebras over these matrix algebras.

For this purpose, one needs Lipschitz-type seminorms on all of the matrix algebras over the underlying algebras, with these seminorms coherent in the sense that they form a "matrix seminorm". In my recent paper [4] the theory of these matrix seminorms was developed, and properties of such matrix seminorms for the setting of coadjoint orbits were obtained. In particular, some general methods were given for obtaining estimates related to how these matrix seminorms mesh with an appropriate quantum analog of the Gromov–Hausdorff distance. The results of that paper will be used here.

Recently Latrémolière introduced an improved version of quantum Gromov–Hausdorff distance [26], which he calls "quantum Gromov–Hausdorff propinquity". In [4] I showed that his propinquity works very well for our setting of coadjoint orbits, and so propinquity is the form of quantum Gromov–Hausdorff distance that we use in the present paper. Latrémolière defines his propinquity in terms of an improved version of the "bridges" that I had used in my earlier papers. For our matrix seminorms we need corresponding "matricial bridges". In [4] natural such matricial bridges were constructed for the setting of coadjoint orbits. They will be used here.

To give somewhat more indication of the nature of our approach, we give now an imprecise version of one of the general theorems that we apply. For simplicity of notation, we express it here only for the C*-algebras, rather than for matrix algebras over the C*-algebras as needed later. Let $(\mathcal{A}, L^{\mathcal{A}})$ and $(\mathcal{B}, L^{\mathcal{B}})$ be compact quantum metric spaces, where $\mathcal{A}$ and $\mathcal{B}$ are unital C*-algebras and $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ are suitable seminorms on them. Let Π be a bridge between $\mathcal{A}$ and $\mathcal{B}$. Then $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ can be used to measure Π. We denote the resulting length of Π by l_Π. Then we will see, imprecisely speaking, that Π together with $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ determine a suitable seminorm, L_Π, on $\mathcal{A} \oplus \mathcal{B}$.

Theorem 1.1 (*Imprecise Version of Theorem 5.7*)**.** *Let $(\mathcal{A}, L^{\mathcal{A}})$ and $(\mathcal{B}, L^{\mathcal{B}})$ be compact quantum metric spaces, and let Π be a bridge between $\mathcal{A}$ and $\mathcal{B}$, with corresponding seminorm L_Π on $\mathcal{A} \oplus \mathcal{B}$. Let l_Π be the length of Π as measured using $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$. Let $p \in \mathcal{A}$ and $q \in \mathcal{B}$ be projections. If $l_\Pi L_\Pi(p, q) < 1/2$, and if q_1 is another projection in $\mathcal{B}$ such that $l_\Pi L_\Pi(p, q_1) < 1/2$, then there is a continuous path, $t \to q_t$ of projections in $\mathcal{B}$ going from q to q_1, so that the projective modules corresponding to q and q_1 are isomorphic. In this way, to the projective $\mathcal{A}$-module determined by p we have associated a uniquely determined isomorphism class of projective $\mathcal{B}$-modules.*

In Sections 7 through 12, theorems of this type are then applied to the specific situation of matrix algebras converging to the 2-sphere, in order to obtain our correspondence between projective modules over the 2-sphere and projective modules over the matrix algebras.

Very recently Latrémolière introduced a fairly different way of saying when two projective modules over compact quantum metric spaces that are close together correspond [27]. For this purpose he equips projective modules with seminorms that play the role of a weak analog of a connection on a vector bundle over a Riemannian manifold, much as our seminorms on a C*-algebra are a weak analog of the total derivative (or of the Dirac operator) of a Riemannian manifold. As experience is gained with more examples it will be interesting to discover the relative strengths and weaknesses of these two approaches.

My next project is to try to understand how the Dirac operator on the 2-sphere is related to "Dirac operators" on the matrix algebras that converge to the 2-sphere, especially since in the quantum physics literature there are at least three inequivalent Dirac operators suggested for the matrix algebras. This will involve results from [28].

2. Matrix Lip-norms and state spaces

As indicated above, the projections representing projective modules are elements of matrix algebras over the basic C*-algebras. In this section we give some useful perspective on the relations between certain types of seminorms on matrix algebras over a given C*-algebra and the metrics on the state spaces of the matrix algebras that come from the seminorms.

Let $\mathcal{A}$ be a unital C*-algebra. For a given natural number d let M_d denote the algebra of $d \times d$ matrices with entries in $\mathbb{C}$, and let $M_d(\mathcal{A})$ denote the C*-algebra of $d \times d$ matrices with entries in $\mathcal{A}$. Thus $M_d(\mathcal{A}) \cong M_d \otimes \mathcal{A}$. Since $\mathcal{A}$ is unital, we can, and will, identify M_d with the subalgebra $M_d \otimes 1_{\mathcal{A}}$ of $M_d(\mathcal{A})$.

We recall from definition 2.1 of [4] that by a "slip-norm" on a unital C*-algebra $\mathcal{A}$ we mean a $*$-seminorm L on $\mathcal{A}$ that is permitted to take the value $+\infty$ and is such that $L(1_A) = 0$. Given a slip-norm $L^{\mathcal{A}}$ on $\mathcal{A}$, we will need slip-norms, $L_d^{\mathcal{A}}$, on each $M_d(\mathcal{A})$ that correspond somewhat to $L^{\mathcal{A}}$. It is reasonable to want these seminorms to be coherent in some sense as d varies. As discussed before definition 5.1 of [4], the appropriate coherence requirement is that the sequence $\{L_d^{\mathcal{A}}\}$ forms a "matrix slip-norm". To recall what this means, for any positive integers m and n we let M_{mn} denote the linear space of $m \times n$ matrices with complex entries, equipped with the norm obtained by viewing such matrices as operators from the Hilbert space $\mathbb{C}^n$ to the Hilbert space $\mathbb{C}^m$. We then note that for any $A \in M_n(\mathcal{A})$, for any $\alpha \in M_{mn}$, and any $\beta \in M_{nm}$, the usual matrix product $\alpha A \beta$ is in $M_m(\mathcal{A})$.

Definition 2.1. A sequence $\{L_d^{\mathcal{A}}\}$ is a *matrix slip-norm* for $\mathcal{A}$ if $L_d^{\mathcal{A}}$ is a $*$-seminorm (with value $+\infty$ permitted) on $M_d(\mathcal{A})$ for each integer $d \geq 1$, and if this family of seminorms has the following properties:

1. For any $A \in M_d(\mathcal{A})$, any $\alpha \in M_{md}$, and any $\beta \in M_{dm}$, we have
$$L_m^{\mathcal{A}}(\alpha A \beta) \leq \|\alpha\| L_d^{\mathcal{A}}(A) \|\beta\|.$$
2. For any $A \in M_m(\mathcal{A})$ and any $C \in M_n(\mathcal{A})$ we have
$$L_{m+n}^{\mathcal{A}}\left(\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}\right) = \max(L_m^{\mathcal{A}}(A), L_n^{\mathcal{A}}(C)).$$
3. $L_1^{\mathcal{A}}$ is a slip-norm, in the sense that $L_1^{\mathcal{A}}(1_{\mathcal{A}}) = 0$. (But $L_1^{\mathcal{A}}$ is also allowed to take value 0 on elements that are not scalar multiples of the identity.)

We will say that such a matrix slip-norm is *regular* if $L_1^{\mathcal{A}}(a) = 0$ only for $a \in \mathbb{C}1_{\mathcal{A}}$.

The properties above imply that for $d \geq 2$ the null-space of $L_d^{\mathcal{A}}$ will contain the subalgebra M_d, not just the scalar multiples of the identity, so that $L_d^{\mathcal{A}}$ is a slip-norm . This is why our definition of a slip-norm does not require that the null-space be exactly the scalar multiples of the identity. When $\{L_d^{\mathcal{A}}\}$ is regular, the properties above imply that for $d \geq 2$ the null-space of $L_d^{\mathcal{A}}$ will be exactly M_d.

In generalization of the relation between $M_d(\mathcal{A})$ and M_d, let now $\mathcal{A}$ be any unital C*-algebra, and let $\mathcal{B}$ be a unital C*-subalgebra of $\mathcal{A}$ ($1_{\mathcal{A}} \in \mathcal{B}$). We let $S(\mathcal{A})$ denote the state space of $\mathcal{A}$, and similarly for $S(\mathcal{B})$. It will be useful for us to view $S(\mathcal{A})$ as fibered over $S(\mathcal{B})$ in the following way. For any $\nu \in S(\mathcal{B})$ let
$$S_\nu(\mathcal{A}) = \{\mu \in S(\mathcal{A}) : \mu|_{\mathcal{B}} = \nu\}.$$
Since each $\nu \in S(\mathcal{B})$ has at least one extension to an element of $S(\mathcal{A})$, no $S_\nu(\mathcal{A})$ is empty, and so the $S_\nu(\mathcal{A})$'s form a partition, or fibration, of $S(\mathcal{A})$ over $S(\mathcal{B})$. We will apply this observation to view $S(M_d(\mathcal{A}))$ as fibered over $S(M_d)$.

Now let $\{L_d^{\mathcal{A}}\}$ be a matrix slip-norm for $\mathcal{A}$. For any given d the seminorm $L_d^{\mathcal{A}}$ determines a metric $\rho^{L_d^{\mathcal{A}}}$ (with value $+\infty$ permitted, so often referred to as an "extended metric") on $S(M_d(\mathcal{A}))$ defined by
$$\rho^{L_d^{\mathcal{A}}}(\mu_1, \mu_2) = \sup\{|\mu_1(A) - \mu_2(A)| : L_d^{\mathcal{A}}(A) \leq 1\}. \tag{2.1}$$
(Notation like $\rho^{L_d^{\mathcal{A}}}$ will be used through most of this paper.) We then observe that if $\mu_1|_{M_d} \neq \mu_2|_{M_d}$ then $\rho^{L_d^{\mathcal{A}}}(\mu_1, \mu_2) = +\infty$, because there will exist an $A \in M_d$ such that $L_d^{\mathcal{A}}(rA) = 0$ for all $r \in \mathbb{R}^+$ while $|\mu_1(rA) - \mu_2(rA)| = r|\mu_1(A) - \mu_2(A)| \neq 0$. Thus $\rho^{L_d^{\mathcal{A}}}$ can be finite only on the fibers of the fibration of $S(M_d(\mathcal{A}))$ over $S(M_d)$.

Consistent with definition 5.1 of [29] (which treats the more general case of order-unit spaces) and theorem 1.8 of [30], we have:

Definition 2.2. Let $\mathcal{A}$ be a unital C*-algebra. By a *Lip-norm* on $\mathcal{A}$ we mean a $*$-seminorm L on $\mathcal{A}$ (with value $+\infty$ allowed) that satisfies

1. L is semifinite, i.e. $\{a \in \mathcal{A} : L(a) < \infty\}$ is dense in $\mathcal{A}$.
2. For any $a \in \mathcal{A}$ we have $L(a) = 0$ if and only if $a \in \mathbb{C}1_{\mathcal{A}}$.
3. L is lower semi-continuous with respect to the norm of $\mathcal{A}$, i.e. for any $r \in \mathbb{R}^+$ the set $\{a \in \mathcal{A} : L(a) \leq r\}$ is norm-closed.
4. The topology on $S(\mathcal{A})$ from the metric ρ^L, defined much as in Eq. (2.1), coincides with the weak-$*$ topology. This is equivalent to the property that the image of
$$\mathcal{L}^1_{\mathcal{A}} = \{a \in \mathcal{A} : a^* = a,\ L(a) \leq 1\}$$
in $\tilde{\mathcal{A}} = \mathcal{A}/\mathbb{C}1_{\mathcal{A}}$ is totally bounded for the quotient norm $\|\cdot\|^{\sim}$ on $\tilde{\mathcal{A}}$. (Or, equivalently, that the image in $\tilde{\mathcal{A}}$ of $\{a \in \mathcal{A} : L(a) \leq 1\}$ is totally bounded.)

Definition 2.3. Let $\mathcal{A}$ be a unital C*-algebra. By a *matrix Lip-norm* on $\mathcal{A}$ we mean a matrix slip-norm $\{L_d^{\mathcal{A}}\}$ for $\mathcal{A}$ which has the property that $L_1^{\mathcal{A}}$ is a Lip-norm for $\mathcal{A}$ and each $L_d^{\mathcal{A}}$ is lower semi-continuous.

We remark that from property (1) of Definition 2.1 it follows then that each $L_d^{\mathcal{A}}$ is semi-finite.

Proposition 2.4. *Let $\mathcal{A}$ be a unital C*-algebra and let $\{L_d^{\mathcal{A}}\}$ be a matrix Lip-norm on $\mathcal{A}$. For each natural number d let*
$$\mathcal{L}^1_{M_d(\mathcal{A})} = \{A \in M_d(\mathcal{A}) : A^* = A,\ L_d^{\mathcal{A}}(A) \leq 1\}.$$
Then the image of $\mathcal{L}^1_{M_d(\mathcal{A})}$ in the quotient $M_d(\mathcal{A})/M_d$ is totally bounded for the quotient norm.

Proof. Let $A \in \mathcal{L}_1^d$, with A the matrix $\{a_{jk}\}$. Then $L_d^{\mathcal{A}}(A) \leq 1$, and so by property (1) of Definition 2.1 we have $L_1^{\mathcal{A}}(a_{jk}) \leq 1$ for all $j,\ k$. Thus for each fixed pair $(j,\ k)$ the set of $(j,\ k)$-entries of all the elements of $\mathcal{L}^1_{M_d(\mathcal{A})}$ lie in $\{a \in \mathcal{A} : L(a) \leq 1\}$, whose image in $\tilde{\mathcal{A}}$ is totally bounded. But the finite product of totally bounded sets is totally bounded for any of the equivalent natural metrics on the product. $\square$

Proposition 2.5. *Let $\mathcal{A}$ be a unital C*-algebra and let $\{L_d^{\mathcal{A}}\}$ be a matrix Lip-norm on $\mathcal{A}$. For each natural number d and each $\nu \in S(M_d)$ the topology on the fiber $S_\nu(M_d(\mathcal{A}))$ in $S(M_d(\mathcal{A}))$ determined by the restriction to $S_\nu(M_d(\mathcal{A}))$ of the metric $\rho^{L_d^{\mathcal{A}}}$, agrees with the weak-$*$ topology restricted to $S_\nu(M_d(\mathcal{A}))$ (and so $S_\nu(M_d(\mathcal{A}))$ is compact).*

Proof. This is an immediate corollary of theorem 1.8 of [30] and Proposition 2.4 when one lets $M_d(\mathcal{A})$ be the normed space $A = \mathcal{L}$ of theorem 1.8 of [30], lets $L_d^{\mathcal{A}}$ be the L of that theorem, lets M_d be the subspace $\mathcal{K}$ of that theorem, and lets the state ν be the η of that theorem. □

As a consequence, even though $\rho^{L_d^{\mathcal{A}}}$ can take value $+\infty$, it is reasonable to talk about whether a subset Y of $S(M_d(\mathcal{A}))$ is ε-dense in $S(M_d(\mathcal{A}))$. This just means that, as usual, for each $\mu \in S(M_d(\mathcal{A}))$ there is an element of Y that is within distance ε of it. This observation will shortly be of importance to us.

A good class of simple examples to keep in mind for all of this is given next. It is the class that is central to the paper [5].

Example 2.6. Let (X, ρ) be a compact metric space, and let $\mathcal{A} = C(X)$. For a fixed natural number d let $L_d^{\mathcal{A}}$ be defined on $M_d(\mathcal{A}) = C(X, M_d)$ by

$$L_d^{\mathcal{A}}(F) = \sup\{\|F(x) - F(y)\|/\rho(x, y) : x, y \in X \quad \text{and} \quad x \neq y\}$$

for $F \in M_d(\mathcal{A})$. Then for $d \geq 2$ the metric on $S(M_d(\mathcal{A}))$ determined by $L_d^{\mathcal{A}}$ will take on value $+\infty$. But $S(M_d(\mathcal{A}))$ will be fibered over $S(M_d)$ and the metric will be finite on each fiber, and the topology it determines on the fiber will coincide with the weak-$*$ topology there.

Different ways of dealing with seminorms that may have a large null-space can be found in definitions 2.1 and 2.3 of [31] and in definition 2.3 of [32], but they do not seem to be useful for our present purposes.

3. Quotients of C*-metric spaces

Let (Z, ρ_Z) be a compact metric space, and let X be a closed subset of Z. Let $\mathcal{C} = C(Z)$ and let $\mathcal{A} = C(X)$. By restricting functions on Z to the subset X we see that $\mathcal{A}$ is a quotient algebra of $\mathcal{C}$. We need to consider the corresponding non-commutative situation. We will mostly use it for the non-commutative analog of the situation in which (X, ρ_X) and (Y, ρ_Y) are two compact metric spaces and Z is the disjoint union of X and Y (with ρ_Z compatible with ρ_X and ρ_Y). Then $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$ where $\mathcal{B} = C(Y)$. We will need the matricial version of this situation.

Accordingly, let $\mathcal{A}$ and $\mathcal{C}$ be unital C*-algebras, and let π be a surjective $*$-homomorphism from $\mathcal{C}$ onto $\mathcal{A}$, so that $\mathcal{A}$ is a quotient of $\mathcal{C}$. Then by composing with π, every state of $\mathcal{A}$ determines a state of $\mathcal{C}$. In this way we obtain a continuous injection of $S(\mathcal{A})$ into $S(\mathcal{C})$, and we will often just view $S(\mathcal{A})$ as a subset of $S(\mathcal{C})$ without explicitly mentioning π.

We think of $\mathcal{A}$ and $\mathcal{C}$ as possibly being matrix algebras over other algebras, and so we will consider a slip-norm, L, on $\mathcal{C}$, not requiring that L take value 0 only on $\mathbb{C}1_{\mathcal{C}}$. Thus the metric ρ^L on $S(\mathcal{C})$ can take value $+\infty$, but we can consider the situation in which, nevertheless, $S(\mathcal{A})$ is ε-dense in $S(\mathcal{C})$ for some given $\varepsilon \in \mathbb{R}^+$.

The main result of this section is the following proposition, which is a generalization of key lemma 4.1 of [5]. A related result in a more restricted setting, relevant to our next section, is emphasized in the paragraph before remark 6.5 of [26]. The inequality obtained in our proposition will be basic for later sections of this paper.

Proposition 3.1. *Let $\mathcal{A}$ and $\mathcal{C}$ be unital C*-algebras, and let π be a surjective $*$-homomorphism of $\mathcal{C}$ onto $\mathcal{A}$, so that $S(\mathcal{A})$ can be viewed as a subset of $S(\mathcal{C})$. Let L be a slip-norm on $\mathcal{C}$, and let there be given $\varepsilon \in \mathbb{R}^+$. If $S(\mathcal{A})$ is ε-dense in $S(\mathcal{C})$ for ρ^L, then for any $c \in \mathcal{C}$ satisfying $c^* = c$ we have*

$$\|c\| \leq \|\pi(c)\| + \varepsilon L(c).$$

Proof. Let $c \in \mathcal{C}$ satisfy $c^* = c$. Then there is a $\mu \in S(\mathcal{C})$ such that $|\mu(c)| = \|c\|$. By assumption there is a $\nu \in S(\mathcal{A})$ such that $\rho^L(\nu \circ \pi, \ \mu) \leq \varepsilon$. This implies that

$$|\nu(\pi(c)) - \mu(c)| \leq \varepsilon L(c),$$

so that

$$\|c\| = |\mu(c)| \leq |\nu(\pi(c))| + \varepsilon L(c) \leq \|\pi(c)\| + \varepsilon L(c),$$

as needed. □

It would be interesting to know whether the converse of this proposition is true, that is, whether the inequality implies the ε-denseness. It is true for ordinary compact metric spaces.

4. Bridges and ε-density

We now recall how in [4] we used slip-norms in connection with Latrémolière's bridges so that we are able to deal also with matricial bridges. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras, and let $\Pi = (\mathcal{D}, \omega)$ be a bridge from $\mathcal{A}$ to $\mathcal{B}$ in the sense of Latrémolière [26,33]. That is, $\mathcal{A}$ and $\mathcal{B}$ are identified as C*-subalgebras of the C*-algebra $\mathcal{D}$ that each contain $1_{\mathcal{D}}$, while $\omega \in \mathcal{D}$, $\omega^* = \omega$, $\|\omega\| \leq 1$, and $1 \in \sigma(\omega)$ (which is more than Latrémolière requires, but which holds for our main examples). Latrémolière calls ω the "pivot" of the bridge. The specific bridges that we will use for the case of $SU(2)$ are described in Section 6.

Fix a positive integer d. We can view $M_d(\mathcal{A})$ and $M_d(\mathcal{B})$ as unital subalgebras of $M_d(\mathcal{D})$. Let $\omega_d = I_d \otimes \omega$, where I_d is the identity element of M_d, so ω_d can be viewed as the diagonal matrix in $M_d(\mathcal{D})$ with ω in each diagonal entry. Then it is easily seen that $\Pi_d = (M_d(\mathcal{D}), \omega_d)$ is a bridge from $M_d(\mathcal{A})$ to $M_d(\mathcal{B})$.

Definition 4.1. For each natural number d let $\omega_d = I_d \otimes \omega$. The bridges $\Pi_d = (M_d(\mathcal{D}), \omega_d)$ are called the *matricial bridges* determined by the bridge Π.

Let $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ be Lip-norms on $\mathcal{A}$ and $\mathcal{B}$. Latrémolière defines [26,33] how to use them to measure bridges from $\mathcal{A}$ to $\mathcal{B}$. We recall here how in section 2 of [4] I adapted his definitions to the case of matricial bridges, using matrix Lip-norms. Let $\{L_d^{\mathcal{A}}\}$ be a matrix Lip-norm on $\mathcal{A}$ and let $\{L_d^{\mathcal{B}}\}$ be a matrix Lip-norm on $\mathcal{B}$. Fix d. Then $L_d^{\mathcal{A}}$ is a slip-norm on $M_d(\mathcal{A})$ and $L_d^{\mathcal{B}}$ is a slip-norm on $M_d(\mathcal{B})$, and they can be used to measure the bridge Π_d, by making only minor modifications to Latrémolière's definition. We review how this is done, but for notational simplicity we will not restrict attention to matrix algebras over algebras. Instead we will work with general unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$, but we will use slip-norms on them. So, let $\mathcal{A}$ and $\mathcal{B}$ be equipped with slip-norms $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$. We use these slip-norms to measure a bridge Π from $\mathcal{A}$ to $\mathcal{B}$, as follows.

Set, much as before,

$$\mathcal{L}^1_{\mathcal{A}} = \{a \in \mathcal{A} : a^* = a \text{ and } L^{\mathcal{A}}(a) \leq 1\},$$

and similarly for $\mathcal{L}^1_{\mathcal{B}}$. We view these as subsets of $\mathcal{D}$.

Definition 4.2. The *reach* of Π is defined by:

$$\mathrm{reach}(\Pi) = \mathrm{Haus}_{\mathcal{D}}\{\mathcal{L}^1_{\mathcal{A}}\omega\ ,\ \omega\mathcal{L}^1_{\mathcal{B}}\},$$

where $\mathrm{Haus}_{\mathcal{D}}$ denotes the Hausdorff distance with respect to the norm of $\mathcal{D}$, and where the product defining $\mathcal{L}^1_{\mathcal{A}}\omega$ and $\omega\mathcal{L}^1_{\mathcal{B}}$ is that of $\mathcal{D}$. We will often write r_Π for $\mathrm{reach}(\Pi)$. Note that r_Π can be $+\infty$.

We now show that when a slip-norm is part of a matrix Lip-norm, as defined in Definition 2.3, its reach is always finite. By definition, the metric on the state space determined by a Lip-norm gives the weak-$*$ topology. Since the state space is compact, it therefore has finite diameter for the metric. Given a unital C*-algebra $\mathcal{A}$ and a Lip-norm $L^{\mathcal{A}}$ on it, we denote the diameter of $S(\mathcal{A})$ for the corresponding metric, $\rho^{\mathcal{A}}$, by $\mathrm{diam}(\mathcal{A})$ (not mentioning $L^{\mathcal{A}}$ unless confusion may arise, as is common practice).

Lemma 4.3. *Let $\mathcal{A}$ be a unital C*-algebra, and let $L^{\mathcal{A}}$ be a Lip-norm on $\mathcal{A}$. Let ν be a state of $\mathcal{A}$. For any $a \in \mathcal{A}$ we have*

$$\|a - \nu(a)1_{\mathcal{A}}\| \leq 2\ \mathrm{diam}(\mathcal{A})L^{\mathcal{A}}(a).$$

Proof. Suppose first that $a \in \mathcal{A}$ with $a^* = a$. For any state μ on $\mathcal{A}$ we have

$$|\mu(a - \nu(a)1_{\mathcal{A}})| = |\mu(a) - \nu(a)| \leq \rho^{\mathcal{A}}(\mu, \nu)L^{\mathcal{A}}(a) \leq \mathrm{diam}(\mathcal{A})L^{\mathcal{A}}(a).$$

Consequently $\|a - \nu(a)1_{\mathcal{A}}\| \leq \mathrm{diam}(\mathcal{A})L^{\mathcal{A}}(a)$. For general $a \in \mathcal{A}$, when we apply this inequality to the real and imaginary parts of a we obtain the desired result. □

Proposition 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras, and let $\Pi = (\mathcal{D}, \omega)$ be a bridge from $\mathcal{A}$ to $\mathcal{B}$. Let $\{L_d^{\mathcal{A}}\}$ be a matrix Lip-norm on $\mathcal{A}$ and let $\{L_d^{\mathcal{B}}\}$ be a matrix Lip-norm on $\mathcal{B}$. Let $\mathrm{diam}(\mathcal{A})$ be the diameter of $\mathcal{A}$ for the Lip-norm $L_1^{\mathcal{A}}$, and similarly for $\mathcal{B}$. Then for any natural number d we have*

$$r_{\Pi_d} \leq 2\,d\ \max\{\mathrm{diam}(\mathcal{A}),\ \mathrm{diam}(\mathcal{B})\}.$$

Proof. By definition, $1 \in \sigma(\omega)$, so we can find a $\psi \in S(\mathcal{D})$ such that $\psi(\omega) = 1$. We fix such a ψ. Let d be given. Let $A \in M_d(\mathcal{A})$ with $A^* = A$ and $L_d^{\mathcal{A}}(A) \leq 1$, and $A = \{a_{jk}\}$. Define $B = \{b_{jk}\}$ in $M_d(\mathcal{B})$ by $b_{jk} = \psi(a_{jk})1_{\mathcal{D}}$. Clearly $B^* = B$, and $L_d(B) = 0$ by conditions 1 and 2 of Definition 2.1. Then

$$\begin{aligned}\|A\omega_d - \omega_d B\| &= \|\{a_{jk}\omega - \omega\psi(a_{jk})\}\| = \|(A - B)\omega_d\| \\ &\leq \|A - B\| \leq d\max\{\|a_{jk} - \psi|_{\mathcal{A}}(a_{jk})\|\} \\ &\leq 2\,d\,\mathrm{diam}(\mathcal{A})\max\{L_1^{\mathcal{A}}(a_{jk})\} \leq 2\,d\,\mathrm{diam}(\mathcal{A}),\end{aligned}$$

where for the next-to-last inequality we have used Lemma 4.3, and for the last inequality we have used condition 1 of Definition 2.1 and the fact that $L_d^{\mathcal{A}}(A) \leq 1$. In this way we see that $A\omega_d$ is within distance $2d$ diam$(\mathcal{A})$ of $\omega_d\mathcal{L}^1_{M_d(\mathcal{B})}$. On reversing the roles of A and B, we see that we have the desired result. □

To define the height of Π we need to consider the state space, $S(\mathcal{A})$, of $\mathcal{A}$, and similarly for $\mathcal{B}$ and $\mathcal{D}$. Even more, we set

$$S_1(\omega) = \{\phi \in S(\mathcal{D}) : \phi(\omega) = 1\},$$

the "level-1 set of ω ". It is not empty because by assumption $1 \in \sigma(\omega)$. The elements of $S_1(\omega)$ are "definite" on ω in the sense [34] that for any $\phi \in S_1(\omega)$ and $d \in \mathcal{D}$ we have

$$\phi(d\omega) = \phi(d) = \phi(\omega d).$$

Let $\rho^{\mathcal{A}}$ denote the metric on $S(\mathcal{A})$ determined by $L^{\mathcal{A}}$, defined, much is in Eq. (2.1), by

$$\rho^{\mathcal{A}}(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in \mathcal{L}^1_{\mathcal{A}}\}. \tag{4.1}$$

(Since we now are not assuming we have Lip-norms, we must permit $\rho^{\mathcal{A}}$ to take the value $+\infty$. Also, it is not hard to see that the supremum can be taken equally well just over all of $\{a \in \mathcal{A} : L^{\mathcal{A}}(a) \leq 1\}$.) Define $\rho^{\mathcal{B}}$ on $S(\mathcal{B})$ similarly.

Notation 4.5. *We denote by $S_1^{\mathcal{A}}(\omega)$ the set of restrictions of the elements of $S_1(\omega)$ to $\mathcal{A}$. We define $S_1^{\mathcal{B}}(\omega)$ similarly.*

Definition 4.6. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras and let $\Pi = (\mathcal{D}, \omega)$ be a bridge from $\mathcal{A}$ to $\mathcal{B}$. Let $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ be slip-norms on $\mathcal{A}$ and $\mathcal{B}$. The *height* of the bridge Π is given by

$$\text{height}(\Pi) = \max\{\text{Haus}_{\rho^{\mathcal{A}}}(S_1^{\mathcal{A}}(\omega), S(\mathcal{A})),\ \text{Haus}_{\rho^{\mathcal{B}}}(S_1^{\mathcal{B}}(\omega), S(\mathcal{B}))\},$$

where the Hausdorff distances are with respect to the indicated metrics determined by $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ (with value $+\infty$ allowed). We will often write h_Π for height(Π). The length of Π is then defined by

$$\text{length}(\Pi) = \max\{\text{reach}(\Pi),\ \text{height}(\Pi)\}.$$

Up to now I have not found a proof of the analog for height of Proposition 4.4, namely that when matrix Lip-norms are involved the height is always finite, though I suspect that this is true. For the "bridges with conditional expectation" that we will use later (with matrix Lip-norms) the height, and so the length, is always finite.

Anyway, we will now just make the quite strong assumption that length$(\Pi) < \infty$. It is shown in section 6 of [4] that this assumption is satisfied for the specific class of examples that we deal with in the present paper. This will be somewhat reviewed later in Section 12. The main consequence of this assumption for our present purposes is a generalization to our present non-commutative setting of key lemma 4.1 of [5]. This generalization will yield for this case the same inequality as just found in Proposition 3.1 but with the added information of a relevant value for ε. The core calculations for this generalization can essentially be found in the middle of the proof of proposition 5.3 of [26]. We will call our generalization again the Key Lemma. The set-up is as follows. As above, let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras, and let $\Pi = (\mathcal{D}, \omega)$ be a bridge between them. Let $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ be slip-norms on $\mathcal{A}$ and $\mathcal{B}$, and let r_Π and h_Π denote the reach and height of Π as measured by $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$. Assume that r_Π and h_Π are both finite. Define a seminorm, N_Π, on $\mathcal{A} \oplus \mathcal{B}$ by

$$N_\Pi(a, b) = \|a\omega\ -\ \omega b\|.$$

Notice that N_Π is in general not a $*$-seminorm. Much as in theorem 6.2 of [3], define a $*$-seminorm, $\hat{N}_\Pi$, on $\mathcal{A} \oplus \mathcal{B}$ by

$$\hat{N}_\Pi(a, b) = N_\Pi(a, b) \vee N_\Pi(a^*, b^*),$$

where $\vee$ means "maximum". Of course, $\hat{N}_\Pi$ agrees with N_Π on self-adjoint elements.

Let $r \geq r_\Pi$ be chosen. (The reason for not just taking $r = r_\Pi$ will be given in the fifth paragraph of the proof of Theorem 12.1.) Define a seminorm, L^r, on the C*-algebra $\mathcal{A} \oplus \mathcal{B}$ by

$$L^r(a, b) = L^{\mathcal{A}}(a) \vee L^{\mathcal{B}}(b) \vee r^{-1}\hat{N}_\Pi(a, b). \tag{4.2}$$

Then L^r is a slip-norm on $\mathcal{A} \oplus \mathcal{B}$, and it determines a metric, ρ^{L^r}, on $S(\mathcal{A} \oplus \mathcal{B})$. Note that $\mathcal{A}$ and $\mathcal{B}$ are both quotients of $\mathcal{A} \oplus \mathcal{B}$ in an evident way, so that we can consider the quotient seminorms on them coming from L^r. As discussed around example 5.4 of [3], there are complications with quotients of $*$-seminorms on non-self-adjoint elements. Accordingly, much as for notation 5.5 of [3], we make:

Definition 4.7. Let $\mathcal{A}$ and $\mathcal{B}$, and $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ be as above. We say that a $*$-seminorm L on $\mathcal{A} \oplus \mathcal{B}$ is *admissible* for $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ if its quotient on $\mathcal{A}$ agrees with $L^{\mathcal{A}}$ on self-adjoint elements of $\mathcal{A}$, and similarly for its quotient on $\mathcal{B}$.

Proposition 4.8. *With notation as above, if $r \geq r_\Pi$ then L^r is admissible for $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$.*

The proof of this proposition is implicit in the proof of theorem 6.3 of [26], and amounts to showing that $(a, b) \to r^{-1}N_\Pi(a, b)$, when restricted to self-adjoint elements, is a "bridge" in the more primitive sense defined in definition 5.1 of [35], and then using the main part of the proof of theorem 5.2 of [35]. For the reader's convenience we give a short direct proof here, in particular because we will need related facts in Section 12.

Proof. Clearly $L^r(a, b) \geq L^{\mathcal{A}}(a)$ for every b. It follows that the quotient of L^r on $\mathcal{A}$ is no smaller than $L^{\mathcal{A}}$. Let $a \in \mathcal{A}$ with $a^* = a$ and $L^{\mathcal{A}}(a) = 1$. Let $\varepsilon > 0$ be given. By the definition of r_Π there is a $b \in \mathcal{B}$ with $b^* = b$ and $L^{\mathcal{B}}(b) \leq 1$ such that $\|a\omega - \omega b\| \leq r_\Pi + \varepsilon$. Since $r \geq r_\Pi$ it follows that

$$L^{\mathcal{B}}(b) \vee r^{-1}\|a\omega - \omega b\| \ \leq \ 1 + r^{-1}\varepsilon = L^{\mathcal{A}}(a) + r^{-1}\varepsilon.$$

Since ε is arbitrary, it follow that on a the quotient of L^r is equal to $L^{\mathcal{A}}(a)$. By scaling it follows that the quotient of L^r on $\mathcal{A}$ agrees with $L^{\mathcal{A}}$ on all self-adjoint elements. Reversing the roles of $\mathcal{A}$ and $\mathcal{B}$, we obtain the corresponding fact for the quotient of L^r on $\mathcal{B}$. □

The following lemma, which is closely related to the comments in the paragraph before remark 6.5 of [26], shows how Proposition 3.1 is relevant to the context of bridges.

Key Lemma 4.9. *With notation as above, let $(a, b) \in \mathcal{A} \oplus \mathcal{B}$ with $a^* = a$ and $b^* = b$. Let $r \geq r_\Pi$ be chosen, and let L^r be defined on $\mathcal{A} \oplus \mathcal{B}$ by* Eq. (4.2)*. Then*

$$\|(a, b)\| \leq \|a\| \ + \ (h_\Pi + r)L^r(a, b),$$

and similarly with the roles of a and b interchanged.

Proof. By scaling, it suffices to prove this under the assumption that $L^r(a, b) = 1$, so we assume this. Let $\nu \in S(\mathcal{B})$. By the definition of h_Π there is a $\psi \in S_1(\omega)$ such that $\rho_{L^{\mathcal{B}}}(\nu, \psi|_{\mathcal{B}}) \leq h_\Pi$. Then, since ψ is definite on ω , and $L^{\mathcal{B}}(b) \leq L^r(a, b) \leq 1$, we have:

$$\begin{aligned}
|\nu(b)| &\leq |\nu(b) - \psi(b)| + |\psi(\omega b)| \\
&\leq \rho_{L^{\mathcal{B}}}(\nu, \psi|_{\mathcal{B}})L^{\mathcal{B}}(b) + |\psi(\omega b) - \psi(a\omega)| + |\psi(a\omega)| \\
&\leq h_\Pi + \|a\omega \ - \omega b\| + |\psi(a)| \\
&\leq h_\Pi + r + \|a\|.
\end{aligned}$$

Since this holds for all $\nu \in S(\mathcal{B})$, and since $b^* = b$, it follows that $\|b\| \leq \|a\| + h_\Pi + r$, and so

$$\|(a, b)\| \leq \|a\| + h_\Pi + r,$$

as needed. □

5. Projections and Leibniz seminorms

We now assume that the slip-norms $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ on $\mathcal{A}$ and $\mathcal{B}$ are lower semi-continuous with respect to the norm topologies on $\mathcal{A}$ and $\mathcal{B}$. It is then clear that L^r, as defined in Eq. (4.2), is lower semi-continuous on $\mathcal{A} \oplus \mathcal{B}$ since N_Π is norm-continuous on $\mathcal{A} \oplus \mathcal{B}$. We now also assume that $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ satisfy the Leibniz inequality, that is,

$$L^{\mathcal{A}}(aa') \leq L^{\mathcal{A}}(a)\|a'\| + \|a\|L^{\mathcal{A}}(a')$$

for any $a, a' \in \mathcal{A}$, and similarly for $\mathcal{B}$. We need to assume in addition that $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ are *strongly* Leibniz, that is, that if $a \in \mathcal{A}$ is invertible in $\mathcal{A}$, then

$$L^{\mathcal{A}}(a^{-1}) \leq \|a^{-1}\|^2 L^{\mathcal{A}}(a),$$

and similarly for $\mathcal{B}$. Then a simple computation, discussed at the beginning of the proof of theorem 6.2 of [3], shows that L^r is strongly Leibniz on $\mathcal{A} \oplus \mathcal{B}$. We will also assume that $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ are semi-finite in the sense that $\{a \in \mathcal{A} : L^{\mathcal{A}}(a) < \infty\}$ is dense in $\mathcal{A}$, and similarly for $\mathcal{B}$. Then L^r is also semi-finite.

With the above structures as motivation, we will now adapt to our non-commutative setting many of the basic results of sections 2, 3 and 4 of [5]. For notational simplicity we first consider a unital C*-algebra $\mathcal{C}$ (such as $\mathcal{A} \oplus \mathcal{B}$) equipped with a semi-finite lower-semi-continuous strongly-Leibniz slip-norm $L^{\mathcal{C}}$ (defined on all of $\mathcal{C}$).

We now consider the relation between the strong Leibniz property and the holomorphic functional calculus, along the lines of section 2 of [5]. Let $c \in \mathcal{C}$ and let θ be a $\mathbb{C}$-valued function defined and holomorphic in some neighborhood of the spectrum, $\sigma(c)$, of c. In the standard way used for ordinary Cauchy integrals, we let γ be a collection of piecewise-smooth oriented closed curves in the domain of θ that surrounds $\sigma(c)$ but does not meet $\sigma(c)$, such that θ on $\sigma(c)$ is represented by its Cauchy integral using γ. Then $z \mapsto (z-c)^{-1}$ will, on the range of γ, be a well-defined and continuous function with values in $\mathcal{C}$. Thus we can define $\theta(c)$ by

$$\theta(c) = \frac{1}{2\pi i}\int_\gamma \theta(z)(z-c)^{-1}dz.$$

For a fixed neighborhood of $\sigma(c)$ containing the range of γ the mapping $\theta \mapsto \theta(c)$ is a unital homomorphism from the algebra of holomorphic functions on this neighborhood of $\sigma(c)$ into $\mathcal{C}$ [34,36]. The following proposition is the generalization of proposition 2.3 of [5] that we need here.

Proposition 5.1. *Let $L^{\mathcal{C}}$ be a lower-semicontinuous strongly-Leibniz slip-norm on $\mathcal{C}$. For $c \in \mathcal{C}$, and for θ and γ as above, we have*

$$L^{\mathcal{C}}(\theta(c)) \leq \left(\frac{1}{2\pi}\int_\gamma |\theta(z)|d|z|\right)(M_\gamma(c))^2 L^{\mathcal{C}}(c),$$

where $M_\gamma(c) = \max\{\|(z-c)^{-1}\| : z \in \mathrm{range}(\gamma)\}$.

Proof. It suffices to prove this for $L^{\mathcal{C}}(c) < \infty$. Now $L^{\mathcal{C}}$ is lower-semicontinuous, and so it can be brought within the integral defining $\theta(c)$, with the evident inequality. (Think of approximating the integral by Riemann sums.) Because $L^{\mathcal{C}}$ is strongly Leibniz, this gives

$$L^{\mathcal{C}}(\theta(c)) \leq \frac{1}{2\pi}\int_\gamma |\theta(z)|\ \|(z-c)^{-1}\|^2\ L^{\mathcal{C}}(c)\ d|z|.$$

On using the definition of $M_\gamma(c)$ we obtain the desired inequality. □

This proposition shows that $\{c \in \mathcal{C} : L^{\mathcal{C}}(c) < \infty\}$ is closed under the holomorphic functional calculus (and is a dense $*$-subalgebra of $\mathcal{C}$ as seen earlier). The next proposition is essentially proposition 3.1 of [5]. It is a known result (see, e.g., section 3.8 of [37]). We do not repeat here the proof of it given in [5].

Proposition 5.2. *Let $\mathcal{C}$ be a unital C^*-algebra, and let C' be a dense $*$-subalgebra closed under the holomorphic functional calculus in $\mathcal{C}$. Let p be a projection in $\mathcal{C}$. Then for any $\delta > 0$ there is a projection p_1 in C' such that $\|p - p_1\| < \delta$. If $\delta < 1$ then p_1 is homotopic to p through projections in $\mathcal{C}$, that is, there is a continuous path of projections in $\mathcal{C}$ going from p_1 to p.*

We apply this result to $\{c \in \mathcal{C} : L^{\mathcal{C}}(c) < \infty\}$. The next proposition is almost exactly proposition 3.3 of [5]. We will not repeat the proof here. It involves Proposition 5.1 and a mildly complicated argument involving contour integrals.

Proposition 5.3. *Let $\mathcal{C}$ be a unital C^*-algebra and let $L^{\mathcal{C}}$ be a strongly-Leibniz lower-semicontinuous slip-norm on $\mathcal{C}$. Let p_0 and p_1 be two projections in $\mathcal{C}$. Suppose that $\|p_0 - p_1\| \leq \delta < 1$, so that there is a norm-continuous path, $t \mapsto p_t$, of projections in $\mathcal{C}$ going from p_0 to p_1 [25,38]. If $L^{\mathcal{C}}(p_0) < \infty$ and $L^{\mathcal{C}}(p_1) < \infty$, then we can arrange that*

$$L^{\mathcal{C}}(p_t) \leq (1-\delta)^{-1}\max\{L^{\mathcal{C}}(p_0), L^{\mathcal{C}}(p_1)\}$$

for every t.

We now let $\mathcal{A}$ be a unital C*-algebra that is a quotient of $\mathcal{C}$, with $\pi : \mathcal{C} \to \mathcal{A}$ the quotient map (such as the evident quotient map from our earlier $\mathcal{A} \oplus \mathcal{B}$ onto $\mathcal{A}$). We let $L^{\mathcal{A}}$ be the quotient of $L^{\mathcal{C}}$ on $\mathcal{A}$, and we assume that $L^{\mathcal{A}}$ is semi-finite, lower semi-continuous, and strongly Leibniz (which is not automatic — see section 5 of [3]). Motivated by Key Lemma 4.9, we will be making hypotheses such as that there is an $\varepsilon > 0$ (such as $h^\Pi + r$) such that

$$\|c\| \leq \|\pi(c)\| + \varepsilon L^{\mathcal{C}}(c)$$

for all $c \in \mathcal{C}$. The next proposition is our non-commutative version of theorem 4.2 of [5].

Theorem 5.4. *Let $\mathcal{C}$, $\mathcal{A}$, π, $L^{\mathcal{C}}$,and $L^{\mathcal{A}}$ be as above. Suppose given an $\varepsilon > 0$ such that for all $c \in \mathcal{C}$ with $c^* = c$ we have*

$$\|c\| \leq \|\pi(c)\| + \varepsilon L^{\mathcal{C}}(c).$$

Let p_0 and p_1 be projections in $\mathcal{A}$, and let q_0 and q_1 be projections in $\mathcal{C}$ such that $\pi(q_0) = p_0$ and $\pi(q_1) = p_1$. Set

$$\delta = \|p_0 - p_1\| + \varepsilon(L^{\mathcal{C}}(q_0) + L^{\mathcal{C}}(q_1)).$$

If $\delta < 1$, then there is a path, $t \mapsto q_t$, through projections in $\mathcal{C}$, from q_0 to q_1, such that

$$L^{\mathcal{C}}(q_t) \leq (1-\delta)^{-1}\max\{L^{\mathcal{C}}(q_0), L^{\mathcal{C}}(q_1)\}$$

for all $t \in [0, 1]$.

Proof. From the hypotheses we see that

$$\begin{aligned}\|q_0 - q_1\| &\leq \|\pi(q_0 - q_1)\| + \varepsilon L^{\mathcal{C}}(q_0 - q_1) \\ &\leq \|p_0 - p_1\| + \varepsilon(L^{\mathcal{C}}(q_0) + L^{\mathcal{C}}(q_1)) = \delta.\end{aligned}$$

Assume now that $\delta < 1$. Then according to Proposition 5.3 applied to q_0 and q_1, there is a path $t \to q_t$ from q_0 to q_1 with the stated properties. □

If $p_0 = p_1$ above then we can obtain some additional information. The following proposition is almost exactly proposition 4.3 of [5]. We do not repeat the proof here.

Proposition 5.5. *With hypotheses as above, let $p \in \mathcal{A}$, and let q_0 and q_1 be projections in $\mathcal{C}$ such that $\pi(q_0) = p = \pi(q_1)$. If $\varepsilon L^{\mathcal{C}}(q_0) < 1/2$ and $\varepsilon L^{\mathcal{C}}(q_1) < 1/2$, then there is a path, $t \to q_t$, through projections in $\mathcal{C}$, from q_0 to q_1, such that $\pi(q_t) = p$ and*

$$L^{\mathcal{C}}(q_t) \leq (1-\delta)^{-1} \max\{L^{\mathcal{C}}(q_0), L^{\mathcal{C}}(q_1)\}$$

for all t, where $\delta = \varepsilon(L^{\mathcal{C}}(q_0) + L^{\mathcal{C}}(q_1))$.

By concatenating paths, we can combine the above results to obtain some information that does not depend on p_0 and p_1 being close together. The next proposition is almost exactly corollary 4.4 of [5].

Corollary 5.6. *Let p_0 and p_1 be projections in $\mathcal{A}$, and let q_0 and q_1 be projections in $\mathcal{C}$ such that $\pi(q_0) = p_0$ and $\pi(q_1) = p_1$. Let K be a constant such that $L^{\mathcal{C}}(q_j) \leq K$ for $j = 0, 1$. Assume further that there is a path p from p_0 to p_1 such that for each t there is a projection $\tilde{q}_t$ in $\mathcal{C}$ such that $\pi(\tilde{q}_t) = p_t$ and $L^{\mathcal{C}}(\tilde{q}_t) \leq K$. Then for any $r > 1$ there is a continuous path $t \mapsto q_t$ of projections in $\mathcal{C}$ going from q_0 to q_1 such that*

$$L^{\mathcal{C}}(q_t) \leq rK$$

for each t. (But we may not have $\pi(q_t) = p_t$ for all t.)

Proof. Given $r > 1$, choose $\delta > 0$ such that $(1-\delta)^{-1} < r$, and then choose an $\varepsilon > 0$ such that $2\varepsilon K < \delta$. Then follow the proof of corollary 4.4 of [5] with $N = K$. □

Let us now see what consequences the above uniqueness results have when we have a bridge between two C*-algebras that are equipped with suitable seminorms. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras and let $\Pi = (\mathcal{D}, \omega)$ be a bridge from $\mathcal{A}$ to $\mathcal{B}$. Let $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ be semi-finite lower-semi-continuous strongly Leibniz slip-norms on $\mathcal{A}$ and $\mathcal{B}$. We use them to measure Π, and we assume that r_Π is finite. Let $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$. Let $r \geq r_\Pi$ be chosen, and define the seminorm L^r on the C*-algebra $\mathcal{C}$ by Eq. (4.2). Note that L^r is admissible for $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ by Proposition 4.8. A projection in $\mathcal{C}$ will now be of the form (p, q) where p and q are projections in $\mathcal{A}$ and $\mathcal{B}$ respectively. Roughly speaking, our main idea is that p and q will correspond if $L^r(p, q)$ is relatively small. Notice that which projections then correspond to each other will strongly depend on the choice of Π (just as in [5], where it was seen that which projections for ordinary compact metric spaces correspond depends strongly on the choice of the metric that is put on the disjoint union of the two metric spaces, as would be expected). We will only consider that projections correspond (for a given bridge Π) if there is some uniqueness to the correspondence. The following theorem gives appropriate expression for this uniqueness. It is our non-commutative generalization of theorem 4.5 of [5], and it is an immediate consequence of Key Lemma 4.9, Proposition 5.5, and then Theorem 5.4. The role of the ε in Proposition 5.5 and Theorem 5.4 is now played by $h_\Pi + r$.

Theorem 5.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras, and let $\Pi = (\mathcal{D}, \omega)$ be a bridge from $\mathcal{A}$ to $\mathcal{B}$. Let $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ be lower semi-continuous strongly-Leibniz slip-norms on $\mathcal{A}$ and $\mathcal{B}$. Assume that the length of Π as measured by $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ is finite. Let $r \geq r_\Pi$ be chosen, and define L^r on $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$ by* Eq. (4.2).

(a) *Let $p \in \mathcal{A}$ and $q \in \mathcal{B}$ be projections, and suppose that*

$$(h_\Pi + r)L^r(p, q) < 1/2.$$

If q_1 is another projection in $\mathcal{B}$ such that $(h_\Pi + r)L^r(p, q_1) < 1/2$, then there is a path $t \mapsto q_t$ through projections in $\mathcal{B}$, going from q to q_1, such that

$$L^r(p, q_t) \leq (1-\delta)^{-1} \max\{L^r(p, q), L^r(p, q_1)\}$$

for all t, where $\delta = (h_\Pi + r)(L^r(p, q) + L^r(p, q_1))$. If instead there is a $p_1 \in \mathcal{A}$ such that $(h_\Pi + r)L^r(p_1, q) < 1/2$ then there is a corresponding path from p to p_1 with corresponding bound for $L^r(p_t, q)$.

(b) *Let p_0 and p_1 be projections in $\mathcal{A}$ and let q_0 and q_1 be projections in $\mathcal{B}$. Set*

$$\delta = \|p_0 - p_1\| + (h_\Pi + r)(L^r(p_0, q_0) + L^r(p_1, q_1)).$$

If $\delta < 1$ then there are continuous paths $t \mapsto p_t$ and $t \mapsto q_t$ from p_0 to p_1 and q_0 to q_1, respectively, through projections, such that

$$L^r(p_t, q_t) \leq (1-\delta)^{-1} \max\{L^r(p_0, q_0), L^r(p_1, q_1)\}$$

for all t.

We remark that a more symmetric way of stating part b) above is to define δ by

$$\delta = \max\{\|p_0 - p_1\|, \|q_0 - q_1\|\} + (h_\Pi + r)(L^r(p_0, q_0), L^r(p_1, q_1)).$$

Let us now examine the consequences of Corollary 5.6. This is best phrased in terms of:

Notation 5.8. *Let $\mathcal{P}(\mathcal{A})$ denote the set of projections in $\mathcal{A}$. For any $s \in \mathbb{R}^+$ let*

$$\mathcal{P}^s(\mathcal{A}) = \{p \in \mathcal{P}(\mathcal{A}) : L^{\mathcal{A}}(p) < s\},$$

and similarly for $\mathcal{B}$ and $\mathcal{C}$.

Now $\mathcal{P}^s(\mathcal{A})$ may have many path components. As suggested by the main results of [5], it may well be appropriate, indeed necessary, to view these different path components as representing *inequivalent* vector bundles, even if algebraically the vector bundles are isomorphic. That is the main idea of [5], and of the present paper. (Some additional perspective on this idea will be given in Section 13.) Let Σ be one of these path components. Let $\Phi_{\mathcal{A}}$ denote the evident restriction map from $\mathcal{P}(\mathcal{C})$ to $\mathcal{P}(\mathcal{A})$ (for $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$). For a given $s' \in \mathbb{R}^+$ with $s' \geq s$ it may be that $\Phi_{\mathcal{A}}(\mathcal{P}^{s'}(\mathcal{C})) \cap \Sigma$ is non-empty. This is an existence question, which we will not deal with here. But at this point, from Corollary 5.6 we obtain our non-commutative version of theorem 4.7 of [5], namely:

Theorem 5.9. *Let notation be as above, and assume that* length$(\Pi) < \varepsilon$. *Let $s \in \mathbb{R}^+$ with $\varepsilon s < 1/2$. Let Σ be a path component of $\mathcal{P}^s(\mathcal{A})$. Let $s' \in \mathbb{R}^+$ with $s' \geq s$ and $\varepsilon s' < 1/2$. Let $p_0,\ p_1 \in \Sigma$ and suppose that there are q_0 and q_1 in $\mathcal{P}^{s'}(\mathcal{B})$ with $L^r(p_j, q_j) \leq s'$ for $j = 0, 1$. Assume, even more, that there is a path $\tilde{p}$ in Σ connecting p_0 and p_1 that lies in $\Phi_{\mathcal{A}}(\mathcal{P}^{s'}(\mathcal{C}))$. Then for any δ with $2\varepsilon s' < \delta < 1$ there exist a path $t \mapsto p_t$ in $\mathcal{P}(\mathcal{A})$ going from p_0 to p_1 and a path $t \mapsto q_t$ in $\mathcal{P}(\mathcal{B})$ going from q_0 to q_1 such that $L^r(p_t, q_t) < (1-\delta)^{-1}s'$ for each t. The situation is symmetric between $\mathcal{A}$ and $\mathcal{B}$, so the roles of $\mathcal{A}$ and $\mathcal{B}$ can be interchanged in the above statement.*

Thus, in the situation described in the theorem, if Σ is a connected path component of $\mathcal{P}^s(\mathcal{A})$ that represents some particular class of projective $\mathcal{A}$-modules, then the projections $q \in \mathcal{P}^s(\mathcal{B})$ paired with ones in Σ by the requirement that $L^r(p, q) < s$, will be homotopic, and in particular will determine isomorphic projective $\mathcal{B}$-modules. We emphasize that the above pairing of projections depends strongly on the choice of Π, and not just on the quantum Gromov–Hausdorff propinquity between $\mathcal{A}$ and $\mathcal{B}$. This reflects the fact that quantum Gromov–Hausdorff propinquity is only a metric on *isometry classes* of quantum compact metric spaces, just as is the case for ordinary Gromov–Hausdorff distance for ordinary compact metric spaces.

Notice that the homotopies obtained above between q_0 and q_1 need not lie in $\mathcal{P}^s(\mathcal{B})$. We can only conclude that they lie in $\mathcal{P}^{s'}(\mathcal{B})$ where $s' = (1-\delta)^{-1}s$. But at least we can say that s' approaches s as ε, and so δ, goes to 0.

The results of this section suggest that the definition of a "C*-metric" given in definition 4.1 of [3] should be modified to use matrix seminorms, and so should be given by:

Definition 5.10. Let $\mathcal{A}$ be a unital C*-algebra. By a *C*-metric* on $\mathcal{A}$ we mean a matrix Lip-norm (as defined in Definition 2.3), $\{L_n^{\mathcal{A}}\}$, such that each $L_n^{\mathcal{A}}$ is strongly Leibniz.

We remark that in contrast to the contents of section 6 of [5], in the present paper we do not include here any *existence* theorems for bundles on quantum spaces that are close together. It appears that existence results in the non-commutative case are more difficult to obtain, but this matter remains to be explored carefully.

6. The algebras and the bridges

In this section we will introduce the specific algebras and the bridges to which we will apply the theory of the previous section. These are described in [4] and in earlier papers on this topic, but in greater generality than we use in the later parts of the present paper. Nevertheless, here we will begin by reviewing this more general setting, since it gives useful context, and the main results of this paper should eventually be generalized to the more general setting.

Let G be a compact group (perhaps even finite at first, but later to be $SU(2)$). Let U be an irreducible unitary representation of G on a (finite-dimensional) Hilbert space $\mathcal{H}$. Let $\mathcal{B} = \mathcal{L}(\mathcal{H})$ denote the C^*-algebra of all linear operators on $\mathcal{H}$ (a "full matrix algebra", with its operator norm). There is a natural action, α, of G on $\mathcal{B}$ by conjugation by U, that is, $\alpha_x(T) = U_x T U_x^*$ for $x \in G$ and $T \in \mathcal{B}$. Because U is irreducible, the action α is "ergodic", in the sense that the only α-invariant elements of $\mathcal{B}$ are the scalar multiples of the identity operator.

Fix a continuous length function, ℓ, on G (so G must be metrizable). Thus ℓ is non-negative, $\ell(x) = 0$ iff $x = e_G$ (the identity element of G), $\ell(x^{-1}) = \ell(x)$, and $\ell(xy) \leq \ell(x) + \ell(y)$. We also require that $\ell(xyx^{-1}) = \ell(y)$ for all $x, y \in G$. Then in terms of α and ℓ we can define a seminorm, $L^{\mathcal{B}}$, on $\mathcal{B}$ by the formula

$$L^{\mathcal{B}}(T) = \sup\{\|\alpha_x(T) - T\|/\ell(x) : x \in G \quad \text{and} \quad x \neq e_G\}. \tag{6.1}$$

Then $(\mathcal{B}, L_{\mathcal{B}})$ is an example of a compact C*-metric-space, as defined in definition 4.1 of [3]. In particular, $L_{\mathcal{B}}$ satisfies the conditions given there for being a Lip-norm, recalled in Definition 2.2.

Let P be a rank-1 projection in $\mathcal{B}$ (soon to be the projection on a highest weight subspace). Let H be the stability subgroup of P for α. Form the quotient space G/H (which later will be the sphere). We let λ denote the action of G on G/H, and so on $\mathcal{A} = C(G/H)$, by left-translation. Then from λ and ℓ we likewise obtain a seminorm, $L^{\mathcal{A}}$, on $\mathcal{A}$ by the evident analog of formula (6.1), except that we must now permit $L^{\mathcal{A}}$ to take the value ∞. It is shown in proposition 2.2 of [30] that the set of functions for which $L^{\mathcal{A}}$ is finite (the Lipschitz functions) is a dense $*$-subalgebra of $\mathcal{A}$. Also, $L^{\mathcal{A}}$ is the restriction to $\mathcal{A}$ of the seminorm on $C(G)$ that we get from ℓ and left translation, when we view $C(G/H)$ as a subalgebra of $C(G)$, as we will do when convenient. From $L^{\mathcal{A}}$ we can use Eq. (4.1) to recover the usual quotient metric [39] on G/H coming from the metric on G determined by ℓ. One can check easily that $L^{\mathcal{A}}$ in turn comes from this quotient metric. Thus $(\mathcal{A}, L^{\mathcal{A}})$ is the compact C*-metric-space associated to this ordinary compact metric space. Then for any bridge from $\mathcal{A}$ to $\mathcal{B}$ we can use $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ to measure the length of the bridge in the way given by Latrémolière [26], which we described in Definitions 4.2 and 4.6.

We now describe the natural bridge, $\Pi = (\mathcal{D}, \omega)$, from $\mathcal{A}$ to $\mathcal{B}$ that was first presented in section 2 of [4]. We take $\mathcal{D}$ to be the C*-algebra

$$\mathcal{D} = \mathcal{A} \otimes \mathcal{B} = C(G/H, \mathcal{B}).$$

We identify $\mathcal{A}$ with the subalgebra $\mathcal{A} \otimes 1_{\mathcal{B}}$ of $\mathcal{D}$, where $1_{\mathcal{B}}$ is the identity element of $\mathcal{B}$. Similarly, we identify $\mathcal{B}$ with the subalgebra $1_{\mathcal{A}} \otimes \mathcal{B}$ of $\mathcal{D}$. In view of many of the calculations done in [1,3] it is not a surprise that we define the pivot ω to be the function in $C(G/H, \mathcal{B})$ defined by

$$\omega(x) = \alpha_x(P)$$

for all $x \in G/H$, where P is the rank-1 projection chosen above. (It is a "coherent state".) We notice that ω is actually a non-zero projection in $\mathcal{D}$, and so it satisfies the requirements for being a pivot.

But projective modules over algebras are in general given by projections in matrix algebras over the given algebra, not just by projections in the algebra itself. This brings us back to the topic of matricial bridges which was introduced early in Section 4. We now apply the general matricial framework discussed there to the more specific situation described just above in which $\mathcal{A} = C(G/H)$, etc., with corresponding natural bridge Π, and then with its associated matricial bridges Π_d defined as in Definition 4.1. We must specify our matrix slip-norms. This is essentially done in example 3.2 of [40] and section 14 of [3]. Specifically:

Notation 6.1. *As above, we have the actions λ and α on $\mathcal{A} = C(G/H)$ and $\mathcal{B} = \mathcal{B}(\mathcal{H})$ respectively. For any natural number d let λ^d and α^d be the corresponding actions $\iota_d \otimes \lambda$ and $\iota_d \otimes \alpha$ on $M_d \otimes \mathcal{A} = M_d(\mathcal{A})$ and $M_d \otimes \mathcal{B} = M_d(\mathcal{B})$, for ι_d denoting the identity operator from M_d to itself. We then use the length function ℓ and formula (6.1) to define seminorms $L_d^{\mathcal{A}}$ and $L_d^{\mathcal{B}}$ on $M_d(\mathcal{A})$ and $M_d(\mathcal{B})$.*

It is easily verified that $\{L_d^{\mathcal{A}}\}$ and $\{L_d^{\mathcal{B}}\}$ are matrix slip-norms. Notice that here $L_1^{\mathcal{A}} = L^{\mathcal{A}}$ and $L_1^{\mathcal{B}} = L^{\mathcal{B}}$ are actually Lip-norms, and so, by property 1 of Definition 2.1, for each d the null-spaces of $L_d^{\mathcal{A}}$ and $L_d^{\mathcal{B}}$ are exactly M_d.

We remark that, as discussed in [4], the bridge $\Pi = (\mathcal{D}, \omega)$ with $\mathcal{D} = C(G/H, \mathcal{B})$ considered above is an example of a "bridge with conditional expectations", and that for such bridges theorem 5.5 of [4] gives upper bounds for the reach and height of Π_d in terms of the choices of ℓ, P, etc. In particular, they are finite.

7. Projections for $\mathcal{A}$

We now restrict our attention to the case in which $G = SU(2)$. We choose our notation in such a way that much of it generalizes conveniently to the setting of general compact semi-simple Lie groups, though we do not discuss that general case here. We let H denote the diagonal subgroup of G, which is a maximal torus in G. The homogeneous space G/H is diffeomorphic to the 2-sphere. As before, we set $\mathcal{A} = C(G/H)$.

It is known that for any 2-dimensional compact space every complex vector bundle is a direct sum of complex line bundles. See theorem 1.2 of chapter 8 of [41]. This applies to the 2-sphere, and so in this and the next few sections we will concentrate on the case of line bundles. In Section 13 we will discuss the situation for direct sums of projective modules. It is not entirely straight-forward.

In this section we seek formulas for projections that represent the line bundles over G/H. We will follow the approach given in [42], where formulas for the projections were first given in the global form that we need. (See also [43,44,28].) But the formulas given in [42] do not seem convenient for obtaining the detailed estimates that we need later, so the specific path that we follow is somewhat different.

We will often view (i.e. parametrize) H as $\mathbb{R}/\mathbb{Z}$. We define the function e on $\mathbb{R}$, and so on H, by $e(t) = e^{2\pi i t}$. Then each irreducible representation of H is of the form $t \mapsto e(kt)$ for some $k \in \mathbb{Z}$. For each $k \in \mathbb{Z}$ let Ξ_k denote the corresponding $\mathcal{A}$-module defined by:

Notation 7.1.

$$\Xi_k = \{\xi \in C(G, \mathbb{C}) : \xi(xs) = \bar{e}(ks)\xi(x) \text{ for all } x \in G, s \in H\},$$

where elements of $\mathcal{A}$ are viewed as functions on G that act on Ξ_k by pointwise multiplication. Then Ξ_k is the module of continuous cross-sections of a fairly evident vector bundle (a complex line bundle) over G/H. For $k \neq 0$ these are the physicists' "monopole bundles". Their "topological charge", or first Chern number, is k (or $-k$ depending on the conventions used). See sections 3.2.1 and 3.2.2 of [42]. We let λ denote the action of G on $\mathcal{A}$, and also on Ξ_k, by left translation. These actions are compatible, so that Ξ_k is a G-equivariant $\mathcal{A}$-module, reflecting the fact that the corresponding vector bundle is G-equivariant.

In order to apply the theory of Section 5 we need to find a suitable projection from a free $\mathcal{A}$-module onto Ξ_k. We do this in the way discussed in section 13 of [5]. The feature that we use to obtain the projections is the well-known fact that the one-dimensional representations of H occur as sub-representations of the restrictions to H of finite-dimensional unitary representations of G. Since H is a maximal torus in $SU(2)$, the integers determining the one-dimensional representations which occur when restricting a representation of G are, by definition, the weights of that representation. We recall [45] that for each non-negative integer m there is an irreducible representation, $(\mathcal{H}^m, U^m)$ of G whose weights are $m, m-2, \dots, -m+2, -m$, each of multiplicity 1, and such a representation is unique up to unitary equivalence. In particular, the dimension of $\mathcal{H}^m$ is $m+1$. The integer m is called the "highest weight" of the representation. For a given integer k (which may be negative) that determines the $\mathcal{A}$-module Ξ_k, we choose to consider the representation $(\mathcal{H}^{|k|}, U^{|k|})$.

Then the one-dimensional subspace $\mathcal{K}$ of $\mathcal{H}^{|k|}$ for the highest weight if k is non-negative, or for the lowest weight if k is negative, is carried into itself by the restriction of $U^{|k|}$ to the subgroup H, and this restricted representation of H is equivalent to the one-dimensional representation of H determining Ξ_k. From now on we simply let V denote this restricted representation of H on $\mathcal{K}$. Set

$$\Xi_k^V = \{\xi \in C(G, \mathcal{K}) : \xi(xs) = V_s^*(\xi(x)) \text{ for } x \in G,\ s \in H\}.$$

Clearly Ξ_k^V is a module over $\mathcal{A} = C(G/H)$ that is isomorphic to Ξ_k.

We want to show that Ξ_k^V is a projective A-module, and to find a projection representing it. Set

$$\Upsilon_k = C(G/H, \mathcal{H}^{|k|}).$$

Then any choice of basis for $\mathcal{H}^{|k|}$ exhibits Υ_k as a free A-module. For $\xi \in \Xi_k^V$ set $(\Phi\xi)(x) = U_x^{|k|}\xi(x)$ for $x \in G$, and notice that $(\Phi\xi)(xs) = (\Phi\xi)(x)$ for $s \in H$ and $x \in G$, so that $\Phi\xi \in \Upsilon_k$. It is clear that Φ is an injective A-module homomorphism from Ξ_V into Υ_k. We show that the range of Φ is projective by exhibiting the projection onto it from Υ_k. This projection is the one that we will use in the later sections to represent the projective module Ξ_k.

Notation 7.2. *We denote the projection from $\mathcal{H}^{|k|}$ onto $\mathcal{K}$ by P^k.*

Note that $U_s^{|k|}P^kU_s^{|k|*} = P^k$ for $s \in H$ by the H-invariance of $\mathcal{K}$. Let $\mathcal{E}_k$ denote the C^*-algebra $C(G/H, \mathcal{L}(\mathcal{H}^{|k|}))$. In the evident way $\mathcal{E}_k = \mathrm{End}_A(\Upsilon_k)$. Define p_k on G by

$$p_k(x) = U_x^{|k|}P^kU_x^{|k|*}, \tag{7.1}$$

and notice that $p_k(xs) = p_k(x)$ for $s \in H$ and $x \in G$, so that $p_k \in \mathcal{E}_k$. Clearly p_k is a projection in $\mathcal{E}_k = \mathrm{End}_A(\Upsilon_k)$.

Proposition 7.3. *As an operator on Υ_k, the range of the projection p_k is exactly the range of the injection Φ.*

Proof. If $\xi \in \Xi_V$, then $p_k(x)(\Phi\xi)(x) = U_x^{|k|}P^kU_x^{|k|*}U_x^{|k|}\xi(x) = (\Phi\xi)(x)$, so that $\Phi\xi$ is in the range of p_k. Suppose, conversely, that $F \in \Upsilon_k$ and that F is in the range of p_k. Set $\eta_F(x) = U_x^{|k|*}F(x) = U_x^{|k|*}p_k(x)F(x) = P^kU_x^{|k|*}F(x)$. Then the range of η_F is in $\mathcal{K}$, and we see easily that $\eta_F(xs) = U_s^{|k|*}\eta_F(x)$. Thus $\eta_F \in \Xi_V$. Furthermore, $(\Phi\eta_F)(x) = F(x)$. Thus F is in the range of Φ. This shows that the range of p_k as a projection on Υ_k is exactly the range of Φ. □

Thus the range of Φ, and so also Ξ_k^V, are projective A-modules that are isomorphic, and p_k is a projection that represents Ξ_k^V, and so represents Ξ_k. It is this projection p_k that we will use in the later parts of this paper.

To express p_k as an element of $M_d(A)$ for $d = |k|+1$ we need only choose an orthonormal basis, $\{e_j\}_{j=1}^d$, for $\mathcal{H}^{|k|}$, and view the corresponding constant functions as a basis (so standard module frame) for Υ_k, and then express p_k in terms of this basis. Furthermore, if we define g_j on G by $g_j(x) = P^kU_x^{|k|*}e_j$, then it is easily seen that each g_j is in Ξ_V, and that $\{g_j\}$ is a standard module frame, as defined in definition 7.1 of [5], for Ξ_V. The basis also gives us an isomorphism of $\mathcal{E}_k$ with $M_d(A)$. But it is more natural and convenient to view p_k as an element of $\mathcal{E}_k = \mathrm{End}_A(\Upsilon_k)$. To summarize:

Notation 7.4. *For p_k defined as in* Eq. (7.1), *we use the identification of $M_d(\mathcal{A})$ with $\mathcal{E}_k = C(G/H, \mathcal{L}(\mathcal{H}^{|k|}))$ to view p_k as an element of $M_d(\mathcal{A})$, and we use p_k as the projection representing the projective $\mathcal{A}$-module Ξ_k.*

8. Projections for $\mathcal{B}^n$

Let $(\mathcal{H}^n, U^n)$ be the irreducible representation of $G = SU(2)$ of highest weight n. Let $\mathcal{B}^n = \mathcal{L}(\mathcal{H}^n)$, and let α be the action of G on $\mathcal{B}^n$ by conjugation, that is, $\alpha_x(T) = U_x^nTU_x^{n*}$ for $T \in \mathcal{B}^n$ and $x \in G$.

Suitable projective modules for our context seem to have been first suggested in [11]. (See the paragraph after Eq. (40) there.) See also Eq. (6.6) of [46]. The formulation closest to that which we use here is found in equation 84 of [21]. For each $k \in \mathbb{Z}$ let Ω_k^n denote the right $\mathcal{B}^n$-module defined by:

Notation 8.1.

$$\Omega_k^n = \mathcal{L}(\mathcal{H}^n, \mathcal{H}^{k+n}),$$

where $(\mathcal{H}^{k+n}, U^{k+n})$ is the irreducible representation of highest weight $k+n$. Thus if $k < 0$ we need n large enough that $k+n \geq 0$.

Then Ω_k^n is a right $\mathcal{B}^n$-module by composing operators in Ω_k^n on the right by operators in $\mathcal{B}^n$.

We want to embed Ω_k^n into a free $\mathcal{B}^n$-module so that we can consider the corresponding projection. Let

$$\Upsilon_k^n = \mathcal{L}(\mathcal{H}^n, \mathcal{H}^{|k|} \otimes \mathcal{H}^n)$$

with its evident right action of $\mathcal{B}^n$ by composing operators. Then Υ_k^n is naturally isomorphic to $\mathcal{H}^{|k|} \otimes \mathcal{B}^n$, so that it is indeed a free $\mathcal{B}^n$-module, of rank the dimension of $\mathcal{H}^{|k|}$, which is $d = |k| + 1$.

If $k > 0$ and if η^k and η^n are highest weight vectors in $\mathcal{H}^k$ and $\mathcal{H}^n$, then $\eta^k \otimes \eta^n$ is a highest weight vector of weight $k+n$ in $\mathcal{H}^k \otimes \mathcal{H}^n$, for the action $U^k \otimes U^n$, and thus $\mathcal{H}^k \otimes \mathcal{H}^n$ contains a (unique) copy of $\mathcal{H}^{k+n}$. If $k < 0$ but $k+n \geq 0$ then $\mathcal{H}^{|k|} \otimes \mathcal{H}^n$ again contains a highest weight vector of weight $k+n$, but the argument is somewhat more complicated, and we give it in Lemma 11.1. Thus again $\mathcal{H}^{|k|} \otimes \mathcal{H}^n$ contains a (unique) copy of $\mathcal{H}^{k+n}$. Consequently, for any k we can, and do, identify $\mathcal{H}^{k+n}$ with the corresponding subspace of $\mathcal{H}^{|k|} \otimes \mathcal{H}^n$. (We always assume that $k + n \geq 0$.) Accordingly, we identify Ω_k^n with a $\mathcal{B}^n$-submodule of Υ_k^n.

We have an evident left action of $\mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$ on Υ_k^n by composing operators in Υ_k^n on the left by operators in $\mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$. In fact, Υ_k^n is a $\mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$-$\mathcal{L}(\mathcal{H}^n)$-bimodule, and because $\mathcal{L}(\mathcal{H}^n) = \mathcal{B}^n$ there is an evident natural isomorphism

$$End_{\mathcal{B}^n}(\Upsilon_k^n) \cong \mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n).$$

(The bimodule Υ_k^n gives a Morita equivalence between $\mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$ and $\mathcal{B}^n$.) But we also have natural isomorphisms

$$\mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n) \cong \mathcal{L}(\mathcal{H}^{|k|}) \otimes \mathcal{L}(\mathcal{H}^n) \cong M_d(\mathcal{B}^n),$$

and we will use these to take $\mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$ as our version of $M_d(\mathcal{B}^n)$.

Let p_k^n denote the projection of $\mathcal{H}^{|k|} \otimes \mathcal{H}^n$ onto its subspace $\mathcal{H}^{k+n}$, and view p_k^n as an element of $\mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$. Then composition with p_k^n on the left gives an evident projection of $\mathcal{L}(\mathcal{H}^n, \mathcal{H}^{|k|} \otimes \mathcal{H}^n)$ onto its submodule $\mathcal{L}(\mathcal{H}^n, \mathcal{H}^{k+n})$, that is, from Υ_k^n onto Ω_k^n, respecting the right action of $\mathcal{B}^n$. We thus see that the projection p_k^n is a projection that represents the projective $\mathcal{B}^n$-module Ω_k^n.

Notation 8.2. *Let p_k^n denote the projection of $\mathcal{H}^{|k|} \otimes \mathcal{H}^n$ onto its subspace $\mathcal{H}^{k+n}$. We use the identification of $M_d(\mathcal{B}^n)$ with $\mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$ to view p_k^n as an element of $M_d(\mathcal{B}^n)$, and we use p_k^n as the projection to represent the projective $\mathcal{B}^n$-module Ω_k^n.*

Suppose now that, as in Section 6, we have chosen a continuous length-function ℓ on G which we then use to define slip-norms for various actions. For the proof of our main theorem (Theorem 12.1), concerning the convergence of modules, we need a bound for $L^{M_d(\mathcal{B}^n)}(p_k^n)$ that is independent of n, where $d = |k| + 1$. That is, we need:

Proposition 8.3. *For fixed k there is a constant c_k (depending in particular on the choice of the length function ℓ) such that*

$$L^{M_d(\mathcal{B}^n)}(p_k^n) \leq c_k$$

for all n.

Proof. Notice that because the representation of G on $\mathcal{H}^{k+n}$ is a subrepresentation of the representation $U^{|k|} \otimes U^n$ on $\mathcal{H}^{|k|} \otimes \mathcal{H}^n$, the operator p_k^n is invariant under the corresponding conjugation action of G on $\mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$. But the action of G used to define $L^{M_d(\mathcal{B}^n)}$, as discussed in Section 6, comes from the action α of G on $\mathcal{B}^n$ using the representation U^n on $\mathcal{H}^n$, and is the action $\beta^n = \iota_d \otimes \alpha$ coming from conjugating elements of $\mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$ by the representation $I_d \otimes U^n$ on $\mathcal{H}^{|k|} \otimes \mathcal{H}^n$. Thus we must consider $\beta_x^n(p_k^n)$, and the action β^n depends strongly on n.

Now

$$\beta_x^n(p_k^n) = (I_d \otimes U_x^n) p_k^n (I_d \otimes U_x^{n*}).$$

But as said above, p_k^n is invariant under conjugation by $U^{|k|} \otimes U^n$, so we can replace p_k^n in the above equation by

$$(U_x^{|k|*} \otimes U_x^{n*}) p_k^n (U_x^{|k|} \otimes U_x^n),$$

from which we find that

$$\beta_x^n(p_k^n) = (U_x^{|k|*} \otimes I^n) p_k^n (U_x^{|k|} \otimes I^n),$$

where I^n is the identity operator on $\mathcal{H}^n$. We can express this as

$$\beta^n_x(p^n_k) = (\alpha^k_{x^{-1}} \otimes \iota^n)(p^n_k)$$

where α^k is the conjugation action of G on $\mathcal{L}(\mathcal{H}^{|k|})$ and ι^n is the identity operator on $\mathcal{L}(\mathcal{H}^n)$ (but this does not work for most other operators on $\mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$ besides p^n_k.) Let γ^k denote the action on $\mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$ defined by $\gamma^k_x = (\alpha^k_x \otimes \iota^n)$. Then we see that we have obtained

$$\beta^n_x(p^n_k) = \gamma^k_{x^{-1}}(p^n_k),$$

(and the action γ^k depends only very weakly on n). It follows that

$$(\beta^n_x(p^n_k) - p^n_k)/\ell(x) = (\gamma^k_{x^{-1}}(p^n_k) - p^n_k)/\ell(x^{-1}),$$

where we have used that $\ell(x) = \ell(x^{-1})$. From this it follows that

$$L^{\beta^n}(p^n_k) = L^{\gamma^k}(p^n_k).$$

Consequently, because $\|p^n_k\| = 1$ for all n, the following lemma will conclude the proof.

Lemma 8.4. *For any k there is a constant, c_k, such that for any n we have*

$$L^{\gamma^k}(T) \leq c_k \|T\|$$

for every $T \in \mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$.

Proof. We use a standard "smoothing"-type argument. Let $f \in C(G)$, and let γ^k_f be the integrated form of γ^k applied to f. Then for any $x \in G$ and $T \in \mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$ we have

$$\begin{aligned}\gamma^k_x(\gamma^k_f(T)) &= \gamma^k_x\Big(\int f(y)\gamma^k_y(T)dy\Big) = \int f(y)\gamma^k_{xy}(T)dy \\ &= \int f(x^{-1}y)\gamma^k_y(T)dy = \int (\lambda_x(f))(y)\gamma^k_y(T)dy,\end{aligned}$$

where λ is the action of left-translation on $C(G)$. Now suppose further that $L^\lambda(f) < \infty$, where L^λ is the Lipschitz seminorm on $C(G)$ using formula (6.1), for the length function ℓ and the action λ. Then

$$(\gamma^k_x(\gamma^k_f(T)) - \gamma^k_f(T))/\ell(x) = \int (((\lambda_x f)(y) - f(y))/\ell(x))\gamma^k_y(T)dy.$$

On taking norms and then supremum over $x \in G$, we find that

$$L^{M_d(\mathcal{B}^n)}(\gamma^k_f(T)) \leq L^\lambda(f)\|T\|. \tag{8.1}$$

It is shown in proposition 2.2 of [30] that the collection of functions f for which $L^\lambda(f) < \infty$ is a norm-dense $*$-subalgebra of $C(G)$. Thus this sub-algebra will contain an approximate identity for the convolution algebra $L^1(G)$. As we let f run through such an approximate identity, α^k_f will converge for the strong operator topology to the identity operator on $\mathcal{L}(\mathcal{H}^{|k|})$. But $\mathcal{L}(\mathcal{H}^{|k|})$ is finite-dimensional, and so the convergence is also for the operator norm. Thus we can find f such that α^k_f is close enough to the identity operator that α^k_f is invertible, which in turn implies that $\gamma^k_f = \alpha^k_f \otimes \iota^n$ is invertible, and that $\|(\gamma^k_f)^{-1}\| = \|(\alpha^k_f)^{-1}\|$. For such a fixed f we have, on using inequality (8.1),

$$\begin{aligned}L^{M_d(\mathcal{B}^n)}(T) &= L^{M_d(\mathcal{B}^n)}(\gamma^k_f((\gamma^k_f)^{-1}(T))) \\ &\leq L^\lambda(f)\|(\gamma^k_f)^{-1}(T)\| \leq L^\lambda(f)\|(\gamma^k_f)^{-1}\|\|T\|.\end{aligned}$$

Thus, with notation as just above, we can set

$$c^k = L^\lambda(f)\|(\gamma^k_f)^{-1}\| \quad \square$$

This concludes the proof of Proposition 8.3. $\square$

9. Bridges and projections

In this section we begin to apply the general results about bridges and projections given in Section 5 to the specific algebras, projections and bridges for $G = SU(2)$ described in Sections 6–8 . We fix the positive integer n and the integer k, to be used as in the sections above, and we set $d = |k| + 1$. We let $\mathcal{D}^n = \mathcal{A} \otimes \mathcal{B}^n$, and we let $\Pi^n_d = (M_d(\mathcal{D}^n), \omega^n_d)$ for ω^n_d defined as in Definition 4.1, so that Π^n_d is a bridge from $M_d(\mathcal{A})$ to $M_d(\mathcal{B}^n)$. We let $p_k \in M_d(\mathcal{A})$ and $p^n_k \in M_d(\mathcal{B}^n)$ be the projections defined in the previous two sections, and we view them as elements of $M_d(\mathcal{D}^n)$ via the injections of $M_d(\mathcal{A})$ and $M_d(\mathcal{B}^n)$ into $M_d(\mathcal{D}^n)$ given earlier. For this purpose we use the identification of $M_d(\mathcal{A})$ with $C(G/H, \mathcal{L}(\mathcal{H}^{|k|}))$, of $M_d(\mathcal{B}^n)$ with $\mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n)$, and the identification of $M_d(\mathcal{D}^n)$ with $C(G/H, M_d \otimes \mathcal{B}^n) = C(G/H, \mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n))$. Thus:

Notation 9.1. *When viewed as elements of* $C(G/H, \mathcal{L}(\mathcal{H}^{|k|} \otimes \mathcal{H}^n))$, *the projection* p_k *is defined by*

$$p_k(x) = U_x^{|k|} P^k U_x^{|k|*} \otimes I^n$$

for $x \in G/H$, *while the projection* p_k^n *is defined as the constant function*

$$p_k^n(x) = p_k^n$$

on G/H.

Then, as discussed in [4] and in Section 5, we need to obtain a useful bound for

$$\|p_k \omega_d^n - \omega_d^n p_k^n\|.$$

Now

$$\begin{aligned} p_k(x)\omega_d^n(x) &= (U_x^{|k|} P^k U_x^{|k|*} \otimes I^n)(I_d \otimes U_x^n P^n U_x^{n*}) \\ &= (U_x^{|k|} \otimes U_x^n)(P^k \otimes P^n)(U_x^{|k|*} \otimes U_x^{n*}), \end{aligned}$$

while

$$\omega_d^n(x) p_k^n(x) = (I_d \otimes U_x^n P^n U_x^{n*}) p_k^n.$$

But the subspace $\mathcal{H}^{k+n}$ of $\mathcal{H}^{|k|} \otimes \mathcal{H}^n$ is carried into itself by the representation $U^{|k|} \otimes U^n$, and so

$$p_k^n = (U_x^{|k|} \otimes U_x^n) p_k^n (U_x^{|k|*} \otimes U_x^{n*}),$$

so that

$$\begin{aligned} &(I_d \otimes U_x^n P^n U_x^{n*}) p_k^n \\ &\quad = (I_d \otimes U_x^n P^n U_x^{n*})(U_x^{|k|} \otimes U_x^n) p_k^n (U_x^{|k|*} \otimes U_x^{n*}) \\ &\quad = (U_x^{|k|} \otimes U_x^n)(I_d \otimes P^n) p_k^n (U_x^{|k|*} \otimes U_x^{n*}) \end{aligned}$$

Thus

$$\begin{aligned} &p_k(x)\omega_d^n(x) - \omega_d^n(x) p_k^n(x) \\ &\qquad = (U_x^{|k|} \otimes U_x^n)(P^k \otimes P^n - (I_d \otimes P^n) p_k^n)(U_x^{|k|*} \otimes U_x^{n*}), \end{aligned}$$

and consequently

$$\|p_k(x)\omega_d^n(x) - \omega_d^n(x) p_k^n(x)\| = \|(P^k \otimes P^n) - (I_d \otimes P^n) p_k^n\|,$$

which is independent of x. Thus

$$\|p_k \omega_d^n - \omega_d^n p_k^n\| = \|(P^k \otimes P^n) - (I_d \otimes P^n) p_k^n\|. \tag{9.1}$$

The next two sections are devoted to obtaining suitable upper bounds for the term on the right.

10. The core calculation for the case of $k \geq 0$

We treat first the case in which $k \geq 1$. The case in which $k \leq -1$ is somewhat more complicated, and we treat it in the next section. (The case for $k = 0$ is trivial.) Fix $k \geq 1$. Let T_k^n be the negative of the operator whose norm is taken on the right side of Eq. (9.1). Notice that $P^k \otimes P^n = (P^k \otimes P^n) p_k^n$ (which is false for $k \leq -1$), and that $P^k \otimes P^n$ commutes with p_k^n. Consequently

$$T_k^n = ((I_d - P^k) \otimes P^n) p_k^n.$$

It is an operator on $\mathcal{H}^k \otimes \mathcal{H}^n$. To understand the structure of this operator we use the weight vectors of the two representations involved. For this purpose we use the ladder operators in the complexified Lie algebra of $SU(2)$. There are many conventions for them used in the literature. Since the calculations in this section are crucial for our main results, we give in Appendix 1 a careful statement of the conventions we use, and of the consequences of our conventions. As explained in more detail there, we let H span the Lie subalgebra of our maximal torus, and we let E and F be the ladder operators that satisfy the relations

$$[E, F] = H, \qquad [H, E] = 2E, \qquad \text{and} \quad [H, F] = -2F.$$

For each n these elements of the complexified Lie algebra of $SU(2)$ act on $\mathcal{H}^n$ via the infinitesimal form of U^n, but as is commonly done we will not explicitly include U^n in our formulas. As operators on $\mathcal{H}^n$ they satisfy the relation $E^* = F$. Let f_n be a highest-weight vector for the representation $(\mathcal{H}^n, U^n)$, with $\|f_n\| = 1$. The weights of this representation are $n, n-2, n-4, \ldots, -n+2, -n$. Set

$$f_{n-2a} = F^a f_n$$

for $a = 0, 1, 2, \dots, n$. These vectors form an orthogonal basis for $\mathcal{H}^n$. As shown in the Appendix, we have

$$\|f_{n-2a}\|^2 = \|F^a f_n\|^2 = a! \Pi_{b=0}^{a-1}(n-b).$$

for $a = 0, 1, 2, \dots, n$.

Much as above, let e_k be a highest-weight vector for the representation $(\mathcal{H}^k, U^k)$, with $\|e_k\| = 1$. Set

$$e_{k-2a} = F^a e_k$$

for $a = 0, 1, 2, \dots, k$. These vectors form an orthogonal basis for $\mathcal{H}^k$.

Set $v_{k+n} = e_k \otimes f_n$. It is a highest weight vector of weight $k+n$ in $\mathcal{H}^k \otimes \mathcal{H}^n$ for the representation $U^k \otimes U^n$. Notice that $\|v_{k+n}\| = 1$. Then set

$$v_{k+n-2a} = F^a v_{k+n}$$

for $a = 0, 1, 2, \dots, k+n$. These vectors form an orthogonal basis for a sub-representation of $(\mathcal{H}^k \otimes \mathcal{H}^n, U^k \otimes U^n)$ that is unitarily equivalent to the irreducible representation $(\mathcal{H}^{k+n}, U^{k+n})$, and we will identify it with the latter. The span of these vectors is the range of the projection p_k^n, and so from the form of T_k^n we see that we only need to calculate the norm of T_k^n on the span of these vectors. For a given a we have

$$F^a(e_k \otimes f_n) = (F^a e_k) \otimes f_n \ + \ \text{lower order terms},$$

where the lower-order terms are of the form $F^{a-b} e_k \otimes F^b f_n$ for some integer $b \geq 1$. But each of these lower-order terms is in the kernel of $(I_d - P^k) \otimes P^n$ because $P^n(F^b f_n) = 0$ for $b \geq 1$. The highest weight vector $e_k \otimes f_n$ is also in that kernel, because P^k is the projection onto the span of e_k. Thus we find that $T_k^n(e_k \otimes f_n) = 0$, while

$$T_k^n(F^a(e_k \otimes f_n)) = (F^a e_k) \otimes f_n$$

for $a = 1, \dots, k$. But the terms $(F^a e_k) \otimes f_n$ are orthogonal to each other for different a's. Because $\|(F^a e_k) \otimes f_n\| = \|F^a e_k\|$ for each a, it follows that

$$\|T_k^n\| = \max\{\|F^a e_k\| / \|F^a(e_k \otimes f_n)\| : a = 1, \dots, k\}.$$

But from Eq. (A.3) of the Appendix we see that for each a we have

$$\|F^a e_k\|^2 = a! \Pi_{b=0}^{a-1}(k-b),$$

while

$$\|F^a(e_k \otimes f_n)\|^2 = a! \Pi_{b=0}^{a-1}(k+n-b).$$

Thus

$$\|F^a e_k\|^2 / \|F^a(e_k \otimes f_n)\|^2 = \Pi_{b=0}^{a-1}(k-b)/(k+n-b).$$

Since $(k-b)/(k+n-b) < 1$ for each $b = 1, \dots, k$, it is clear that the maximum of these products depending on a occurs when $a = 1$ and so we find that

$$\|T_k^n\| = (k/(k+n))^{1/2}.$$

We thus obtain:

Proposition 10.1. *In terms of our notation in* Section 9, *and by* Eq. (9.1), *for any $k \geq 0$ we have*

$$\|p_k(x)\omega_d(x) - \omega_d(x)p_k^n(x)\| = \|(P^k \otimes P^n) - (I_d \otimes P^n)p_k^n\| = (k/(k+n))^{1/2}$$

for each x.

Crucially, this goes to 0 as $n \to \infty$, for fixed k.

11. The core calculation for the case of $k \leq -1$

We now treat the case in which $k \leq -1$, for which we must assume that $k+n \geq 1$. Again we set

$$T_k^n = (P^k \otimes P^n) - (I_d \otimes P^n)p_k^n.$$

We use the same basis vectors e_i and f_j for $\mathcal{H}^{|k|}$ and $\mathcal{H}^n$ as in the previous section. But now $k+n < n$, and p_k^n is the projection on the subspace of $\mathcal{H}^{|k|} \otimes \mathcal{H}^n$ generated by the highest weight vector v_{k+n} of weight $k+n$. This vector has a more complicated expression in terms of the basis vectors $e_i \otimes f_j$ than for the case of $k \geq 0$. Specifically, v_{k+n} will be a linear combination of those basis vectors $e_i \otimes f_j$ that are of weight $k+n$.

Lemma 11.1. *For $k \le -1$ a highest weight vector of weight $k+n$ in $\mathcal{H}^{|k|} \otimes \mathcal{H}^n$ is given by*

$$v_{k+n} = \sum_{b=0}^{-k} \alpha_b \, e_{-k-2b} \otimes f_{n+2k+2b} \ ,$$

where

$$\alpha_b = (-1)^b \frac{(n+k+b)!}{(n+k)!b!}$$

for $0 \le b \le -k$.

Proof. We must determine the coefficients α_b for v_{k+n} of the general form given in the statement of the lemma. In order for v_{k+n} to be a highest weight vector it must satisfy $Ev_{k+n} = 0$, that is,

$$0 = Ev_{k+n} = \sum_{b=0}^{-k} \alpha_b \, (Ee_{-k-2b} \otimes f_{n+2k+2b} + e_{-k-2b} \otimes Ef_{n+2k+2b}).$$

For each b with $0 \le b < -k$ the term in this sum that is a multiple of

$$e_{-k-2b} \otimes f_{n+2k+2(b+1)}$$

is

$$\alpha_{b+1} \, Ee_{-k-2(b+1)} \otimes f_{n+2k+2(b+1)} \ + \ a_b \, e_{-k-2b} \otimes Ef_{n+2k+2b} \ .$$

By Eq. (A.1) in the Appendix,

$$Ee_{-k-2(b+1)} = (b+1)(-k-(b+1)+1)e_{-k-2(b+1)+2},$$

while

$$Ef_{n+2k+2b} = Ef_{n-2(-k-b)} = (-k-b)(n-(-k-b)+1)f_{n+2k+2(b+1)} \ .$$

It follows that

$$0 = \alpha_{b+1}(b+1)(-k-b) \ + \ \alpha_b(-k-b)(n+k+b+1).$$

Thus for $0 \le b \le -k-1$ we have

$$\alpha_{b+1} = -(n+k+b+1)(b+1)^{-1}\alpha_b \ ,$$

that is, if $1 \le b \le -k$ then

$$\alpha_b = -(n+k+b)b^{-1}\alpha_{b-1}.$$

We are free to set $\alpha_0 = 1$. On doing that, we find by induction that

$$\alpha_b = (-1)^b \frac{(n+k+b)!}{(n+k)!b!}$$

for $0 \le b \le -k$. $\square$

Much as in the case in which $k \ge 0$, we set

$$v_{k+n-2a} = F^a v_{k+n}$$

for $a = 0, 1, \dots, n+k$. These vectors form an orthogonal basis for a sub-representation of $(\mathcal{H}^{|k|} \otimes \mathcal{H}^n, U^{|k|} \otimes U^n)$ that is unitarily equivalent to the irreducible representation $(\mathcal{H}^{k+n}, U^{k+n})$, and we will identify the latter with this sub-representation. The span of these vectors is, by definition, the range of the projection p_k^n.

We seek to determine $\|T_k^n\|$, and to show that, for fixed k, it goes to 0 as n goes to ∞. Recall that P^k is the rank-one projection on e_k, which for $k \le -1$ is the *lowest* weight vector in $\mathcal{H}^{|k|}$ (and is not of norm 1). Since the range of $P^k \otimes P^n$ is spanned by $e_k \otimes f_n$ whereas the only vectors in the range of p_k^n that are of weight $k+n$ are multiples of v_{k+n}, it is clear that the range of $P^k \otimes P^n$ is not included in the range of p_k^n, in contrast to what happens for $k \ge 0$. Let W be the subspace of $\mathcal{H}^{|k|} \otimes \mathcal{H}^n$ spanned by the vectors $v_{k+n}, \dots, v_{-k-n}$ together with $e_k \otimes f_n$. If u is any vector in $\mathcal{H}^{|k|} \otimes \mathcal{H}^n$ that is orthogonal to W, then both $P^k \otimes P^n$ and p_k^n take u to 0, and thus so does T_k^n. Consequently in order to determine $\|T_k^n\|$ it suffices to view T_k^n as an operator from W into $\mathcal{H}^{|k|} \otimes \mathcal{H}^n$.

We consider now the action of T_k^n on the vectors v_{k+n-2a}. Let us assume first that $a \ge 1$. Since each term in the formula for v_{k+n-2a} must involve an elementary tensor of weight $k+n-2a$, it is clear that $(P^k \otimes P^n)(v_{k+n-2a}) = 0$ for $a \ge 1$, and so

$$T_k^n(v_{k+n-2a}) = (I_d \otimes P^n)(F^a(v_{k+n})).$$

Since F lowers weights, the only term in the formula for v_{k+n} given in Lemma 11.1 on which $(I_d \otimes P^n)F^a$ has a possibility of being non-zero is the term for $b = -k$, that is $\alpha_{-k}\, e_k \otimes f_n$. But because e_k is the lowest weight vector in $\mathcal{H}^{|k|}$, we see that $F(e_k \otimes f_n) = e_k \otimes f_{n-2}$, which is in the kernel of $I_d \otimes P^n$. We conclude that for all $a \geq 1$ we have $T_k^n(v_{k+n-2a}) = 0$.

Thus it suffices to determine the norm of the restriction of T_k^n to the subspace spanned by v_{k+n} and $e_k \otimes f_n$. Now

$$\begin{aligned} T_k^n(v_{k+n}) &= (P^k \otimes P^n)(v_{k+n}) \; - \; (I_d \otimes P^n)p_k^n(v_{k+n}) \\ &= ((P^k - I_d) \otimes I_{n+1})(I_d \otimes P^n)(v_{k+n}) \\ &= ((P^k - I_d) \otimes I_{n+1})(\alpha_{-k} e_k \otimes f_n) \; = \; 0. \end{aligned}$$

Thus, finally, it comes down to determining $T_k^n(e_k \otimes f_n)$. Now clearly

$$T_k^n(e_k \otimes f_n) = e_k \otimes f_n \; - \; (I_d \otimes P^n)p_k^n(e_k \otimes f_n).$$

The weight vectors v_{k+n-2a} form an orthogonal basis for the range of p_k^n, and all of these vectors except the one for $a = 0$ are of different weight than the weight of $e_k \otimes f_n$ and so are orthogonal to $e_k \otimes f_n$. It follows that

$$p_k^n(e_k \otimes f_n) \; = \; \frac{\langle e_k \otimes f_n, \; v_{k+n}\rangle}{\|v_{k+n}\|^2} v_{k+n}.$$

But from the formula for v_{k+n} given in Lemma 11.1 we see that

$$(I_d \otimes P^n)(v_{k+n}) \; = \; \alpha_{-k}\, e_k \otimes f_n \; = \; \frac{\langle e_k \otimes f_n, \; v_{k+n}\rangle}{\|e_k \otimes f_n\|^2} e_k \otimes f_n.$$

Thus

$$T_k^n(e_k \otimes f_n) = \big(1 \; - \; \frac{\langle e_k \otimes f_n, \; v_{k+n}\rangle^2}{\|e_k \otimes f_n\|^2 \|v_{k+n}\|^2}\big) e_k \otimes f_n,$$

so that

$$\|T_k^n\| = \frac{\|v_{k+n}\|^2 \; - \; \langle \frac{e_k \otimes f_n}{\|e_k \otimes f_n\|}, \, v_{k+n}\rangle^2}{\|v_{k+n}\|^2} = \|v'_{k+n}\|^2 / \|v_{k+n}\|^2,$$

where v'_{k+n} denotes v_{k+n} with its last term (involving $e_k \otimes f_n$) removed. We want to show that the above expression goes to 0 as $n \to \infty$.

Now

$$\|v_{k+n}\|^2 = \sum_{b=0}^{-k} \alpha_b^2 \; \|e_{-k-2b}\|^2 \|f_{n-2(-k-b)}\|^2,$$

while $\|v'_{k+n}\|^2$ is the same sum but with the upper limit of summation being $-k-1$. Since each $\|e_{-k-2b}\|$ is independent of n, to show that $\|v'_{k+n}\|^2 / \|v_{k+n}\|^2$ converges to 0 as $n \to \infty$, it suffices to show that for each b with $0 \leq b \leq -k-1$ the term

$$\alpha_b^2 \; \|f_{n-2(-k-b)}\|^2 / \|v_{k+n}\|^2$$

goes to 0 as $n \to \infty$. (Note that α_b does depend on n.) To show this it suffices to show that this holds when v_{k+n} is replaced by the $b = -k$ term in its expansion, which is the term missing in v'_{k+n}. On noting that e_k is independent of n and that $\|f_n\| = 1$ for all n, we see that we must show that for each b with $0 \leq b \leq -k-1$ the term

$$\alpha_b^2 \; \|f_{n-2(-k-b)}\|^2 / \alpha_{-k}^2$$

goes to 0 as $n \to \infty$.

From the formula in Lemma 11.1 we find that for each b with $0 \leq b \leq -k-1$ we have

$$|\alpha_b| / |\alpha_{-k}| = \frac{(n+k+b)!}{(n+k)!b!} \Big/ \frac{n!}{(n+k)!(-k)!} = \frac{(-k)!}{b!} \frac{(n+k+b)!}{n!} \;,$$

while from formula (A.3) of the Appendix we have

$$\|f_{n-2(-k-b)}\|^2 = \frac{(-k-b)!n!}{(n+k+b)!} \;\; .$$

Thus

$$\begin{aligned} \alpha_b^2 \; \|f_{n-2(-k-b)}\|^2 / \alpha_{-k}^2 &= \big(\frac{(-k)!}{b!}\big)^2 (-k-b)! \big(\frac{(n+k+b)!}{n!}\big)^2 \frac{n!}{(n+k+b)!} \\ &\leq (-k)^3 \frac{(n+k+b)!}{n!} \leq \frac{(-k)^3}{n} \end{aligned}$$

since $(k+b) \leq -1$. The power of n can not be improved, as seen by considering the case $b = -k-1$.

We conclude from these estimates that $\|T_k^n\| \to 0$ as $n \to \infty$, and consequently that:

Proposition 11.2. *In terms of our earlier notation in* Section 9*, and by* Eq. (9.1)*, for any fixed* $k \leq -1$

$$\|p_k(x)\omega_d(x) - \omega_d(x)p_k^n(x)\| = \|(P^k \otimes P^n) - (I_d \otimes P^n)p_k^n\|$$

converges to 0 as $n \to \infty$*.*

I have not managed to extend the results of this section to the case of general coadjoint orbits of compact semisimple Lie groups.

12. The main theorem and its proof

We are now in position to state and prove the main theorem of this paper. We will recall some of the notation at the beginning of the proof.

Theorem 12.1. *Let notation be as above, for* $G = SU(2)$ *and a chosen continuous length function* ℓ*, etc. Fix the integer* k *(and set* $d = |k| + 1$*). Let* L_k^n *be defined as in* Eq. (4.2) *for* $r = r^n = l_{\Pi_d^n}$*. Then* $L_k^n(p_k,\ p_k^n)$ *goes to 0 as* $n \to \infty$*. Furthermore, we can find a natural number* N_k *large enough that for every* $n \geq N_k$ *we have*

$$(h_{\Pi_d^n} + l_{\Pi_d^n})L_k^n(p_k,\ p_k^n) < 1/2\ ,$$

so that if q *is any other projection in* $M_d(\mathcal{B}^n)$ *that satisfies this same inequality when* p_k^n *is replaced by* q*, then there is a continuous path of projections in* $M_d(\mathcal{B}^n)$ *going from* p_k^n *to* q*, which implies that the projective* $\mathcal{B}^n$ *modules determined by* p_k^n *and* q *are isomorphic. In this sense the projective* $\mathcal{B}^n$*-module* Ω_k^n *is associated by the bridge* Π_d^n *to the projective* $\mathcal{A}$*-module* Ξ_k*.*

Proof. We recall some of our earlier notation and results. We have $\mathcal{A} = C(G/H)$ and $\mathcal{B}^n = \mathcal{L}(\mathcal{H}^n)$. Then we let $\mathcal{D}^n = \mathcal{A} \otimes \mathcal{B}^n$, and we let $\Pi^n = (\mathcal{D}^n, \omega^n)$, a bridge from $\mathcal{A}$ to $\mathcal{B}^n$. Let r_{Π^n} denote the reach of the bridge Π^n as measured by $L^{\mathcal{A}}$ and $L^{\mathcal{B}^n}$, as defined in Definition 4.2. Define a seminorm, N_{Π^n}, and then a $*$-seminorm $\hat{N}_{\Pi^n}$ on $\mathcal{A} \oplus \mathcal{B}^n$ much as done just before Eq. (4.2), and then, much as in Eq. (4.2), define, for some $r^n \geq r_{\Pi^n}$, a seminorm $L_{r^n}^n$ on $\mathcal{A} \oplus \mathcal{B}^n$ by

$$L_{r^n}^n(a, b) = L^{\mathcal{A}}(a) \vee L^{\mathcal{B}}(b) \vee (r^n)^{-1}\hat{N}_{\Pi^n}(a, b).$$

Then $L_{r^n}^n$ is an admissible seminorm on $\mathcal{A} \oplus \mathcal{B}^n$, according to Proposition 4.8.

We need the matricial version of this seminorm. We set $\Pi_d^n = (M_d(\mathcal{D}^n), \omega_d^n)$, where ω_d^n is defined, much as in Definition 4.1, by

$$\omega_d^n(x) = I_d \otimes \alpha_x(P^n)$$

for all $x \in G$. Then Π_d^n is a bridge from $M_d(\mathcal{A})$ to $M_d(\mathcal{B}^n)$. We measure it with the seminorms $L_d^{\mathcal{A}}$ and $L_d^{\mathcal{B}^n}$, defined much as at the end of Section 6. We now denote the resulting reach and height of Π_d^n by $r_{\Pi_d^n}$ and $h_{\Pi_d^n}$.

Define, on $M_d(\mathcal{A}) \oplus M_d(\mathcal{B}^n)$, a seminorm, N_d^n, by $N_d^n(a, b) = \|a\omega_d^n - \omega_d^n b\|$, and then a $*$-seminorm $\hat{N}_d^n$, much as done just before Eq. (4.2). Then, for any $r^n \geq r_{\Pi_d^n}$ define a seminorm, L_{d,r^n}^n, by

$$L_{d,r^n}^n(a, b) = L_d^{\mathcal{A}}(a) \vee L_d^{\mathcal{B}^n}(b) \vee (r^n)^{-1}\hat{N}_d^n(a, b). \tag{12.1}$$

Then L_{d,r^n}^n is an admissible seminorm for $L_d^{\mathcal{A}}$ and $L_d^{\mathcal{B}^n}$, by Proposition 4.8, because $r^n \geq r_d^n$.

Let p_k and p_k^n be the projections defined in Notation 9.1 for the projective modules Ξ_k and Ω_k^n. Then

$$L_{d,r^n}^n(p_k, p_k^n) = L_d^{\mathcal{A}}(p_k) \vee L_d^{\mathcal{B}^n}(p_k^n) \vee (r^n)^{-1}\hat{N}_d^n(p_k, p_k^n).$$

According to part (a) of Theorem 5.7, in order for p_k^n to be a projection associated to p_k up to path connectedness, we need that

$$(h_{\Pi_d^n} + r^n)L_{d,r^n}^n(p_k, p_k^n)\ <\ 1/2.$$

Thus each of the three main terms in the formula for $L_{k,r^n}^n(p_k, p_k^n)$ must satisfy the corresponding inequality.

We examine the third term first. Because p_k and p_k^n are self-adjoint, this term is equal to

$$(h_{\Pi_d^n} + r^n)(r^n)^{-1}\|p_k\omega_d^n - \omega_d^n p_k^n\|\ .$$

We now use theorem 6.10 of [4] (where q there is our k, and m is our n), which is one of the two main theorems of [4]. It tells us the quite un-obvious fact that $l_{\Pi_d^n}$, and so both $r_{\Pi_d^n}$ and $h_{\Pi_d^n}$, go to 0 as $n \to \infty$, for fixed k. This theorem furthermore gives quantitative upper bounds for $l_{\Pi_d^n}$ in terms of the length function ℓ chosen for G.

We also now see the reason for allowing r^n to possibly be different from $r_{\Pi_d^n}$ in defining L_{d,r^n}^n, namely that if $r_{\Pi_d^n}$ goes to 0 more rapidly than does $h_{\Pi_d^n}$, then their ratio goes to $+\infty$, so that the term $(h_{\Pi_d^n} + r^n)(r^n)^{-1}$ in the displayed expression above goes to $+\infty$. There are many ways to choose r^n to avoid this problem, but the simplest is probably just to set $r^n = \max\{r_{\Pi_d^n}, h_{\Pi_d^n}\}$, which is just the definition of $l_{\Pi_d^n}$. We now make this choice, and for this choice we write L_d^n instead of L^n_{d,r^n}. Then we have $1 + h_{\Pi_d^n}/l_{\Pi_d^n} \leq 2$, and so

$$(h_{\Pi_d^n} + l_{\Pi_d^n})(l_{\Pi_d^n})^{-1}\|p_k\omega_d^n - \omega_d^n p_k^n\| \leq 2\|p_k\omega_d^n - \omega_d^n p_k^n\|.$$

From Propositions 10.1 and 11.2 it follows that for fixed k this term goes to 0 as n goes to ∞.

We examine next the first term, $(h_{\Pi_d^n} + l_{\Pi_d^n})L_d^{\mathcal{A}}(p_k)$. Since $L_d^{\mathcal{A}}(p_k)$ is independent of n, and we have seen that $(h_{\Pi_d^n} + l_{\Pi_d^n})$ goes to 0 as $n \to \infty$, it follows that this first term too goes to 0 as $n \to \infty$, for fixed k.

Finally, we examine the second term,

$$(h_{\Pi_d^n} + l_{\Pi_d^n})L_d^{\mathcal{B}^n}(p_k^n).$$

In examining above the first term we have seen that $h_{\Pi_d^n}$ and $l_{\Pi_d^n}$ go to 0 as $n \to \infty$, so the only issue is the growth of $L_d^{\mathcal{B}^n}(p_k^n)$ as $n \to \infty$. But Proposition 8.3 tells us exactly that, for fixed k, there is a common bound for the $L_d^{\mathcal{B}^n}(p_k^n)$'s.

We now apply Theorem 5.7 to the present situation, and this concludes the proof. $\square$

As mentioned at the end of Section 6, upper bounds for $h_{\Pi_d^n}$ and $l_{\Pi_d^n}$ in terms of just the data for Π, $L^{\mathcal{A}}$, and $L^{\mathcal{B}}$ are given in theorem 5.5 of [4].

13. Bridges and direct sums of projective modules

In this section we discuss how to deal with direct sums of projective modules, and we indicate in what sense it is sufficient for us to deal in detail here only with the line-bundles on the 2-sphere.

It is well known that every complex vector bundle over the 2-sphere is isomorphic to the direct sum of a line bundle with a trivial bundle. (Use e.g. Proposition 1.1 of chapter 8 of [41].) We can see this in part as follows.

Proposition 13.1. *With notation as in Notation 7.1, for any $j, k \in \mathbb{Z}$ we have a natural module isomorphism*

$$\Xi_j \oplus \Xi_k \cong \Xi_{j+k} \oplus \Xi_0.$$

Proof. To simplify notation, we identify $SU(2)$ with the 3-sphere in the usual way, so that $(z, w) \in S^3 \subseteq \mathbb{C}^2$ corresponds to the matrix $\left(\begin{smallmatrix} z & -\bar{w} \\ w & \bar{z}\end{smallmatrix}\right)$. If we set $e_t = e(t)$ for each $t \in \mathbb{R}$, then right multiplication of elements of $SU(2)$ by the matrix $\left(\begin{smallmatrix} e_t & 0 \\ 0 & \bar{e}_t\end{smallmatrix}\right)$ corresponds to sending (z, w) to (ze_t, we_t). Thus elements of Ξ_k can be viewed as continuous functions ξ on S^3 that satisfy

$$\xi(ze_t, we_t) = \bar{e}_t^k \xi(z, w)$$

for all $t \in \mathbb{R}$.

Let $j, k \in \mathbb{Z}$ be given. Define a $GL(2, \mathbb{C})$-valued function M on S^3 by

$$M(z, w) = \begin{pmatrix} \bar{z}^k & -\bar{w}^j \\ w^j & z^k \end{pmatrix}.$$

For any $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \Xi_j \oplus \Xi_k$ set $\Phi\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) = M\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$, so that

$$\Phi\begin{pmatrix} f \\ g \end{pmatrix}(z, w) = \begin{pmatrix} \bar{z}^k f(z) - \bar{w}^j g(w) \\ w^j f(z) + z^k g(w). \end{pmatrix}$$

It is easily checked that $\Phi\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$ is in $\Xi_{j+k} \oplus \Xi_0$, and that Φ is an $\mathcal{A}$-module homomorphism. Furthermore, Φ has an inverse, obtained by using the inverse of M. Thus Φ is an isomorphism, as needed. $\square$

This proposition can be used inductively to show that the direct sum of any finite number of the Ξ_k's is isomorphic to the direct sum of a single Ξ_j with a trivial (i.e. free) module.

The corresponding result for the Ω_k^n's is even easier:

Proposition 13.2. *With notation as in Notation 8.1, for any $j, k \in \mathbb{Z}$ and for any $n \geq 1$ with $n+j \geq 0$, $n+k \geq 0$, and $n+j+k \geq 0$, we have a natural module isomorphism*

$$\Omega_j^n \oplus \Omega_k^n \cong \Omega_{j+k}^n \oplus \Omega_0^n.$$

Proof. We have

$$\begin{aligned}\Omega_j^n \oplus \Omega_k^n &= \mathcal{L}(\mathcal{H}^n, \mathcal{H}^{j+n}) \oplus \mathcal{L}(\mathcal{H}^n, \mathcal{H}^{k+n}) \\ &\cong \mathcal{L}(\mathcal{H}^n, \mathcal{H}^{j+k+2n}) \cong \mathcal{L}(\mathcal{H}^n, \mathcal{H}^{j+k+n}) \oplus \mathcal{L}(\mathcal{H}^n, \mathcal{H}^n) \\ &= \Omega_{j+k}^n \oplus \Omega_0^n. \quad \square\end{aligned}$$

But if one wants to show these correspondences by using projections associated to the projective modules, there are substantial complications. Let us consider the general case first.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras, and let $\Pi = (\mathcal{D}, \omega)$ be a bridge from $\mathcal{A}$ to $\mathcal{B}$. Let $\{L_n^{\mathcal{A}}\}$ and $\{L_n^{\mathcal{B}}\}$ be matrix Lip-norms on $\mathcal{A}$ and $\mathcal{B}$ (as defined in Definition 2.3). For a given d we can use $L_d^{\mathcal{A}}$ and $L_d^{\mathcal{B}}$ to measure the length of Π_d. One significant difficulty is that it seems to be hard in general to obtain an upper bound for the length of Π_d in terms of the length of Π (though we will see that for the case of the "bridges with conditional expectation" that are discussed in [4] we can get some useful information). But the following little result will be useful below.

Proposition 13.3. *Let $\Pi = (\mathcal{D}, \omega)$ be a bridge from $\mathcal{A}$ to $\mathcal{B}$, and let $\{L_n^{\mathcal{A}}\}$ and $\{L_n^{\mathcal{B}}\}$ be matrix slip-norms on $\mathcal{A}$ and $\mathcal{B}$, used to measure the length of Π_d for any d. If e is another natural number such that $d < e$ then $r_{\Pi_d} \le r_{\Pi_e}$.*

Proof. Let $A \in M_d(\mathcal{A})$ with $A^* = A$ and $L_d^{\mathcal{A}}(A) \le 1$, so that $A \in \mathcal{L}_d^1(\mathcal{A})$. Then $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$, with the 0's of correct sizes, is in $\mathcal{L}_e^1(\mathcal{A})$ by property (2) of Definition 2.1. Let $\delta > 0$ be given. Then by the definition of r_{Π_e} there is a $C \in \mathcal{L}_e^1(\mathcal{B})$ such that

$$\| \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_d & 0 \\ 0 & \omega_{e-d} \end{bmatrix} - \begin{bmatrix} \omega_d & 0 \\ 0 & \omega_{e-d} \end{bmatrix} C \| \le r_{\Pi_e} + \delta.$$

Compress the entire term inside the norm symbols by the matrix $E = \left(\begin{smallmatrix} I_d & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and define B by $ECE = \left(\begin{smallmatrix} B & 0 \\ 0 & 0 \end{smallmatrix}\right)$ to obtain

$$\|A\omega_d - \omega_d B\| \le r_{\Pi_e} + \delta.$$

Note that $B^* = B$ and that $L_d^{\mathcal{B}}(B) \le 1$ by property (1) of Definition 2.1, so that $B \in \mathcal{L}_d^1(\mathcal{B})$. Since δ is arbitrary, it follows that the distance from $A\omega_d$ to $\omega_d \mathcal{L}_d^1(\mathcal{B})$ is no greater than r_{Π_e}. In the same way we show that for any $B \in \mathcal{L}_d^1(\mathcal{B})$ there is an $A \in \mathcal{L}_d^1(\mathcal{A})$ such that the distance from $\omega_d B$ to $\mathcal{L}_d^1(\mathcal{A})\omega_d$ is no greater than r_{Π_e}. It follows that $r_{\Pi_d} \le r_{\Pi_e}$ as desired. $\square$

For any natural number d let N_d be the seminorm on $M_d(\mathcal{A}) \oplus M_d(\mathcal{B})$ defined much as done shortly before Eq. (4.2), by

$$N_d(A, B) = \|A\omega_d - \omega_d B\|$$

for $A \in M_d(\mathcal{A})$ and $B \in M_d(\mathcal{B})$. Suppose now that for natural numbers d and e we have $a_1 \in M_d(\mathcal{A})$ and $a_2 \in M_e(\mathcal{A})$ as well as $b_1 \in M_d(\mathcal{B})$ and $b_2 \in M_e(\mathcal{B})$, so that

$$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in M_{d+e}(\mathcal{A}) \qquad \text{and} \qquad \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \in M_{d+e}(\mathcal{B}).$$

Then

$$\begin{aligned} N_{d+e}(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}) &= \| \begin{bmatrix} a_1\omega_d - \omega_d b_1 & 0 \\ 0 & a_2\omega_e - \omega_e b_2 \end{bmatrix} \| \\ &= N_d(a_1, b_1) \vee N_e(a_2, b_2). \end{aligned}$$

Next, much as done shortly before Eq. (4.2), for each d we define a $*$-seminorm, $\hat{N}_d$, by

$$\hat{N}_d(A, B) = N_d(A, B) \vee N_d(A^*, B^*),$$

for $A \in M_d(\mathcal{A})$ and $B \in M_d(\mathcal{B})$. It then follows from the calculation done just above that

$$\hat{N}_{d+e}(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}) = \hat{N}_d(a_1, b_1) \vee \hat{N}_e(a_2, b_2),$$

for $a_1 \in M_d(\mathcal{A})$ and $a_2 \in M_e(\mathcal{A})$, and $b_1 \in M_d(\mathcal{B})$ and $b_2 \in M_e(\mathcal{B})$.

Next, assume that $r_{\Pi_d} < \infty$ for each d, as is the case for matrix Lip-norms as seen in Proposition 4.4. Let some choice of finite $r_d \ge r_{\Pi_d}$ be given for each d. Set, much as in Eq. (4.2),

$$L_d^{r_d}(A, B) = L_d^{\mathcal{A}}(A) \vee L_d^{\mathcal{B}}(B) \vee r_d^{-1} \hat{N}_{\Pi_d}(A, B)$$

for $A \in M_d(\mathcal{A})$ and $B \in M_d(\mathcal{B})$. It then follows from the calculations done above that

$$L_{d+e}^{r_{d+e}}(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}) = L_d^{r_{d+e}}(a_1, b_1) \vee L_e^{r_{d+e}}(a_2, b_2),$$

for $a_1 \in M_d(\mathcal{A})$ and $a_2 \in M_e(\mathcal{A})$, and $b_1 \in M_d(\mathcal{B})$ and $b_2 \in M_e(\mathcal{B})$. Because $r_{\Pi_{d+e}} \ge r_{\Pi_d} \vee r_{\Pi_e}$ according to Proposition 13.3, one can check quickly that $L_d^{r_{d+e}}$ is admissible (Definition 4.7) for $L_d^{\mathcal{A}}$ and $L_d^{\mathcal{B}}$, and similarly for $L_e^{r_{d+e}}$.

Suppose now that $p_1 \in M_d(\mathcal{A})$ and $p_2 \in M_e(\mathcal{A})$ are projections representing projective $\mathcal{A}$-modules, and that $q_1 \in M_d(\mathcal{B})$ and $q_2 \in M_e(\mathcal{B})$ are projections representing projective $\mathcal{B}$-modules. Then

$$L_{d+e}^{r_{d+e}}(\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}, \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}) = L_d^{r_{d+e}}(p_1, q_1) \vee L_e^{r_{d+e}}(p_2, q_2).$$

From Theorem 5.7a we then obtain:

Proposition 13.4. *Let notation be as just above, and assume that $l_{\Pi_{d+e}} < \infty$. If*

$$(h_{\Pi_{d+e}} + r_{d+e}) \max\{(L_d^{r_{d+e}}(p_1, q_1), L_e^{r_{d+e}}(p_2, q_2))\} < 1/2,$$

and if there is a projection $Q \in M_{d+e}(\mathcal{B})$ such that

$$(h_{\Pi_{d+e}} + r_{d+e}) L_{d+e}^{r_{d+e}}(\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}, Q) < 1/2,$$

then there is a path through projections in $M_{m+n}(\mathcal{B})$ going from Q to $\left(\begin{smallmatrix} q_1 & 0 \\ 0 & q_2 \end{smallmatrix}\right)$.

This uniqueness result means that $\left(\begin{smallmatrix} q_1 & 0 \\ 0 & q_2 \end{smallmatrix}\right)$ is a projection in $M_{d+e}(\mathcal{B})$ corresponding to the projection $\left(\begin{smallmatrix} p_1 & 0 \\ 0 & p_2 \end{smallmatrix}\right)$. Thus the consequence of this proposition is that for suitable bounds, if the projective modules for p_1 and q_1 correspond, and if those for p_2 and q_2 correspond, then the direct sum modules correspond. This suggests a further reason for saying that when considering complex vector bundles over the 2-sphere it suffices for our purposes to consider only the line bundles.

The difficulty with using this proposition is that I have not found a good way of bounding in general the reach and height of Π_d in terms of those of Π. However, the specific bridges that we have been using for matrix algebras converging to the sphere are examples of "bridges with conditional expectations", as defined in [4]. For such a bridge, bounds are obtained in terms of the conditional expectations. More specifically, there is a constant, $\mathring{\gamma}_\Pi$, (equal to $2\max\{\gamma^{\mathcal{A}}, \gamma^{\mathcal{B}}\}$ in the notation of theorem 5.4 of [4]) such that

$$r_{\Pi_d} \leq d\mathring{\gamma}_\Pi$$

for all d, and there is a constant, $\mathring{\delta}_\Pi$, (equal to $2\max\{\min\{\delta^{\mathcal{A}}, \hat{\delta}^{\mathcal{A}}\}\}, \min\{\delta^{\mathcal{B}}, \hat{\delta}^{\mathcal{B}}\}$ in the notation of theorem 5.4 of [4]) such that

$$h_{\Pi_d} \leq d\mathring{\delta}_\Pi$$

for all d. Then if we choose $s \geq \mathring{\gamma}_\Pi$ we have

$$(d+e)s \geq (d+e)\mathring{\gamma}_\Pi \geq r_{\Pi_{d+e}},$$

so we can set $r_{d+e} = (d+e)s$ and apply Proposition 13.4 to obtain:

Proposition 13.5. *Assume that Π is a bridge with conditional expectations, and let notation be as just above. Choose $s \geq \mathring{\gamma}_\Pi$. If*

$$(d+e)(\mathring{\delta}_\Pi + s) \max\{(L_d^{(d+e)s}(p_1, q_1), L_e^{(d+e)s}(p_2, q_2))\} < 1/2,$$

and if there is a projection $Q \in M_{d+e}(\mathcal{B})$ such that

$$(d+e)(\mathring{\delta}_\Pi + s) L_{d+e}^{(d+e)s}(\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}, Q) < 1/2, \tag{13.1}$$

then there is a path through projections in $M_{d+e}(\mathcal{B})$ going from Q to $\left(\begin{smallmatrix} q_1 & 0 \\ 0 & q_2 \end{smallmatrix}\right)$.

Thus again, if the inequality (13.1) is satisfied, then if the projective modules for p_1 and q_1 correspond, and if those for p_2 and q_2 correspond, then the direct sum modules correspond. But notice that the factor $d+e$ at the beginning means that as d or e get bigger, the remaining term must be smaller in order for the product to be $< 1/2$. Thus, for example, suppose that p and q correspond. This will not in general imply that $\left(\begin{smallmatrix} p & 0 \\ 0 & 0_e \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} q & 0 \\ 0 & 0_e \end{smallmatrix}\right)$ correspond, where 0_e denotes the 0 matrix of size e.

We can now apply the above results to our basic example in which $\mathcal{A} = C(G/H)$ and $\mathcal{B}^n = \mathcal{L}(\mathcal{H}^n)$, and we have the bridge Π^n between them, as in the previous section and earlier. From the discussion leading to propositions 6.3 and 6.7 of [4], which is strongly based on the results of [3], it can be seen that the constants $\mathring{\gamma}_{\Pi^n}$ and $\mathring{\delta}_{\Pi^n}$ converge to 0 as $n \to \infty$. If in Proposition 13.5 one sets $s_n = \max\{\mathring{\gamma}_{\Pi^n}, \mathring{\delta}_{\Pi^n}\}$, then for fixed d and e the term $(d+e)(\mathring{\delta}_\Pi^n + s_n)$ in inequality (13.1) will converge to 0 as $n \to \infty$. Consequently, for projections p_1 and p_2 as above, one can find a sufficiently large N that if $n \geq N$ and if q_1 and q_2 are corresponding projections in $M_d(\mathcal{B}^n)$ and $M_e(\mathcal{B}^n)$ respectively, then inequality (13.1) is satisfied, so that $\left(\begin{smallmatrix} p_1 & 0 \\ 0 & p_2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} q_1 & 0 \\ 0 & q_2 \end{smallmatrix}\right)$ correspond.

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Appendix. Weights

We give here a precise statement of the conventions we use concerning weights, weight vectors, etc., and of their properties, including some proofs (all basically well-known, e.g. in section VIII.4 of [45]).

At first we assume only that $\mathcal{H}$ is a finite-dimensional vector space over $\mathbb{C}$, and that H is a non-zero operator on $\mathcal{H}$ while E and F are operators on $\mathcal{H}$ satisfying the relations

$$[E, F] = H, \qquad [H, E] = 2E, \qquad \text{and} \quad [H, F] = -2F.$$

Let ξ be an eigenvector of H with eigenvalue r. Then

$$H(E\xi) = E(H\xi) + [H, E]\xi = rE\xi + 2E\xi = (r + 2)E\xi,$$

so that if $E\xi \neq 0$ then $E\xi$ is an eigenvector for H of eigenvalue r+2. In the same way, $F\xi$, if $\neq 0$, is an eigenvector of H of eigenvalue $r-2$. (So E and F are often called "ladder operators".) Since $\mathcal{H}$ is finite-dimensional, it follows that there must be an eigenvector, ξ_*, for H such that $E\xi_* = 0$ (a highest weight). Fix such a ξ_*, and let r be its eigenvalue. Then for each natural number a we see that $F^a\xi_*$ will be an eigenvector of eigenvalue $r - 2a$ unless $F^a\xi_* = 0$. Since $\mathcal{H}$ is finite-dimensional, there will be a natural number m such that $F^a\xi_* \neq 0$ for $a \leq m$ but $F^{m+1}\xi_* = 0$. Now $E(F\xi_*) = [E, F]\xi_* = r\xi_*$. In a similar way we find by induction that

$$\begin{aligned} E(F^a\xi_*) &= EF(F^{a-1}\xi_*) = (H + FE)(F^{a-1}\xi_*) \\ &= a(r - a + 1)F^{a-1}\xi_*. \end{aligned}$$

Consequently, since $F^{m+1}\xi_* = 0$, we have

$$H(F^m\xi_*) = -F(EF^m\xi_*) = -m(r - m + 1)F^m\xi_*.$$

But also $H(F^m\xi_*) = (r - 2m)F^m\xi_*$. Since $F^m\xi_* \neq 0$, it follows that $r = m$, and so it is appropriate to denote ξ_* by e_m. With this notation, set

$$e_{m-2a} = F^a e_m$$

for $a = 0, \ldots, m$. Each of these vectors is an eigenvector of H with corresponding eigenvalue $m - 2a$. These eigenvectors span a subspace of $\mathcal{H}$ of dimension $m+1$. From the calculations done above, it is clear that this subspace is carried into itself by the operators H, E and F, and that furthermore, we have

$$E(e_{m-2a}) = a(m - a + 1)e_{m-2a+2}. \tag{A.1}$$

From this we immediately obtain

$$FE(e_{m-2a}) = a(m - a + 1)e_{m-2a}.$$

Since $EF = H + FE$, it then follows that

$$EF(e_{m-2a}) = (a - 1)(m - a)e_{m-2a}. \tag{A.2}$$

Now assume that $\mathcal{H}$ is a finite-dimensional Hilbert space, and let e_m be a highest weight vector of weight m as above, and for each $a = 1, \ldots, m$ define e_{m-2a} as above. Assume now that $F = E^*$ (the adjoint of E). This implies that H is self-adjoint, so that its eigenvectors of different eigenvalue are orthogonal. Consequently the e_{m-2a}'s are orthogonal vectors, that span a subspace of $\mathcal{H}$ that is carried into itself by the operators H, E, and F (giving an irreducible unitary representation of $SU(2)$). Assume further that $\|e_m\| = 1$. We need to know the norms of the vectors e_{m-2a}. Notice that from Eq. (A.1) we see that for $a \geq 1$ we have

$$\begin{aligned} \|e_{m-2a}\|^2 &= \langle Fe_{m-2a+2}, e_{m-2a}\rangle = \langle e_{m-2a+2}, Ee_{m-2a}\rangle \\ &= a(m - a + 1)\|e_{m-2a+2}\|^2. \end{aligned}$$

A simple induction argument then shows that

$$\|F^a e_m\|^2 = \|e_{m-2a}\|^2 = a!\Pi_{b=0}^{a-1}(m - b) = \frac{a!m!}{(m - a)!} \tag{A.3}$$

for $a = 1, \ldots, m$. (We find it convenient not to normalize the e_m's to length 1.)

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