Journal of Geometry and Physics 157 (2020) 103846

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Characterization of three-dimensional Riemannian manifolds with a type of semi-symmetric metric connection admitting Yamabe soliton

Sudhakar K. Chaubey [a,*], Uday Chand De [b]

[a] Section of Mathematics, Department of Information Technology, Shinas College of Technology, Shinas, P.O. Box 77 Postal Code 324, Oman

[b] Department of Mathematics, University of Calcutta, West Bengal, India

ARTICLE INFO

Article history:
Received 24 April 2020
Received in revised form 20 July 2020
Accepted 25 July 2020
Available online 5 August 2020

MSC:
53D10
53C25
53D15

Keywords:
Riemannian manifolds
Yamabe solitons
Semi-symmetric metric connection
Conformal vector field

ABSTRACT

We characterize the three-dimensional Riemannian manifolds equipped with a semi-symmetric metric ρ-connection under the assumption that the Riemannian metric is a Yamabe soliton. It is shown that a three-dimensional Riemannian manifold endowed with a semi-symmetric ρ-connection, whose metric is Yamabe soliton, is a manifold of constant sectional curvature -1 and the Yamabe soliton is expanding with soliton constant -6. Finally, we give an example of a three-dimensional Riemannian manifold and validate our findings.

© 2020 Elsevier B.V. All rights reserved.

1. Introduction

Let M be a Riemannian manifold of dimension n and g denotes the Riemannian metric of M. Then a linear connection $\tilde{\nabla}$ defined on M is said to be symmetric or torsion free if its torsion tensor $\tilde{T}$ defined as $\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$ is zero for all vector fields X and Y on M, where $[\cdot\,,\,\cdot]$ denotes the Lie bracket of g. If the torsion tensor $\tilde{T}$ is non-zero on M, then the linear connection $\tilde{\nabla}$ is said to be non-symmetric [6]. The linear connection $\tilde{\nabla}$ on M is said to be semi-symmetric if $\tilde{T}$ takes the form

$$\tilde{T}(X,Y) = \pi(Y)X - \pi(X)Y, \quad \forall\, X,\ Y \in \chi(M). \tag{1.1}$$

This notion has been introduced by Friedmann and Schouten [19] on a differentiable manifold. Here π is a 1-form and satisfies $\pi(X) = g(X,\rho)$ for a vector field ρ on M. In this series, Hayden [21] introduced the idea of metric connection on a Riemannian manifold (M,g). According to Hayden, a linear connection $\tilde{\nabla}$ is said to be metric if the Riemannian metric is parallel with respect to $\tilde{\nabla}$, that is, $\tilde{\nabla} g = 0$. Otherwise, it is non-metric. Pak [25] has combined the idea of Friedmann et al. and Hayden and proved that $\tilde{\nabla}$ satisfies (1.1) and $\tilde{\nabla} g = 0$ is a semi-symmetric metric connection. The systemic study of semi-symmetric metric connection on a Riemannian manifold has been given by Yano [34]. Since then,

* Corresponding author.
E-mail addresses: sk22_math@yahoo.co.in (S.K. Chaubey), uc_de@yahoo.com (U.C. De).

https://doi.org/10.1016/j.geomphys.2020.103846
0393-0440/© 2020 Elsevier B.V. All rights reserved.

the properties of semi-symmetric metric connection have been studied by many geometers. For instance, we cite [1,13,23] and the references therein. Recently, Chaubey et al. [5] introduced the notion of semi-symmetric metric ρ-connection on a Riemannian manifold.

A Riemannian metric g of a complete Riemannian manifold M of dimension $n \geq 2$ is said to be Yamabe soliton [20] if it satisfies

$$\mathcal{L}_V g = (\lambda - r)g \tag{1.2}$$

for a real constant λ (soliton constant) and a smooth vector field V (soliton vector field) on M, where $\mathcal{L}$ denotes the Lie derivative operator and r is the scalar curvature of M. In particular, if we choose λ as positive, negative, or zero, then the Yamabe solitons, respectively, become shrinking, expanding, or steady.

Yamabe's conjecture (a metric of a complete Riemannian manifold is conformally related to a metric with constant scalar curvature) motivates the Hamilton [20] to introduce the notion of Yamabe flow. The Yamabe flow,

$$\frac{\partial}{\partial t} g(t) = -rg(t), \quad g_0 = g(0),$$

introduced by Hamilton [20], is a technique to solve the Yamabe problems. Notice that the problem of fast diffusion case of porous medium equation (the plasma equation), in mathematical physics, may be solved by Yamabe flow. It is well known that a Ricci soliton is a special solution of Ricci flow [12,20,29]. Similarly, the Yamabe solitons are the special solutions of the Yamabe flows. If the soliton vector field of the Yamabe soliton is replaced by the gradient of a smooth function f on M, then the Yamabe soliton becomes gradient Yamabe soliton and Eq. (1.2) takes the form

$$\nabla^2 f = \frac{1}{2}(\lambda - r)g,$$

where $\nabla^2 = Hess$, the Hessian operator of g and f is the potential function of the gradient Yamabe soliton. A Yamabe soliton (gradient Yamabe soliton) is said to be trivial when V is Killing (f is constant) on M. Sharma [26] studied Yamabe solitons on a three-dimensional Sasakian manifold, although Wang [31] and Erken [18] characterized the three-dimensional Kenmotsu and normal almost paracontact metric manifolds with the same solitons, respectively. In this series, Suh et al. [27,28] characterized the almost co-Kähler and three-dimensional $N(k)$-paracontact metric manifolds with Yamabe solitons, respectively. The Yamabe solitons within the framework of Riemannian manifolds have been studied by B. Y. Chen and his co-authors (see [8,17]). Also, the deep results on Yamabe solitons have been studied by many geometers but few are [2–4,9–11,22,30].

The above studies motivate us to characterize the three-dimensional Riemannian manifolds equipped with a semi-symmetric metric ρ-connection if its metric is a Yamabe soliton. In this series, we prove the following:

Theorem 1.1. *If the Riemannian metric of a three-dimensional Riemannian manifold with a semi-symmetric metric ρ-connection is a Yamabe soliton, then the manifold is of constant sectional curvature -1 and the Yamabe soliton is expanding with $\lambda = -6$.*

2. Semi-symmetric metric ρ-connection and basic definitions

Let M be a Riemannian manifold of dimension n. Then a linear connection $\tilde{\nabla}$ on M is defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X, Y)\rho \tag{2.1}$$

for all vector fields X and Y is known as a semi-symmetric metric connection [34] if it satisfies Eq. (1.1) and $\tilde{\nabla} g = 0$, where ∇ denotes the Levi-Civita connection and π is the 1-form associated to the unit vector field ρ by

$$\pi(X) = g(X, \rho). \tag{2.2}$$

Moreover, if $\tilde{\nabla}$ satisfies the relation

$$\tilde{\nabla}\rho = 0 \Longleftrightarrow \nabla\rho = \pi \otimes \rho - I, \tag{2.3}$$

which is equivalent to

$$(\nabla_X \pi)(Y) = \pi(X)\pi(Y) - g(X, Y)$$

for all vector fields X and Y on M, then $\tilde{\nabla}$ is called a semi-symmetric metric ρ-connection [5]. A Riemannian manifold M of dimension n equipped with a semi-symmetric metric ρ-connection satisfies the following:

$$R(X, Y)\rho = \pi(X)Y - \pi(Y)X, \tag{2.4}$$

$$R(\rho, X)Y = \pi(Y)X - g(X, Y)\rho,$$

$$\pi(R(X,Y)Z) = \pi(Y)g(X,Z) - \pi(X)g(Y,Z),$$

$$S(Y,\rho) = -(n-1)\pi(Y) \Longleftrightarrow Q\rho = -(n-1)\rho \tag{2.5}$$

for arbitrary vector fields X, Y and Z [5].

A smooth vector field V on an n-dimensional Riemannian manifold M is said to be a conformal vector field if its flow consists of conformal transformations, or equivalently, there exists a smooth function σ (called the potential function of V) that satisfies the equation

$$\mathcal{L}_V g = 2\sigma g. \tag{2.6}$$

The conformal vector field V is said to be non-trivial on M if V is a non-Killing vector field. A conformal vector field V is Killing if the potential function $\sigma = 0$ (or the flow generated by V consists of isometries of the Riemannian manifold). For more detail about conformal vector field, we refer [14–16,24,35] and the references therein. From Eqs. (1.2) and (2.6), it is obvious that

$$\sigma = \frac{\lambda - r}{2}. \tag{2.7}$$

Now, we recall some basic integrability conditions for conformal vector field in the following:

Lemma 2.1. *On an n-dimensional Riemannian manifold (M,g) endowed with a conformal vector field V, the following identities*

$$(\mathcal{L}_V S)(X,Y) = -(n-2)g(\nabla_X D\sigma, Y) + (\triangle\sigma)g(X,Y), \tag{2.8}$$

$$\mathcal{L}_V r = -2\sigma r + 2(n-1)\triangle\sigma \tag{2.9}$$

hold, where $\triangle := -div D$, the Laplacian operator of g and D represents the gradient operator [33].

A conformal curvature tensor C on an $n(\geq 3)$-dimensional Riemannian manifold M is defined as

$$\begin{aligned} C(X,Y)Z = {} & R(X,Y)Z - \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ & - g(X,Z)QY - \frac{r}{(n-1)}[g(Y,Z)X - g(X,Z)Y]\} \end{aligned}$$

for arbitrary vector fields X, Y and Z on M [32]. The manifold M is said to be conformally flat if and only if $C = 0$.

3. Main results

This section deals with the study of Yamabe soliton on a three-dimensional Riemannian manifold M^3 with a semi-symmetric metric ρ-connection.

Lemma 3.1. *A three-dimensional Riemannian manifold equipped a semi-symmetric metric ρ-connection satisfies*

$$\rho(r) = 2(r+6). \tag{3.1}$$

Proof. It is well-known that a three-dimensional Riemannian manifold is conformally flat and therefore the curvature tensor R of M^3 holds the following:

$$\begin{aligned} R(X,Y)Z = {} & S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \\ & - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{3.2}$$

for all vector fields X, Y and Z on M^3. Changing Y and Z by ρ and then using (2.2) and (2.4), we get

$$QX = \left(\frac{r}{2}+1\right)X - \left(\frac{r}{2}+3\right)\pi(X)\rho. \tag{3.3}$$

This shows that M^3 is a quasi Einstein manifold. The covariant derivative of (3.3) along the vector field Y gives

$$(\nabla_Y Q)(X) = -\left(\frac{r}{2}+3\right)\{(2\pi(X)\pi(Y) - g(Y,X))\rho - \pi(X)Y\} + \frac{Y(r)}{2}\{X - \pi(X)\rho\}. \tag{3.4}$$

Contracting (3.4) for Y and using $(trace\{Y \longrightarrow (\nabla_Y Q)(X)\} = \frac{1}{2}X(r))$ we infer

$$2\left(\frac{r}{2}+3\right)\pi(X) - \frac{\rho(r)}{2}\pi(X) = 0.$$

With the help of (2.2), the above equation gives (3.1). Hence Lemma 3.1 is proved. □

Proposition 3.2. *If the Riemannian metric of a three-dimensional Riemannian manifold M^3 equipped with a semi-symmetric metric ρ-connection is Yamabe soliton, then the scalar curvature of M^3 is harmonic and the Yamabe soliton is expanding with soliton constant $\lambda = -6$.*

Proof. The Lie derivatives of $g(\rho, \rho) = \pi(\rho) = 1$ along the vector field V, follows with (2.6), give

$$\pi(\mathcal{L}_V \rho) = -(\mathcal{L}_V \pi)(\rho) = \frac{r - \lambda}{2}. \tag{3.5}$$

From (2.7), it is obvious that

$$\triangle\sigma = -\frac{1}{2}\triangle r, \quad D\sigma = -\frac{1}{2}Dr. \tag{3.6}$$

On M^3, Eqs. (2.8), (2.9) and (3.6) give

$$(\mathcal{L}_V S)(X, Y) = \frac{1}{2}g(\nabla_X Dr, Y) - \frac{1}{2}(\triangle r)g(X, Y) \tag{3.7}$$

and

$$\mathcal{L}_V r = r(r - \lambda) - 2\triangle r. \tag{3.8}$$

The inner product of (3.3) with Y gives

$$S(X, Y) = \left(\frac{r}{2} + 1\right) g(X, Y) - \left(\frac{r}{2} + 3\right) \pi(X)\pi(Y). \tag{3.9}$$

Taking Lie derivative of (3.9) along the soliton vector field V and then using (1.2), (3.7) and (3.8), we find

$$\begin{aligned} g(\nabla_X Dr, Y) = {} & [2(\lambda - r) - \triangle r]g(X, Y) + [2\triangle r - r(r - \lambda)]\pi(X)\pi(Y) \\ & -(r + 6)\{(\mathcal{L}_V \pi)(X)\pi(Y) + \pi(X)(\mathcal{L}_V \pi)(Y)\}. \end{aligned} \tag{3.10}$$

Setting $Y = X = \rho$ in (3.10) and using (2.2), (3.1) and (3.5), we lead

$$\rho(\rho(r)) = \triangle r - 4(\lambda - r) \tag{3.11}$$

In consequence of (3.1), Eq. (3.11) takes the form

$$\triangle r = 4(\lambda + 6). \tag{3.12}$$

Changing Y with ρ in Eq. (3.10) and using (2.2) and (3.5), we get

$$(r + 6)(\mathcal{L}_V \pi)(X) = [4(\lambda + 6) + (\lambda - r)\left(\frac{r}{2} - 1\right)]\pi(X) - g(\nabla_X Dr, \rho). \tag{3.13}$$

The covariant derivative of $g(Dr, \rho) = \rho(r)$ along the vector field X, follows with Eqs. (2.2), (2.3) and (3.1), gives

$$g(\nabla_X Dr, \rho) = 3X(r) - 2(r + 6)\pi(X). \tag{3.14}$$

In view of (3.12) and (3.14), Eq. (3.13) assumes the form

$$(r + 6)(\mathcal{L}_V \pi)(X) = [4(\lambda + 6) + (\lambda - r)\left(\frac{r}{2} - 1\right) + 2(r + 6)]\pi(X) - 3X(r).$$

Using the last equation in (3.10) and then using (3.12), we find

$$\begin{aligned} g(\nabla_X Dr, Y) = {} & -2(r + \lambda + 12)g(X, Y) + 2(\lambda - 3r - 12)\pi(X)\pi(Y) \\ & +3X(r)\pi(Y) + 3\pi(X)Y(r), \end{aligned}$$

which gives

$$\nabla_X Dr = -2(r + \lambda + 12)X + 2(\lambda - 3r - 12)\pi(X)\rho + 3X(r)\rho + 3\pi(X)Dr. \tag{3.15}$$

Replacing X by ρ in Eq. (3.15) and using Eq. (2.2), we infer

$$\nabla_\rho Dr = -2(r + 6)\rho + 3Dr. \tag{3.16}$$

Also, from Eqs. (2.2) and (2.5) we have

$$S(\rho, Dr) = g(Q\rho, Dr) = -2\rho(r) = -4(r + 6). \tag{3.17}$$

Taking covariant derivative of (3.16) along the vector field X, we get

$$\nabla_X \nabla_\rho Dr = -2X(r)\rho - 2(r + 6)\pi(X)\rho + 2(r + 6)X + 3\nabla_X Dr, \tag{3.18}$$

where Eq. (2.3) is used. The covariant derivative of (3.15) along the vector field ρ gives

$$\begin{aligned}\nabla_\rho\nabla_X Dr &= 2(\lambda-3r-12)\eta(\nabla_\rho X)\rho-2(r+\lambda+12)\nabla_\rho X-12(r+6)\eta(X)\rho\\ &\quad -4(r+6)X+3\nabla_\rho(X(r))\rho+3\eta(\nabla_\rho X)Dr+3\eta(X)\nabla_\rho Dr,\end{aligned}\tag{3.19}$$

because Eqs. (2.2) and (2.3) have been used. Also, from (3.15) we infer

$$\nabla_{[X,\rho]}Dr=-2(r+\lambda+12)[X,\rho]+2(\lambda-3r-12)\eta([X,\rho])\rho+3[X,\rho](r)\rho+3\eta([X,\rho])Dr.\tag{3.20}$$

Using Eqs. (3.18)–(3.20) in

$$R(X,\rho)Dr=\nabla_X\nabla_\rho Dr-\nabla_\rho\nabla_X Dr-\nabla_{[X,\rho]}Dr,$$

we find

$$\begin{aligned}R(X,\rho)Dr &= -2X(r)\rho-2(r+6)\pi(X)\rho+2(r+6)X+3\nabla_X Dr+4(r+6)X\\ &\quad +2(r+\lambda+12)\nabla_\rho X+12(r+6)\eta(X)\rho-2(\lambda-3r-12)\eta(\nabla_\rho X)\rho\\ &\quad -2(\lambda-3r-12)\eta(\nabla_\rho X)\rho-3\nabla_\rho(X(r))\rho-3\eta(\nabla_\rho X)Dr\\ &\quad -3\eta(X)\nabla_\rho Dr+2(r+\lambda+12)[X,\rho]-2(\lambda-3r-12)\eta([X,\rho])\rho\\ &\quad -3[X,\rho](r)\rho-3\eta([X,\rho])Dr.\end{aligned}$$

Contracting the above equation for X, we lead

$$\begin{aligned}S(\rho,Dr) &= 24(r+6)-3\nabla_\rho(2(r+6))-6\eta(\nabla_\rho Dr)-4(r+\lambda+12)\\ &\quad -3\eta(\nabla_{Dr}\rho-\nabla_\rho Dr)+3\{trace\ X\longrightarrow\nabla_X Dr\}.\end{aligned}\tag{3.21}$$

From (3.15), we have

$$\{trace\ X\longrightarrow\nabla_X Dr\}=-4(\lambda-3r+6).\tag{3.22}$$

Using (3.16) and (3.22) in (3.21), we obtain

$$S(\rho,Dr)=-4(r+4\lambda+30).\tag{3.23}$$

From (3.17) and (3.23) we have

$$\lambda=-6.\tag{3.24}$$

This shows that the Yamabe soliton is expanding. Again using (3.24) in (3.12), we get

$$\triangle r=0.$$

Hence, we conclude that the scalar curvature of M^3 is harmonic. Thus, Proposition 3.2 is proved. □

Proof of Theorem 1.1. The covariant derivative of (1.2) along an arbitrary vector field X gives

$$\nabla_X\mathcal{L}_V g=-X(r)g.\tag{3.25}$$

In [33], it is proved that

$$(\mathcal{L}_V\nabla_X g-\nabla_X\mathcal{L}_V g-\nabla_{[V,X]}g)(Y,Z)=-g((\mathcal{L}_V\nabla)(X,Y),Z)-g((\mathcal{L}_V\nabla)(X,Z),Y),$$

which becomes

$$(\nabla_X\mathcal{L}_V g)(Y,Z)=g((\mathcal{L}_V\nabla)(X,Y),Z)+g((\mathcal{L}_V\nabla)(X,Z),Y),\tag{3.26}$$

because $\nabla g=0$. Since the operator $(\mathcal{L}_V\nabla)$ is symmetric, that is, $(\mathcal{L}_V\nabla)(X,Y)=(\mathcal{L}_V\nabla)(Y,X)$. Making use of (3.25) in (3.26), we have

$$(\mathcal{L}_V\nabla)(X,Y)=\frac{1}{2}g(X,Y)Dr-\frac{1}{2}X(r)Y-\frac{1}{2}Y(r)X,\tag{3.27}$$

where D denotes the gradient operator on M^3. Now, the covariant derivative of (3.27) along the vector field Z gives

$$(\nabla_Z\mathcal{L}_V\nabla)(X,Y)=\frac{1}{2}g(X,Y)\nabla_Z Dr-\frac{1}{2}g(X,\nabla_Z Dr)Y-\frac{1}{2}g(Y,\nabla_Z Dr)X.\tag{3.28}$$

It is well-known that

$$\begin{aligned}(\mathcal{L}_V R)(X,Y)Z &= (\nabla_X \mathcal{L}_V \nabla)(Y,Z) - (\nabla_Y \mathcal{L}_V \nabla)(X,Z)\\ &= \frac{1}{2}g(Y,Z)\nabla_X Dr - \frac{1}{2}g(Z,\nabla_X Dr)Y - \frac{1}{2}g(X,Z)\nabla_Y Dr\\ &\quad + \frac{1}{2}g(Z,\nabla_Y Dr)X - \frac{1}{2}\{g(Y,\nabla_X Dr) - g(X,\nabla_Y Dr)\}Z, \end{aligned} \tag{3.29}$$

where Eq. (3.28) is used. Substituting $Z = \rho$ in (3.29) and then using Eqs. (2.2), (3.15) and (3.24), we get

$$\begin{aligned}(\mathcal{L}_V R)(X,Y)\rho &= \{\frac{3}{2}Y(r) - (2r+\lambda+18)\pi(Y)\}X + \{(2r+\lambda+18)\pi(X)\\ &\quad -\frac{3}{2}X(r)\}Y + \frac{3}{2}\{X(r)\pi(Y) - Y(r)\pi(X)\}\rho. \end{aligned} \tag{3.30}$$

The Lie derivative of Eq. (2.4) along the soliton vector V gives

$$\begin{aligned}(\mathcal{L}_V R)(X,Y)\rho &= \{g(X,\mathcal{L}_V\rho) - (r+6)\pi(X)\}Y - \{g(Y,\mathcal{L}_V\rho)\\ &\quad -(r+6)\pi(Y)\}X - R(X,Y)\mathcal{L}_V\rho. \end{aligned} \tag{3.31}$$

In consequence of (3.30) and (3.31), we obtain

$$\begin{aligned}R(X,Y)\mathcal{L}_V\rho &= \{g(X,\mathcal{L}_V\rho) - (3r+\lambda+24)\pi(X) + \frac{3}{2}X(r)\}Y - \frac{3}{2}\{X(r)\pi(Y)\\ &\quad -Y(r)\pi(X)\}\rho + \{(3r+\lambda+24)\pi(Y) - g(Y,\mathcal{L}_V\rho) - \frac{3}{2}Y(r)\}X. \end{aligned}$$

Contracting the above equation along X, we find that

$$S(Y,\mathcal{L}_V\rho) = -2g(Y,\mathcal{L}_V\rho) + (3r+2\lambda+30)\pi(Y) - \frac{3}{2}Y(r). \tag{3.32}$$

Again, from Eqs. (2.2), (3.3), (3.5) and (3.24) we infer

$$S(Y,\mathcal{L}_V\rho) = \left(\frac{r}{2}+1\right)g(Y,\mathcal{L}_V\rho) - \left(\frac{r}{2}+3\right)\left(\frac{r+6}{2}\right)\pi(Y).$$

The above equation along with (3.32) should be

$$(r+6)\mathcal{L}_V\rho = \frac{1}{2}\{r^2+24r+156+8\lambda\}\rho - 3Dr.$$

If possible, we suppose that $r \neq -6$ on M^3, then the above equation assumes the form

$$\mathcal{L}_V\rho = \frac{r+18}{2}\rho - \frac{3}{r+6}Dr. \tag{3.33}$$

Replacing Y with ρ in the well-known formula of Yano [33]

$$(\mathcal{L}_V\nabla)(X,Y) = \mathcal{L}_V\nabla_X Y - \nabla_X\mathcal{L}_V Y - \nabla_{[V,X]}Y$$

and using Eqs. (2.2), (2.3), (3.15), (3.24) and (3.33), we lead

$$\begin{aligned}(\mathcal{L}_V\nabla)(X,\rho) &= -\left(\frac{r}{2}+15\right)\pi(X)\rho - \frac{(r-6)}{2(r+6)}X(r)\rho\\ &\quad + \frac{6}{r+6}\pi(X)Dr + \frac{r+6}{2}X - \frac{3}{(r+6)^2}X(r)Dr. \end{aligned} \tag{3.34}$$

Again, setting $Y = \xi$ in (3.27) we have

$$(\mathcal{L}_V\nabla)(X,\rho) = \frac{1}{2}\pi(X)Dr - \frac{1}{2}X(r)\rho - \frac{1}{2}\rho(r)X, \tag{3.35}$$

where Eq. (2.2) has been used. From Eqs. (3.34) and (3.35), we have

$$\begin{aligned}\pi(X)Dr - X(r)\rho - \rho(r)X &= -(r+30)\,\pi(X)\rho - \frac{(r-6)}{(r+6)}X(r)\rho\\ &\quad + \frac{12}{r+6}\pi(X)Dr + (r+6)X - \frac{6}{(r+6)^2}X(r)Dr. \end{aligned}$$

Changing X with ρ and then using Eq. (2.2) we get

$$\rho(r)\rho - 24\rho - \frac{r-6}{r+6}Dr - \frac{6}{(r+6)^2}\rho(r)Dr = 0. \tag{3.36}$$

Using (3.1) in (3.36) we infer that

$$Dr = 2(r-6)\rho. \tag{3.37}$$

Differentiating equation (3.37) covariantly along the vector field X and using (2.3) we have

$$\nabla_X Dr = 2X(r)\rho + 2(r-6)\pi(X)\rho - 2(r-6)X.$$

Again Eqs. (3.15) and (3.24) give

$$\nabla_X Dr = -2(r+6)X - 72\pi(X)\rho + 3X(r)\rho.$$

The last two equations with $X = \rho$ give

$$r = -6$$

and therefore Eq. (3.3) takes the form

$$QX = -2X \Longleftrightarrow S(X,Y) = -2g(X,Y).$$

Using the last two equations in (3.2), we find

$$R(X,Y)Z = -g(Y,Z)X + g(X,Z)Y.$$

This means that M^3 is of constant sectional curvature -1. Thus, we have the statement of Theorem 1.1.

4. Example

Example 4.1. Let $\mathcal{R}^n$ denote a real space of dimension n and

$$M^3 = \{(x,y,z) \in \mathcal{R}^3 : x,y,z(\neq 0) \in \mathcal{R}\}$$

is a differentiable manifold of dimension three. Define a positive definite metric g on M^3 as $g = e^{-2z}[dx \otimes dx + dy \otimes dy] + dz \otimes dz$ and $g(U,\rho) = \pi(U)$, where π is the 1-form and ρ is a unit vector field on M^3 [7]. Let us define a set of linearly independent vector fields on M^3 as $e_1 = e^z \frac{\partial}{\partial x}$, $e_2 = e^z \frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z}$ and therefore it forms a basis for the tangent space $T(M^3)$ of M^3. For $\rho = e_3$, (M^3, g) forms a Riemannian manifold of dimension three. By direct computations, we can observe that the non-zero components of Lie bracket are given by $[e_1, e_3] = -e_1$, $[e_2, e_3] = -e_2$. Motivated from the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ & -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned}$$

we can easily find the non-zero components as

$$\nabla_{e_1} e_1 = e_3, \quad \nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_2} e_3 = -e_2,$$

where ∇ denotes the Levi-Civita connection corresponding to the metric g. By straight forward computations, we have

$$\begin{aligned} & R(e_1,e_2)e_1 = e_2, \quad R(e_1,e_3)e_1 = e_3, \quad R(e_1,e_2)e_2 = -e_1, \quad R(e_2,e_3)e_2 = e_3, \\ & R(e_1,e_3)e_3 = -e_1, \quad R(e_2,e_3)e_3 = -e_2, \quad S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = -2 \end{aligned}$$

and other components can be easily obtained by the symmetric properties. From the above results, we have $r = -6$.

Let X be an arbitrary vector field of M^3, then it can be written as $X = a_1e_1 + a_2e_2 + a_3e_3$. For $\rho = e_3$, we have

$$\begin{aligned} & \tilde{\nabla}_X \rho = \nabla_X \rho + X - \eta(X)\rho \\ & = \nabla_{a_1e_1+a_2e_2+a_3e_3} e_3 + (a_1e_1 + a_2e_2 + a_3e_3) - g(a_1e_1 + a_2e_2 + a_3e_3, e_3)e_3 = 0. \end{aligned}$$

If $X = a_1e_1 + a_2e_2 + a_3e_3$, $Y = b_1e_1 + b_2e_2 + b_3e_3$ and $Z = c_1e_1 + c_2e_2 + c_3e_3$ are arbitrary vector fields of M^3, then by direct calculations we have $\tilde{T}(X,Y) = g(Y,e_3)X - g(X,e_3)Y$ and $(\tilde{\nabla}_X g)(Y,Z) = 0$ for all vector fields X, Y and Z. Thus, the linear connection $\tilde{\nabla}$ defined by Eq. (2.1) is a semi-symmetric metric ρ-connection on M^3. Here a_i, b_i, $c_i \in \mathcal{R}$ for $i = 1,2,3$. Again, let $V = \sum d_i e_i$ for all $i = 1,2,3$, $d_i \in \mathcal{R}$ and a_i, b_i and d_i satisfy the condition $2(a_1b_1 + a_2b_2)d_3 = (a_1b_3 + a_3b_1)d_1 + (a_2b_3 + a_3b_2)d_2$. Then by straight forward calculations, we have

$$g(\nabla_X V, Y) = a_1b_3d_1 - a_1d_3b_1 + a_2d_2b_3 - a_2d_3b_2$$

and

$$g(X, \nabla_Y V) = b_1d_1a_3 - b_1d_3a_1 + b_2d_2a_3 - b_2d_3a_2$$

By considering the above facts, we conclude that

$$(\mathcal{L}_V g)(X,Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V) = 0.$$

This shows that the soliton vector field V on M^3 is Killing.

We can easily observe that the Riemannian curvature tensor and the Riemannian metric of M^3 take the following form

$$R(X,Y)Z = [a_2c_2b_1 + a_3c_3b_1 - b_2c_2a_1 - c_3b_3a_1]\,e_1 + [a_1b_2c_1 - a_2b_1c_1 - a_2b_3c_3 + a_3b_2c_3]e_2 + [a_1b_3c_1 + a_2b_3c_2 - a_3b_1c_1 - a_3b_2c_2]\,e_3$$

and

$$g(Y,Z)X - g(X,Z)Y = [b_2c_2a_1 + c_3b_3a_1 - a_2c_2b_1 - a_3c_3b_1]\,e_1 + [a_2b_1c_1 - a_1b_2c_1 - a_3b_2c_3 + a_2b_3c_3]e_2 + [-a_1b_3c_1 - a_2b_3c_2 + a_3b_1c_1 + a_3b_2c_2]\,e_3.$$

From the above discussions, it is observed that $R(X,Y)Z = -\{g(Y,Z)X - g(X,Z)Y\}$ and $r = -6$ and thus Theorem 1.1 is verified.

Acknowledgment

The authors express their sincere thanks to the anonymous referees for providing the valuable suggestions in the improvement of the paper. The first author acknowledges authority of Shinas College of Technology for their continuous support and encouragement to carry out this research work.

References

[1] S. Altay, F. Özen, Some examples of recurrent Riemannian manifolds with semi-symmetric metric connection, Studia Sci. Math. Hungar. 44 (2007) 15–25.
[2] E. Barbosa, Jr.E. Ribeiro, On conformal solutions of the Yamabe flow, Arch. Math. 101 (2013) 79–89.
[3] G. Calvaruso, A. Zaeim, A complete classification of Ricci and yamabe solitons of non-reductive homogeneous 4-spaces, J. Geom. Phys. 80 (2014) 15–25.
[4] H.D. Cao, X.F. Sun, Y.Y. Zhang, On the structure of gradient Yamabe solitons, Math. Res. Lett. 19 (4) (2012) 767–774.
[5] S.K. Chaubey, J.W. Lee, S. Yadav, Riemannian manifolds with a semi-symmetric metric P-connection, J. Korean Math. Soc. 56 (4) (2019) 1113–1129.
[6] S.K. Chaubey, R.H. Ojha, On a semi-symmetric non-metric connection, Filomat 26 (2) (2012) 269–275.
[7] S.K. Chaubey, A. Yildiz, Riemannian manifolds admitting a new type of semisymmetric nonmetric connection, Turkish J. Math. 43 (2019) 1887–1904.
[8] B.Y. Chen, S. Deshmukh, Yamabe and quasi-Yamabe solitons on Euclidean submanifolds, Mediterr. J. Math. 15 (2018) 194–202, http://dx.doi.org/10.1007/s00009-018-1237-2.
[9] B. Chow, The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature, Comm. Pure Appl. Math. 45 (8) (1992) 1003–1014.
[10] P. Daskalopoulos, N. Sesum, The classification of locally conformally flat Yamabe solitons, Adv. Math. 240 (2013) 346–369.
[11] U.C. De, S.K. Chaubey, Y.J. Suh, A note on almost co-Kähler manifolds, Int. J. Geom. Methods Mod. Phys. (2020) http://dx.doi.org/10.1142/S0219887820501534.
[12] U.C. De, Y. Matsuyama, Ricci solitons and gradient Ricci solitons in a Kenmotsu manifold, Southeast Asian Bull. Math. 37 (2013) 691–697.
[13] S.Altay Demirbag, On weakly Ricci symmetric manifolds admitting a semi symmetric metric connection, Hacet. J. Math. Stat. 41 (2012) 507–513.
[14] S. Deshmukh, Conformal vector fields and Eigenvectors of Laplace operator, Math. Phys. Anal. Geom. 15 (2012) 163–172.
[15] S. Deshmukh, F. Alsolamy, Conformal vector fields on a Riemannian manifold, Balkan J. Geom. Appl. 19 (2) (2014) 86–93.
[16] S. Deshmukh, F. Alsolamy, A note on conformal vector fields on a Riemannian manifold, Colloq. Math. 136 (1) (2014) 65–73.
[17] S. Deshmukh, B.Y. Chen, A note on Yamabe solitons, Balkan J. Geom. Appl. 23 (1) (2018) 37–43.
[18] I.K. Erken, Yamabe solitons on three-dimensional normal almost paracontact metric manifolds, Period Math. Hungar. (2019) http://dx.doi.org/10.1007/s10998-019-00303-3.
[19] A. Friedmann, J.A. Schouten, Uber die geometry der halbsymmetrischen ubertragung, Math. Z. 21 (1924) 211–223.
[20] R. Hamilton, The Ricci flow on surfaces, Contemp. Math. 71 (1988) 237–261.
[21] H.A. Hayden, Subspace of space with torsion, Proc. Lond. Math. Soc. 34 (1932) 27–50.
[22] S.Y. Hsu, A note on compact gradient Yamabe solitons, J. Math. Anal. Appl. 388 (2012) 725–726.
[23] C. Murathan, C. Özgür, Riemannian manifolds with a semi-symmetric metric connection satisfying some semisymmetry conditions, Proc. Est. Acad. Sci. 57 (4) (2008) 210–216.
[24] M. Okumura, On infinitesimal conformal and projective transformations of normal contact spaces, Tohoku Math. J. 14 (1962) 398–412.
[25] E. Pak, On the pseudo-Riemannian spaces, J. Korean Math. Soc. 6 (1969) 23–31.
[26] R. Sharma, A 3-dimensional sasakian metric as a Yamabe soliton, Int. J. Geom. Methods Mod. Phys. 9 (2012) 1220003.
[27] Y.J. Suh, U.C. De, Yamabe solitons and Ricci solitons on almost co-Kähler manifolds, Canad. Math. Bull. 62 (3) (2019) 653–661.
[28] Y.J. Suh, K. Mandal, Yamabe solitons on three-dimensional $N(k)$-paracontact metric manifolds, Bull. Iran. Math. Soc. (2018) http://dx.doi.org/10.1007/s41980-018-0013-1.
[29] M. Turana, C. Yetima, S.K. Chaubey, On quasi-Sasakian 3-manifolds admitting η-Ricci solitons, Filomat 33 (15) (2019) 4923–4930.
[30] V. Venkatesha, D.M. Naik, Yamabe solitons on 3-dimensional contact metric manifolds with $Q\phi = \phi Q$, Int. J. Geom. Methods Mod. Phys. 16 (3) (2019) 1950039.
[31] Y. Wang, Yamabe solitons on three-dimensional Kenmotsu manifolds, Bull. Belg. Math. Soc. Simon Stevin 23 (3) (2016) 345–355.
[32] H. Weyl, Reine infinitesimalgeometrie, Math. Z. 2 (1918) 384–411.
[33] K. Yano, Integral Formulas in Riemannian Geometry, in: Pure and Applied Mathematics, vol. I, Marcel Dekker, New York, 1970.
[34] K. Yano, On semi-symmetric metric connections, Rev. Roumaine Math. Pures Appl. 15 (1970) 1579–1586.
[35] K. Yano, T. Nagano, Einstein spaces admitting a one-parameter group of conformal transformations, Ann. of Math. 69 (1959) 451–461.