



Singularities of renormalization group flows[☆]

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ABSTRACT

We discuss singularity formation in certain renormalization group flows. Special cases are the Ricci Yang–Mills and *B*-field flows. We point out some results suggesting that topological hypotheses can make RG flows much less singular than Ricci flow. In particular we show that for rotationally symmetric initial data on $S^2 \times S^1$ one gets long time existence and convergence of RYM flow, in stark contrast to the case for Ricci flow [S. Angenent, D. Knopf, An example of neckpinching for Ricci flow on S^{n+1} , *Math. Res. Lett.* 11 (4) (2004) 493–518]. Other results are given which allow one to rule out many singularity models under strictly topological hypotheses. A conjectural picture of singularity formation for RG flow on 3-manifolds is given.

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1. Introduction

Ricci flow has attracted much interest in the physics community, partially because of its role as the simplest model of renormalization group flow. Our physical understanding of spacetime goes beyond just the Ricci tensor, and so renormalization group flows arise which couple the Ricci flow with flows on certain external fields. Specifically, let (M^n, g) be a Riemannian manifold. Let $K \rightarrow E \rightarrow M$ denote a principal K -bundle over M , and let A be a connection on this bundle. Furthermore, let B denote a local 2-form defined up to the addition of an exact 2-form, and let $H = dB$ be a well-defined 3-form on M . For more detailed information on *B*-fields see [5,11]. Finally, let

$$\eta_{ij} = \text{Tr} (g^{kl} F_{ik} F_{jl})$$

where here Tr denotes the trace over the Lie algebra \mathfrak{k} , and let

$$\mathcal{H} = g^{kl} g^{mn} H_{ikm} H_{jln}.$$

Our main object of study will be the following system of equations:

$$\begin{aligned} \frac{\partial}{\partial t} g &= -2 \text{Rc} + \eta + \frac{1}{2} \mathcal{H} \\ \frac{\partial}{\partial t} A &= -d_{g(t)}^* F \\ \frac{\partial}{\partial t} B &= -d_{g(t)}^* H. \end{aligned} \tag{1.1}$$

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For convenience in this paper, we will refer to this system as *renormalization group flow*, or *RG flow*. The case $F = 0$, which is preserved by (1.1), yields the system studied in [7], and we will refer to this as *B-field flow*. Note that we also studied this equation under the name connection Ricci flow in [12]. Thus this special case strictly arises as an instance of physical renormalization group flow. Also, the case $H \equiv 0$, which is preserved by (1.1), yields the system of equations studied in [13–15], which is most naturally named *Ricci Yang–Mills flow*, or *RYM flow*. Note also that on a surface one automatically has $H \equiv 0$ so that RG flow reduces to RYM flow. We will at points below reduce to one of these special cases.

We hasten to point out that RYM flow and moreover the $F \neq 0$ case of our RG flow are *not* strictly physical RG flows. The system considered for instance in [3], line 15, suggests a flow equation for metrics given by

$$\frac{\partial}{\partial t} g = -\alpha \operatorname{Rc} + \frac{\alpha}{4} \mathcal{H} - \alpha^2 R_{ipqr} R_j^{pqr} + \frac{\alpha^2}{2} \eta$$

where α is the string coupling constant. Note that the η term appears at second order in α . One really should include all terms of a given order to call a flow a physical RG flow. Thus our flow has (up to constants) removed an Rm^2 term from the proper physical RG flow. There are two reasons for doing this. First, one does not even have a general short time existence theory for this flow (although one can get one by normalizing the metric to have curvature norm 1 and then choosing α sufficiently small). Second, our goal is to emphasize how the η and \mathcal{H} terms in our RG flow have clear topological consequences for the behavior of the related flows. These consequences will not always strictly carry over to the physical RG flow, but can be thought of as what will happen to the flow if the principal bundle has high twisting, i.e. the F term is very large in comparison to the curvature of g .

Our main goal is to point out certain qualitative differences between the behavior of RG flow and Ricci flow. In particular, we give some evidence that RG flow can be much less singular than Ricci flow. While the singular behavior of Ricci flow is essential to drawing topological conclusions, for building physical theories a less singular flow can be useful. We start by pointing out part of the main result of [14].

Theorem 1.1 ([14]). *Consider the 2-sphere S^2 with Riemannian metric g . Let $E \rightarrow S^2$ denote the total space of a nontrivial $U(1)$ -bundle over S^2 , say for instance the unit tangent bundle, and let A be a connection on E . Then the solution for RYM flow with initial condition (g, A) exists for all time and converges to the round sphere.*

Recall that in the case of Ricci flow on S^2 , for any initial condition the flow goes singular in finite time and is collapsing to a round sphere. Since Ricci flow occurs as a special case of RYM flow, specifically RYM flow on a trivial bundle, we deduce that the *topological hypothesis of the nontriviality of the bundle is producing qualitatively different behavior in the RG flow*. One might say that the Yang–Mills term in the RYM flow is acting as a volume-renormalization term and so, unsurprisingly, one gets similar behavior to the volume-normalized Ricci flow, which on S^2 always exists for all time and converges to the round sphere. However, this interpretation is not correct, and in fact as we will see below one should think of the Yang–Mills term as preventing collapse along certain homology classes.

We now outline the paper and the results within. In Section 2 we briefly discuss the existence and regularity theory of RG flow. In Section 3 we examine RG flow on $S^2 \times S^1$. In the homogeneous case, if the Yang–Mills field is nontrivial, one gets long time existence and convergence of RG flow to a product metric. However, if the Yang–Mills field is trivial, even with H nontrivial one gets a finite time singularity. We then prove that the first statement holds for rotationally symmetric initial conditions. In particular we prove the following:

Proposition 1.2. *Given g , a rotationally symmetric metric on $M = S^2 \times S^1$, and A , a connection on a $U(1)$ bundle over M , such that F is the (unique up to scaling) rotationally symmetric representative of $[(S^2)^*]$, the cohomology class dual to the nontrivial S^2 , then the solution for RYM flow with initial condition (g, A, B) exists for all time and converges to a multiple of the product metric with F parallel.*

Note that this is in stark contrast to the Ricci flow case, where an open set of initial conditions produces local neckpinch or cap singularities. Showing that rotationally symmetric RG flow with $F = 0$ and $H \neq 0$ still encounters a neckpinch singularity will be the subject of a later paper.

In Section 4 we prove a result ruling out singularities of RG flow on 3-manifolds along homology classes which pair nontrivially with F .

Theorem 1.3. *Let (M^3, g) be a 3-manifold and $U(1) \rightarrow E \rightarrow M$ be a principal $U(1)$ -bundle over M with connection A and curvature 2-form F . Let Σ be a weakly stable minimal surface with orientable normal bundle. Then the area of Σ satisfies*

$$\frac{dA}{dt} \geq -4\pi\chi + \frac{(\int_{\Sigma} F)^2}{4A}.$$

This allows one to rule out neckpinch singularities along the topologically nontrivial S^2 . Furthermore we point out how the main result of [6] extends to RG flow, and prove a theorem which allows us to rule out any compact blow-up limit for broad topological hypotheses.

Theorem 1.4. Let $(M^n, g(t), A(t), B(t))$ be a solution for RG flow which goes singular at $t = T < \infty$ and satisfies an injectivity radius estimate on the scale of maximum curvature. Suppose that the bundle $E \rightarrow M$ is nontrivial, or $[H] \neq 0$. Then the underlying base manifold of the blow-up limit is noncompact.

These results suggest certain conjectures which we discuss and give heuristic arguments for in Section 5.

2. Existence and regularity

Short time existence for RG flow follows as in the case of Ricci flow using the method of DeTurck [1]. Indeed short time existence for RYM flow and Bernstein–Shi-type derivative estimates were established independently in [13,15]. These results were established for B -field flow in [12]. Establishing the derivative estimates for RYM flow is achieved by examining the evolution of the curvature of a Kaluza–Klein-type metric on the total space E of the principal bundle. This metric takes the form

$$G = g_{ij} dx^i dx^j + (e^\theta + A_k^\theta dx^k)^2 \quad (2.1)$$

where g is the metric on the base manifold and e^θ is a left-invariant coframe for K , the group of our principal bundle. Similarly, the derivative estimates for B -field flow are established by looking at the evolution of the curvature of the connection given by the sum of the Levi-Civita connection of g and H with one index raised acting as torsion. That is, one considers

$$\bar{\Gamma}_{ij}^k = \Gamma_{ij}^k + g^{kl} H_{ijl} \quad (2.2)$$

where Γ is the Levi-Civita connection. Indeed these two viewpoints can be combined to yield regularity theory for the full RG flow as well. In particular, let $\bar{\nabla}$ denote the connection defined by (2.2), but this time using the Levi-Civita connection of G where G is as in (2.1). Under RG flow, the curvature of this connection satisfies a heat-type equation, and one can derive basic regularity bounds as for Ricci flow.

Theorem 2.1. Given (g, A, H) as above, there exists a solution for RG flow with this initial condition. Moreover, for each $\alpha > 0$ and every $m \in \mathbb{N}$ there exists a constant C_m depending only on m, n and $\max\{\alpha, 1\}$ such that if

$$\begin{aligned} |\text{Rm}(\bar{\nabla})|_{C^0(M_t)} &\leq K \\ |H|_{C^0(M_t)}^2 &\leq K \end{aligned}$$

for all $x \in M$ and $t \in [0, \frac{\alpha}{K}]$, then

$$|\bar{\nabla}^m \text{Rm}|_{C^0(M_t)} \leq \frac{C_m K}{t^{m/2}}$$

for all $x \in M$ and $t \in (0, \frac{\alpha}{K}]$.

Proof. For more details see [9,10,12,13,15]. \square

Moreover, extending the Perelman result, one can show that RG flow is a gradient flow, namely the gradient flow of

$$\mu(g, A, B) = \inf_{\int_M e^{-f} dV = 1} \int_M \left(R - \frac{1}{12} |H|^2 - \frac{1}{4} |F|^2 + |\nabla f|^2 \right) e^{-f} dV.$$

See [2,7,8,13,15] for more details.

3. Renormalization group flow on $S^2 \times S^1$

We begin by computing the behavior of renormalization group flow on homogeneous $M = S^2 \otimes S^1$. First of all $\bigwedge^3 T^*M$ is one-dimensional, so for any 3-form H we have

$$H = \pm |H| dV.$$

Also, it is clear that

$$\mathcal{H} = \frac{1}{3} |H|^2 g.$$

In the homogeneous setting on M we can furthermore make the decomposition

$$H = \pm |H| dV_{S^2} \wedge d\theta$$

where dV_{S^2} and $d\theta$ are the volume forms on S^2 and S^1 respectively. Furthermore let $E \rightarrow S^2 \times S^1$ be the pullback of the unit tangent bundle on S^2 to $S^2 \times S^1$ and fix the usual connection on E with F given by a nonzero multiple of the usual round area form on S^2 . Consider the family of metrics on M

$$g := \alpha g_{S^2} \oplus \beta g_{S^1}$$

for positive constants α, β . Thus

$$\mathcal{H} = \frac{1}{3} |H|^2 g = \frac{|[H]|}{3\alpha^2\beta} g$$

where $[H]$ denotes the cohomology class of H . Also, since $F = |F| dV_{S^2}$ we have

$$\eta = \frac{1}{2} |F|^2 g_{S^2} = \frac{|[F]|}{2\alpha^2} g_{S^2}$$

where here F is the pairing of F with the S^2 homology cycle. Finally, observe that both F and H are parallel, and so they do not change under RG flow. Collecting these observations allows us to reduce RG flow in this setting to the system of equations

$$\dot{\alpha} = -2 + \frac{|[F]|}{2\alpha} + \frac{|[H]|}{6\alpha\beta}$$

$$\dot{\beta} = \frac{|[H]|}{6\alpha^2}.$$

First consider the case of Ricci flow, which corresponds to $[F] = [H] = 0$. Then the S^2 factor collapses homothetically in finite time. If $[F] \neq 0$, we immediately see that α satisfies an a priori lower bound; therefore one has a priori upper and lower bounds for β and $\dot{\alpha}$. Furthermore, if $[H] = 0$ one in fact gets smooth convergence at infinity, but if $[H] \neq 0$ then the length of the S^1 factor goes to infinity as time does. Thus we see that the presence of the Yang–Mills term prevents the collapse of the S^2 factor and ensures long time existence.

However, in the case $[F] = 0, [H] \neq 0$, one still encounters a finite time singularity. To see this, first assume that we consider the model case where $\frac{|[H]|}{6} = \alpha(0) = \beta(0) = 1$. Then the solution is given explicitly by

$$\alpha(t) = 1 - t$$

$$\beta(t) = \frac{1}{1 - t}.$$

To see the more general case, first one checks the formula

$$\ddot{\alpha} = \frac{-2(\dot{\alpha})^2 - 3\dot{\alpha} - 1}{\alpha}.$$

One also observes the obvious bound $\beta(t) \geq \beta(0)$, which in turn implies an upper bound $\alpha(t) \leq C$. From this we conclude that if $\dot{\alpha} \geq -\frac{1}{4}$, one has $\ddot{\alpha} \leq -\frac{1}{4C}$, so for finite time one concludes the (preserved) bound $\dot{\alpha} \leq -\frac{1}{8C}$, which in turn implies $\alpha \rightarrow 0$ in finite time.

Now we relax the hypothesis of full homogeneity and examine rotationally symmetric solutions for RG flow on $S^2 \times S^1$. We will specifically look at RYM flow for the next proposition, although a similar result holds for the full RG flow.

Proposition 3.1. *Given g , a rotationally symmetric metric on $M = S^2 \times S^1$, and A , a connection on a $U(1)$ bundle over M , such that F is the (unique up to scaling) rotationally symmetric representative of $[(S^2)^*]$, the cohomology class dual to the nontrivial S^2 , then the solution for RYM flow with initial condition (g, A, B) exists for all time and converges to a multiple of the product metric with F parallel.*

Proof. Let us first set up the equations. We can write our rotationally symmetric metric as

$$g = \phi^2 dx^2 + \psi^2 g_{S^2} \quad (3.1)$$

on $[-1, 1]$ with $\phi(1) = \phi(-1)$ and moreover ϕ should extend to a smooth function on S^1 identifying -1 with 1 . Our explicit formulation has a selected point at 0 and so we can introduce

$$s(x) := \int_0^x \phi(x) dx.$$

Then $\frac{\partial}{\partial s} = \frac{1}{\phi} \frac{\partial}{\partial x}$ and $ds = \phi dx$. This allows us to write a metric of the form (3.1) as

$$g = ds^2 + \psi^2 g_{S^2} \quad (3.2)$$

We can easily compute the Christoffel symbols in coordinates which consist of s and a normal coordinate system for g_{S^2} at a point. Then

$$\Gamma_{ss}^s = \Gamma_{ss}^\theta = 0$$

$$\Gamma_{s\theta}^s = 0$$

$$\Gamma_{s\theta}^\rho = \frac{\psi_s}{\psi} \delta_\theta^\rho$$

$$\Gamma_{\theta\rho}^s = -\psi \psi_s \delta_{\theta\rho}$$

$$\Gamma_{\theta\rho}^\mu = (\Gamma_{S^2}^\mu)_{\theta\rho}.$$

The sectional curvatures of the planes perpendicular to the S^2 cross sections are

$$K_0 = -\frac{\psi_{ss}}{\psi}$$

whereas the sectional curvature of the plane tangent to the sphere is

$$K_1 = \frac{1 - \psi_s^2}{\psi^2}.$$

We can assume without loss of generality that $[F] > 0$ and so we will write

$$F = \pi^*(dV_{S^2}) \quad (3.3)$$

where $\pi : S^2 \times S^1 \rightarrow S^2$ is the projection map. If F is to be closed and rotationally symmetric this is the only choice for F . Indeed we note that with respect to any rotationally symmetric metric we have

$$(d^*F)_k = g^{ij}(\partial_i F_{jk} - \Gamma_{ij}^m F_{mk} - \Gamma_{ik}^m F_{jm}).$$

Thus

$$(d^*F)_s = -g^{ij}(\Gamma_{is}^m F_{jm}) = 0$$

and

$$(d^*F)_\theta = g^{\mu\rho}\partial_\mu F_{\rho\theta} = 0$$

so that the Yang–Mills portion of the evolution equation drops out completely. Thus we reduce to the system of equations

$$\phi_t = 2\frac{\psi_{ss}}{\psi}\phi \quad (3.4)$$

and

$$\psi_t = \psi_{ss} - \frac{1 - \psi_s^2}{\psi} + \frac{[F]^2}{\psi^2}. \quad (3.5)$$

Lemma 3.2. *Under the above hypotheses, there exists $C = C(g(0), [F])$ such that the inequality*

$$\frac{1}{C} \leq \psi(\cdot, t) \leq C$$

holds as long as the solution exists.

Proof. We note that if a minimum of ψ satisfies

$$2\psi(x) \leq [F]^2$$

then

$$\psi_t \geq \psi_{ss} + \frac{1}{\psi}.$$

Therefore by the maximum principle we see that $\psi_t \geq 0$. Likewise, at a maximum for ψ such that $\psi \geq 2[F]^2$ we see that

$$\psi_t \leq \psi_{ss} - \frac{1}{2\psi}$$

and an upper bound for ψ follows by the maximum principle. \square

Now that we have a C^0 bound for ψ , completing the proof of the proposition is basically a regularity proof. We bootstrap our bound on ψ to a bound on ψ_{ss} which will give us a curvature bound. First we note the equation

$$[\partial_t, \partial_s] = -2\frac{\psi_{ss}}{\psi}\partial_s.$$

Let $v := \psi_s$. We can compute the evolution equation

$$v_t = v_{ss} + \frac{vv_s}{\psi} + \frac{v(1 - v^2)}{\psi^2} - \frac{2[F]^2 v}{\psi^3}.$$

Applying the maximum principle and Lemma 3.2 we immediately conclude that $|v| \leq C$. Next let $w = \psi_{ss}$. We can calculate the time evolution of w :

$$w_t = w_{ss} + \frac{vw_s}{\psi} - 2\frac{w^2}{\psi} - 3\frac{v^2}{\psi^2}w + \frac{w}{\psi^2} - 2\frac{v^2(1-v^2)}{\psi^3} + 6[F]^2\frac{v^2}{\psi^4} - 2[F]^2\frac{w}{\psi^3}. \quad (3.6)$$

Using the uniform bounds on v and ψ we immediately see the inequality

$$w_t \leq w_{ss} + \frac{vw_s}{\psi} - \epsilon w^2 + Cw + C.$$

The maximum principle yields an a priori upper bound for w . To get the lower bound, consider the quantity $\Phi = w - \alpha v^2$ for some constant α to be determined. To calculate the time evolution of Φ we first calculate

$$(v^2)_t = (v^2)_{ss} - 2w^2 + \frac{v}{\psi}(v^2)_s + \frac{v^2(1-v^2)}{\psi^2} - \frac{2[F]^2v^2}{\psi^3}. \quad (3.7)$$

Collecting (3.6) and (3.7) and applying the uniform bounds on v and ψ yields

$$\Phi_t \geq \Phi_{ss} + \frac{v}{\psi}\Phi_s - Cw - C'w^2 + 2\alpha w^2 \geq \Phi_{ss} + \frac{v}{\psi}\Phi_s - Cw + \epsilon w^2$$

for α chosen large. Thus by the maximum principle one concludes an a priori lower bound for Φ which in turn implies an a priori lower bound for w .

These uniform bounds already give uniform bounds on the curvature for all times. Taking a convergent subsequence it is clear from (3.5) that ψ converges to a constant function. Therefore both v and w converge uniformly to zero. By reparameterizing ϕ we get convergence to the product metric, and thus the proposition follows. \square

4. Renormalization group flow on 3-manifolds

In this section we consider renormalization group flow on a general 3-manifold. We prove a result ruling out singularities along homology classes pairing nontrivially with F , and mention that a result of Knopf–Ilmanen extends to RG flow.

Theorem 4.1. *Let (M^3, g) be a 3-manifold and $U(1) \rightarrow E \rightarrow M$ be a principal $U(1)$ -bundle over M with connection A and curvature 2-form F . Let Σ be a weakly stable minimal surface with orientable normal bundle. Then the area of Σ satisfies*

$$\frac{dA}{dt} \geq -4\pi\chi + \frac{(\int_{\Sigma} F)^2}{4A}.$$

Proof. The calculation is based on the one appearing in [4]. Fix $\Sigma^2 \subset M^3$. First we note the general variation formula for the area of Σ under a change of metric:

$$\frac{\partial A}{\partial t} = \frac{1}{2} \int_{\Sigma} \text{tr}_{g_{\Sigma}} h d\Sigma$$

where h is the variation of g and g_{Σ} is the induced metric on Σ . For a solution for RG flow this yields

$$\frac{\partial A}{\partial t} = \int_M \left[-2\text{Rm}(T) - \text{Rc}(N) + \text{tr}_{g_{\Sigma}} \left(\frac{1}{2}\eta + \frac{1}{4}\mathcal{H} \right) \right] d\Sigma.$$

Note that we may also flow the surface Σ so as to stay minimal; hence $\frac{dA}{dt} = \frac{\partial A}{\partial t}$. We first write this as a heat equation. Given a variation of surfaces parameterized by r with the given minimal Σ corresponding to $r = 0$, we can compute the second variation of Σ at $r = 0$ using

$$\frac{\partial^2 A}{\partial r^2} = \int_{\Sigma} (2\det B - \text{Rc}(N)) d\Sigma$$

where B is the second fundamental form of Σ . We also have a formula for the Gauss curvature K of Σ , namely

$$K = \det B + \text{Rm}(T).$$

Collecting these calculations, applying the Gauss–Bonnet theorem and throwing away the positive $\text{tr}_{g_{\Sigma}} \mathcal{H}$ term yields

$$\frac{\partial A}{\partial t} \geq \frac{\partial^2 A}{\partial r^2} - 4\pi\chi + \frac{1}{2} \int_M \text{tr}_{g_{\Sigma}} \eta d\Sigma.$$

Now, fix a point $p \in \Sigma$. We can choose coordinates and a basis $\{e_i\}$ for TM_p so that at p we have $g_{\Sigma}(e_1, e_1) = g_{\Sigma}(e_2, e_2) = 1$, $g_{\Sigma}(e_3, e_3) = g_{\Sigma}(e_2, e_2) = 0$, and $g(e_3, e_3) = 1$. Suppose that we let the matrix for F at p be given by

$$F = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

Then one can easily compute that

$$\eta = \begin{pmatrix} a^2 + b^2 & bc & -ac \\ bc & a^2 + c^2 & ab \\ -ac & ab & b^2 + c^2 \end{pmatrix}.$$

Therefore

$$\mathrm{tr}_\Sigma \eta = 2a^2 + b^2 + c^2 \geq \frac{1}{2} |F|^2.$$

Now we proceed to make an integral estimate using Hölder's inequality. First note that for a 2-form F on a surface, for any metric one has $F = \pm |F| \, dV$. Using this we see that

$$\begin{aligned} \left| \int_\Sigma F \right| &= \left| \int_\Sigma \pm |F| \, dV \right| \\ &\leq \int_\Sigma |F| \, dV \\ &\leq \left(\int_\Sigma 1 \, dV \right)^{1/2} \left(\int_M |F|^2 \, dV \right)^{1/2} \\ &= A^{1/2} \left(\int_M |F|^2 \, dV \right)^{1/2} \end{aligned}$$

which implies the inequality

$$\int_M |F|^2 \, dV \geq \frac{(\int_\Sigma F)^2}{A}.$$

Plugging this into our formula above yields

$$\frac{\partial A}{\partial t} \geq \frac{\partial^2 A}{\partial r^2} - 4\pi \chi + \frac{(\int_\Sigma F)^2}{4A}. \quad \square$$

Corollary 4.2. *In the situation above, suppose $\int_\Sigma F \neq 0$. Note that this condition is independent of the choice of $\Sigma \in [\Sigma]$ and also preserved by RG flow since the cohomology class of F is preserved. Then there exists a constant $\epsilon = \epsilon([F])$ such that $A \leq \epsilon$ implies*

$$\frac{\partial A}{\partial t} \geq \frac{\epsilon}{2A}.$$

Proof. This follows immediately from the previous theorem, using that the last term in the differential inequality has a nonzero coefficient. \square

Recall that in the Ricci flow case ($F = 0$), the preceding theorem tells you that one cannot have toroidal necks pinching off, only spherical ones. In the RG flow case, when $F \neq 0$, the theorem tells you not to expect any neck pinching off which has nontrivial pairing with F . In particular, in the case of $S^2 \times S^1$, one can rule out a neckpinch singularity along the nontrivial S^2 . While this is very nice and very suggestive, we still have a hard time saying anything about topologically trivial necks. Next we briefly discuss how to extend the main result of [6] to RG flow, indeed to any supersolution for Ricci flow.

Proposition 4.3. *Given $(M^n, g(t))$ a compact supersolution Ricci flow and $\alpha \in H_1(M, \mathbb{Z})$ an element of infinite order, there exists a constant $C = C(\alpha, g(0))$ such that*

$$L_\alpha(g(t)) \geq C > 0$$

where $L_\alpha(g(t))$ is the infimum of lengths of curves representing α measured with respect to $g(t)$.

Proof. We may proceed exactly as in [6] since the key monotonicity formula in line (2.1) of that paper still holds. In particular, let $\phi(t)$ be a one-parameter family of 1-forms evolving via

$$\frac{\partial}{\partial t} \phi = \Delta_{d,g(t)} \phi$$

where $\Delta_{d,g(t)}$ is the Hodge–de Rham Laplacian with respect to the time-dependent metric. Using the Böchner formula we can compute

$$\begin{aligned}
\frac{\partial}{\partial t} |\phi|^2 &= \frac{\partial}{\partial t} (g^{ij} \phi_i \phi_j) \\
&\leq 2 \operatorname{Rc}^{ij} \phi_i \phi_j + 2 \phi^i \frac{\partial}{\partial t} \phi_i \\
&= 2 \phi^i \Delta \phi_i \\
&= \Delta |\phi|^2 - 2 |\nabla \phi|^2 \\
&\leq \Delta |\phi|^2.
\end{aligned}$$

Applying the maximum principle gives

$$|\phi(t)|_{C^0(g(t))} \leq |\phi|_{C^0(g(0))}$$

as required. One can now carry out the proof in [6] word for word to get the result. \square

First note that solutions for RG flow are of course supersolutions for Ricci flow. The hypotheses of the above theorem are satisfied for instance on a $2n + 1$ -manifold where $[F^{\wedge n}] \neq 0$, as this gives us an element of $H_1(M, \mathbb{R})$ by Poincaré duality. This result for instance allows one to rule out $(S^1 \times S^{n-1})$ as a blow-up limit of a finite time singularity. To end this section we prove another theorem which rules out compact blow-up limits for RYM flow under certain topological hypotheses on the bundle E .

Theorem 4.4. *Let $(M^n, g(t), A(t), B(t))$ be a solution for RG flow which goes singular at $t = T < \infty$ and satisfies an injectivity radius estimate on the scale of maximum curvature. Suppose that the bundle $E \rightarrow M$ is nontrivial, or $[H] \neq 0$. Then the underlying base manifold of the blow-up limit is noncompact.*

Proof. We will spell out full details in the case $[F] \neq 0$, the other case being similar. First we briefly outline the construction of the blow-up limit. Choose a sequence of points (x_i, t_i) realizing the spacetime supremum of curvature. Let

$$\begin{aligned}
g_i &:= \lambda_i g \left(t_i + \frac{t}{\lambda_i} \right) \\
F_i &:= \sqrt{\lambda_i} F \left(t_i + \frac{t}{\lambda_i} \right)
\end{aligned}$$

where $\lambda_i = |\operatorname{Rm}(g)|_{g(x_i, t_i)}$. It was shown in [13] that a solution for RYM flow goes singular only if the curvature of g blows up. It is straightforward to extend this result to RG flow. Note also that we have ignored the connection and restricted attention to the curvature F , which we may scale at will. By the assumption of noncollapsing and the compactness of solutions proved in [13], this sequence will converge to an ancient solution with bounded curvature and moreover bounded F . Call the limiting data $(M_\infty, g_\infty, F_\infty)$.

If M_∞ is compact then $M_\infty = M$. Now, the nontriviality of the bundle implies the existence of another cohomology class $[\alpha] \in H^{n-2} \otimes \mathfrak{k}$ such that

$$\int_M \operatorname{Tr} F \wedge \alpha > 0.$$

(Clearly one can make this integral nonzero, then just change the sign of α to make it positive.) Note that the value of this integral is fixed along the RG flow since the cohomology class of F is unchanged along the flow. But the convergence, Cheeger–Gromov smooth on compact sets, implies convergence of integrals of functions with uniform bounds. Therefore in particular we see that

$$\begin{aligned}
\int \operatorname{Tr} F_\infty \wedge \alpha &= \int \langle F, \star \alpha \rangle dV \\
&\leq |F|_{C^0(g_\infty)} |\alpha|_{C^0(g_\infty)} \operatorname{Vol}(g_\infty) \\
&< \infty.
\end{aligned}$$

However, in taking the blow-up limit, it is clear that

$$\int_M \operatorname{Tr} F_i \wedge \alpha = \sqrt{\lambda_i} \int_M \operatorname{Tr} F \wedge \alpha \xrightarrow{i \rightarrow \infty} \infty.$$

This contradicts our assumption that M_∞ was compact, and so the result follows. \square

5. Conjectural picture of RG flow on 3-manifolds

At the outset of this section describing a conjectural picture for the singularity formation of this flow, we must state that a general no-local-collapsing result for RG flow is not known. Being a supersolution for Ricci flow, it seems very unlikely

that RG flow could encounter local collapse in finite time, but as yet this is unproven. In the discussion that follows, we will essentially assume that all finite time singularities satisfy a no-local-collapsing estimate. On the basis of the results of the previous section, we make a conjecture.

Conjecture 5.1. Consider $(S^2 \times S^1, g)$ and let $U(1) \rightarrow E \rightarrow M$ be a principal $U(1)$ -bundle over M with connection A satisfying $[F] = [S^2]^*$. Then the solution for RYM flow with initial condition (g, A) exists for all time and converges to a multiple of the product metric with F parallel.

The content of Theorem 4.1 is that for any solution for RG flow with the bundle as in the conjecture, one cannot encounter a neckpinch-like singularity of RG flow along the topologically nontrivial S^2 . This is quite suggestive; however our methods say nothing about the possibility of topologically trivial necks (i.e. a copy of S^3 bubbling off), which might be hard to rule out.

Heuristic Proof of Conjecture 5.1. Suppose the flow goes singular at a time $T < \infty$. As shown in [13], we know that the curvature of g must blow up for this to happen. So take a sequence of points $(x_i, t_i) \rightarrow (x_T, T)$ realizing the supremum of curvature at time t_i . We break this into two cases. First suppose that F goes to zero near x_T . Then by examining the evolution equation for F using the Böchner formula, one sees that the Ricci curvature must be strictly positive and bounded away from zero near x_T . Assuming that the singularity is not locally collapsed, one can take a blow-up limit which yields an ancient solution for Ricci flow since F is going to zero. Moreover, the Ricci curvature is strictly positive so we expect this limit to be modeled on the shrinking sphere, which clearly cannot arise as a blow-up limit of $S^2 \times S^1$.

In the second case, assume there is a plane V in $G_2(T^*M_{x_T})$, the Grassmannian of 2-planes at x_T , such that $\lim_{t \rightarrow T} F(V_{x_T}) \neq 0$. Again by examining the evolution of F and applying the Böchner formula, one sees this time that the Ricci curvature of the vector orthogonal to V should be nonpositive. Now the evolution equation for g suggests that the lengths of all vectors at x_T are bounded away from zero. This is because the η term prevents the lengths of vectors in V from going to zero, while the nonpositive Ricci curvature for the vector orthogonal to V prevents that vector's length from going to zero. Since the curvature of g must be blowing up at a singularity, this would contradict the singularity being not locally collapsed. \square

The argument given above is basically completely local. One important part is ruling out a compact blow-up limit, but as we saw in Theorem 4.4 this can be done in quite a general setting. Therefore this suggests a broader conjecture.

Conjecture 5.2. Let (M^3, g) be a Riemannian manifold and let $U(1) \rightarrow E \rightarrow M$ be a principal $U(1)$ -bundle over M with connection A satisfying $[F] \neq 0$. Furthermore let $H \in \Lambda^3(T^*M)$. Then the solution for RG flow with initial condition (g, A, H) exists for all time.

Finally, we would like to point out a related conjecture on 4-manifolds. Following the heuristic proof above and using the intuition gained from Theorem 4.4, the following conjecture is plausible:

Conjecture 5.3. Let (M^4, g) be a Riemannian manifold and let $SU(2) \rightarrow E \rightarrow M$ be a principal $SU(2)$ -bundle with $c_2(E) \neq 0$. Let A denote a connection on E . Then the solution for RYM flow with initial condition (g, A) exists for all time.

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References

- [1] D. DeTurck, Deforming metrics in the direction of their Ricci tensors, *J. Differential Geom.* 18 (1) (1983) 157–162.
- [2] M. Feldman, T. Ilmanen, L. Ni, Entropy and reduced distance for Ricci expanders, [arXiv:math/0405036](https://arxiv.org/abs/math/0405036).
- [3] A. Giverson, E. Rabinovici, A.A. Tseytlin, Heterotic string solutions and coset conformal field theories, *Nuclear Phys. B* 409 (2) (1993) 339–362.
- [4] R. Hamilton, Singularities of Ricci Flow, in: *Surveys in Differential Geometry*, vol. 2, International Press, 1993, pp. 7–136.
- [5] N. Hitchin, Lectures on special Lagrangian submanifolds, in: *Winter School on Mirror Symmetry, Vector Bundles, and Lagrangian Submanifolds* (Cambridge, MA, 1999), Amer. Math. Soc., Providence, RI, 2001, pp. 151–182.
- [6] T. Ilmanen, D. Knopf, A lower bound for the diameter of solutions to the Ricci flow with nonzero $H^1(M^n; \mathbb{R})$, *Math. Res. Lett.* 10 (2–3) (2003) 161–168.
- [7] T. Oliynyk, V. Suneeta, E. Woolgar, A gradient flow for worldsheet nonlinear sigma models, *Nuclear Phys. B* 739 (2006) 441–458.
- [8] G. Perelman, The entropy formula for Ricci flow and its geometric applications, [arXiv:math/0211159](https://arxiv.org/abs/math/0211159).
- [9] W.X. Shi, Deforming the metric on complete Riemannian manifolds, *J. Differential Geom.* 29 (1989) 353–360.
- [10] W.X. Shi, Ricci deformation of the metric on complete noncompact Riemannian manifolds, *J. Differential Geom.* 20 (2) (1989) 303–394.
- [11] M. Stern, B -fields from a Luddite perspective, [arXiv:hep-th/0401099v1](https://arxiv.org/abs/hep-th/0401099v1).
- [12] J. Streets, Regularity and expanding entropy for connection Ricci flow, *J. Geom. Phys.*
- [13] J. Streets, Ricci Yang–Mills flow, Ph.D. Thesis, Duke University, 2007.
- [14] J. Streets, Ricci Yang–Mills flow on surfaces, preprint 2007.
- [15] A. Young, Ricci flow on principal bundles, Ph.D. Thesis, University of Texas–Austin, 2008.