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Centroaffine surfaces with degenerate center map

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ABSTRACT

We study and classify the centroaffine surfaces with degenerate center map. The centroaffine surface with definite induced centroaffine metric has degenerate center map if and only if it is a proper affine sphere. In particular, we investigate the degenerate center map of the centroaffine ruled surfaces and consider its relationship with the centroaffine space curve.

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1. Introduction

Center map was firstly introduced for affine hypersurfaces by Furuhata and Vrancken as a generalization of the center of proper affine spheres [1]. They studied affine hypersurfaces whose center map is centroaffinely congruent with the original hypersurface, called to be self congruent. In particular, they showed that the center map of a definite centroaffine surface in the 3-space which is not a proper affine sphere centered at the origin is self congruent if and only if the centroaffine Tchebychev operator vanishes.

In this paper, we study the degenerate center map for centroaffine surfaces in $\mathbb{R}^3$. If the induced centroaffine metric is definite, we can prove the center map is degenerate if and only if the centroaffine surface is a proper affine sphere (Theorem 3.2). If the induced centroaffine metric is indefinite, besides the proper affine sphere, we can get two kinds of surfaces, one of which can be written as a centroaffine minimal ruled surface with centroaffine scalar curvature 1. In the last section, we consider the degenerate center map for centroaffine ruled surfaces, and obtain the following theorem:

Theorem 1.1. *Let $f : M \to \mathbb{R}^3$ be a non-degenerate centroaffine ruled surface given by $f(u, v) = a(u) + vb(u)$. The center map c of f is degenerate to a curve if and only if $a(u)$ is written as $a(u) = \phi(u)b'(u)$, where $\phi(u)$ is a nonzero function such that $3\phi'(u) + \phi(u)\kappa_2(u) \neq 0$, $\kappa_2(u)$ is the centroaffine second curvature of $b(u)$. Moreover, the center map of f is given by*

$$c(u, v) = -\frac{1}{2}(3\phi'(u) + \phi(u)\kappa_2(u))b(u).$$

As a corollary, for any nondegenerate centroaffine space curve, we can construct a centroaffine ruled surface whose center map is precisely degenerate to the given curve (Corollary 4.4).

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2. Basic theory of affine hypersurfaces

Let $f : M \to \mathbb{R}^{n+1}$ be an immersion of an n-dimensional oriented manifold M into $\mathbb{R}^{n+1}$. We denote by $\Gamma(TM)$ the space of sections of the tangent bundle TM, by D the standard flat affine connection in $\mathbb{R}^{n+1}$, and by Det the standard parallel volume form of $\mathbb{R}^{n+1}$. Throughout this paper, we assume f is nondegenerate. Let ξ be the *Blaschke normal vector field* of f. By definition, the following hold (see [2,3]):

(1) At each point x of M, the tangent space $T_{f(x)}\mathbb{R}^{n+1}$ is decomposed as

$$T_{f(x)}\mathbb{R}^{n+1} = f_*T_xM \oplus \mathbb{R}\xi_x.$$

(2) The $(0, 2)$-tensor field h^e, defined by

$$D_Xf_*Y = f_*\nabla^e_XY + h^e(X, Y)\xi$$

for $X, Y \in \Gamma(TM)$, is a semi-Riemannian metric on M.

(3) The 1-form τ, defined by

$$D_X\xi = -f_*SX + \tau(X)\xi$$

for $X \in \Gamma(TM)$, vanishes identically.

(4) The volume form θ defined by

$$\theta(X_1, \ldots, X_n) := Det(f_*X_1, \ldots, f_*X_n, \xi)$$

for $X_1, \ldots, X_n \in \Gamma(TM)$, is compatible with the orientation of M.

(5) θ coincides with the volume form

$$\mathrm{Vol}_{h^e}(X_1, \ldots, X_n) := |\det(h^e(X_i, X_j))|^{1/2}$$

determined by h^e, where $X_1, \ldots, X_n$ are vector fields on M.

It is well known that such a ξ is uniquely determined. We then call ∇^e and h^e the *equiaffinely-induced connection* and the *equiaffine metric* of f, respectively.

The line through each point of f in the direction of ξ is called the *equiaffine normal line*. As we know, the affine surface f is a proper affine sphere if and only if the equiaffine normals meet at one point, *the center*, which can be generalized to a map for affine hypersurfaces as follows. For more details, we refer to [1].

Let r be the equiaffine support function of f with respect to the origin $o \in \mathbb{R}^{n+1}$. By definition, it is a function on M written as

$$f(x) = Z_x + r(x)\xi_x,$$

where Z is an $\mathbb{R}^{n+1}$-valued function tangent to f.

Definition 2.1. For an immersion $f : M \to \mathbb{R}^{n+1}$, we set $c : M \to \mathbb{R}^{n+1}$ by

$$c(x) := c_f(x) := f(x) - r(x)\xi_x, \quad \text{for } x \in M,$$

and call it the *center map* of f.

Proposition 2.2. *An immersion $f : M \to \mathbb{R}^{n+1}$ is a proper affine hypersphere if and only if the center map c of f is constant (see Fig. 1).*

Proposition 2.3. *The center map c of an immersion $f : M \to \mathbb{R}^{n+1}$ is an immersion if and only if*

$$\ker(\mathrm{id} + rS) \cap \ker dr = \{0\}.$$

Remark 2.4. The center map of an improper affine hypersphere is an immersion (see Fig. 2).

We assume that $f : M \to \mathbb{R}^{n+1}$ is a *centroaffine* immersion as well (see [4,5]). By definition it means for each point the position vector is transversal to the tangent space, and the symmetric $(0, 2)$-tensor field h defined by

$$D_Xf_*Y = f_*\nabla_XY + h(X, Y)f$$

is nondegenerate. We call ∇ and h the *centroaffinely-induced connection* and the *centroaffine metric* of f, respectively. We denote the difference tensor of the centroaffinely-induced connection ∇ and the Levi-Civita connection $\widetilde{\nabla}$ of the centroaffine metric h by

$$K := \nabla - \widetilde{\nabla} \in \Gamma(TM^{(1,2)}),$$

and define the *centroaffine Tchebychev vector field* T and the *centroaffine Tchebychev operator* $\Im$ by

$$T := tr_hK \in \Gamma(TM),$$
$$\Im := \widetilde{\nabla}T \in \Gamma(TM^{(1,1)}).$$

We formulize the center map for centroaffine immersion.

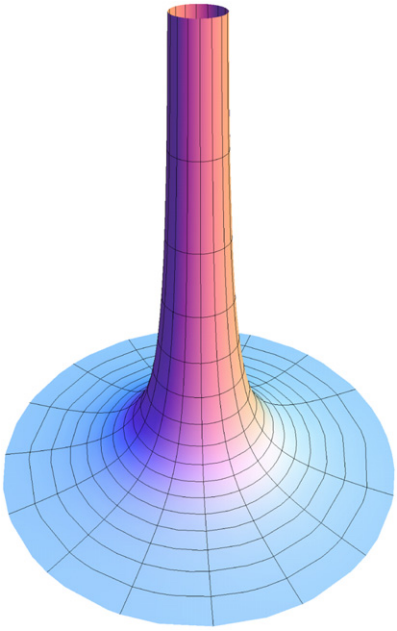


Fig. 1. Proper affine sphere $f(u, v) = (e^{-2u}, e^u \cos v, e^u \sin v)$ with center at the origin.

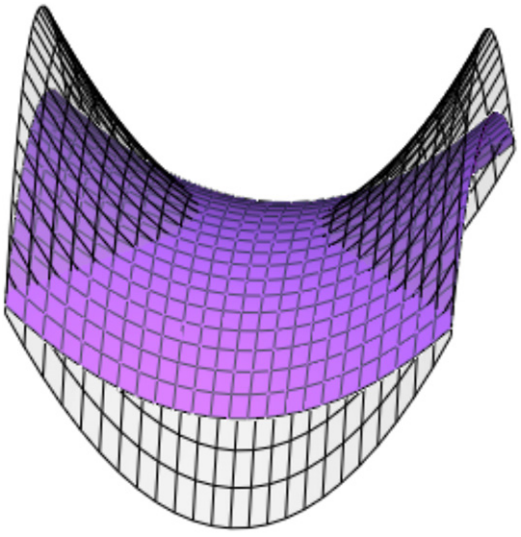


Fig. 2. Hyperbolic paraboloid (improper affine sphere) and its center map.

Proposition 2.5. *Let c be the center map of an immersion $f : M \to \mathbb{R}^{n+1}$. Then the following formula holds:*

$$c = -\frac{2}{n+2} f_* T,$$

where T is the centroaffine Tchebychev vector field of f.

In the following, we consider when the center map c is degenerate. By definition, we have

$$c_* X = -\frac{2}{n+2} D_X f_* T = -\frac{2}{n+2} (f_* \nabla_X T + h(X, T) f)$$

for $X \in \Gamma(TM)$, so that c is degenerate if and only if there is a nonvanishing vector $X \in \Gamma(TM)$ such that

$$\begin{cases} h(X, T) = 0, \\ \nabla_X T = 0. \end{cases} \tag{2.1}$$

3. Classification of centroaffine surfaces with degenerate center map

We study the centroaffine surfaces with definite and indefinite centroaffine metric, respectively.

3.1. Definite case

Lemma 3.1 ([6]). *Suppose X_1 and X_2 are tangential vector fields on $f : U \to \mathbb{R}^3$ which are linearly independent at each $u \in U$. Then in a neighborhood U_0 of each u_0 we can change variables, $\psi : V_0 \to U_0$, so that $f \circ \psi = \tilde{f}$ has coordinate vector fields $\tilde{f}_{v_i}$ proportional to X_i.*

Let $f : M \to \mathbb{R}^3$ be a centroaffine surface with definite centroaffine metric h and nonvanishing centroaffine Tchebychev vector field T. Then using the above lemma, for any point $x \in M$, we can take local coordinates (u, v) such that

$$T = \alpha \partial_u, \tag{3.1}$$

and

$$h(\partial_u, \partial_u) = \lambda, \qquad h(\partial_v, \partial_v) = \mu, \qquad h(\partial_u, \partial_v) = 0$$

for some nonvanishing functions $\alpha(u, v)$, $\lambda(u, v)$ and $\mu(u, v)$.

If we set the Gauss equation as

$$\begin{cases} f_{uu} = \Gamma_{11}^1 f_u + \Gamma_{11}^2 f_v + \lambda f, \\ f_{uv} = \Gamma_{12}^1 f_u + \Gamma_{12}^2 f_v, \\ f_{vv} = \Gamma_{22}^1 f_u + \Gamma_{22}^2 f_v + \mu f, \end{cases} \tag{3.2}$$

we can get

$$\begin{cases} f_{uuv} = ((\Gamma_{11}^1)_v + \Gamma_{11}^1\Gamma_{12}^1 + \Gamma_{11}^2\Gamma_{22}^1) f_u + (\Gamma_{11}^1\Gamma_{12}^2 + (\Gamma_{11}^2)_v + \Gamma_{11}^2\Gamma_{22}^2 + \lambda) f_v + (\Gamma_{11}^2\mu + \lambda_v) f, \\ f_{uvu} = ((\Gamma_{12}^1)_u + \Gamma_{12}^1\Gamma_{11}^1 + \Gamma_{12}^2\Gamma_{12}^1) f_u + (\Gamma_{12}^1\Gamma_{11}^2 + (\Gamma_{12}^2)_u + \Gamma_{12}^2\Gamma_{12}^2) f_v + \Gamma_{12}^1\lambda f, \\ f_{uvv} = ((\Gamma_{12}^1)_v + \Gamma_{12}^1\Gamma_{12}^1 + \Gamma_{12}^2\Gamma_{22}^1) f_u + (\Gamma_{12}^1\Gamma_{12}^2 + (\Gamma_{12}^2)_v + \Gamma_{12}^2\Gamma_{22}^2) f_v + \Gamma_{12}^2\mu f, \\ f_{vvu} = ((\Gamma_{22}^1)_u + \Gamma_{11}^1\Gamma_{22}^1 + \Gamma_{12}^1\Gamma_{22}^2 + \mu) f_u + (\Gamma_{22}^1\Gamma_{11}^2 + (\Gamma_{22}^2)_u + \Gamma_{12}^2\Gamma_{22}^2) f_v + (\Gamma_{22}^1\lambda + \mu_u) f. \end{cases} \tag{3.3}$$

Comparing the coefficients of f by using $f_{uuv} = f_{uvu}$ and $f_{uvv} = f_{vvu}$, we get integrability conditions

$$\begin{cases} \Gamma_{11}^2 = \lambda\mu^{-1}\Gamma_{12}^1 - \mu^{-1}\lambda_v, \\ \Gamma_{22}^1 = \lambda^{-1}\mu\Gamma_{12}^2 - \lambda^{-1}\mu_u. \end{cases} \tag{3.4}$$

From

$$0 = T^2 = h^{ij}K_{ij}^2 = \lambda^{-1}\Gamma_{11}^2 + \mu^{-1}\Gamma_{22}^2 + \frac{1}{2}\mu^{-1}(\lambda^{-1}\lambda_v - \mu^{-1}\mu_v),$$

we get

$$\Gamma_{22}^2 = -\lambda^{-1}\mu\Gamma_{11}^2 - \frac{1}{2}(\lambda^{-1}\lambda_v - \mu^{-1}\mu_v). \tag{3.5}$$

From

$$\alpha = T^1 = h^{ij}K_{ij}^1 = \lambda^{-1}\Gamma_{11}^1 + \mu^{-1}\Gamma_{22}^1 + \frac{1}{2}\lambda^{-1}(\mu^{-1}\mu_u - \lambda^{-1}\lambda_u),$$

we get

$$\Gamma_{11}^1 = \lambda\alpha - \lambda\mu^{-1}\Gamma_{22}^1 - \frac{1}{2}(\mu^{-1}\mu_u - \lambda^{-1}\lambda_u). \tag{3.6}$$

Using the second equation of (2.1), and considering the coefficients of ∂_u and ∂_v, we get

$$\begin{cases} \Gamma_{12}^2 = 0, \\ \Gamma_{12}^1 = -\alpha^{-1}\alpha_v = -(\log(\alpha))_v. \end{cases} \tag{3.7}$$

Substituting this into (3.4), we get

$$\begin{cases} \Gamma_{11}^2 = -\alpha^{-1}\alpha_v\lambda\mu^{-1} - \lambda_v\mu^{-1} = -\lambda\mu^{-1}(\log(\lambda\alpha))_v, \\ \Gamma_{22}^1 = -\lambda^{-1}\mu_u. \end{cases} \tag{3.8}$$

Substituting this into (3.5) and (3.6), respectively, we get

$$\begin{cases} \Gamma^2_{22} = \alpha^{-1}\alpha_v + \dfrac{1}{2}(\lambda^{-1}\lambda_v + \mu^{-1}\mu_v) = (\log(\alpha))_v + \dfrac{1}{2}(\log(\lambda\mu))_v, \\ \Gamma^1_{11} = \lambda\alpha + \dfrac{1}{2}(\lambda^{-1}\lambda_u + \mu^{-1}\mu_u) = \lambda\alpha + \dfrac{1}{2}(\log(\lambda\mu))_u. \end{cases} \tag{3.9}$$

Then we can rewrite the Gauss formulas as

$$\begin{cases} f_{uu} = (\lambda\alpha + \dfrac{1}{2}(\log(\lambda\mu))_u)f_u + (-\lambda\mu^{-1}(\log(\lambda\alpha))_v)f_v + \lambda f, \\ f_{uv} = -(\log(\alpha))_v f_u, \\ f_{vv} = (-\lambda^{-1}\mu_u)f_u + ((\log(\alpha))_v + \dfrac{1}{2}(\log(\lambda\mu))_v)f_v + \mu f. \end{cases} \tag{3.10}$$

Putting $\Gamma^2_{12} = 0$ into (3.3), we can get integrability conditions as

$$\begin{cases} (\Gamma^1_{11})_v + \Gamma^2_{11}\Gamma^1_{22} = (\Gamma^1_{12})_u, \\ (\Gamma^2_{11})_v + \Gamma^2_{11}\Gamma^2_{22} + \lambda = \Gamma^1_{12}\Gamma^2_{11}, \\ (\Gamma^1_{12})_v + (\Gamma^1_{12})^2 = (\Gamma^1_{22})_u + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} + \mu, \\ \Gamma^1_{22}\Gamma^2_{11} + (\Gamma^2_{22})_u = 0. \end{cases} \tag{3.11}$$

Comparing the first and the fourth equation in (3.11), we can get

$$(\Gamma^1_{11})_v = (\Gamma^1_{12} + \Gamma^2_{22})_u, \tag{3.12}$$

that is

$$(\lambda\alpha)_v = 0. \tag{3.13}$$

Then we will get

$$\Gamma^2_{11} = -\lambda\mu^{-1}(\log(\lambda\alpha))_v = 0. \tag{3.14}$$

Substituting this into the second equation in (3.11), we will get

$$\lambda = 0, \tag{3.15}$$

which is a contradiction with our assumption.

Concluding the above discussion, we will get the following theorem.

Theorem 3.2. *Let $f : M \to \mathbb{R}^3$ be a centroaffine surface with definite centroaffine metric. The center map c of f is degenerate if and only if f is a proper affine sphere.*

3.2. Indefinite case

In this section, we assume that f is a centroaffine surface with indefinite centroaffine metric. We take local asymptotic coordinates (u, v) such that the centroaffine metric h satisfies

$$h_{11} = h_{22} = 0,\ h_{12} = h_{21} = -\varphi$$

for some non-zero function φ.

The Gauss formulas for the centroaffine surface f are given by

$$\begin{cases} f_{uu} = \left(\dfrac{\varphi_u}{\varphi} + \rho_u\right)f_u + \dfrac{\alpha}{\varphi}f_v, \\ f_{uv} = -\varphi f + \rho_v f_u + \rho_u f_v, \\ f_{vv} = \left(\dfrac{\varphi_v}{\varphi} + \rho_v\right)f_v + \dfrac{\beta}{\varphi}f_u \end{cases} \tag{3.16}$$

with the integrability conditions

$$\begin{cases} (\log|\varphi|)_{uv} = -\varphi - \dfrac{\alpha\beta}{\varphi^2} + \rho_u\rho_v, \\ \alpha_v + \rho_u\varphi_u = \rho_{uu}\varphi, \\ \beta_u + \rho_v\varphi_v = \rho_{vv}\varphi, \end{cases} \tag{3.17}$$

where $\rho = -\frac{1}{4}\log(-\frac{K}{d^4})$, K is the Euclidean Gauss curvature, $d = \langle f, n\rangle$ is the Euclidean support function, and

$$\begin{cases} \alpha = \varphi \dfrac{\det[f, f_u, f_{uu}]}{\det[f, f_u, f_v]}, \\ \beta = \varphi \dfrac{\det[f, f_v, f_{vv}]}{\det[f, f_v, f_u]}. \end{cases} \tag{3.18}$$

Then the centroaffine Tchebychev vector field T is computed as

$$T = 2\frac{\rho_v}{\varphi}\partial_u + 2\frac{\rho_u}{\varphi}\partial_v. \tag{3.19}$$

Case 1. If $T = 0$, then f is a proper affine sphere, its center map is constant. In the following, we assume T does not vanish.
Case 2. If $T = T^1\partial_u + T^2\partial_v$ is a null-vector field with respect to the centroaffine metric, that is

$$h(T, T) = 0,$$

from which we can get $T^1 = 0$ or $T^2 = 0$. Without loss of generality, we set $T^2 = 0$, so that $T = T^1\partial_u \neq 0$. If the center map is degenerate and $X = X^1\partial_u + X^2\partial_v$ satisfies (2.1) we will get

$$X = X^1\partial_u = \frac{X^1}{T^1}T, \qquad \nabla_T T = 0,$$

which means an integral curve of the centroaffine Tchebychev vector field T is a ∇-geodesic. Conversely, if $\nabla_T T = 0$, we can easily get a vector field $X = \psi T$ for an arbitrary nonzero function ψ satisfies that $h(X, T) = \psi h(T, T) = 0$, $\nabla_X T = \psi\nabla_T T = 0$.

Using the asymptotic coordinates, we can get $0 = h(T, T) = -8\rho_u\rho_v\varphi^{-1}$. Remark that ρ_u and ρ_v can not be zero at the same time, because T does not vanish. Without loss of generality, we can assume $\rho_u \neq 0$ and $\rho_v = 0$, then we get

$$0 = \nabla_T T = 4\rho_u^2\varphi^{-3}\beta\partial_u + 4\rho_u\varphi^{-1}((\rho_u\varphi^{-1})_v + \rho_u\varphi_v\varphi^{-2})\partial_v,$$

that is

$$\begin{cases} \beta = 0, \\ \rho_{uv} = 0. \end{cases} \tag{3.20}$$

The second equation above means f is a centroaffine minimal surface, which was originally defined by Wang [7] and was characterized as a centroaffine surface whose trace of the centroaffine Tchebychev operator $\Im$ vanishes, that is,

$$\operatorname{trace}\Im = \frac{4\rho_{uv}}{\varphi} = 0.$$

Using the first equation in the integrability conditions (3.17), we can easily get $(\log|\varphi|)_{uv} = -\varphi$ which shows the centroaffine scalar curvature

$$\kappa = -\frac{(\log|\varphi|)_{uv}}{\varphi} = 1.$$

Lemma 3.3 (*[8]*). *Let f be an indefinite centroaffine minimal surface with centroaffine scalar curvature $\kappa = 1$. Then f is given by a ruled surface:*

$$f = b_u + vb,$$

where b is an $\mathbb{R}^3$-valued function of u such that

$$\det[b, b_u, b_{uu}] \neq 0, \qquad \frac{d}{du}\det[b, b_u, b_{uu}] \neq 0.$$

Using Lemma 3.3, we get f is a centroaffine minimal surfaces expressed as $f = b_u + vb$.

Case 3. If T is not a null-vector, then $T^i \neq 0 (i = 1, 2)$. From the first equation of (2.1), we get $X^1T^2 + X^2T^1 = 0, X^i \neq 0 (i = 1, 2)$, and $\frac{X^1}{X^2} = -\frac{T^1}{T^2} = -\frac{\rho_v}{\rho_u}$ by (3.19). Setting $\frac{X_1}{\rho_v} = -\frac{X_2}{\rho_u} = \lambda \neq 0$, we can write

$$X^1 = \rho_v\lambda,\ X^2 = -\rho_u\lambda.$$

From the second equation of (2.1), comparing the coefficient of ∂_u and ∂_v, we will get

$$\begin{cases} X^1((T^1)_u + T^1\Gamma^1_{11} + T^2\Gamma^1_{12}) + X^2((T^1)_v + T^1\Gamma^1_{12} + T^2\Gamma^1_{22}) = 0, \\ X^1((T^2)_u + T^1\Gamma^2_{11} + T^2\Gamma^2_{12}) + X^2((T^2)_v + T^1\Gamma^2_{12} + T^2\Gamma^2_{22}) = 0, \end{cases}$$

that is

$$\begin{cases}\rho_v(\rho_{uv}+\rho_u\rho_v)\varphi-\rho_u(\beta_u+\rho_u\beta)=0,\\ \rho_v(\alpha_v+\rho_v\alpha)-\rho_u(\rho_{uv}+\rho_u\rho_v)\varphi=0.\end{cases}$$

Then

$$\rho_v^2(\alpha_v+\rho_v\alpha)=\rho_u^2(\beta_u+\rho_u\beta)=\rho_u\rho_v(\rho_{uv}+\rho_u\rho_v)\varphi.$$

Concluding all of the above cases, we will get the following

Proposition 3.4. *Let $f : M \to \mathbb{R}^3$ be an indefinite centroaffine surface. Its center map is degenerate if and only if one of the following holds.*

(i) *f is a proper affine sphere, if T vanishes,*
(ii) *f is a centroaffine minimal surfaces with scalar curvature 1 expressed as $f = b_u + vb$, if nonvanishing T is a null vector for h,*
(iii) *$\rho, \alpha, \beta, \varphi$ for f satisfy $\rho_v^2(\alpha_v+\rho_v\alpha)=\rho_u^2(\beta_u+\rho_u\beta)=\rho_u\rho_v(\rho_{uv}+\rho_u\rho_v)\varphi$, otherwise.*

Moreover, a centroaffine minimal surface does not belong to the class (iii).

4. Degenerate center map for centroaffine ruled surfaces

Let $a(u)$, $b(u)$ be linearly independent $\mathbb{R}^3$-valued functions in one variable. Suppose that $b(u)$ is a nondegenerate centroaffine space curve and u is the centroaffine arc-length parameter of $b(u)$, which is defined by

$$\varepsilon(u) := \frac{\det[b'(u), b''(u), b'''(u)]}{\det[b(u), b'(u), b''(u)]} = \pm 1.$$

Denote the *centroaffine first curvature* by

$$\kappa_1(u) := -\frac{\det[b(u), b''(u), b'''(u)]}{\det[b(u), b'(u), b''(u)]},$$

and the *centroaffine second curvature* by

$$\kappa_2(u) := \frac{\det[b(u), b'(u), b'''(u)]}{\det[b(u), b'(u), b''(u)]}$$

(see[9]). We then give the proof of Theorem 1.1.

Proof. Let $f : M \to \mathbb{R}^3$ be a non-degenerate centroaffine ruled surface given by $f(u, v) = a(u) + vb(u)$ [10]. Because

$$\begin{cases}f_u = a' + vb',\\ f_v = b,\end{cases}$$

we have $\det[f_u, f_v, f] = \det[a', b, a] + v\det[b', b, a] \neq 0$. We define functions in u as follows:

$$s_1 := \det[b', b, a], \qquad s_2 := \det[a', b, a], \qquad s_3 := \det[a'', a', b],$$
$$s_4 := \det[a'', b', b] + \det[b'', a', b], \qquad s_5 := \det[b'', b', b] \neq 0,$$
$$s_6 := \det[b', a', b] \neq 0, \qquad s_7 := \det[b', a', a].$$

(1) If $s_1 \neq 0$, the induced centroaffine metric is given by

$$\begin{cases}h_{11} = \dfrac{s_5v^2+s_4v+s_3}{s_1v+s_2},\\ h_{12} = h_{21} = \dfrac{s_6}{s_1v+s_2},\\ h_{22} = 0,\end{cases}$$

the centroaffinely-induced connection as

$$\begin{cases}\Gamma_{12}^1 = \dfrac{s_1}{s_1v+s_2},\\ \Gamma_{12}^2 = -\dfrac{s_6v+s_7}{s_1v+s_2},\\ \Gamma_{22}^1 = 0,\\ \Gamma_{22}^2 = 0,\end{cases}$$

the Levi-Civita connection $\widetilde{\nabla}$ with respect to the induced centroaffine metric as

$$\begin{cases} \widetilde{\Gamma}^1_{12} = 0, \\ \widetilde{\Gamma}^2_{12} = \dfrac{s_1s_5v^2 + 2s_2s_5v + s_2s_4 - s_1s_3}{2s_6(s_1v + s_2)}, \\ \widetilde{\Gamma}^1_{22} = 0, \\ \widetilde{\Gamma}^2_{22} = -\dfrac{s_1}{s_1v + s_2}, \end{cases}$$

and the difference tensor K as

$$\begin{cases} K^1_{12} = \dfrac{s_1}{s_1v + s_2}, \\ K^2_{12} = -\dfrac{-s_1s_5v^2 - 2(s_2s_5 + s_6^2)v - s_2s_4 + s_1s_3 - 2s_6s_7}{2s_6(s_1v + s_2)}, \\ K^1_{22} = 0, \\ K^2_{22} = \dfrac{s_1}{s_1v + s_2}. \end{cases}$$

Then we can calculate the coefficients of the Tchebychev vector field as

$$\begin{cases} T^1 = \dfrac{2s_1}{s_6}, \\ T^2 = -\dfrac{2s_1s_5v^2 + (2s_2s_5 + 2s_6^2 + s_1s_4)v + s_2s_4 + 2s_6s_7}{s_6^2}. \end{cases}$$

We can see the center map is given by

$$\begin{aligned} c &= -\frac{1}{2}f_*T = -\frac{1}{2}(T^1f_u + T^2f_v) \\ &= -\frac{s_1}{s_6}(a' + vb') + \frac{2s_1s_5v^2 + (2s_2s_5 + 2s_6^2 + s_1s_4)v + s_2s_4 + 2s_6s_7}{2s_6^2}b. \end{aligned}$$

We then define the functions in u as follows.

$$t_1 := -\frac{s_1}{s_6}, \qquad t_2 := -\frac{s_1s_5}{s_6^2},$$

$$t_3 := \frac{2s_2s_5 + 2s_6^2 + s_1s_4}{2s_6^2}, \qquad t_4 := \frac{s_2s_4 + 2s_6s_7}{2s_6^2}.$$

We can rewrite c as

$$c = t_1a' + t_1vb' + (t_2v^2 + t_3v + t_4)b,$$

and then

$$\begin{cases} c_u = t_1'a' + t_1a'' + (t_2'v^2 + t_3'v + t_4')b + (t_2v^2 + (t_3 + t_1')v + t_4)b' + t_1vb'', \\ c_v = t_1b' + (2t_2v + t_3)b. \end{cases}$$

Then

$$\begin{aligned} \det[c_u, c_v, c] = \det[&t_1'a' + t_1a'' + (t_2'v^2 + t_3'v + t_4')b + (t_2v^2 + (t_3 + t_1')v + t_4)b' \\ &+ t_1vb'', t_1b' + (2t_2v + t_3)b, t_1a' + t_1vb' + (t_2v^2 + t_3v + t_4)b]. \end{aligned}$$

Noticing the coefficients of v^3, that is,

$$\det[t_2b', 2t_2b, t_1a'] + \det[t_1b'', t_1b', -t_2b] = \frac{3s_1^3s_5^2}{s_6^4} \neq 0,$$

we have that the center map is not degenerate.

(2) If $s_1 = 0$, it is easy to prove f is centroaffinely equivalent to $f(u, v) = \phi(u)b'(u) + vb(u)$, for some nonzero function $\phi(u)$, the centroaffine metric is given by

$$\begin{cases} h_{11} = -\phi^{-2}v^2 - (3\phi'\phi^{-2} + \kappa_2\phi^{-1})v + \kappa_1 - 2(\phi')^2\phi^{-2} + \phi''\phi^{-1} - \phi'\phi^{-1}\kappa_2, \\ h_{12} = h_{21} = \phi^{-1}, \\ h_{22} = 0. \end{cases}$$



Fig. 3. Ruled surface $f(u, v) = b'(u) + vb(u)$ and its center map, where $b(u) = (u^2, u^3, u^4)$.

Through a serious of calculation, the difference tensor is given by

$$\begin{cases} K^1_{12} = K^1_{22} = K^2_{22} = 0, \\ K^2_{12} = \dfrac{1}{2}(3\phi'\phi^{-1} + \kappa_2). \end{cases}$$

Then we can get the center map

$$c = -\frac{1}{2}f_*T = -\frac{1}{2}(3\phi' + \phi\kappa_2)b(u).$$

The center map is degenerate to be a curve if the nonzero function ϕ satisfy $3\phi' + \phi\kappa_2 \neq 0$ (see Fig. 3). □

Remark 4.1. (1) Centroaffine ruled surfaces with $3\phi'(u) + \phi(u)\kappa_2(u) = 0$ are proper affine spheres.

(2) Centroaffine ruled surface $f(u, v) = \phi(u)b'(u) + vb(u)$ is minimal and its centroaffine scalar curvature 1. It can be reparameterized as $f(u, v) = b_u + vb$ mentioned in Lemma 3.3, that is, it belongs to (ii) in Proposition 3.4.

Example. For a cubic curve $b(u) = (u, u^2, u^3)$, we can change the parameter to be the centroaffine arc-length parameter s and rewrite the curve as

$$b(s) = (e^{6^{-1/3}s}, e^{2\cdot 6^{-1/3}s}, e^{3\cdot 6^{-1/3}s}).$$

The second centroaffine curvature is given by

$$\kappa_2 = 6^{2/3}.$$

Then we can get the center map of $f(s, v) = b'(s) + vb(s)$ as follows,

$$c(s, v) = -\frac{1}{2}6^{2/3}(e^{6^{-1/3}s}, e^{2\cdot 6^{-1/3}s}, e^{3\cdot 6^{-1/3}s}) = -\frac{1}{2}\kappa_2 b(s).$$

Corollary 4.2. *The center map of the centroaffine ruled surface $f(u, v) = b'(u) + vb(u)$ is centroaffinely equivalent to $b(u)$ if and only if the second centroaffine curvature of $b(u)$ is constant.*

Proof. By Theorem 1.1, the center map of $f(u, v) = b'(u) + vb(u)$ is given by

$$c = -\frac{1}{2}\kappa_2(u)b(u).$$

It is obvious that c is centroaffinely equivalent to $b(u)$ if κ_2 is constant. Conversely, if the center map c is centroaffinely equivalent to $b(u)$, that is, there exists a matrix $A \in \mathbb{GL}(3; \mathbb{R})$ such that $c = Ab(u)$, then from $\det(A + \frac{1}{2}\kappa_2 I) = 0$, where I is the identity matrix, we get κ_2 is constant. □

Corollary 4.3. *The center map of the centroaffine ruled surface $f(u, v) = b'(u) + vb(u)$ is projectively equivalent to $b(u)$.*

Corollary 4.4. *Given a nondegenerate centroaffine space curve $b(u)$ with centroaffine arc-length parameter u and centroaffine second curvature $\kappa_2(u)$, we can construct a centroaffine ruled surface f whose center map is $b(u)$. In fact, the center map of $f(u, v) = \phi(u)b'(u) + vb(u)$ is $b(u)$, for $\phi(u)$ satisfying $\phi(u) = -\frac{2}{3}\mu^{-1}(u)\int \mu(u)du$, where $\mu(u) = e^{\frac{1}{3}\int \kappa_2(u)du}$.*

Proof. Solving the ODE $3\phi'(u) + \phi(u)\kappa_2(u) = -2$. □

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References

[1] H. Furuhata, L. Vrancken, The center map of an affine immersion, Result. Math. 49 (2006) 201–217.
[2] K. Nomizu, T. Sasaki, Affine Differential Geometry, Cambridge Univ. Press, 1994.
[3] U. Simon, A. Schwenk-Schellschmidt, H. Viesel, Introduction to the Affine Differential Geometry of Hypersurfaces, in: Lecture Notes, Science University Tokyo, 1991.
[4] H.L. Liu, C.P. Wang, Centroaffinely homogeneous surfaces in $\mathbb{R}^3$, Contrib. Algebra. Geom. 35 (1994) 109–117.
[5] H.L. Liu, C.P. Wang, The centroaffine Tchebychev operator, Result. Math. 27 (1995) 77–92.
[6] W. Klingenberg, A Course in Differential Geometry, Springer-Verlag, 1978.
[7] C.P. Wang, Centroaffine minimal hypersurfaces in $\mathbb{R}^{n+1}$, Geom. Dedicata 51 (1994) 63–74.
[8] A. Fujioka, Centroaffine minimal surfaces with non-semicemisimple centroaffine Tchebychev operator, Result. Math. 56 (2009) 177–195.
[9] N. Hu, Centroaffine space curves with constant curvatures and homogeneous surfaces, J. of Geom. 102 (2011) 103–114.
[10] Y.H. Yu, Y. Yang, H.L. Liu, Centroaffine ruled surfaces in $\mathbb{R}^3$, J. Math. Anal. Appl. 365 (2010) 683–693.