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A fundamental theorem for hypersurfaces in semi-Riemannian warped products

Marie-Amélie Lawn [a], Miguel Ortega [b,*]

[a] Mathematics Department, The University of Texas at Austin, 2515 Speedway Stop C1200, Austin, TX 78712-1202, USA
[b] Department of Geometry and Topology, Faculty of Science, University of Granada, 18071 Granada, Spain

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ABSTRACT

We find necessary and sufficient conditions for nondegenerate arbitrary signature manifolds to be realized as hypersurfaces in a large class of warped products manifolds. As an application, we give conditions for a 3-dimensional hypersurface in a 4-dimensional Robertson–Walker spacetime to be foliated by surfaces with lightlike or zero mean curvature and hence describe a way to study horizons in such spacetimes.

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1. Introduction

Given an immersed hypersurface in a spacetime, the Einstein equations satisfied on the ambient space impose relationships on the extrinsic and intrinsic curvatures, the so-called constraint equations. In this sense, Y. Choquet-Bruhat and R. Geroch obtained one of the main results in General Relativity, namely, if a Riemannian 3-manifold equipped with a given $(2, 0)$-tensor K satisfies the constraint equations, then it can be embedded as a Cauchy hypersurface in a unique maximal 4-dimensional globally hyperbolic Lorentzian manifold whose Riemann tensor is trace-free, in such a way that the given tensor K is its second fundamental form (see [1]). This is the so-called Cauchy problem.

In addition, this can be seen as constructing solutions to Einstein's equation, for which there are four well established techniques, namely

1. a spacelike Cauchy problem,
2. a boundary–initial value problem,
3. a characteristic Cauchy problem on two transverse hypersurfaces, or
4. a characteristic Cauchy problem on the light-cone.

* Corresponding author. Tel.: +34 958 243282; fax: +34 958 243281.
E-mail addresses: mlawn@math.utexas.edu (M.-A. Lawn), miortega@ugr.es (M. Ortega).

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(See [2] and references therein.) These techniques can be regarded as particular cases of a more general, very interesting problem of Submanifold Theory, namely, to determine when a (semi-)Riemannian manifold (M^n, g) can be immersed into a (semi-)Riemannian manifold $(\bar{M}^{n+p}, \bar{g})$. This problem is two-fold, depending on what you fix. On one hand, the target is to find in which ambient space can be immersed a specific manifold. This is the case of the above mentioned ones, like the Cauchy Problem. On the other hand, it is also possible to set the ambient space and try to find necessary and sufficient conditions for the existence of submanifolds with certain properties, known as *Fundamental Theorems*. In either case, the Gauß, Ricci and Codazzi equations must be satisfied by any submanifold, and their rule consists of relating the intrinsic and extrinsic curvatures. In fact, the classical Fundamental Theorem of Submanifolds states that they are necessary and sufficient conditions for a semi-Riemannian n-dimensional manifold to admit a (local) immersion into spaces of constant sectional curvature of dimension $n + d$. If the ambient space is not of constant sectional curvature, proving fundamental theorems is technically difficult and there are only few other results known by the authors. For instance, B. Daniel obtained in [3] a fundamental theorem for hypersurfaces in the Riemannian products $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, looking for tools to work with minimal surfaces in such manifolds when $n = 2$. J. Roth generalized B. Daniel's theorem to spacelike hypersurfaces in some Lorentzian products (see [4] and also [5,6] or [7] for related works). It is also worth pointing out that this geometric framework has been studied by discussing the causal character of the submanifold, namely timelike, lightlike, or spacelike. However, in a recent work, M. Mars showed a common formulation for all causal characters (see [8]). In addition, the classical moving frame technique by E. Cartan allows to construct immersions from a smooth manifold into a Lie group with associated Lie algebra $\mathfrak{g}$, if and only if, there exists a $\mathfrak{g}$-valuated 1-form satisfying the so-called Maurer–Cartan equation. Sometimes, the Maurer–Cartan equation holds due to the structure equations (see for instance [9,6]).

Now, we recall that the spaces of constant sectional curvature are examples of Einstein manifolds, i.e., manifolds for which the metric is a solution to the vacuum Einstein field equations (with cosmological constant). In order to get a deeper understanding of the global characteristics of the universe including its topology, a generalization of Einstein manifolds is needed, especially to the non-vacuum case. First generalizations are quasi-Einstein manifold. Examples of such manifolds include Robertson–Walker-spaces, which play an important role in standard models of cosmology and arise as solutions of the non-vacuum Einstein equations. Geometrically speaking, they are special cases of semi-Riemannian warped products.

Our main result in this paper is a *fundamental theorem* for non-degenerate hypersurfaces in a family of semi-Riemannian warped products $\varepsilon I \times_a \mathbb{M}^n_k(c)$, where $\varepsilon = \pm 1$, $a : I \subset \mathbb{R} \to \mathbb{R}^+$ is the scale factor and $\mathbb{M}^n_k(c)$ is the semi-Riemannian space form of index k and constant curvature $c \in \{1, 0, -1\}$. In this way, this allows to study existence problems of spacelike and timelike hypersurfaces in this family of warped products. As a particular case, we are describing a way to study 3-dimensional spacelike or timelike surfaces in 4-dimensional Robertson–Walker spacetimes. Thus, we are not following M. Mars' approach, and thus we are discarding the timelike case. Since our computations are made for any dimension, they might also allow to study surfaces in toy-models or supergravities, but this has to be explored.

For a hypersurface M in $\varepsilon I \times_a \mathbb{M}^n_k(c)$, the vector field ∂_t $(t \in I)$ decomposes in its tangent and normal parts, i.e., $\partial_t = T + \varepsilon_{n+1} T_{n+1} e_{n+1}$ where e_{n+1} is a (local) normal unit vector field, $\varepsilon_{n+1} = \pm 1$ shows its causal character and T_{n+1} is the corresponding coordinate. It turns out that the Gauss and Codazzi equations depend not only on the shape operator A, but also on the vector field T and on the warping function a, as well as on the function T_{n+1}. Recall that the Ricci equation provides no information for hypersurfaces (i.e., codimension 1). Moreover, in order to construct the immersion, a third equation is needed, namely the covariant derivative of T, *which cannot be obtained from Gauß and Codazzi equations*. Based on these three necessary conditions, we state in Definition 1 all needed tools. Theorem 1 states the necessary and sufficient conditions for the existence of a (local) metric immersion $\chi : \mathcal{U} \subset M \to \varepsilon I \times_a \mathbb{M}^n_k(c)$. Technically, we are using the previously mentioned technique of moving frames. As an application, in Corollaries 2 and 3, we obtain conditions for such a 3-dimensional hypersurface to be foliated by surfaces whose mean curvature vector is either lightlike or zero, in Robertson–Walker spacetimes. In this way, we are including surfaces with vanishing mean curvature vector, MOTS and mixed cases. This can be helpful for the study of horizons on Robertson–Walker spacetimes with spacelike or timelike causal character, including Marginally Outer Trapped Tubes.

2. Preliminaries

Let (P, g_P) be a semi-Riemannian manifold of dimension $\dim P = m$. All our manifolds will be connected and of class C^∞, unless otherwise stated. We consider a smooth function $a : I \subset \mathbb{R} \to \mathbb{R}^+$, a (sign) constant $\varepsilon = \pm 1$ and the warped product

$$\bar{P}^{m+1} = \varepsilon I \times_a P, \qquad \langle , \rangle = \varepsilon \mathrm{d}t^2 + a^2(t) g_P.$$

Clearly, the unit vector field $\partial/\partial t = \partial_t$ will play a crucial role on the manifold $\bar{P}^{n+1}$. We set our convention for the curvature operator $\mathcal{R}$ of a connection $\mathcal{D}$ as

$$\mathcal{R}(X, Y)Z = \mathcal{D}_X \mathcal{D}_Y Z - \mathcal{D}_Y \mathcal{D}_X Z - \mathcal{D}_{[X,Y]} Z.$$

Let $\bar{R}_P$ and R_P be the curvature operator of $\bar{P}^{n+1}$ and P, respectively. Since there is no confusion, we will use the same notation for the associated curvature tensors along the paper. Let $\mathbf{D}$ be the Levi-Civita connection of $\bar{P}^{n+1}$. We recall the following formulae, which can be checked in [10], bearing in mind the change of sign.

Lemma 1 (*Cf. [10]*). *On the semi-Riemannian manifold $\bar{P}^{n+1}$, the following statements hold, for any V, W lifts of vector fields tangent to P:*

1. $\mathbf{D}_{\partial_t}\partial_t = 0$, $\mathbf{D}_V\partial_t = \frac{a'}{a}V$,
2. $\operatorname{grad}(a) = \varepsilon a'\partial_t$.
3. $\mathbf{D}_V W = \nabla^P_V W - \frac{\varepsilon a'}{a}\langle V, W\rangle\partial_t$.
4. $\bar{R}_P(V,\partial_t)\partial_t = -\frac{a''}{a}V$, $\bar{R}_P(\partial_t, V)W = -\varepsilon\frac{a''}{a}\langle V, W\rangle\partial_t$, $\bar{R}_P(V, W)\partial_t = 0$.

Now, let $\mathcal{M}$ be a non-degenerate hypersurface of $\bar{P}^{m+1}$, with $\nabla^{\mathcal{M}}$ its Levi-Civita connection, σ the second fundamental form and $R_{\mathcal{M}}$ the curvature operator of $\mathcal{M}$, respectively. Given a (local) unit normal vector field ν of $\mathcal{M}$ in $\bar{P}^{m+1}$, with $\delta = \langle\nu,\nu\rangle = \pm 1$, let $\mathcal{A}$ be the shape operator associated with ν. The Gauß and Weingarten's formulae are

$$\mathbf{D}_X Y = \nabla^{\mathcal{M}}_X Y + \sigma(X,Y), \qquad \mathbf{D}_X\nu = -\mathcal{A}X,$$

for any $X, Y \in T\mathcal{M}$. The second fundamental form can be written as

$$\sigma(X,Y) = \delta\langle\mathcal{A}X, Y\rangle\nu, \quad \text{for any } X, Y \in T\mathcal{M}.$$

Recall that the *mean curvature vector* of $\mathcal{M}$ is defined by

$$\vec{H} = \frac{1}{\dim(\mathcal{M})}\operatorname{Tr}(\sigma).$$

Next, the Codazzi equation of $\mathcal{M}$ takes the general form $(\bar{R}_P(X,Y)Z)^\perp = (\mathbf{D}_X\sigma)(Y,Z) - (\mathbf{D}_Y\sigma)(X,Z)$, for any X, Y, Z tangent to $\mathcal{M}$, equivalently,

$$\bar{R}_P(X,Y,Z,\nu) = \langle(\nabla^{\mathcal{M}}_X\mathcal{A})Y - (\nabla^{\mathcal{M}}_Y\mathcal{A})X, Z\rangle, \tag{1}$$

for any $X, Y, Z \in T\mathcal{M}$. Further, the general Gauß equation is given by

$$\begin{aligned}\bar{R}_P(X,Y,Z,W) &= R_{\mathcal{M}}(X,Y,Z,W) - \langle\sigma(Y,Z),\sigma(X,W)\rangle + \langle\sigma(Y,W),\sigma(X,Z)\rangle\\ &= R_{\mathcal{M}}(X,Y,Z,W) - \delta\langle\mathcal{A}Y,Z\rangle\langle\mathcal{A}X,W\rangle + \delta\langle\mathcal{A}Y,W\rangle\langle\mathcal{A}X,Z\rangle,\end{aligned} \tag{2}$$

with X, Y, Z, W tangent to $\mathcal{M}$.

We also need to set the standard Euclidean semi-Riemannian space of dimension $l+1 \geq 3$ and index k, $\mathbb{E}^{l+1}_k = \mathbb{E}^{l+1}$ with its metric

$$g = \sum_{i=0}^{l-k} dx_i^2 - \sum_{i=l-k+1}^{l+1} dx_i^2.$$

Following the previous notation, we construct $\widetilde{P}^{n+2} = \varepsilon I \times_a \mathbb{E}^{n+1}$. Let $\widetilde{R}$ be the curvature tensor of $\widetilde{P}^{n+2}$. The following result can be found in [10].

Proposition 1. *Let $X, Y, Z, W \in \Gamma(T\widetilde{P}^{n+2})$.*

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = {}& \varepsilon\frac{(a')^2}{a^2}\Big(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\Big)\\ &+ \left(\frac{a''}{a} - \frac{(a')^2}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,\partial_t\rangle\langle W,\partial_t\rangle - \langle Y,Z\rangle\langle X,\partial_t\rangle\langle W,\partial_t\rangle\\ &- \langle X,W\rangle\langle Y,\partial_t\rangle\langle Z,\partial_t\rangle + \langle Y,W\rangle\langle X,\partial_t\rangle\langle Z,\partial_t\rangle\Big).\end{aligned}$$

Let $\mathbb{M}^n_k(c)$ be the semi-Riemannian space form of constant sectional curvature $c \in \{0, 1, -1\}$ and index k, with metric g. We put

$$\mathbb{E}^{n+1} = \begin{cases}\mathbb{R}^{n+1}_k, & \text{if } \mathbb{M}^n_k(c) = \mathbb{S}^n_k,\ c = +1,\\ \mathbb{R}^{n+1}_{k+1}, & \text{if } \mathbb{M}^n_k(c) = \mathbb{H}^n_k,\ c = -1,\end{cases}$$

with its standard pseudo-Riemannian flat metric g_o and Levi-Civita connection ∇^o. We are considering $\mathbb{S}^n_k$ and $\mathbb{H}^n_k$, respectively, as one of the connected components of

$$\mathbb{S}^n_k \subset \{p \in \mathbb{E}^{n+1} : g_o(p,p) = +1\}, \qquad \mathbb{H}^n_k \subset \{p \in \mathbb{E}^{n+1} : g_o(p,p) = -1\}.$$

Next, we construct $\bar{P}^{n+1} = \varepsilon I \times_a \mathbb{M}^n_k(c)$, with metric $\langle,\rangle$. We denote by $\bar{R}$ the curvature operator of $\varepsilon I \times_a \mathbb{M}^n_k(c)$.

For a moment, we assume $c \neq 0$. From the usual totally umbilical embedding $\Xi : \mathbb{M}^n_k(c) \to \mathbb{E}^{n+1}$, we construct the following isometric embedding

$$\widetilde{\Xi} : (\varepsilon I \times_a \mathbb{M}^n_k(c), \langle,\rangle) \longrightarrow (\varepsilon I \times_a \mathbb{E}^{n+1}, \langle,\rangle_2), \quad (t,p) \mapsto (t, \Xi(p)).$$

In the sequel, for the sake of simplicity, we will also use the notation $\langle , \rangle_2 = \langle , \rangle$. Let $\widetilde{\nabla}$ and $\bar{\nabla}$ be the Levi-Civita connection on $\varepsilon I \times_a \mathbb{E}^{n+1}$ and $\varepsilon I \times_a \mathbb{M}^n_k(c)$, respectively. It is well-known that $\xi = \Xi/c$ is a unit normal vector field satisfying $\nabla^o_X \xi = X/c$ for any X tangent to $T_p\mathbb{M}^n_k(c)$. Thus, we can consider the normal vector field of $\tilde{\Xi} : \varepsilon I \times_a \mathbb{M}^n_k(c) \to \varepsilon I \times_a \mathbb{E}^{n+1}$ as $e_0(t,p) = (0, \xi(p)/a(t)) = \big(0, p/(c\,a(t))\big)$, forany$(t,p) \in \varepsilon I \times_a \mathbb{M}^n_k(c)$. We also set $\varepsilon_0 = \langle e_0, e_0\rangle = \pm 1$. Since $\mathbb{M}^n_k(c)$ lies naturally in $\mathbb{E}^{n+1}$, the normal vector field ξ satisfies $g_o(\xi,\xi) = c$. In addition, $\varepsilon_0 = \langle e_0, e_0\rangle = \langle (0, p/(ac)), (0, p/(ac))\rangle = a^2 g_o(p,p)/(a^2c^2) = c$. In this way, by Lemma 1,

$$\widetilde{\nabla}_{\partial_t} e_0 = \frac{-a'}{a^2}(0,\xi) + \frac{1}{a}\widetilde{\nabla}_{\partial_t}(0,\xi) = \frac{-a'}{a^2}(0,\xi) + \frac{1}{a}\frac{a'}{a}(0,\xi) = 0.$$

Moreover, if $(0,Z) \perp e_0$, then $Z \perp \xi$, so that

$$\widetilde{\nabla}_{(0,Z)} e_0 = \frac{1}{a}\widetilde{\nabla}_{(0,Z)}(0,\xi) = \frac{1}{a}\Big(\nabla^0_Z \xi - \frac{\langle (0,Z),(0,\xi)\rangle}{a}\mathrm{grad} a\Big) = \frac{1}{ac}(0,Z).$$

This means that the Weingarten operator S associated with e_0 is

$$SY = \frac{-1}{ac}(Y - \varepsilon\langle Y, \partial_t\rangle \partial_t), \quad \text{for any } Y \in T\bar{P}^{n+1}.$$

Now, we let $c = 0$. We consider a totally geodesic embedding $\Xi : \mathbb{E}^n_k \to \mathbb{E}^{n+1}_k$ and we make the corresponding totally geodesic embedding $\tilde{\Xi}$ as in the cases $c = \pm 1$, with a unit spacelike normal vector field is e_0. Thus, $S = 0$. Therefore, it makes sense to let c take the values $\{0, 1, -1\}$, and then

$$SY = \frac{-c}{a}(Y - \varepsilon\langle Y, \partial_t\rangle \partial_t), \quad \text{for any } Y \in T\bar{P}^{n+1}. \tag{3}$$

Proposition 2. *The curvature tensor of $\bar{P}^{n+1} = \varepsilon I \times_a \mathbb{M}^n_k(c)$ is*

$$\begin{aligned}\bar{R}(X,Y,Z,W) = &\left(\varepsilon\frac{(a')^2}{a^2} - \frac{c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\Big)\\ &+ \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,\partial_t\rangle\langle W,\partial_t\rangle - \langle Y,Z\rangle\langle X,\partial_t\rangle\langle W,\partial_t\rangle\\ &- \langle X,W\rangle\langle Y,\partial_t\rangle\langle Z,\partial_t\rangle + \langle Y,W\rangle\langle X,\partial_t\rangle\langle Z,\partial_t\rangle\Big),\end{aligned} \tag{4}$$

for any X, Y, Z, W in $T\bar{P}^{n+1}$.

Proof. For $c \neq 0$, we recall $\varepsilon_0 = c$, and we just need to resort to (2), Proposition 1 and (3). For $c = 0$, $\varepsilon I \times_a \mathbb{M}^n_k(c)$ is totally geodesic in $\varepsilon I \times_a \mathbb{E}^{n+1}$, so that the formula trivially holds since $c = 0$. □

3. Hypersurfaces

Let M^n, $n \geq 2$, be an immersed, non-degenerate hypersurface in $\bar{P}^{n+1}$. Let ∇ be the Levi-Civita connections on M. Let e_{n+1} be a (locally defined) normal unit vector field to M, with $\varepsilon_{n+1} = \langle e_{n+1}, e_{n+1}\rangle = \pm 1$. Along M, the vector field ∂_t can be decomposed in its tangent and normal parts, i.e.,

$$\partial_t = T + f\, e_{n+1},$$

where T is tangent to M and $f = \varepsilon_{n+1}\langle \partial_t, e_{n+1}\rangle$. We also define the 1-form on M given by $\eta(X) = \langle X, T\rangle$, for any $X \in \Gamma(TM)$. Given a tangent vector X to M, we again decompose it in the part tangent to $\{t\} \times \mathbb{M}^n_k(c)$ and its component in the direction of ∂_t as $X = \tilde{X} + x\partial_t$. Similarly, $e_{n+1} = \tilde{e}_{n+1} + n\partial_t$.

Lemma 2. *Under the previous conditions,*

1. $\varepsilon n = \langle e_{n+1}, \partial_t\rangle = \varepsilon_{n+1} f$, $\varepsilon x = \langle X, \partial_t\rangle = \langle X, T\rangle$.
2. $\langle \tilde{X}, \tilde{e}_{n+1}\rangle = -\varepsilon\varepsilon_{n+1} f \langle X, T\rangle$,

Proof. From $e_{n+1} = \tilde{e}_{n+1} + n\partial_t$, it is immediate that $\varepsilon n = \langle \partial_t, e_{n+1}\rangle = \varepsilon_{n+1} f$. Further, $\langle X, \partial_t\rangle = \langle X, T + f e_{n+1}\rangle = \langle X, T\rangle = \langle \tilde{X} + x\partial_t, \partial_t\rangle = \varepsilon x$. Next, $0 = \langle X, e_{n+1}\rangle = \langle \tilde{X}, \tilde{e}_{n+1}\rangle + xn\varepsilon = \langle \tilde{X}, \tilde{e}_{n+1}\rangle + \varepsilon\varepsilon_{n+1} f \langle X, T\rangle$. □

Let A be the shape operator of M associated with e_{n+1}. By using (2), Proposition 2 and Lemma 2, we can prove:

Proposition 3. *The curvature tensor R of M in $\varepsilon I \times_a \mathbb{M}^n_k(c)$ is*

$$\begin{aligned} R(X,Y,Z,W) = &\left(\varepsilon\frac{(a')^2}{a^2} - \frac{c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\Big)\\ &+\left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,T\rangle\langle W,T\rangle - \langle Y,Z\rangle\langle X,T\rangle\langle W,T\rangle\\ &-\langle X,W\rangle\langle Y,T\rangle\langle Z,T\rangle + \langle Y,W\rangle\langle X,T\rangle\langle Z,T\rangle\Big) + \varepsilon_{n+1}\Big(\langle AY,Z\rangle\langle AX,W\rangle - \langle AY,W\rangle\langle AX,Z\rangle\Big), \end{aligned}$$

for any $X, Y, Z, W \in TM$.

Proposition 4. *The Codazzi equation of M in $\varepsilon I \times_a \mathbb{M}^n_k(c)$ is given by*

$$(\nabla_X A)Y - (\nabla_Y A)X = \varepsilon_{n+1} f\left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\Big(\langle Y,T\rangle X - \langle X,T\rangle Y\Big), \tag{5}$$

for any X, Y tangent to M.

Proof. By (1), we have to compute $\bar{R}(X,Y,Z,e_{n+1})$ for any tangent vectors X, Y, Z to M. To do so, we recall Proposition 2. Thus,

$$\begin{aligned} \bar{R}(X,Y,Z,e_{n+1}) = &\left(\varepsilon\frac{(a')^2}{a^2} - \frac{c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,e_{n+1}\rangle - \langle Y,Z\rangle\langle X,e_{n+1}\rangle\Big)\\ &+\left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,\partial_t\rangle\langle e_{n+1},\partial_t\rangle - \langle Y,Z\rangle\langle X,\partial_t\rangle\langle e_{n+1},\partial_t\rangle\\ &-\langle X,e_{n+1}\rangle\langle Y,\partial_t\rangle\langle Z,\partial_t\rangle + \langle Y,e_{n+1}\rangle\langle X,\partial_t\rangle\langle Z,\partial_t\rangle\Big)\\ = &\,\varepsilon_{n+1} f\left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\right)\Big(\langle Y,T\rangle\langle X,Z\rangle - \langle X,T\rangle\langle Y,Z\rangle\Big). \end{aligned}$$

This yields the result. □

Lemma 3. *The following equations hold for any X tangent to M:*

1. $\bar{\nabla}_X\partial_t = \frac{a'}{a}(X - \varepsilon\langle X,T\rangle\partial_t)$.
2. $\nabla_X T = \frac{a'}{a}(X - \varepsilon\langle X,T\rangle T) + fAX$.
3. $X(f) = -\varepsilon_{n+1}\langle AT,X\rangle - \frac{a'}{a}\varepsilon\langle X,T\rangle f$.

Proof. First, we recall that $X = \tilde{X} + \varepsilon\eta(X)\partial_t$. Therefore, $\bar{\nabla}_X\partial_t = \bar{\nabla}_{\tilde{X}}\partial_t = \frac{a'}{a}\tilde{X} = \frac{a'}{a}(X - \varepsilon\langle X,T\rangle\partial_t)$. Next, we compute $\bar{\nabla}_X T = \bar{\nabla}_X(\partial_t - fe_{n+1}) = \bar{\nabla}_X\partial_t - X(f)e_{n+1} - f\bar{\nabla}_X e_{n+1} = \frac{a'}{a}(X - \varepsilon\langle X,T\rangle\partial_t) - X(f)e_{n+1} + fAX = \frac{a'}{a}(X - \varepsilon\langle X,T\rangle(T + fe_{n+1})) - X(f)e_{n+1} + fAX = \frac{a'}{a}(X - \varepsilon\langle X,T\rangle T) - \frac{\varepsilon fa'}{a}\langle X,T\rangle e_{n+1})) - X(f)e_{n+1} + fAX$. Now, each equation is just the tangential and the normal part of $\bar{\nabla}_X T = \nabla_X T + \varepsilon_{n+1}\langle AX,T\rangle e_{n+1}$. □

4. Moving frames

Élie Cartan developed the *moving frame* technique. Definitions, basic results and some other details can be found in [9, p. 18]. We will use the following convention on the ranges of indices, unless mentioned otherwise:

$$1 \le i,j,k,l \le n; \qquad 1 \le u,v,w,\ldots \le n+1; \qquad 0 \le \alpha,\beta,\gamma,\ldots \le n+1.$$

We recall that M is a hypersurface of $\bar{P}^{n+1}$, hence $n = \dim M$. Consider a local orthonormal frame $(e_0, e_1, \ldots, e_n, e_{n+1})$ defined on an open subset $\mathcal{U}$ of M, such that $e_1, \ldots, e_n$ are tangent to M and e_{n+1} is normal to M in $\bar{P}^{n+1}$, with $\varepsilon_\alpha = \langle e_\alpha, e_\alpha\rangle = \pm 1$. We define the matrix $G = (\varepsilon_\alpha\delta_{\alpha\beta})$. Let $(\omega_0, \ldots, \omega_{n+1})$ be the dual basis of e_α, i.e. $\omega_\alpha(e_\beta) = \delta_{\alpha\beta}$, where $\omega_r|_{TM} = 0$, $r \in \{0, n+1\}$. The dual 1-forms ω_α can be obtained as $\omega_\alpha(X) = \varepsilon_\alpha\langle e_\alpha, X\rangle$.

Given $(E_0, \ldots, E_n)$ a parallel orthonormal frame of $\mathbb{E}^{n+1}$, we construct $(\bar{E}_0, \ldots, \bar{E}_{n+1}) = (E_0/a, \ldots, E_n/a, \partial_t)$, which is an orthonormal frame of $\varepsilon I \times_a \mathbb{E}^{n+1}$. If necessary, we reorder the basis $(\bar{E}_0, \ldots, \bar{E}_{n+1})$ to obtain $\langle\bar{E}_\alpha, \bar{E}_\alpha\rangle = \varepsilon_\alpha$, $\bar{E}_{n+1} = \partial_t$. Next, we define the functions $B_{\alpha\beta} := \langle\bar{E}_\alpha, e_\beta\rangle$ and the matrix $B = (B_{\alpha\beta})$. With them we compute $\sum_\mu \varepsilon_\mu B_{\mu\alpha}B_{\mu\beta} = \sum_\mu \varepsilon_\mu \langle\bar{E}_\mu, e_\alpha\rangle\langle\bar{E}_\mu, e_\beta\rangle = \langle e_\alpha, e_\beta\rangle = \varepsilon_\alpha\delta_{\alpha\beta}$. This equation reduces to $B^t G B = G$, which implies $B^{-1} = GB^t G$, where B^t is the transpose of B and $B^{-1} = (B^{\alpha\beta})$. Next, from the fact that $B^t G B = G$, we recall the Lie group $\{Z \in \mathcal{M}_{n+2}(\mathbb{R}) | Z^t G Z = G, \ \det Z = 1\}$,

where $\mathcal{M}_{n+2}(\mathbb{R})$ is the usual set of all matrices of order $(n+2)\times(n+2)$. For simplicity, we call $\mathbf{S}$ the connected component containing the identity matrix,

$$I_{n+2}\in\mathbf{S}\subset\{Z\in\mathcal{M}_{n+2}(\mathbb{R})|Z^tGZ=G,\ \det Z=1\}.$$

In addition, $\mathfrak{s}=\{H\in\mathcal{M}_{n+2}(\mathbb{R})|H^tG+GH=0\}$ is the Lie algebra associated with $\mathbf{S}$. As usual, we identify the tangent space of $\mathbf{S}$ at the identity matrix with $\mathfrak{s}$. Next, we point out that we have constructed a map $B:\mathcal{U}\to\mathbf{S}$, and therefore, we immediately obtain the *$\mathfrak{s}$-valued 1-form $B^{-1}dB$*. We explain this notion a little. Given a smooth manifold $\mathbf{M}$ of dimension n, $\Lambda^1(\mathbf{M})$ is the set of all 1-forms defined on $\mathbf{M}$. Then, we introduce the set $\mathfrak{s}(\Lambda^1(\mathbf{M}))$ of all matrices Υ or order $(n+2)\times(n+2)$ whose entries are 1-forms defined on $\mathbf{M}$, and satisfying the equation $\Upsilon^tG+G\Upsilon=0$. In other words, $\mathfrak{s}(\Lambda^1(\mathbf{M}))$ is the set of the so-called $\mathfrak{s}$-valued 1-forms on $\mathbf{M}$. In particular, $B^{-1}dB\in\mathfrak{s}(\Lambda^1(\mathcal{U}))$. As a remark, given matrices $Z\in\mathcal{M}_{n+2}(\mathbb{R})$ and $\Upsilon\in\mathfrak{s}(\Lambda^1(M))$, it makes sense to compute $Z\Upsilon$.

Let us now define the connection 1-forms $\Omega=(\omega_{\alpha\beta})$,

$$\omega_{\alpha\beta}(X)=\varepsilon_\alpha\langle e_\alpha,\widetilde{\nabla}_Xe_\beta\rangle,\quad\text{for any } X\in T\mathcal{U},$$

or equivalently $\omega_{\beta\alpha}=-\varepsilon_\alpha\varepsilon_\beta\omega_{\alpha\beta}$. This means $\Omega\in\mathfrak{s}(\Lambda^1(\mathcal{U}))$. In particular,

$$\nabla e_i=\sum_k\omega_{ki}e_k,\qquad\bar{\nabla}e_u=\sum_v\omega_{vu}e_v,\qquad\widetilde{\nabla}e_\alpha=\sum_\gamma\omega_{\gamma\alpha}e_\gamma,\tag{6}$$

We now define the 1-form $\eta(X)=\langle T,X\rangle$ and functions $T_k=\langle e_k,T\rangle$, $T_{n+1}=\varepsilon_{n+1}f$ and $T_0=0$. Clearly, $\sum_kT_k\omega_k=\eta$. Obviously, we can recover the vectors $e_\beta=\sum_\gamma\varepsilon_\gamma B_{\gamma\beta}\overline{E}_\gamma$. Consequently, by (6),

$$\begin{aligned}\widetilde{\nabla}_{e_\alpha}e_\beta&=\sum_\mu\omega_{\mu\beta}(e_\alpha)e_\mu=\sum_\gamma\varepsilon_\gamma\Big(\sum_\mu\omega_{\mu\beta}(e_\alpha)B_{\gamma\mu}\Big)\overline{E}_\gamma\\&=\widetilde{\nabla}_{e_\alpha}\Big(\sum_\gamma\varepsilon_\gamma B_{\gamma\beta}\overline{E}_\gamma\Big)=\sum_\gamma\varepsilon_\gamma dB_{\gamma\beta}(e_\alpha)\overline{E}_\gamma+\sum_{\mu,\gamma}\varepsilon_\gamma\varepsilon_\mu B_{\mu\alpha}B_{\gamma\beta}\widetilde{\nabla}_{\overline{E}_\mu}\overline{E}_\gamma.\end{aligned}$$

By now, we just care for the last summand. In these particular computations, we take $\gamma,\mu\neq n+1$, obtaining

$$\begin{aligned}&\widetilde{\nabla}_{\overline{E}_{n+1}}\overline{E}_{n+1}=\widetilde{\nabla}_{\partial_t}\partial_t=0,\qquad\widetilde{\nabla}_{\overline{E}_\mu}\overline{E}_{n+1}=\frac{1}{a}\widetilde{\nabla}_{E_\mu}\partial_t=\frac{a'}{a}\overline{E}_\mu,\\&\widetilde{\nabla}_{\overline{E}_{n+1}}\overline{E}_\gamma=-\frac{a'}{a^2}E_\gamma+\frac{1}{a}\widetilde{\nabla}_{\partial_t}E_\gamma=-\frac{a'}{a^2}E_\gamma+\frac{a'}{a^2}E_\gamma=0,\\&\widetilde{\nabla}_{\overline{E}_\mu}\overline{E}_\gamma=\frac{1}{a^2}\widetilde{\nabla}_{E_\mu}E_\gamma=\frac{1}{a^2}\nabla^0_{E_\mu}E_\gamma-\frac{\langle\overline{E}_\mu,\overline{E}_\gamma\rangle}{a}\mathrm{grad}(a)=-\frac{\varepsilon_\mu\delta_{\mu\gamma}\varepsilon a'}{a}\partial_t.\end{aligned}$$

Consequently, by the fact that the terms for $\mu=n+1$ vanish, now $\gamma,\mu\leq n+1$, we have

$$\begin{aligned}\widetilde{\nabla}_{e_\alpha}e_\beta-\sum_\gamma\varepsilon_\gamma dB_{\gamma\beta}(e_\alpha)\overline{E}_\gamma&=\sum_{\mu,\gamma}\varepsilon_\gamma\varepsilon_\mu B_{\mu\alpha}B_{\gamma\beta}\widetilde{\nabla}_{\overline{E}_\mu}\overline{E}_\gamma\\&=\sum_{\mu,\gamma<n+1}\varepsilon_\gamma\varepsilon_\mu B_{\mu\alpha}B_{\gamma\beta}\widetilde{\nabla}_{\overline{E}_\mu}\overline{E}_\gamma+\sum_{\mu<n+1}\varepsilon_{n+1}\varepsilon_\mu B_{\mu\alpha}B_{n+1\,\beta}\widetilde{\nabla}_{\overline{E}_\mu}\overline{E}_{n+1}\\&=\frac{\varepsilon_{n+1}a'}{a}\sum_{\mu<n+1}\varepsilon_\mu B_{\mu\alpha}B_{n+1\beta}\overline{E}_\mu-\frac{\varepsilon a'}{a}\sum_{\gamma<n+1}\varepsilon_\gamma B_{\gamma\alpha}B_{\gamma\beta}\overline{E}_{n+1}.\end{aligned}$$

By comparing coordinates, we get for $\gamma=n+1$

$$\sum_\mu B_{n+1\,\mu}\omega_{\mu\beta}(e_\alpha)=dB_{n+1\,\beta}(e_\alpha)-\frac{a'}{a}\sum_{\gamma<n+1}\varepsilon_\gamma B_{\gamma\alpha}B_{\gamma\beta}\tag{7}$$

and for $\gamma=0,\dots,n$,

$$\sum_\mu B_{\gamma\mu}\omega_{\mu\beta}(e_\alpha)=dB_{\gamma\beta}(e_\alpha)+\frac{\varepsilon a'}{a}B_{\gamma\alpha}B_{n+1\,\beta}.\tag{8}$$

Since $B_{\mu\alpha}=\sum_\gamma B_{\mu\gamma}\omega_\gamma(e_\alpha)$, we get for Eq. (7), $\sum_\mu B_{n+1\,\mu}\omega_{\mu\beta}-dB_{n+1\,\beta}=-\frac{a'}{a}\sum_\gamma\sum_{\mu<n+1}\varepsilon_\mu B_{\mu\beta}B_{\mu\gamma}\omega_\gamma=-\frac{a'}{a}\sum_{\gamma,\mu}\varepsilon_\mu B_{\mu\beta}B_{\mu\gamma}\omega_\gamma+\varepsilon\frac{a'}{a}\sum_\gamma B_{n+1\,\beta}B_{n+1\,\gamma}\omega_\gamma$, and for Eq. (8), $\sum_\mu B_{\gamma\mu}\omega_{\mu\beta}=dB_{\gamma\beta}+\frac{\varepsilon a'}{a}\sum_\kappa B_{n+1\,\beta}B_{\gamma\kappa}\omega_\kappa$, for any $\gamma=0,\dots,n+1$. Summing up, for all $\gamma=\alpha$

$$\sum_\mu B_{\alpha\mu}\omega_{\mu\beta}=dB_{\alpha\beta}+\frac{\varepsilon a'}{a}B_{n+1\,\beta}\sum_\gamma B_{\alpha\gamma}\omega_\gamma-\frac{a'}{a}\varepsilon_\beta\delta_{\alpha\,n+1}\omega_\beta.$$

Moreover, we have $\sum_\mu B^{\alpha\mu}\delta_{\mu 0}\varepsilon_\beta\omega_\beta = B^{\alpha 0}\varepsilon_\beta\omega_\beta = \varepsilon_\beta\varepsilon_\alpha\varepsilon B_{0\alpha}\omega_\beta$, that is, $\omega_{\alpha\beta} = \sum_\mu B^{\alpha\mu}dB_{\mu\beta} + \frac{\varepsilon a'}{a}\big(B_{n+1\,\beta}\omega_\alpha - \varepsilon_\beta\varepsilon_\alpha B_{n+1\,\alpha}\omega_\beta\big)$. Finally, we obtain

$$\Omega - \mathbf{X} = B^{-1}dB, \qquad \mathbf{X}_{\alpha\beta} = \frac{\varepsilon a'}{a}\Big(B_{n+1\,\beta}\omega_\alpha - \varepsilon_\beta\varepsilon_\alpha B_{n+1\,\alpha}\omega_\beta\Big). \tag{9}$$

We point out that $B_{n+1\,\alpha} = \langle \overline{E}_{n+1}, e_\alpha\rangle = \langle \partial_t, e_\alpha\rangle = T_\alpha$.

5. Main theorem

Let $(M, \langle,\rangle)$ be a semi-Riemannian manifold with its Levi-Civita connection ∇, its Riemann tensor R. We choose numbers $\varepsilon, \varepsilon_0, \varepsilon_{n+1} \in \{-1, 1\}$ and $c \in \{-1, 0, 1\}$ such that either $c = \varepsilon_0 \neq 0$, or $c = 0$ and $\varepsilon_0 = 1$. Consider smooth functions $a : I \subset \mathbb{R} \to \mathbb{R}^+$, $T_{n+1} : M \to \mathbb{R}$ and $\pi : M \to I$. We construct the vector field $T \in \mathfrak{X}(M)$ by $T = \varepsilon\,\mathrm{grad}(\pi)$, with its 1-form $\eta(X) = \langle X, T\rangle$. Also, let A be a tensor of type (1, 1) on M.

Definition 1. Under the previous conditions, we will say that M satisfies the *structure conditions* if the following conditions hold:

(A) A is $\langle,\rangle$-self adjoint;
(B) $\varepsilon = \langle T, T\rangle + \varepsilon_{n+1}T_{n+1}^2$;
(C) $\nabla_X T = \frac{a'\circ\pi}{a\circ\pi}(X - \varepsilon\eta(X)T) + \varepsilon_{n+1}T_{n+1}AX$, for any $X \in TM$;
(D) $X(T_{n+1}) = -\langle AT, X\rangle - \varepsilon\frac{a'\circ\pi}{a\circ\pi}T_{n+1}\eta(X)$, for any $X \in TM$;
(E) *Codazzi equation:* for any $X, Y \in TM$, it holds

$$(\nabla_X A)Y - (\nabla_Y A)X = T_{n+1}\,b\,\big(\eta(Y)X - \eta(X)Y\big),$$
$$b = \frac{a''\circ\pi}{a\circ\pi} - \frac{(a'\circ\pi)^2}{(a\circ\pi)^2} + \frac{\varepsilon c}{(a\circ\pi)^2};$$

(F) *Gauß equation:* for any $X, Y, Z, W \in TM$, it holds

$$\begin{aligned} R(X,Y,Z,W) = {} & \left(\varepsilon\frac{(a'\circ\pi)^2}{(a\circ\pi)^2} - \frac{c}{(a\circ\pi)^2}\right)\Big(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\Big) \\ & + b\Big(\langle X,Z\rangle\eta(Y)\eta(W) - \langle Y,Z\rangle\eta(X)\eta(W) - \langle X,W\rangle\eta(Y)\eta(Z) \\ & + \langle Y,W\rangle\eta(X)\eta(Z)\Big) + \varepsilon_{n+1}\big(\langle AY,Z\rangle\langle AX,W\rangle - \langle AY,W\rangle\langle AX,Z\rangle\big). \end{aligned}$$

We recall the warped product $(\bar{P}^{n+1} = I \times \mathbb{M}_k^n(c), \langle,\rangle_1 = \varepsilon dt^2 + a^2 g_o)$.

Theorem 1. *Let $(M, \langle,\rangle)$ a semi-Riemannian manifold satisfying the structure conditions. Then, for each point $p \in M$, there exists a neighborhood $\mathcal{U}$ of p on M, a metric immersion $\chi : (\mathcal{U}, \langle,\rangle) \to (\bar{P}^{n+1}, \langle,\rangle_1)$ and a normal unit vector field e_{n+1} along χ such that:*

1. $\varepsilon_{n+1} = \langle e_{n+1}, e_{n+1}\rangle_1$;
2. $\pi_I \circ \chi = \pi$, *where* $\pi_I : \varepsilon\, I \times_a \mathbb{M}_k^n(c) \to I$ *is the projection;*
3. *The shape operator associated with* e_{n+1} *is* A;
4. (E) *is the Codazzi equation, and* (F) *is the Gauß equation,*
5. *and along* χ, *it holds* $\partial_t = T + \varepsilon_{n+1}T_{n+1}e_{n+1}$.

Proof. Given a point $x \in M$, around it we consider a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M, with their signs $\varepsilon_i = g(e_i, e_i) = \pm 1$, and its corresponding dual basis of 1-forms $\{\omega_1, \ldots, \omega_n\}$. We point out that an alternative definition for these 1-forms is $\omega_i(X) = \varepsilon_i\langle e_i, X\rangle$, for any $X \in TM$. We also need to define $\omega_{n+1} = \omega_0 = 0$. With the help of the tensor $SY = -c\big(Y - \varepsilon\eta(Y)T\big)/a$, for any $Y \in TM$, we construct the following 1-forms

$$\begin{aligned} &\omega_{ij}(X) = \varepsilon_i\langle e_i, \nabla_X e_j\rangle, \qquad \omega_{i\,n+1}(X) = -\varepsilon_i\langle e_i, AX\rangle, \\ &\omega_{i0}(X) = -\varepsilon_i\langle e_i, SX\rangle, \qquad \omega_{n+1,0} = -\frac{\varepsilon\varepsilon_{n+1}c}{a\circ\pi}T_{n+1}\eta, \qquad \omega_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta\omega_{\beta\alpha}, \end{aligned} \tag{10}$$

for any $X \in TM$, known as the *connection 1-forms*. In this way, we consider the $\mathfrak{s}$-valued matrix $\Omega = (\omega_{\alpha\beta})$. As a consequence, we get $\nabla e_i = \sum_k \omega_{ki}e_k$. Now, we define the functions $T_i = \eta(e_i)$, $i \in \{1, \ldots, n\}$, $T_0 = 0$. We point out that by condition (B), we have $\varepsilon = \sum_\gamma \varepsilon_\gamma T_\gamma^2$. Next, we also construct the matrices $\mathbf{X} = (\mathbf{X}_{\alpha\beta})$ and Υ as

$$\mathbf{X}_{\alpha\beta} = \frac{\varepsilon a'}{a}\big(T_\beta\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta T_\alpha\omega_\beta\big), \quad \Upsilon = \Omega - \mathbf{X}. \tag{11}$$

A simple computation shows

$$d\Upsilon + \Upsilon \wedge \Upsilon = d\Omega - d\mathbf{X} + \Omega \wedge \Omega - \Omega \wedge \mathbf{X} - \mathbf{X} \wedge \Omega + \mathbf{X} \wedge \mathbf{X}.$$

Thus, our target consists of proving that the second half of this equality vanishes. Since the computation is rather lengthy, we will split it in some lemmata.

Lemma 4. $d\eta = 0$.

Proof. Since $T = \varepsilon \operatorname{grad}(\pi)$, we obtain that $\eta = \varepsilon d\pi$. Therefore, $d\eta = 0$. □

We define the matrices $\varpi = (\omega_\alpha)$ and $\Gamma = (\Gamma_{\alpha\beta}) = d\Omega + \Omega \wedge \Omega$.

Lemma 5. $d\varpi = -\Omega \wedge \varpi$, $\Gamma_{\alpha\beta} = -\varepsilon_\alpha \varepsilon_\beta \Gamma_{\beta\alpha}$,

$$\Gamma_{ij} = \varepsilon\varepsilon_j \frac{(a')^2}{a^2}\omega_j \wedge \omega_i - \left(\frac{a''}{a} - \frac{(a')^2}{a^2}\right)\left(T_j\omega_i - \varepsilon_i\varepsilon_j T_i\omega_j\right) \wedge \eta,$$

$$\Gamma_{i\,n+1} = T_{n+1}\left(\frac{a''}{a} - \frac{(a')^2}{a^2}\right)\eta \wedge \omega_i, \quad \Gamma_{u0} = 0.$$

Proof. Given $X, Y \in TM$, since $\omega_{n+1} = \omega_0 = 0$, we have

$$\begin{aligned} d\omega_i(X,Y) &= X(\omega_i(Y)) - Y(\omega_i(X)) - \omega_i([X,Y]) \\ &= \varepsilon_i\langle \nabla_X e_i, Y\rangle + \varepsilon_i\langle e_i, \nabla_X Y\rangle - \varepsilon_i\langle \nabla_Y e_i, X\rangle - \varepsilon_i\langle e_i, \nabla_Y X\rangle - \varepsilon_i\langle e_i, [X,Y]\rangle \\ &= \varepsilon_i \sum_k \omega_{ki}(X)\langle e_k, Y\rangle - \varepsilon_i \sum_k \omega_{ki}(Y)\langle e_k, X\rangle = -\sum_\gamma \omega_{i\gamma} \wedge \omega_\gamma(X,Y). \end{aligned}$$

On the other hand, by (10),

$$\begin{aligned} \sum_\gamma \omega_{n+1\,\gamma} \wedge \omega_\gamma(X,Y) &= \sum_k \omega_{n+1\,k} \wedge \omega_k(X,Y) = \sum_k \big(\varepsilon_{n+1}\langle e_k, AX\rangle\varepsilon_k\langle e_k, Y\rangle \\ &\quad - \varepsilon_{n+1}\langle e_k, AY\rangle\varepsilon_k\langle e_k, X\rangle\big) = \varepsilon_{n+1}\big(\langle Y, AX\rangle - \langle X, AY\rangle\big) = 0 = -d\omega_{n+1}(X,Y). \end{aligned}$$

We recall that whenever $c \neq 0$, then $c = \varepsilon_0$. Then,

$$\begin{aligned} \sum_\gamma \omega_{0\gamma} \wedge \omega_\gamma(X,Y) &= \sum_k \omega_{0k} \wedge \omega_k(X,Y) \\ &= \frac{c}{a}\sum_k \Big(-\varepsilon_0\big(\langle e_k, X\rangle - \varepsilon T_k\eta(X)\big)\omega_k(Y) + \varepsilon_0\big(\langle e_k, Y\rangle - \varepsilon T_k\eta(Y)\big)\omega_k(X)\Big) \\ &= \frac{c\,\varepsilon_0}{a}\Big(-\langle Y, X\rangle + \varepsilon\eta(Y)\eta(X) + \langle X, Y\rangle - \varepsilon\eta(X)\eta(Y)\Big) = 0 = -d\omega_0(X,Y). \end{aligned}$$

Now, given $X, Y \in TM$, we make the computation $d\omega_{ij}(X,Y) = X(\omega_{ij}(Y)) - Y(\omega_{ij}(X)) - \omega_{ij}([X,Y]) = \varepsilon_i X(\langle e_i, \nabla_Y e_j\rangle) - \varepsilon_i Y(\langle e_i, \nabla_X e_j\rangle) - \varepsilon_i\langle e_i, \nabla_{[X,Y]} e_j\rangle = \varepsilon_i\langle \nabla_X e_i, \nabla_Y e_j\rangle - \varepsilon_i\langle \nabla_Y e_i, \nabla_X e_j\rangle + \varepsilon_i R(X,Y,e_j,e_i)$. Next, by Eq. (6), $\langle \nabla_X e_i, \nabla_Y e_j\rangle = \sum_k \varepsilon_k \omega_{ki}(X)\omega_{kj}(Y) = -\sum_k \varepsilon_i\omega_{ik}(X)\omega_{kj}(Y)$. Therefore, we obtain $d\omega_{ij}(X,Y) = -\sum_k \omega_{ik} \wedge \omega_{kj}(X,Y) - \varepsilon_i R(X,Y,e_i,e_j)$, which implies

$$\begin{aligned} &d\omega_{ij}(X,Y) + \sum_\gamma \omega_{i\gamma} \wedge \omega_{\gamma j}(X,Y) \\ &= \omega_{i0} \wedge \omega_{0j}(X,Y) + \omega_{i\,n+1} \wedge \omega_{n+1\,j}(X,Y) + \varepsilon_i R(X,Y,e_j,e_i) \\ &= -\varepsilon_i\varepsilon_0\langle e_i, SX\rangle\langle e_j, SY\rangle + \varepsilon_i\varepsilon_0\langle e_i, SY\rangle\langle e_j, SX\rangle - \varepsilon_i\varepsilon_{n+1}\langle e_i, AX\rangle\langle e_j, AY\rangle \\ &\quad + \varepsilon_i\varepsilon_{n+1}\langle e_i, AY\rangle\langle e_j, AX\rangle + \varepsilon_i\Big(\Big(\varepsilon\frac{(a')^2}{a^2} - \frac{c}{a^2}\Big)\big(\langle X, e_j\rangle\langle Y, e_i\rangle - \langle Y, e_j\rangle\langle X, e_i\rangle\big) \\ &\quad + \Big(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon c}{a^2}\Big)\big(\langle X, e_j\rangle\eta(Y)\eta(e_i) - \langle Y, e_j\rangle\eta(X)\eta(e_i) \\ &\quad - \langle X, e_i\rangle\eta(Y)\eta(e_j) + \langle Y, e_i\rangle\eta(X)\eta(e_j)\big) + \varepsilon_{n+1}\Big(\langle AY, e_j\rangle\langle AX, e_i\rangle - \langle AY, e_i\rangle\langle AX, e_j\rangle\Big)\Big) \\ &= \varepsilon\varepsilon_j\frac{(a')^2}{a^2}\omega_j \wedge \omega_i(X,Y) - \Big(\frac{a''}{a} - \frac{(a')^2}{a^2}\Big)\big(T_j\omega_i - \varepsilon_i\varepsilon_j T_i\omega_j\big) \wedge \eta(X,Y). \end{aligned}$$

Next, given $X, Y \in TM$, we compute

$$\begin{aligned} d\omega_{in+1}(X,Y) &= X(\omega_{in+1}(Y)) - Y(\omega_{in+1}(X)) - \omega_{in+1}([X,Y]) \\ &= -\varepsilon_i\langle\nabla_X e_i, AY\rangle + \varepsilon_i\langle\nabla_Y e_i, AX\rangle + \varepsilon_i\langle e_i, (\nabla_Y A)X - (\nabla_X A)Y\rangle \\ &= -\varepsilon_i\sum_k \omega_{ki}(X)\langle e_k, AY\rangle + \varepsilon_i\sum_k \omega_{ki}(Y)\langle e_k, AX\rangle + \varepsilon_i\langle e_i, (\nabla_Y A)X - (\nabla_X A)Y\rangle \\ &= -\sum_\gamma \omega_{i\gamma}\wedge\omega_{\gamma\, n+1}(X,Y) + T_{n+1}\Big(\frac{a''}{a} - \frac{(a')^2}{a^2}\Big)\eta\wedge\omega_i(Y,X). \end{aligned}$$

Moreover, $d\omega_{i0}(X,Y) = X(\omega_{i0}(Y)) - Y(\omega_{i0}(X)) - \omega_{i0}([X,Y]) = \varepsilon_i\big(\langle\nabla_Y e_i, SX\rangle - \langle\nabla_X e_i, SY\rangle - \langle e_i, (\nabla_Y S)X - (\nabla_X S)Y\rangle\big)$. On one hand, for any $U \in \mathfrak{X}(M)$, it holds $\nabla_Y\big(-U/a\big) = \varepsilon a'\eta(Y)U/(a^2) - (\nabla_Y U)/a$, so that we can see

$$\begin{aligned} (\nabla_Y S)X - (\nabla_X S)Y &= \nabla_Y SX - S\nabla_Y X - \nabla_X SY + S\nabla_X Y \\ &= \frac{\varepsilon c a'}{a^2}\big(\eta(Y)X - \eta(X)Y\big) + \frac{\varepsilon}{ac}\Big(\eta(X)\nabla_Y T - \eta(Y)\nabla_X T\Big) \\ &= \frac{\varepsilon\varepsilon_{n+1}cT_{n+1}}{a}(\eta(X)AY - \eta(Y)AX). \end{aligned}$$

By using this, we obtain

$$\begin{aligned} \varepsilon_i d\omega_{i0}(X,Y) &= \langle\nabla_Y e_i, SX\rangle - \langle\nabla_X e_i, SY\rangle + \frac{\varepsilon\varepsilon_{n+1}cT_{n+1}}{a}\langle e_i, \eta(X)AY - \eta(Y)AX\rangle \\ &= \sum_k\big(\omega_{ki}(Y)\langle e_k, SX\rangle - \omega_{ki}(X)\langle e_k, SY\rangle\big) + \frac{\varepsilon\varepsilon_{n+1}cT_{n+1}}{a}\langle e_i, \eta(X)AY - \eta(Y)AX\rangle \\ &= -\varepsilon_i\sum_\gamma \omega_{i\gamma}\wedge\omega_{\gamma 0}(X,Y) + \varepsilon_i\omega_{i\,n+1}\wedge\omega_{n+1,0}(X,Y) \\ &\quad - \frac{\varepsilon_i\varepsilon\varepsilon_{n+1}cT_{n+1}}{a}\eta\wedge\omega_{i\,n+1}(X,Y) = -\varepsilon_i\sum_\gamma \omega_{i\gamma}\wedge\omega_{\gamma 0}(X,Y). \end{aligned}$$

Next, we easily see $d\omega_{n+1,0} = -\varepsilon\varepsilon_{n+1}(c/a)\,dT_{n+1}\wedge\eta$. This implies

$$\begin{aligned} d\omega_{n+1,0}(X,Y) &= -\frac{\varepsilon\varepsilon_{n+1}c}{a}dT_{n+1}\wedge\eta(X,Y) = -\frac{\varepsilon\varepsilon_{n+1}c}{a}\Big(X(T_{n+1})\eta(Y) - Y(T_{n+1})\eta(X)\Big) \\ &= \varepsilon_{n+1}\sum_k \varepsilon_k\big(\langle AX, e_k\rangle\langle SY, e_k\rangle - \langle AY, e_k\rangle\langle SX, e_k\rangle\big) \\ &= -\sum_k \omega_{n+1\,k}\wedge\omega_{k0}(X,Y) = -\sum_\gamma \omega_{n+1\,\gamma}\wedge\omega_{\gamma 0}(X,Y). \quad \square \end{aligned}$$

Lemma 6. $d\mathbf{X}_{\alpha\beta} = \varepsilon_\alpha\varepsilon_\beta d\mathbf{X}_{\beta\alpha}$, $d\mathbf{X}_{u0} = 0$,

$$\begin{aligned} d\mathbf{X}_{ij} &= \frac{a''a - 2(a')^2}{a^2}\eta\wedge(T_j\omega_i - \varepsilon_i\varepsilon_j T_i\omega_j) + \frac{\varepsilon a'}{a}\Big(T_j d\omega_i - \varepsilon_i\varepsilon_j T_i d\omega_j\Big) \\ &\quad + 2\varepsilon\varepsilon_j\left(\frac{a'}{a}\right)^2\omega_j\wedge\omega_i + \frac{\varepsilon a'}{a}\sum_u T_u\Big(\omega_{uj}\wedge\omega_i - \varepsilon_i\varepsilon_j\omega_{ui}\wedge\omega_j\Big), \\ d\mathbf{X}_{i\,n+1} &= \frac{a''a - 2(a')^2}{a^2}T_{n+1}\eta\wedge\omega_i + \frac{\varepsilon a'}{a}\Big(\sum_k T_k\omega_{k\,n+1}\wedge\omega_i + T_{n+1}d\omega_i\Big). \end{aligned}$$

Proof. Since $\mathbf{X}_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta\mathbf{X}_{\beta\alpha}$, we trivially have $d\mathbf{X}_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta d\mathbf{X}_{\beta\alpha}$. Next,

$$\begin{aligned} d\mathbf{X}_{\alpha\beta} &= d\Big(\frac{\varepsilon a'}{a}\big(T_\beta\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta T_\alpha\omega_\beta\big)\Big) = \sum_k e_k\Big(\frac{\varepsilon a'}{a}\Big)\omega_k\wedge(T_\beta\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta T_\alpha\omega_\beta) \\ &\quad + \frac{\varepsilon a'}{a}\Big(dT_\beta\wedge\omega_\alpha + T_\beta d\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta dT_\alpha\wedge\omega_\beta - \varepsilon_\alpha\varepsilon_\beta T_\alpha d\omega_\beta\Big) \\ &= \frac{a''a - (a')^2}{a^2}\eta\wedge(T_\beta\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta T_\alpha\omega_\beta) + \frac{\varepsilon a'}{a}\Big(dT_\beta\wedge\omega_\alpha + T_\beta d\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta dT_\alpha\wedge\omega_\beta - \varepsilon_\alpha\varepsilon_\beta T_\alpha d\omega_\beta\Big). \end{aligned}$$

For $\beta = 0$ and any $u > 0$, since $T_0 = 0$ and $\omega_0 = 0$, then $d\mathbf{X}_{u0} = 0$. For $\beta = n+1$ and $i < n+1$, since $\omega_{n+1} = 0$, we have

$$d\mathbf{X}_{i\,n+1} = \frac{a''a - (a')^2}{a^2} T_{n+1}\eta \wedge \omega_i + \frac{\varepsilon a'}{a}\Big(dT_{n+1} \wedge \omega_i + T_{n+1} d\omega_i\Big).$$

By condition (C), and the fact $T = \sum_k \varepsilon_k T_k e_k$, we see $dT_{n+1}(X) = -\langle AX, T\rangle - \varepsilon \frac{a'}{a} T_{n+1}\eta(X) = -\sum_k \varepsilon_k T_k \langle e_k, AX\rangle - \varepsilon T_{n+1}\frac{a'}{a}\eta(X)$, and then $dT_{n+1}(X) = \sum_k T_k \omega_{k\,n+1}(X) - \varepsilon T_{n+1}\frac{a'}{a}\eta(X)$. With this,

$$\begin{aligned} d\mathbf{X}_{i\,n+1} &= \frac{a''a - (a')^2}{a^2} T_{n+1}\eta \wedge \omega_i + \frac{\varepsilon a'}{a}\Big(\sum_k T_k \omega_{k\,n+1} \wedge \omega_i - \varepsilon \frac{a'}{a} T_{n+1}\eta \wedge \omega_i + T_{n+1} d\omega_i\Big) \\ &= \frac{a''a - 2(a')^2}{a^2} T_{n+1}\eta \wedge \omega_i + \frac{\varepsilon a'}{a}\Big(\sum_k T_k \omega_{k\,n+1} \wedge \omega_i + T_{n+1} d\omega_i\Big). \end{aligned}$$

Next, for $\alpha = i$ and $\beta = j$, we have $d\mathbf{X}_{ij} = \frac{a''a-(a')^2}{a^2}\eta \wedge (T_j\omega_i - \varepsilon_i\varepsilon_j T_i\omega_j) + \frac{\varepsilon a'}{a}\Big(dT_j \wedge \omega_i + T_j d\omega_i - \varepsilon_i\varepsilon_j dT_i \wedge \omega_j - \varepsilon_i\varepsilon_j T_i d\omega_j\Big)$. We need the following computation $dT_k = \sum_l e_l(\langle T, e_k\rangle)\omega_l = \sum_l \big(\langle \nabla_{e_l} T, e_k\rangle + \langle T, \nabla_{e_l} e_k\rangle\big)\omega_l = \sum_l \Big(\langle \frac{a'}{a}(e_l - \varepsilon T_l T) + \varepsilon_{n+1} T_{n+1} A e_l, e_k\rangle + \sum_j \omega_{jk}(e_l) T_j\Big)\omega_l$, which implies

$$dT_k = \frac{a'}{a}\varepsilon_k \omega_k - \varepsilon \frac{a'}{a} T_k \eta + \sum_u T_u \omega_{uk}. \tag{12}$$

In this way, a straight forward computation yields

$$\begin{aligned} d\mathbf{X}_{ij} &= \frac{a''a - 2(a')^2}{a^2}\eta \wedge (T_j\omega_i - \varepsilon_i\varepsilon_j T_i\omega_j) + \frac{\varepsilon a'}{a}\Big(T_j d\omega_i - \varepsilon_i\varepsilon_j T_i d\omega_j\Big) \\ &\quad + 2\frac{\varepsilon\varepsilon_j a'}{a}\omega_j \wedge \omega_i + \frac{\varepsilon a'}{a}\left(\sum_u T_u(\omega_{uj} \wedge \omega_i - \varepsilon_i\varepsilon_j \omega_{ui} \wedge \omega_j)\right). \quad \square \end{aligned}$$

Lemma 7.

$$(\mathbf{X} \wedge \mathbf{X})_{\alpha\beta} = \left(\frac{a'}{a}\right)^2 \Big(\big(T_\beta\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta T_\alpha\omega_\beta\big) \wedge \eta - \varepsilon\varepsilon_\beta\omega_\alpha \wedge \omega_\beta\Big).$$

Proof. We recall condition (B). Also, given $X \in TM$, we have $\sum_\gamma T_\gamma\omega_\gamma(X) = \sum_k T_k\omega_k(X) = \eta(X)$. Then,

$$\begin{aligned} (\mathbf{X} \wedge \mathbf{X})_{\alpha\beta} &= \sum_\gamma \Big(\frac{\varepsilon a'}{a}\big(T_\gamma\omega_\alpha - \varepsilon_\alpha\varepsilon_\gamma T_\alpha\omega_\gamma\big) \wedge \frac{\varepsilon a'}{a}\big(T_\beta\omega_\gamma - \varepsilon_\gamma\varepsilon_\beta T_\gamma\omega_\beta\big)\Big) \\ &= \left(\frac{a'}{a}\right)^2 \sum_\gamma \Big(T_\gamma T_\beta\omega_\alpha \wedge \omega_\gamma - \varepsilon_\beta\varepsilon_\gamma T_\gamma^2\omega_\alpha \wedge \omega_\beta - \varepsilon_\alpha\varepsilon_\gamma T_\alpha T_\beta\omega_\gamma \wedge \omega_\gamma \\ &\quad + \varepsilon_\gamma\varepsilon_\gamma\varepsilon_\alpha\varepsilon_\beta T_\alpha T_\gamma\omega_\gamma \wedge \omega_\beta\Big) \left(\frac{a'}{a}\right)^2 \Big(\big(T_\beta\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta T_\alpha\omega_\beta\big) \wedge \eta - \varepsilon\varepsilon_\beta\omega_\alpha \wedge \omega_\beta\Big). \quad \square \end{aligned}$$

Now, we put $\Phi = \Omega \wedge \mathbf{X} + \mathbf{X} \wedge \Omega$.

Lemma 8. $\Phi_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta\Phi_{\beta\alpha}$, $\Phi_{u0} = 0$,

$$\Phi_{i\,n+1} = \frac{\varepsilon a'}{a}\big(-T_{n+1} d\omega_i + \sum_k T_k\omega_i \wedge \omega_{k\,n+1}\big), \tag{13}$$

$$\Phi_{ij} = \frac{\varepsilon a'}{a}\Big(\varepsilon_i\varepsilon_j T_i d\omega_j - T_j d\omega_i + \sum_u T_u\big(\omega_i \wedge \omega_{uj} - \varepsilon_j\varepsilon_u\omega_{iu} \wedge \omega_j\big)\Big). \tag{14}$$

Proof. By construction, $\mathbf{X}_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta\mathbf{X}_{\beta\alpha}$. This shows $\Phi_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta\Phi_{\beta\alpha}$. In general, we get

$$\begin{aligned} \Phi_{\alpha\beta} &= \frac{\varepsilon a'}{a}\sum_\gamma \omega_{\alpha\gamma} \wedge \big(T_\beta\omega_\gamma - \varepsilon_\beta\varepsilon_\gamma T_\gamma\omega_\beta\big) + \sum_\gamma \frac{\varepsilon a'}{a}\big(T_\gamma\omega_\alpha - \varepsilon_\gamma\varepsilon_\alpha T_\alpha\omega_\gamma\big) \wedge \omega_{\gamma\beta} \\ &= \frac{\varepsilon a'}{a}\sum_\gamma \big(T_\beta\omega_{\alpha\gamma} \wedge \omega_\gamma - \varepsilon_\beta\varepsilon_\gamma T_\gamma\omega_{\alpha\gamma} \wedge \omega_\beta + T_\gamma\omega_\alpha \wedge \omega_{\gamma\beta} - \varepsilon_\alpha\varepsilon_\gamma T_\alpha\omega_\gamma \wedge \omega_{\gamma\beta}\big). \end{aligned}$$

By Lemma 5,

$$\Phi_{\alpha\beta} = \frac{\varepsilon a'}{a}\Big(\varepsilon_\alpha\varepsilon_\beta T_\alpha d\omega_\beta - T_\beta d\omega_\alpha + \sum_\gamma \big(T_\gamma \omega_\alpha \wedge \omega_{\gamma\beta} - \varepsilon_\beta\varepsilon_\gamma T_\gamma \omega_{\alpha\gamma} \wedge \omega_\beta\big)\Big).$$

For the case $\alpha = i, \beta = n+1$, the result is immediate due to the fact $\omega_{n+1} = 0$. For the case $\alpha = i, \beta = 0$, since $\omega_0 = 0$ and $T_0 = 0$, we begin by writing down $\Phi_{i0} = \frac{\varepsilon a'}{a}\sum_\gamma T_\gamma \omega_i \wedge \omega_{\gamma 0}$. However, by Condition (C), $\sum_\gamma T_\gamma \omega_{\gamma 0}(X) = \sum_i T_i\omega_{i0}(X) + T_{n+1}\omega_{n+1,0}(X) = \sum_i \frac{T_i}{ac}\big(\varepsilon_i\langle e_i, X\rangle - \varepsilon\varepsilon_i T_i\eta(X)\big) - \varepsilon\varepsilon_{n+1}\frac{T_{n+1}^2}{ac}\eta(X) = \big(\sum_i \varepsilon_i\langle e_i, T\rangle\langle e_i, X\rangle - \varepsilon\big(\sum_i \varepsilon_i T_i^2\big)\eta(X) - \varepsilon\varepsilon_{n+1}T_{n+1}^2\eta(X)\big)/(ac)$, and hence

$$\sum_\gamma T_\gamma \omega_{\gamma 0}(X) = 0. \tag{15}$$

The case $\alpha = n+1$ and $\beta = 0$ trivially vanishes due to $\omega_{n+1} = \omega_0 = 0$. □

Lemma 9. $d\Upsilon + \Upsilon \wedge \Upsilon = 0$.

Proof. This is equivalent to prove $d\mathbf{X} - d\Omega - \Omega\wedge\Omega - X\wedge X + X\wedge\Omega + \Omega\wedge X = 0$. The case $\alpha = u, \beta = 0$ is trivial. For $\alpha = i, \beta = n+1$, we compute

$$\begin{aligned}
&(d\mathbf{X} - d\Omega - \Omega\wedge\Omega - X\wedge X + X\wedge\Omega + \Omega\wedge X)_{i\,n+1}\\
&= \frac{a''a - 2(a')^2}{a^2}T_{n+1}\eta\wedge\omega_i + \frac{\varepsilon a'}{a}\Big(\sum_k T_k\omega_{k\,n+1}\wedge\omega_i + T_{n+1}d\omega_i\Big)\\
&\quad - T_{n+1}\left(\frac{a''}{a} - \frac{(a')^2}{a^2}\right)\eta\wedge\omega_i + \frac{\varepsilon a'}{a}\Big(-T_{n+1}d\omega_i + \sum_k T_k\omega_i\wedge\omega_{k\,n+1}\Big)\\
&\quad - \left(\frac{a'}{a}\right)^2\Big(\big(T_{n+1}\omega_i - \varepsilon_i\varepsilon_{n+1}T_i\omega_{n+1}\big)\wedge\eta - \varepsilon\varepsilon_{n+1}\omega_i\wedge\omega_{n+1}\Big)\\
&= \frac{a''a - 2(a')^2}{a^2}T_{n+1}\eta\wedge\omega_i - T_{n+1}\left(\frac{a''}{a} - \frac{(a')^2}{a^2}\right)\eta\wedge\omega_i - \left(\frac{a'}{a}\right)^2\big(T_{n+1}\omega_i\big)\wedge\eta\\
&= 0.
\end{aligned}$$

A similar computation holds for the case $\alpha = i$ and $\beta = j$, but we have to use the previous Lemmata. Thus, we also obtain $(d\mathbf{X} - d\Omega - \Omega\wedge\Omega - X\wedge X + X\wedge\Omega + \Omega\wedge X)_{ij} = 0$. □

We recall $\mathbb{E}^{n+2}$ as in Section 2, with its standard flat pseudo-Riemannian metric. Now, it is clear that the map $s : \mathbf{S} \to \mathbb{S}(\mathbb{E}^{n+2}) = \{X \in \mathbb{E}^{n+2} | \langle X, X\rangle\} = \varepsilon_{n+1}\}, Z \mapsto (Z_{n+1,0}, \ldots, Z_{n+1\,n+1})^t$, is a submersion. Given a point $x \in M$, we define the set

$$\mathcal{Z}(x) = \{Z \in \mathbf{S} | Z_{n+1\beta} = T_\beta(x),\ \beta = 0, \ldots, n+1\}.$$

Now, we prove the following

Lemma 10. *Let $(M, \langle,\rangle)$ be a semi-Riemannian manifold satisfying the structure conditions. For each $x_0 \in M$ and $B_0 \in \mathcal{Z}(x_0)$, there exists a neighborhood $\mathcal{U}$ of x_0 in M and a unique map $B : \mathcal{U} \to \mathbf{S}$, such that $B^{-1}dB = \Omega - \mathbf{X}$, $B_0 = B(x_0)$, and $B(x) \in \mathcal{Z}(x)$ for all $x \in \mathcal{U}$. Also, in case $c = 0$, we can assume $B_{0\alpha} = \delta_{0\alpha}$.*

Proof. Given $\mathcal{U}$ be an open neighborhood of $x_0 \in M$, we define the set

$$\mathcal{F} = \{(x, Z) \in \mathcal{U}\times\mathbf{S} | Z \in \mathcal{Z}(x)\}.$$

Since the map s is a submersion, $\mathcal{F}$ is a submanifold of $M\times\mathbf{S}$ with $\dim\mathcal{F} = n + (n+1)(n+2)/2 - (n+1) = n(n+1)/2 + n$. Moreover, given $(x, Z) \in \mathcal{F}$,

$$T_{(x,Z)}\mathcal{F} = \{(U, V) \in T_x\mathcal{U}\oplus T_Z\mathbf{S} | V_{n+1\beta} = (dT_\beta)_x(U),\ \beta = 0, \ldots, n+1\}.$$

We recall the definition of $\mathfrak{s}$ and $\mathfrak{s}(\Lambda^1(\mathcal{F}))$ in Section 4. Then, we introduce the $\mathfrak{s}$-valued 1-form $\Theta = \Upsilon - Z^{-1}dZ = \Omega - \mathbf{X} - Z^{-1}dZ$, defined on $\mathcal{F}$. In other words, given $(U, V) \in T_{(x,Z)}\mathcal{F}$, we have $\Theta_{(x,Z)}(U, V) = \Omega_x(U) - \mathbf{X}_x(U) - Z^{-1}V$. Needless to say, $\Theta \in \mathfrak{s}(\Lambda^1(\mathcal{F}))$.

Next, we consider on $\mathcal{F}$ the distribution $\mathfrak{D}(x, Z) = \ker\Theta_{(x,Z)}$, and we would like to see that $\dim\mathfrak{D} = n$. To do so, we choose any point $(x, Z) \in \mathcal{F}$. Since $\mathbf{S}$ is a Lie group, the way to move from $\mathfrak{s}$ to the tangent space at a given point $Z \in \mathbf{S}$ is by using the left translation L_Z, which satisfies $(L_Z)_* = L_Z$. However, we will not use this notation to make the computations a bit simpler. Now, we consider the associated space $\mathcal{H} = \{H \in \mathfrak{s} | (ZH)_{n+1\beta} = 0,\ \beta = 0, \ldots, n+1\}$.

Next, it is clear that $H \in \mathcal{H}$ if and only if $H \in \ker(ds)_{l_{n+2}}$. But the map s is a submersion, hence $\dim(\ker(ds)_{l_{n+2}}) = \dim \mathcal{H} = (n+1)n/2$. We notice that $(Z\Theta)_{n+1\beta} = (Z\Omega)_{n+1\beta} - (Z\mathbf{X})_{n+1\beta} - (dZ)_{n+1\beta} = (Z\Omega)_{n+1\beta} - (Z\mathbf{X})_{n+1\beta} - dT_\beta$. Consequently, by Eq. (12) and $T_0 = 0$, we get

$$\begin{aligned}(Z\Theta)_{n+1k} &= \sum_\gamma Z_{n+1\gamma}\omega_{\gamma k} - \sum_\gamma Z_{n+1\gamma}\mathbf{X}_{\gamma k} - \frac{a'}{a}\varepsilon_k\omega_k + \varepsilon\frac{a'}{a}T_k\eta - \sum_u T_u\omega_{uk}\\ &= \sum_\gamma T_\gamma\omega_{\gamma k} - \sum_\gamma T_\gamma\frac{\varepsilon a'}{a}\left(T_k\omega_\gamma - \varepsilon_\gamma\varepsilon_k T_\gamma\omega_k\right) - \frac{a'}{a}\varepsilon_k\omega_k + \varepsilon\frac{a'}{a}T_k\eta - \sum_u T_u\omega_{uk}\\ &= -\frac{\varepsilon a'}{a}\left(T_k\eta - \sum_\gamma \varepsilon_\gamma\varepsilon_k T_\gamma T_\gamma\omega_k\right) - \frac{a'}{a}\varepsilon_k\omega_k + \varepsilon\frac{a'}{a}T_k\eta\\ &= \frac{\varepsilon a'}{a}\sum_\gamma \varepsilon_\gamma\varepsilon_k T_\gamma T_\gamma\omega_k - \frac{a'}{a}\varepsilon_k\omega_k = 0,\end{aligned}$$

since $\varepsilon = \langle T, T\rangle + \varepsilon_{n+1}T_{n+1}^2$. Similarly,

$$\begin{aligned}(Z\Theta)_{n+1\,n+1} &= \sum_\gamma T_\gamma\omega_{\gamma\,n+1} - \sum_\gamma T_\gamma\frac{\varepsilon a'}{a}\left(T_{n+1}\omega_\gamma - \varepsilon_\gamma\varepsilon_{n+1}T_\gamma\omega_{n+1}\right) - dT_{n+1}\\ &= \sum_\gamma T_\gamma\omega_{\gamma\,n+1} - \frac{\varepsilon a'}{a}T_{n+1}\eta - \sum_k T_k\omega_{k\,n+1} + \varepsilon T_{n+1}\frac{a'}{a}\eta = 0,\end{aligned}$$

and with Eq. (15), it is clear $(Z\Theta)_{n+1,0} = \sum_\gamma T_\gamma\omega_{\gamma 0} - \sum_\gamma \varepsilon T_\gamma a'\left(T_0\omega_\gamma - \varepsilon_\gamma\varepsilon_0 T_\gamma\omega_0\right)/a - dT_0 = 0$. Hence, $\mathrm{Im}(\Theta) \subset \mathcal{H} = \ker(ds)_{l_{n+2}}$. Now, given the space $\{(0, ZH)|H \in \mathcal{H}\} \subset T_{(x,Z)}\mathcal{F}$, we have $\Theta_{(x,Z)}(0, ZH) = -Z^{-1}(ZH) = -H$, which means that $\Theta_{(x,Z)}$ is a submersion unto $\mathcal{H}$, and $\mathrm{Im}(\Theta) = \mathcal{H}$. Now, we get $\dim \mathfrak{D}(x, Z) = \dim\ker\Theta_{(x,Z)} = \dim T_{(x,Z)}\mathcal{F} - \dim \mathrm{Im}\Theta_{(x,Z)} = n$.

Next, we prove that $\mathfrak{D}$ is integrable. On one hand, since $\mathfrak{D} = \ker\Theta$ and $d\Upsilon + \Upsilon\wedge\Upsilon = 0$, we compute $d\Theta = d\Upsilon + Z^{-1}dZ\wedge Z^{-1}dZ = d\Upsilon + (\Upsilon - \Theta)\wedge(\Upsilon - \Theta) = -\Upsilon\wedge\Theta - \Theta\wedge\Upsilon + \Theta\wedge\Theta$. Therefore, given $U, V \in \mathfrak{D}$, $d\Theta(U, V) = U(\Theta(V)) - V(\Theta(U)) - \Theta([U, V]) = -\Theta([U, V]) = (-\Upsilon\wedge\Theta - \Theta\wedge\Upsilon + \Theta\wedge\Theta)(U, V) = 0$, which implies $[U, V] \in \mathfrak{D}$.

Next, let $\mathcal{L} \subset \mathcal{F}$ be an integral manifold through (x_0, B_0). For each $(0, V) \in \mathfrak{D}_{(x_0,B_0)} = T_{B_0}\mathcal{L}$, we have $\Theta_{(x_0,B_0)}(V) = B_0^{-1}V = 0$, and hence $V = 0$, since $B_0 \in \mathbf{S}$. This implies $\mathfrak{D}_{(x_0,B_0)} \cap [\{0\} \times T_{B_0}\mathbf{S}] = \{0\}$. In particular, by shrinking $\mathcal{U}$ if necessary, $\mathcal{L}$ is the graph of a unique map $B : \mathcal{U} \to \mathbf{S}$. Also, since $\mathcal{L} \subset \mathcal{F}$, then for each $x \in \mathcal{U}$, $B(x) \in \mathcal{Z}(x)$. Finally since $\Theta \equiv 0$ on $\mathcal{L}$, B satisfies by definition $B^{-1}dB = \Omega - \mathbf{X}$.

As a final remark for $c = 0$, since $\omega_{0\alpha} = \mathbf{X}_{0\alpha} = 0$, we have $(B^{-1}dB)_{0\alpha} = 0$. This means that we can assume $B_{0\alpha} = \delta_{0\alpha}$. □

Lemma 11. *For $c = 0$, given a point $x \in \mathcal{U}$ and the matrix*

$$C = Diag\left(1, \frac{\varepsilon_0\varepsilon_1}{a\circ\pi}, \ldots, \frac{\varepsilon_0\varepsilon_n}{a\circ\pi}, \varepsilon_{n+1}\right),$$

there exists a map $\chi : \mathcal{U} \to \varepsilon I \times_a \mathbb{E}^{n+1}$, such that $d\chi = CB\varpi$, $\chi_0 = 0$, $\chi_{n+1} = \pi$.

Proof. A simple computation shows $dC = \frac{\varepsilon\varepsilon_0 a'}{a^2}Diag(0, \varepsilon_1, \ldots, \varepsilon_n, 0)\eta$. Now, given a point $(x, (t, p)) \in \mathcal{U} \times \left(I \times \mathbb{E}^n\right)$, we define the map

$$\begin{aligned}&\Lambda_{(x,(p,t))} : T_x\mathcal{U} \oplus T_{(t,p)}I \times \mathbb{E}^n \to \mathbb{R}^{n+2},\\ &\Lambda_{(x,(p,t))}(Y, W) = -d\tilde{\Xi}_{(p,t)}(W) + C_xB_x\varpi_x(Y).\end{aligned}$$

In other words, we have the 1-form $\Lambda = -d\tilde{\Xi} + CB\varpi \in \Lambda^1(\mathcal{U} \times (I \times \mathbb{E}^n), \mathbb{R}^{n+2})$. By using $dB = B\Omega - B\mathbf{X}$ and Lemma 5,

$$d\Lambda = dC\wedge B\varpi + CdB\wedge\varpi + CBd\varpi = dC\wedge B\varpi - CB\mathbf{X}\wedge\varpi.$$

Given $i \in \{1, \ldots, n\}$, since $C_{i0} = C_{in+1} = 0$, we compute

$$\begin{aligned}d\Lambda_i &= (dC\wedge B\varpi - CBX\wedge\varpi)_i = \sum_{\alpha\beta} dC_{i\alpha}\wedge B_{\alpha\beta}\omega_\beta - \sum_{\alpha\beta\gamma} C_{i\gamma}B_{\gamma\alpha}\mathbf{X}_{\alpha\beta}\wedge\omega_\beta\\ &= \sum_{j\beta}\frac{-\varepsilon\varepsilon_0\varepsilon_i a'}{a^2}\delta_{ij}\eta\wedge B_{j\beta}\omega_\beta - \sum_{\alpha\beta j}\frac{\varepsilon_0\varepsilon_i}{a}\delta_{ij}B_{j\alpha}\mathbf{X}_{\alpha\beta}\wedge\omega_\beta\\ &= -\frac{\varepsilon\varepsilon_0\varepsilon_i a'}{a^2}\sum_\beta \eta\wedge B_{i\beta}\omega_\beta - \frac{\varepsilon_0\varepsilon_i}{a}\sum_{\alpha\beta} B_{i\alpha}\frac{\varepsilon a'}{a}\left(T_\beta\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta T_\alpha\omega_\beta\right)\wedge\omega_\beta\end{aligned}$$

$$= -\frac{\varepsilon\varepsilon_0\varepsilon_i a'}{a^2}\sum_\beta \eta\wedge B_{i\beta}\omega_\beta - \frac{\varepsilon_0\varepsilon_i\varepsilon a'}{a^2}\sum_{\alpha\beta} B_{i\alpha}T_\beta\omega_\alpha\wedge\omega_\beta$$
$$= -\frac{\varepsilon\varepsilon_0\varepsilon_i a'}{a^2}\Big(\sum_\beta \eta\wedge B_{i\beta}\omega_\beta + \sum_\alpha B_{i\alpha}\omega_\alpha\wedge\eta\Big) = 0.$$

Similar but simpler computations show $d\Lambda_0 = d\Lambda_{n+1} = 0$. To do so, we just need to copy the previous computations, but recalling that $B_{0\alpha} = \delta_{0\alpha}$, $B_{n+1\,\alpha} = T_\alpha$ and $\omega_0 = \omega_{n+1} = 0$. Next, we see that the distribution $\ker\Lambda$ is integrable. Indeed, given $Y, W \in \ker\Lambda$, $0 = d\Lambda(Y,W) = Y(\Lambda(W)) - W(\Lambda(Y)) - \Lambda([Y,W]) = -\Lambda([Y,W])$. From here, we can repeat the final part of the proof of Lemma 10. For the sake of simplicity, we also denote by $\mathcal{U}$ the final neighborhood of $x\in M$. □

Lemma 12. *For $c=\pm1$, given a point $x\in\mathcal{U}$ and the matrix*

$$C = \text{Diagonal}\Big(c\varepsilon_0/(a\circ\pi),\dots,c\varepsilon_n/(a\circ\pi),\varepsilon_{n+1}\Big),$$

there exists a map $\chi:\mathcal{U}\to\varepsilon I\times_a\mathbb{M}^n_k(c)$, such that $d\chi = CB\varpi$, $\chi_{n+1}=\pi$.

Proof. Define now the map $\chi:\mathcal{U}\to\mathbb{R}^{n+2}$ by $\chi_0 = \varepsilon_0 B_{00}$, $\chi_i = \varepsilon_i B_{i0}$, $\chi_{n+1}=\pi$. Notice that, since $B(x)\in\mathcal{Z}(x)\subset\mathbf{S}$, then $B_{n+1,0} = T_0 = 0$, which implies $\varepsilon_0\chi_0^2 + \sum_{i=1}^n \varepsilon_i\chi_i^2 = \sum_{\alpha=0}^n \varepsilon_\alpha B_{\alpha 0}^2 = \varepsilon_0 = c$, thus obtaining that $(\chi_0,\dots,\chi_n)$ lies in $\mathbb{M}^n_k(c)$, which means $\text{Im}(\chi)\subset\varepsilon I\times_a\mathbb{M}^n_k(c)$. Now, we have by definition $dB = B\Omega - B\mathbf{X}$. Hence

$$\begin{aligned}
d\chi_i(e_k) &= \varepsilon_i dB_{i0}(e_k) = \sum_{\alpha=0}^{n+1}\varepsilon_i(B_{i\alpha}\omega_{\alpha 0}(e_k) - B_{i\alpha}\mathbf{X}_{\alpha 0}(e_k))\\
&= \varepsilon_i\sum_{\alpha=1}^{n+1}B_{i\alpha}\omega_{\alpha 0}(e_k) - \varepsilon_i\sum_{\alpha=1}^{n+1}B_{i\alpha}\Big(\frac{\varepsilon a'}{a}(T_0\omega_\alpha(e_k) - \varepsilon_0\varepsilon_\alpha T_\alpha\omega_0(e_k))\Big)\\
&= \varepsilon_i\sum_{j=1}^n B_{ij}\frac{c}{a}(\varepsilon_j\langle e_j,e_k\rangle - \varepsilon\varepsilon_j T_j\eta(e_k)) - \varepsilon_i B_{in+1}\frac{\varepsilon\varepsilon_{n+1}c}{a}T_{n+1}\eta(e_k)\\
&= \frac{\varepsilon_i}{ac}\sum_{j=1}^n B_{ij}\varepsilon_j\langle e_j,e_k\rangle - \frac{\varepsilon\varepsilon_i c\eta(e_k)}{a}\sum_{j=1}^n B_{ij}\varepsilon_j T_j - \frac{\varepsilon\varepsilon_i c}{a}B_{in+1}\varepsilon_{n+1}\eta(e_k)\\
&= \frac{\varepsilon_i c}{a}B_{ik} - \frac{\varepsilon\varepsilon_i c\eta(e_k)}{a}\sum_{\alpha=0}^{n+1}B_{i\alpha}\varepsilon_\alpha B_{n+1\,\alpha} = \frac{\varepsilon_i}{a}B_{ik} - \frac{\varepsilon\varepsilon_i\eta(e_k)c}{a}(B^tGB)_{in+1}\\
&= \frac{\varepsilon_i c}{a}B_{ik}.
\end{aligned}$$

A similar computation yields $d\chi_0(e_k) = \varepsilon_0 cB_{0k}/a$ and $d\chi_{n+1}(e_k) = \varepsilon\eta(e_k) = \varepsilon T_k = \varepsilon B_{n+1k}$. Hence, we have that $d\chi = CB\varpi$. □

Now, we can finish the proof of Theorem 1. From Lemmatta 11 and 12, we see that $d\chi(e_k)_{\underline{\alpha}} = (CB)_{\alpha k}$, meaning that in the frame $\frac{\partial}{\partial x_\alpha}$ the vector $d\chi(e_k)$ is given by the k-th column of the matrix CB and in the frame $\bar{E}_\alpha$ by the k-th column of the matrix B. In other words, $d\chi(e_k) = \sum_\alpha \varepsilon_\alpha B_{\alpha k}\bar{E}_\alpha$. Since B and C are invertible, $d\chi$ has maximal rank n and χ is an immersion. Moreover, for any i,j, since $B\in\mathbf{S}$, we have

$$\langle d\chi(e_i), d\chi(e_j)\rangle = \Big\langle\sum_\alpha\varepsilon_\alpha B_{\alpha i}\bar{E}_\alpha, \sum_\gamma\varepsilon_\gamma B_{\gamma j}\bar{E}_\gamma\Big\rangle = \sum_\gamma\varepsilon_\alpha B_{\alpha i}B_{\alpha j} = \varepsilon_i\delta_{ij}.$$

Hence, χ is isometric. Next, we would like to compute the shape operator of the immersion. Recall $\bar{E}_{n+1} = \partial_t$. We show that the shape operator of the immersion is exactly what we need, namely $d\chi\circ A\circ d\chi^{-1}$. Indeed,

$$\begin{aligned}
\langle\bar{\nabla}_{d\chi_{(e_i)}}d\chi_{(e_j)}, e_{n+1}\rangle &= \Big\langle\sum_{\alpha\beta}\bar{\nabla}_{\varepsilon_\alpha B_{\alpha i}\bar{E}_\alpha}\varepsilon_\beta B_{\beta j}\bar{E}_\beta, e_{n+1}\Big\rangle\\
&= \sum_\beta\varepsilon_\beta dB_{\beta j}(e_i)B_{\beta n+1} + \sum_{\alpha\beta\gamma}\varepsilon_\alpha\varepsilon_\beta\varepsilon_\gamma B_{\alpha i}B_{\beta j}B_{\gamma\,n+1}\langle\widetilde{\nabla}_{\bar{E}_\alpha}\bar{E}_\beta,\bar{E}_\gamma\rangle\\
&= \sum_\beta\varepsilon_\beta dB_{\beta j}(e_i)B_{\beta n+1} + \sum_{uv\gamma}\Big[\varepsilon_u\varepsilon_v\varepsilon_\gamma B_{ui}B_{vj}B_{\gamma\,n+1}\Big\langle-\frac{\varepsilon_u\delta_{uv}\varepsilon a'}{a}\partial_t,\bar{E}_\gamma\Big\rangle\\
&\quad + \varepsilon_u\varepsilon_{n+1}\varepsilon_\gamma B_{ui}B_{n+1j}B_{\gamma\,n+1}\Big\langle\frac{a'}{a}\bar{E}_u,\bar{E}_\gamma\Big\rangle\Big]
\end{aligned}$$

$$= -\frac{\varepsilon a'}{a} B_{n+1n+1} \sum_u \varepsilon_u B_{ui} B_{uj} + \frac{\varepsilon_{n+1} a'}{a} B_{n+1j} \sum_u \varepsilon_u B_{ui} B_{un+1} + \varepsilon_{n+1} (B^{-1} dB)_{n+1j}(e_i)$$
$$= \varepsilon_{n+1} \left[(B^{-1} dB)_{n+1j}(d\chi_{(e_i)}) - \frac{\varepsilon a}{a'} [\varepsilon_{n+1} \varepsilon_j T_{n+1} \omega_j(e_i) - T_j \omega_{n+1}(e_i)] \right]$$
$$= \varepsilon_{n+1} [(B^{-1} dB)_{n+1j} + X_{n+1j}] = \varepsilon_{n+1} \omega_{n+1j}(e_i) = \langle e_j, A e_i \rangle.$$

Moreover, along χ, we obtain

$$\partial_t = \sum_{i=1}^{n} \varepsilon_i \langle \partial_t, d\chi(e_i) \rangle d\chi(e_i) + \varepsilon_{n+1} \langle \partial_t, e_{n+1} \rangle e_{n+1} = T + \varepsilon_{n+1} T_{n+1} e_{n+1}.$$

Finally, the uniqueness of the local immersion follows from the uniqueness of the map B in Lemma 10. □

This concludes the proof of Theorem 1.

Corollary 1. 1. *If the hypersurface M satisfies $\eta = 0$ everywhere, then M is a slice of $\varepsilon\, I \times_a \mathbb{M}^n_k(c)$.*
2. *If $\eta \neq 0$ everywhere, then M admits a foliation of codimension* 1.

Proof. Item 1 is an immediate consequence of item 5 of Theorem 1. For item 2, by Lemma 4, we know $d\eta = 0$. This implies that $\ker \eta$ is integrable. Indeed, given $X, Y \in \ker \eta$, $0 = d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) = -\eta([X, Y])$, which shows $[X, Y] \in \ker \eta$. In other words, it has to admit a foliation whose leaves are of codimension 1 in M. In fact, T is a normal vector field to the leaves. □

Remark. Under the same assumption on $(M, \langle , \rangle, \nabla, R)$, we can find another equivalent formulation for Theorem 1. In fact, consider again a $\langle , \rangle$-self adjoint (1, 1)-tensor A on M, a nowhere vanishing vector field $T \in \mathfrak{X}(M)$ and its associated 1-form $\eta(X) = \langle X, T \rangle$ for any $X \in TM$. We also assume the existence of smooth functions $\rho, \tilde{\rho}, \bar{\rho}, T_{n+1} : M \to \mathbb{R}$. Let the following conditions be satisfied:

(a) $\rho > 0$, $d\rho = \varepsilon \tilde{\rho}\, \eta$, $d\tilde{\rho} = \varepsilon \bar{\rho}\, \eta$;
(b) $\varepsilon = \langle T, T \rangle + \varepsilon_{n+1} T^2_{n+1}$;
(c) $\nabla_X T = \frac{\tilde{\rho}}{\rho}(X - \varepsilon \eta(X) T) + \varepsilon_{n+1} T_{n+1} AX$, for any $X \in TM$;
(d) $X(T_{n+1}) = -\langle AT, X \rangle - \varepsilon \frac{\tilde{\rho}}{\rho} T_{n+1} \eta(X)$, for any $X \in TM$;
(e) (Codazzi equation). For any $X, Y, Z, W \in TM$,

$$(\nabla_X A)Y - (\nabla_Y A)X = T_{n+1} \frac{\rho\bar{\rho} - \tilde{\rho}^2 + \varepsilon\varepsilon_0}{\rho^2} \big(\eta(Y)X - \eta(X)Y\big).$$

(f) (Gauß equation). For any $X, Y, Z, W \in TM$

$$R(X, Y, Z, W) = \frac{\varepsilon\tilde{\rho}^2 - \varepsilon_0}{\rho^2} \Big(\langle X, Z \rangle \langle Y, W \rangle - \langle Y, Z \rangle \langle X, W \rangle \Big)$$
$$+ \frac{\rho\bar{\rho} - \tilde{\rho}^2 + \varepsilon\varepsilon_0}{\rho^2} \Big(\langle X, Z \rangle \eta(Y)\eta(W) - \langle Y, Z \rangle \eta(X)\eta(W) - \langle X, W \rangle \eta(Y)\eta(Z)$$
$$+ \langle Y, W \rangle \eta(X)\eta(Z) \Big) + \varepsilon_{n+1} \Big(\langle AY, Z \rangle \langle AX, W \rangle - \langle AY, W \rangle \langle AX, Z \rangle \Big).$$

Then Theorem 1 can be reformulated in the following way:

Theorem 2. *Let $(M, \langle , \rangle)$ a simply connected semi-Riemannian manifold satisfying the previous conditions. Then, there exists smooth functions $\pi : M \to I$, I an interval, $a : I \subset \mathbb{R} \to \mathbb{R}^+$, a metric immersion $\chi : (M, \langle , \rangle) \to (\varepsilon\, I \times_a \mathbb{M}^n_k(c), \langle , \rangle_1)$ and a normal unit vector field e_{n+1} along χ such that:*

1. $\varepsilon_{n+1} = \langle e_{n+1}, e_{n+1} \rangle_1$;
2. $\pi_I \circ \chi = \pi$, *where* $\pi_I : \varepsilon\, I \times_a \mathbb{M}^n_k(c) \to I$ *is the projection;*
3. $\rho = a \circ \pi$, $\tilde{\rho} = a' \circ \pi$ *and* $\bar{\rho} = a'' \circ \pi$;
4. *The shape operator associated with e_{n+1} is A;*
5. (e) *is the Codazzi equation and* (f) *is the Gauß equation;*
6. *and along χ, $\partial_t = T + \varepsilon_{n+1} T_{n+1} e_{n+1}$ holds.*

Proof. First of all, from the expression $\nabla_X T = \frac{\tilde{\rho}}{\rho}(X - \varepsilon\eta(X)T) + \varepsilon_{n+1}T_{n+1}AX$, for any $X \in TM$, we are going to check that the 1-form η satisfies $d\eta = 0$.

$$d\eta(e_i, e_j) = \langle e_j, \nabla_{e_i}T\rangle - \langle e_i, \nabla_{e_j}T\rangle = \left\langle e_j, \frac{\tilde{\rho}}{\rho}(e_i - \varepsilon\eta(e_i)T) + \varepsilon_{n+1}T_{n+1}Ae_i\right\rangle$$
$$- \left\langle e_i, \frac{\tilde{\rho}}{\rho}(e_j - \varepsilon\eta(e_j)T) + \varepsilon_{n+1}T_{n+1}Ae_j\right\rangle = \frac{\tilde{\rho}}{\rho}\big(\langle e_j, e_i\rangle - \varepsilon\eta(e_i)\eta(e_j)\big)$$
$$+ \varepsilon_{n+1}T_{n+1}\langle e_j, Ae_i\rangle - \frac{\tilde{\rho}}{\rho}\big(\langle e_i, e_j\rangle - \varepsilon\eta(e_i)\eta(e_j)\big) - \varepsilon_{n+1}T_{n+1}\langle e_i, Ae_j\rangle = 0.$$

Since M is simply connected, we can obtain a new function $\pi : M \to \mathbb{R}$ such that $\eta = \varepsilon d\pi$, which implies that $T = \varepsilon\,\mathrm{grad}(\pi)$. As M is connected, $I = \pi(M)$ is an interval. Moreover, since $T \neq 0$, each $t \in I$ is a regular value of π, which means that each level set $\pi^{-1}(t) \subset M$, $t \in I$, is a hypersurface of M. Choose $t \in I$ and $X \in T\pi^{-1}(t)$. π is constant along its level subsets, so that $d\rho(X) = \varepsilon\tilde{\rho}\,\eta(X) = \tilde{\rho}\,d\pi(X) = 0$. In other words, function ρ is constant along the level sets of π. This allows us to define $a : I \to \mathbb{R}^+$ as follows. Given $t \in I$, there exists $p \in M$ such that $t = \pi(p)$, so that $a(t) := \rho(p)$. Clearly, $\rho = a \circ \pi$. In addition, $d\rho = (a' \circ \pi)d\pi = (a' \circ \pi)\varepsilon\eta = \varepsilon\tilde{\rho}\eta$, and therefore $\tilde{\rho} = a' \circ \pi$. Similarly, $a'' \circ \pi = \bar{\rho}$. Now, we just need to resort to Theorem 1. □

6. An application to horizons in RW 4-spacetimes

In General Relativity, an oriented, spacelike, 2-dimensional surface $\tilde{M}^2$ in a 4-dimensional time-oriented spacetime admits two lightlike normal vector fields, one *inward* and one *outward*. The traces of the associated shape operators are called the inner and outer expansions of $\tilde{M}^2$. Then, the surface $\tilde{M}$ is known as a Marginally Trapped Outer Surface (MOTS) when the outward expansion identically vanish. In fact, the original definition asked the surface to be compact and without boundary, but it seems that these topological conditions are too restrictive, so many authors do not consider them. In addition, the mean curvature vector $\vec{H}$ of $\tilde{M}$ in the ambient space satisfies the more general condition $\langle\vec{H}, \vec{H}\rangle = 0$. Then, we will focus on it.

We put $\varepsilon_4 = \pm 1$ and either $\varepsilon = -1$, $\varepsilon_0 = c = \pm 1$, or $\varepsilon_0 = 1$, $c = 0$. Also, consider smooth functions $a : I \subset \mathbb{R} \to \mathbb{R}^+$, $T_4 : M \to \mathbb{R}$ and $\pi : M \to I$. We construct the vector field $T \in \mathfrak{X}(M)$ by $T = -\mathrm{grad}(\pi)$, with its 1-form $\eta(X) = \langle X, T\rangle$. Also, consider a tensor A of type (1, 1) on M. We assume the above data satisfy the structure conditions of Definition 1. Given the simply connected Riemannian 3-dimensional space $\mathbb{M}^3(c)$ of constant sectional curvature $c \in \{-1, 0, 1\}$, we construct the well-known Robertson–Walker space–time as $(\bar{P}^4 = I \times \mathbb{M}^3(c), \langle , \rangle_1 = -dt^2 + a^2 g_o)$, hence a special case of the warped products considered in this paper. From Theorem 1, we get immediately the following

Corollary 2. *Let $(M, \langle , \rangle)$ a semi-Riemannian manifold of $\dim M = 3$, satisfying the previous conditions. Then, for each point $p \in M$, there exists a neighborhood $\mathcal{U}$ of p on M, a metric immersion $\chi : (\mathcal{U}, \langle , \rangle) \to (\bar{P}^4, \langle , \rangle_1)$ and a normal unit vector field e_4 along χ such that $\varepsilon_4 = \langle e_4, e_4\rangle_1$, A is the shape operator associated with the immersion, T is the projection of ∂_t on TM and $\pi_I \circ \chi = \pi$, where $\pi_I : I \times \mathbb{M}^3(c) \to I$ is the projection.*

In addition, if $T \neq 0$ everywhere, the family $\{\chi(\mathcal{U}) \cap \pi^{-1}\{t\} : t \in \mathbb{R}\}$ provides a foliation of $\chi(\mathcal{U})$ by space-like surfaces.

Next, let L be one of the leafs of $\mathcal{U}$. Let σ be its second fundamental form in $\bar{P}^4$. Clearly, $T^\perp L = \mathrm{Span}\{\mathbf{T}, e_4\}$, where $\mathbf{T} = T/\sqrt{|\langle T, T\rangle|}$. We take $\varepsilon_T = sign(\langle T, T\rangle)$. Since the leaves are spacelike and $\langle e_4, e_4\rangle = \varepsilon_4 = \pm 1$ is constant, $\varepsilon_T = \pm 1$ is constant, with $\varepsilon_4\varepsilon_T = -1$. Then, for any $X, Y \in TL$,

$$\sigma(X, Y) = \varepsilon_T\langle\widetilde{\nabla}_X Y, \mathbf{T}\rangle\mathbf{T} + \varepsilon_4\langle\widetilde{\nabla}_X Y, e_4\rangle e_4 = \frac{-1}{\langle T, T\rangle}\langle Y, \nabla_X T\rangle T + \varepsilon_4\langle Y, AX\rangle e_4.$$

Given a local orthonormal frame $\{u_1, u_2\}$ of L, its mean curvature vector $\vec{H}$ is

$$2\vec{H} = \sum_i \sigma(u_i, u_i) = \sum_i\Big(\frac{-1}{\langle T, T\rangle}\langle u_i, \nabla_{u_i}T\rangle T + \varepsilon_4\langle u_i, Au_i\rangle e_4\Big)$$
$$= \frac{-1}{\langle T, T\rangle}\sum_i\left\langle u_i, \frac{a'}{a}u_i + \varepsilon_4 T_4 Au_i\right\rangle T + \varepsilon_4\Big(\mathrm{trace}(A) - \frac{\langle AT, T\rangle}{\langle T, T\rangle}\Big)e_4$$
$$= \frac{-1}{\langle T, T\rangle}\Big(2\frac{a'}{a} + \varepsilon_4 T_4\Big(\mathrm{trace}(A) - \frac{\langle AT, T\rangle}{\langle T, T\rangle}\Big)\Big)T + \varepsilon_4\Big(\mathrm{trace}(A) - \frac{\langle AT, T\rangle}{\langle T, T\rangle}\Big)e_4.$$

We put $h = \mathrm{trace}(A) - \langle AT, T\rangle/\langle T, T\rangle$. As a result, we obtain $4\langle\vec{H}, \vec{H}\rangle = \big(2a'/a + \varepsilon_4 T_4 h\big)^2/\langle T, T\rangle + \varepsilon_4 h^2$. Then $\vec{H}$ is null if, and only if, $-\varepsilon_4 h^2 = \big(2a'/a + \varepsilon_4 T_4 h\big)^2/\langle T, T\rangle = \big(4(a')^2/a^2 + T_4^2 h^2 + 4\varepsilon_4 a' T_4 h/a\big)/\langle T, T\rangle$, which is equivalent to $h^2 - 4a'T_4 h/a - 4\varepsilon_4(a')^2/a^2 = 0$. Since $\varepsilon_T\varepsilon_4 = -1$, we see that $-\varepsilon_4\langle T, T\rangle > 0$, and so, by solving this equation, we obtain the following

Corollary 3. *The leaves of* $\ker\eta$ *satisfy* $\langle\vec{H},\vec{H}\rangle = 0$ *in* $-I\times_a\mathbb{M}^3(c)$ *if, and only if, the immersion* $\chi : M \to -I\times_a\mathbb{M}^3(c)$ *satisfies the following equality:*

$$\operatorname{trace}(A) - \frac{\langle AT,T\rangle}{\langle T,T\rangle} = \frac{2a'T_4}{a} \pm 2\varepsilon_T\left|\frac{a'}{a}\right|\sqrt{|\langle T,T\rangle|}.$$

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