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$SO(4, 2)$ and derivatively coupled dRGT massive gravity

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ABSTRACT

In this paper we study the possibility of assigning a geometric structure to the Lie groups. It is shown the Poincaré and Weyl groups have geometrical structure of the Riemann–Cartan and Weyl space–time respectively. The geometric approach to these groups can be carried out by considering the most general (non)metricity conditions, or equivalently, tetrad postulates which we show that can be written in terms of the group's gauge fields. By focusing on the conformal group we apply this procedure to show that a nontrivial 3-metrics geometry may be extracted from the group's Maurer–Cartan structure equations. We systematically obtain the general characteristics of this geometry, i.e. its most general nonmetricity conditions, tetrad postulates and its connections. We then deal with the gravitational theory associated to the conformal group's geometry. By proposing an Einstein–Hilbert type action, we conclude that the resulting gravity theory has the form of quintessence where the scalar field derivatively coupled to massive gravity building blocks.
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1. Introduction

It is well known that present-day understanding of structure of space–time is based on the Einstein's theory of general relativity (GR) which some principles such as principle of equivalence, general covariance and isometry group, form its symmetry properties. In its first formulation done by Einstein gravity is linked to the geometrical properties of space–time which in turn, is fully determined by the (dynamical) metric tensor $g_{\mu\nu}$. In addition, a metric compatible torsion-free connection, usually called Levi-Civita connection will appear which may be extracted from metric by the metricity condition $\nabla_\rho g_{\mu\nu} = 0$. Then, equations of motion for metric are the result of variation of Einstein–Hilbert action with respect to the metric. However, in general, these two objects are completely independent from each other and a manifold may be equipped with metric and connection as independent variables. Although the variation of the Einstein–Hilbert action of GR (in Palatini formalism) with respect to connection gives no more information other than the metricity condition mentioned above, this is not the case for the generalized theories of gravity that their actions are more complicated than GR [18].

In addition, the formulation of gravity seems to be somehow different from other known interactions which are described by Yang–Mills gauge theories. The action of all interactions but gravity is constructed by means of the invariance under the act of some internal Lie groups, some example are $U(1), SU(2)\times U(1)$ and $SU(3)$ gauge theories for quantum electrodynamics, electroweak and quantum chromodynamics respectively. These gauge theories have a natural formulation within the fiber bundle framework. GR may also be formulated within this framework as a gauge theory, however, the most important difference between gravitation and other gauge theories is due to the solder form which relates the bundle and base manifold.

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The term of gauge first appeared in the works of Hermann Weyl [21], in which he attempted to unify gravitation and electromagnetism, see also [8]. Weyl formulated his theory in a geometry without metricity condition. Therefore, the first prototype model of gauge theories was indeed a gravitational theory, although after that the other forces have been formulated and known as a gauge theory. Another pioneering attempt to formulate gravity in the framework of a gauge theory has been done by Élie Cartan who tried to generalize the space–time structure by assuming the metric and the non-symmetric affine connection as independent quantities [5]. Since then, many efforts have been made in this regard and their results have been followed and developed by a number of works, for instance [14,16,19]. A notable point about the gravity as a gauge theory is that instead of the Yang–Mills theories of internal symmetries, here one should deal with some external (space–time) gauge symmetry such as Poincaré, Conformal and de-Sitter gauge theories; in which the usual diffeomorphisms and gauge transformations are linked to each other in the corresponding fiber bundle.

In gauge theories, we can distinguish two approaches. On one hand, as mentioned above, a gauge theory is considered as a theory in which the action is invariant under a continuous symmetry Lie group that depends on space–time. Then, gauge symmetries introduce gauge fields to the theory which mediate a force. The gauge fields are massless unless spontaneous symmetry breaking occurs. In this approach, one first chooses a gauge group and then tries to construct a Lagrangian in such a way that it respects to the gauge symmetry. On the other hand, we can begin with a Lagrangian with a symmetry and then try to localize it. As an example, in the local Poincaré gauge theory, the gauge fields are obtained by requiring invariance of the Lagrangian density under the local Poincaré transformations. From the point of this view, it can be shown that there is a geometric interpretation corresponding to the present symmetry Lie group. In the cases of the Poincaré and Poincaré+Scale gauge groups, for instants, the corresponding geometries are Riemann–Cartan and Weyl–Cartan respectively [3].

In this regard, we may still go further and raise a question: Can a generalized theory of gravity be developed beyond the geometric structure derived from the gauge theory? Inspired by some recent works, [2] for instance, in which a bimetric theory of gravity is derived from the $SO(1,5)$ gauge theory and [12] in which the relation between extended Gauss–Bonnet gravities and Weyl geometry has been investigated, it seems that this question has a positive answer, see also [15] for a review on the geometrical foundation of gravity. In this paper, we are going to deal with this question in the framework of the $SO(4,2)$ conformal group. This group has already been used to study of some infrared modifications of gravity [9]. To do this, we begin with a Lie group G for which a distinguished differential 1-form ω, usually called Maurer–Cartan form exists that carries the basic infinitesimal information of the structure of the group G. Maurer–Cartan form is actually a Cartan connection of the Cartan geometry and is a special case of the principal connection. This means that in the gauge gravity models, the Maurer–Cartan form may be identified with the vector potential A. After writing the Maurer–Cartan structure equations corresponding to the selected group we use the building blocks and symmetries of the groups to propose a geometric interpretation. By examining this strategy on the Poincaré and Weyl groups we get the corresponding geometric structure and then we apply the mentioned method on the conformal group. Interestingly, we encountered a three-metric geometry constructed from the group's gauge fields with non-trivial metricity conditions and showed that the gravity theory corresponds to it has an additional scalar field which is coupled to metric via massive gravity building blocks.

The paper is organized as follows. In Section 2, we will provide the general framework of our work and describe how the Cartan connection can be written in terms of the gauge fields and structure constants of a Lie group. The general form of the Maurer–Cartan structure equations is also introduced in this section and for two special cases of Poincaré and Weyl groups the detail are presented. Section 3 is devoted to our main goal, that is, the conformal group and its geometrical properties which are deduced based on the method developed in Section 2. In Section 4, we presented a gravity theory according to the space–time geometry of the conformal group. Finally, we summarize the results in Section 5.

2. Lie groups and their associated geometry

In this section we consider a Lie group G with $\mathbf{g}$ being its Lie algebra. The generators T_a of the group form the closed commutation relations $[T_a, T_b] = f^c_{ab}T_c$, where $f^c_{ab} = -f^c_{ba}$ are the structure functions. With each generator we associate a dual 1-form θ^a in such a way that $\langle\theta^a, T_b\rangle = \delta^a_b$. While the elements of the set $\{T_a\}$ are linearly independent left-invariant vector fields at each point in G, the elements of the set $\{\theta^a\}$ form the basis for the left-invariant 1-forms. The Maurer–Cartan form $\omega = \theta^a T_a$ is a $\mathbf{g}$-valued 1-form which satisfies the Maurer–Cartan structure equation

$$d\omega + \omega \wedge \omega = 0, \tag{1}$$

which can be written in the following equivalent form

$$d\theta^c + \frac{1}{2} f^c_{ab}\theta^a \wedge \theta^b = 0. \tag{2}$$

We may generalize the connection in the Maurer–Cartan equation by adding the curvature to it, as

$$d\omega + \omega \wedge \omega = \Omega, \tag{3}$$

where Ω is the curvature 2-form. In the following two subsections by considering the Poincaré and Weyl groups, we will review how starting from the Maurer–Cartan equations leads us to the corresponding geometric structure of each of these groups.

2.1. Poincaré group

The Poincaré group consists of two Lie groups: the group of Lorentz transformation with generators M_{ab} and the group of translation with generators P_a. It can be verified that they satisfy the following Lie algebra

$$\begin{cases} [M_{ab}, M_{cd}] = \eta_{ad}M_{bc} + \eta_{bc}M_{ad} + \eta_{ac}M_{db} + \eta_{bd}M_{ca}, \\ [P_a, M_{bc}] = \eta_{ba}P_c - \eta_{ca}P_b, \\ [P_a, P_b] = 0. \end{cases} \tag{4}$$

Now, the Maurer–Cartan 1-form takes the form $\omega = \theta^a P_a + \omega^a{}_b M^b{}_a$, where $\theta^a = \theta_\mu{}^a dx^\mu$ and $\omega^a{}_b = \omega_\mu{}^a{}_b dx^\mu$ are the non-coordinate (tetrad) basis and spin connection 1-forms respectively [23]. The Maurer–Cartan equations then read

$$\begin{cases} T^a = d\theta^a + \omega^a{}_c \wedge \theta^c, \\ \Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \end{cases} \tag{5}$$

in which, in terms of Cartan's terminology, T^a is the torsion 1-form and $\Omega^a{}_b$ denotes the (Riemann) curvature 2-form. By means of the tetrad field one may now construct a metric $g_{\mu\nu}$ as

$$g_{\mu\nu} = \theta_\mu{}^a \theta_\nu{}^b \eta_{ab}. \tag{6}$$

In addition, considering $\omega^a{}_b$ as a spin connection, namely as the non-coordinate based connection corresponding to the Levi-Civita connection $\Gamma^\nu_{\rho\mu}$, leads to

$$\begin{aligned} 0 &= \nabla_\rho g_{\mu\nu} \\ &= D_\rho(\eta_{ab}\theta_\mu{}^a\theta_\nu{}^b) \\ &= (\eta_{ab}\theta_\nu{}^b)D_\rho\theta_\mu{}^a + (\eta_{ab}\theta_\mu{}^a)D_\rho\theta_\nu{}^b \end{aligned} \tag{7}$$

in which by D_ρ we mean the covariant derivative operating on both internal and spacetime indices. One possible solution to the above relation is $D_\rho\theta_\mu{}^a = 0$ which is

$$D_\rho\theta_\mu{}^a = \partial_\rho\theta_\mu{}^a + \omega_\rho{}^a{}_b\theta_\mu{}^b - \Gamma^\nu_{\rho\mu}\theta_\nu{}^a = 0. \tag{8}$$

This relation in the literature is mentioned as the tetrad postulate and is indeed the result of consistency of frames [23]. Although, as it is shown here, it can be derived from metricity condition. It can easily be shown that this argument can be reversed, that is, starting from the tetrad postulate we may get the metricity condition:

$$\nabla_\rho g_{\mu\nu} = 0 \Leftrightarrow D_\rho\theta_\mu{}^a = 0. \tag{9}$$

The next step is to use the Eqs. (6) and (8) for θ^a and $\omega^a{}_b$, in favor of g and Γ and substitute the results into the Maurer–Cartan equations. By doing the calculation, while the first relation yields the torsion tensor, the second one gives us the Riemann curvature tensor. This demonstrates that Poincaré gauge theory has the geometrical structure of the Riemann–Cartan space–time which in the case where torsion vanishes reduces to the usual Riemann space–time of GR.

2.2. Weyl group

In addition to the transformations of the Poincaré group, the Weyl group has an additional transformation called dilatation which is responsible for the resize of a coordinate: $x^a \to e^\lambda x^a$. It can be shown that its generator has the form: $D = x^a\partial_a$ [20]. The commutation relations between the generators of the Weyl group generate the following Lie algebra [4]

$$\begin{cases} [M_{ab}, M_{cd}] = \eta_{ad}M_{bc} + \eta_{bc}M_{ad} + \eta_{ac}M_{db} + \eta_{bd}M_{ca}, \\ [P_a, M_{bc}] = \eta_{ba}P_c - \eta_{ca}P_b, \\ [P_a, D] = P_a, \\ [P_a, P_b] = [M_{ab}, D] = [D, D] = 0. \end{cases} \tag{10}$$

Following the same steps as in the previous subsection, we can define the Maurer–Cartan 1-form as

$$\omega = \theta^a P_a + \omega^a{}_b M^b{}_a + \kappa D, \tag{11}$$

where κ is a gauge field corresponding to the dilatation usually called Weyl vector. So, the Maurer–Cartan structure equations for the Weyl group take the form

$$\begin{cases} T^a = d\theta^a + \omega^a{}_c \wedge \theta^c - \kappa \wedge \theta^a, \\ \Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \\ K = d\kappa, \end{cases} \tag{12}$$

in which the dilatation curvature K is introduced. It is seen that in Weyl group one has three building blocks θ, ω and κ. While as before a metric $g_{\mu\nu}$ may be constructed by using of θs, the field κ leads $\omega^a{}_b$ not to be a usual spin connection. In other words, in Weyl group, $\omega^a{}_b$ cannot be considered as associated connection to the Levi-Civita connection. Therefore, as a generalization for the relation (8), we assume

$$D_\rho \theta_\mu{}^a = \frac{1}{2}\kappa_\rho \theta_\mu{}^a, \tag{13}$$

which in terms of the coordinate language reads

$$\partial_\rho \theta_\mu{}^a + \omega_\rho{}^a{}_b \theta_\mu{}^b - \Gamma^\nu_{\rho\mu}\theta_\nu{}^a = \frac{1}{2}\kappa_\rho \theta_\mu{}^a. \tag{14}$$

With a straightforward calculation one can show that this relation is equivalent to the familiar non-metricity condition in Weyl geometry [22]

$$\nabla_\rho g_{\mu\nu} = \kappa_\rho g_{\mu\nu}, \tag{15}$$

which as the case of the previous subsection may be considered as the starting point from which the tetrad postulate (13) can be obtained, that is

$$\nabla_\rho g_{\mu\nu} = \kappa_\rho g_{\mu\nu} \Leftrightarrow D_\rho \theta_\mu{}^a = \frac{1}{2}\kappa_\rho \theta_\mu{}^a. \tag{16}$$

If torsion vanishes the above relations result

$$\Gamma^\nu_{\rho\mu} = \{g\}^\nu_{\rho\,\mu} + C^\nu_{\rho\mu}, \tag{17}$$

where $\{g\}^\nu_{\rho\,\mu}$ is the Christoffel symbol of the metric g and

$$C^\nu_{\rho\mu} = \frac{-1}{2}\left(\delta^\nu_\rho \kappa_\mu + \delta^\nu_\mu \kappa_\rho - g_{\rho\mu}\kappa^\nu\right). \tag{18}$$

By using Eqs. (6) and (14) for θ^a and $\omega^a{}_b$, these quantities can be expressed in terms of g and Γ. The second relation of the Maurer–Cartan structure equations then gets the curvature tensor of Weyl geometry which means that one can consider the Weyl geometry as associated geometry of Weyl group.

3. Conformal group and its related geometry

In this section we will address the issue of the relation between the Lie groups and their corresponding geometric structure in the framework of the conformal group. The conformal group includes the most general transformations preserving the ratios of the lengths. Aside from the transformations in the Weyl group, this group has an additional transformation which in the literature is called special conformal transformation or co-translation with generator $L_a = -2x_a x^b \partial_b + x^2 \partial_a$ [20]. By taking of all possible commutation relations between the generators of the conformal group we are led to the following Lie algebra [4,20]

$$\begin{cases} [M_{ab}, M_{cd}] = \eta_{ad}M_{bc} + \eta_{bc}M_{ad} + \eta_{ac}M_{db} + \eta_{bd}M_{ca}, \\ [P_a, M_{bc}] = \eta_{ba}P_c - \eta_{ca}P_b, \\ [P_a, D] = P_a, \\ [D, L_a] = L_a, \\ [L_a, M_{bc}] = \eta_{ba}L_c - \eta_{ca}L_b, \\ [P_a, L_b] = 2\left(-\eta_{ab}D + M_{ab}\right), \\ [P_a, P_b] = [M_{ab}, D] = [D, D] = [L_a, L_b] = 0. \end{cases} \tag{19}$$

By means of the structure functions of the above Lie algebra the Cartan connection can be written as

$$\omega = \theta^a P_a + \sigma^a L_a + \omega^a{}_b M^b{}_a + \kappa D, \tag{20}$$

where the co-solder form σ^a, are the gauge fields of the co-translations. Having all these at hand we arrive at the Maurer–Cartan equations as

$$\begin{cases} T^a = d\theta^a + \omega^a{}_c \wedge \theta^c - \kappa \wedge \theta^a, \\ S^a = d\sigma^a + \omega^a{}_c \wedge \sigma^c + \kappa \wedge \sigma^a, \\ \Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b + \sigma^{[a} \wedge \theta_{b]}, \\ K = d\kappa + \sigma_a \wedge \theta^a, \end{cases} \tag{21}$$

where S^a may be called special conformal curvature. As we did in the previous section, we are now ready to make a metric using the gauge fields. However, it should be noted that with an extra field σ^a, we will face three different possibilities to arrive a metric, that is

- The metric g: as before may be constructed using the gauge fields θ^a as

$$g_{\mu\nu} = \theta_\mu{}^a \theta_\nu{}^b \eta_{ab}. \tag{22}$$

- The metric f: can be constructed using the gauge fields σ^a as

$$f_{\mu\nu} = \sigma_\mu{}^a \sigma_\nu{}^b \eta_{ab}. \tag{23}$$

- The metric X: which may be defined by a symmetric combination of the fields θ and σ, as

$$X_{\mu\nu} = \theta_{(\mu}{}^a \sigma_{\nu)}{}^b \eta_{ab}. \tag{24}$$

Before going any further, let us see how can we make the metric X symmetric and how its inverse may be well-defined. To do these, we use the Deser–Van Nieuwenhuizen condition [7]

$$\theta^\mu{}_a \sigma_{b\mu} = \theta^\mu{}_b \sigma_{a\mu}, \tag{25}$$

$$\theta_\mu{}^a \sigma_{a\nu} = \theta_\nu{}^a \sigma_{a\mu}, \tag{26}$$

from which we get

$$\begin{aligned} X_{\mu\nu} &= \theta_\mu{}^a \sigma_\nu{}^b \eta_{ab} \\ &= \theta_\nu{}^a \sigma_\mu{}^b \eta_{ab} \\ &= X_{\nu\mu}. \end{aligned} \tag{27}$$

Also, at the first glance, definition of the inverse of the metric X does not seem to be straightforward. Namely, if one assumes

$$\left(X^{-1}\right)^{\mu\nu} = \theta^{(\mu}_a \sigma^{\nu)}_b \eta^{ab}, \tag{28}$$

then

$$\left(X^{-1}\right)^{\mu\beta} X_{\mu\nu} = \frac{1}{2}\delta^\beta_\nu + \frac{1}{4}\left(\theta_\mu{}^a \sigma_{a\nu} \theta^\beta_c \sigma^{\mu c} + \theta_\nu{}^a \sigma_{a\mu} \theta^\mu_c \sigma^{\beta c}\right). \tag{29}$$

However, by using of the Deser–Van Nieuwenhuizen condition, this relation reduces to $\left(X^{-1}\right)^{\mu\beta} X_{\mu\nu} = \delta^\beta_\nu$. Also, note that the structure of the conformal group does not change under

$$\theta \to \sigma, \qquad \sigma \to \theta, \qquad \kappa \to -\kappa, \tag{30}$$

or equivalently

$$g \to f, \qquad f \to g, \qquad X \to X, \qquad \kappa \to -\kappa. \tag{31}$$

Let us now see if a particular geometric structure can be attributed to the conformal group. To begin, we suggest the following tetrad postulate-like relations

$$\begin{cases} D_\rho \theta_\mu{}^a = \kappa_\rho \left(M \theta_\mu{}^a + N \sigma_\mu{}^a\right), \\ D_\rho \sigma_\mu{}^a = \kappa_\rho \left(A \theta_\mu{}^a + B \sigma_\mu{}^a\right), \end{cases} \tag{32}$$

where M, N, A and B are some coefficients. The requirement that these relations should satisfy the symmetries in Eq. (30) yields $N = -A$ and $B = -M$. So, the most general tetrad postulates which respect the mentioned symmetries are

$$\begin{cases} D_\rho \theta_\mu{}^a = \kappa_\rho \left(M\theta_\mu{}^a - A\sigma_\mu{}^a \right), \\ D_\rho \sigma_\mu{}^a = \kappa_\rho \left(A\theta_\mu{}^a - M\sigma_\mu{}^a \right). \end{cases} \tag{33}$$

In order to obtain the nonmetricity conditions associated to the metrics X, f and g, we do as follows. First, it is not difficult to see that the most general nonmetricity condition for the metric X that also respects the relations (30) and (31) is of the form

$$\nabla_\rho X_{\mu\nu} = A\kappa_\rho \left(g_{\mu\nu} - f_{\mu\nu} \right). \tag{34}$$

On the other hand, by multiplying the first equation of (33) by θ and the second one by σ, one gets

$$\begin{cases} \theta_\nu{}^b D_\rho \theta_\mu{}^a = \kappa_\rho \left(M\theta_\nu{}^b \theta_\mu{}^a - A\theta_\nu{}^b \sigma_\mu{}^a \right), \\ \sigma_\nu{}^b D_\rho \sigma_\mu{}^a = \kappa_\rho \left(A\sigma_\nu{}^b \theta_\mu{}^a - M\sigma_\nu{}^b \sigma_\mu{}^a \right), \end{cases} \tag{35}$$

where by changing the indices as $a \leftrightarrow b$ and $\mu \leftrightarrow \nu$ read

$$\begin{cases} \theta_\mu{}^a D_\rho \theta_\nu{}^b = \kappa_\rho \left(M\theta_\mu{}^a \theta_\nu{}^b - A\theta_\mu{}^a \sigma_\nu{}^b \right), \\ \sigma_\mu{}^a D_\rho \sigma_\nu{}^b = \kappa_\rho \left(A\sigma_\mu{}^a \theta_\nu{}^b - M\sigma_\mu{}^a \sigma_\nu{}^b \right). \end{cases} \tag{36}$$

By summing up the sides of the relations (35) and (36) we arrive at

$$\nabla_\rho g_{\mu\nu} = 2\kappa_\rho \left(Mg_{\mu\nu} - AX_{\mu\nu} \right), \tag{37}$$

and

$$\nabla_\rho f_{\mu\nu} = 2\kappa_\rho \left(AX_{\mu\nu} - Mf_{\mu\nu} \right). \tag{38}$$

Hence, the structure of the conformal group imposes a 3-metric geometrical structure whose nonmetricity conditions are given in (34), (37) and (38). These are also the most general relations which respect the symmetries (30) and (31). Now, a question may be: what is the connection associated to these relations. To answer, let us expand, for instance, the relation (34) and then apply the cyclic permutation $\rho \to \mu \to \nu \to \rho$ on its indices. The result is

$$\partial_\rho X_{\mu\nu} - \Gamma^\alpha_{\rho\mu} X_{\alpha\nu} - \Gamma^\alpha_{\rho\nu} X_{\alpha\mu} = A\kappa_\rho \left(g_{\mu\nu} - f_{\mu\nu} \right), \tag{39}$$

$$\partial_\mu X_{\nu\rho} - \Gamma^\alpha_{\mu\nu} X_{\alpha\rho} - \Gamma^\alpha_{\mu\rho} X_{\nu\alpha} = A\kappa_\mu \left(g_{\nu\rho} - f_{\nu\rho} \right), \tag{40}$$

$$\partial_\nu X_{\rho\mu} - \Gamma^\alpha_{\nu\rho} X_{\alpha\mu} - \Gamma^\alpha_{\nu\mu} X_{\rho\alpha} = A\kappa_\nu \left(g_{\mu\rho} - f_{\mu\rho} \right), \tag{41}$$

where the connection is assumed to be symmetric in its lower indices. The combination [(39) − (40) + (41)] gives

$$\begin{aligned} &(\partial_\rho X_{\mu\nu} + \partial_\nu X_{\rho\mu} - \partial_\mu X_{\nu\rho}) - (\Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu}) X_{\alpha\rho} - (\Gamma^\alpha_{\nu\rho} + \Gamma^\alpha_{\rho\nu}) X_{\alpha\mu} - (\Gamma^\alpha_{\rho\mu} - \Gamma^\alpha_{\mu\rho}) X_{\alpha\nu} \\ &= A\left[\kappa_\rho (g_{\mu\nu} - f_{\mu\nu}) + \kappa_\nu (g_{\mu\rho} - f_{\mu\rho}) - \kappa_\mu (g_{\nu\rho} - f_{\nu\rho}) \right]. \end{aligned} \tag{42}$$

Multiplying both sides of the above relation by $\frac{1}{2}\left(X^{-1}\right)^{\mu\beta}$ and assuming that the connection Γ is symmetric results

$$\Gamma^\beta_{\nu\rho} = \{X\}^\beta_{\nu\ \rho} - \frac{A}{2}\left[\kappa_\rho X^{-1\mu\beta} (g_{\mu\nu} - f_{\mu\nu}) + \kappa_\nu X^{-1\mu\beta} (g_{\mu\rho} - f_{\mu\rho}) - \kappa_\mu X^{-1\mu\beta} (g_{\nu\rho} - f_{\nu\rho}) \right], \tag{43}$$

where $\{X\}^\beta_{\nu\ \rho}$ is the Christoffel symbols of the metric X. If we set $X^\mu_{\ \nu} = \theta^\mu_{\ b} \sigma_\nu{}^b$,[1] the above relation can be simplified as

$$\Gamma^\beta_{\nu\rho} = \{X\}^\beta_{\nu\ \rho} - \frac{A}{2}\left[\kappa_\rho (X_\nu{}^\beta - X^\beta{}_\nu) + \kappa_\nu (X_\rho{}^\beta - X^\beta{}_\rho) - \kappa_\mu (X^{-1})^{\mu\beta} (g_{\rho\nu} - f_{\rho\nu}) \right]. \tag{44}$$

By repeating of the same procedure for the nonmetricity conditions of the metrics g and f, we will obtain their corresponding connections as

$$\Gamma^\beta_{\nu\rho} = \{g\}^\beta_{\nu\ \rho} - M\left[\kappa_\rho \delta^\beta_\nu + \kappa_\nu \delta^\beta_\rho - \kappa_\mu g^{\mu\beta} g_{\rho\nu} \right] + A\left[\kappa_\rho X^\beta{}_\nu + \kappa_\nu X^\beta{}_\rho - \kappa_\mu g^{\mu\beta} X_{\rho\nu} \right], \tag{45}$$

and

$$\Gamma^\beta_{\nu\rho} = \{f\}^\beta_{\nu\ \rho} + M\left[\kappa_\rho \delta^\beta_\nu + \kappa_\nu \delta^\beta_\rho - \kappa_\mu f^{\mu\beta} f_{\rho\nu} \right] - A\left[\kappa_\rho X_\nu{}^\beta + \kappa_\nu X_\rho{}^\beta - \kappa_\mu f^{\mu\beta} X_{\rho\nu} \right]. \tag{46}$$

Our set-up for writing the geometrical properties of the conformal group is now complete. In the next section we will investigate how a gravitation theory may be extracted from this set-up.

[1] This and some other useful relations that may be used in the following calculations can be obtained sequentially from $X_{\mu\nu} = \theta_\mu{}^a \sigma_{\nu a}$. One can show that $(X^{-1})^{\mu\beta} g_{\mu\nu} = X_\nu{}^\beta$, where $X_\nu{}^\beta = \sigma^\beta_{\ b} \theta_\nu{}^b$. Also, $(X^{-1})^{\mu\beta} f_{\mu\nu} = X^\beta{}_\nu$ where $X^\beta{}_\nu = \theta^\beta_{\ b} \sigma_\nu{}^b$. In addition: $(g^{-1})^{\mu\beta} X_{\mu\nu} = X^\beta{}_\nu$, $(f^{-1})^{\mu\beta} X_{\mu\nu} = X_\nu{}^\beta$, $(f^{-1})^{\mu\nu} g_{\mu\beta} = X_\beta{}^\mu X_\mu{}^\nu$ and $(g^{-1})^{\mu\nu} f_{\mu\beta} = X^\nu{}_\mu X^\mu{}_\beta$.

4. Conformal group as an arena for extension of massive gravity

An active topic of research in gravity is modifying Einstein–Hilbert to a theoretically consistent model of gravity. By theoretically consistent we mean absence of any instabilities including ghosts. The construction of modified gravity models are interesting by their own but they can also be checked with the cosmological observations. We should recall that a scalar and a vector field cannot explain the gravity force because of the absence of interaction with light and anisotropic behavior respectively. Hence the first chance is a tensor field. It is well-known, according to Lovelock theorem, that a theoretically consistent theory of a massless spin-2 particle (graviton) is the unique Einstein–Hilbert action. This means any modified gravity should have additional degrees of freedom e.g. scalar–tensor or scalar–vector–tensor theories.[2] A very beautiful extension of gravity is massive gravity which has five degrees of freedom instead of two for massless graviton [11]. On the other hand, as we mentioned Poincaré group corresponds to Einstein–Hilbert model and it is natural to think any more complicated gauge group with additional symmetry generators should be corresponded to a gravity model with more degrees of freedom.

In this section, we discuss the question of whether a theory of gravity could correspond to the above geometric structure. To do this, we need to assign an action to the model described above. In a gauge theory point of view, such an action is usually in the form of the Yang–Mills type actions. In [1], for example, a Yang–Mills type theory of gravity is written in the framework of the biconformal gauging of conformal group. Here, we do not do so, but instead are looking for a gravity theory beyond the Einstein–Hilbert action in the framework of the geometric structure of the conformal group. However, in the presence of three metrics which are not independent according to (44)–(46), the process of writing the action may be ambiguous. As the first candidate, we may think of an action which is built by summing up Einstein–Hilbert actions for every three metric, that is

$$\int \left[M_g^2\sqrt{-g}R(g)+M_f^2\sqrt{-f}R(f)+M_X^2\sqrt{-X}R(X)\right]d^4x. \tag{47}$$

The above action is not unique in the sense that one can suggest another types of models. To give a clue that what we can do with $SO(4,2)$ geometric structure, we propose a simpler model which still is non-trivial as

$$\mathcal{S}=M_g^2\int\sqrt{-g}\mathcal{R}(\Gamma)d^4x, \tag{48}$$

where Γ is defined in (45). Note that with the above assumption we break the symmetry between metrics g and f. By writing Eq. (45) in the form of

$$\Gamma^{\beta}_{\nu\rho}=\{g\}^{\beta}_{\nu\ \rho}+C^{\beta}_{\nu\rho}, \tag{49}$$

we can deduce the Riemann tensor as

$$\begin{aligned}\mathcal{R}^{\beta}_{\nu\rho\alpha}&=R^{\beta}_{\nu\rho\alpha}+\partial_\rho C^{\beta}_{\nu\alpha}-\partial_\alpha C^{\beta}_{\nu\rho}+C^{\beta}_{\mu\rho}C^{\mu}_{\nu\alpha}-C^{\beta}_{\mu\alpha}C^{\mu}_{\nu\rho}\\&+\{g\}^{\beta}_{\mu\ \rho}C^{\mu}_{\nu\alpha}+\{g\}^{\mu}_{\nu\ \alpha}C^{\beta}_{\mu\rho}-\{g\}^{\beta}_{\mu\ \alpha}C^{\mu}_{\nu\rho}-\{g\}^{\mu}_{\nu\ \rho}C^{\beta}_{\mu\alpha},\end{aligned} \tag{50}$$

from which the Ricci scalar reads

$$\mathcal{R}=R+g^{\alpha\nu}\left[\nabla_\beta C^{\beta}_{\nu\alpha}-\nabla_\alpha C^{\beta}_{\nu\beta}\right]+g^{\alpha\nu}\left(C^{\beta}_{\mu\beta}C^{\mu}_{\nu\alpha}-C^{\beta}_{\mu\alpha}C^{\mu}_{\nu\beta}\right),$$

where $R^{\beta}_{\nu\rho\alpha}$ and R are the Riemann tensor and Ricci scalar correspond to the g's Christoffel symbol. Also, we used the metric $g_{\mu\nu}$ to take traces and the covariant derivatives, ∇s, which are compatible with metric $g_{\mu\nu}$. If now, we return to the definition of $C^{\beta}_{\nu\rho}$, the last term in the Ricci scalar takes the form

$$\begin{aligned}g^{\alpha\nu}\left(C^{\beta}_{\mu\beta}C^{\mu}_{\nu\alpha}-C^{\beta}_{\mu\alpha}C^{\mu}_{\nu\beta}\right)&=-6M^2\kappa^2+4AM\left(\kappa^2X^{\alpha}_{\ \alpha}-\kappa^{\mu}\kappa^{\nu}X_{\mu\nu}\right)\\&-(A^2\kappa^2)\left(X^{\alpha}_{\ \alpha}X^{\beta}_{\ \beta}-X^{\beta}_{\ \alpha}X^{\alpha}_{\ \beta}\right)+2A^2\left(\kappa^{\mu}\kappa^{\nu}X_{\mu\nu}X^{\alpha}_{\ \alpha}-\kappa^{\mu}\kappa^{\beta}X_{\mu\nu}X^{\nu}_{\ \beta}\right)\\&=-6M^2\kappa^2+2AM\,\epsilon^{\rho\sigma\mu\alpha}\epsilon_{\rho\sigma\nu\beta}\kappa_\mu\kappa^{\nu}X^{\beta}_{\ \alpha}-A^2\,\epsilon^{\gamma\mu\rho\alpha}\epsilon_{\gamma\nu\sigma\beta}\kappa_\mu\kappa^{\nu}X^{\sigma}_{\ \rho}X^{\beta}_{\ \alpha}.\end{aligned} \tag{51}$$

Hence, the action (48) up to a total derivative can be written as

$$\mathcal{S}=M_g^2\int d^4x\sqrt{-g}\left[R-6M^2\kappa^2+2AM\,\epsilon^{\rho\sigma\mu\alpha}\epsilon_{\rho\sigma\nu\beta}\kappa_\mu\kappa^{\nu}X^{\beta}_{\ \alpha}-A^2\,\epsilon^{\gamma\mu\rho\alpha}\epsilon_{\gamma\nu\sigma\beta}\kappa_\mu\kappa^{\nu}X^{\sigma}_{\ \rho}X^{\beta}_{\ \alpha}\right]. \tag{52}$$

The above action is a kind of vector-tensor model with massive gravity building block i.e. $X^{\beta}_{\ \nu}=\sqrt{g^{\beta\mu}f_{\mu\nu}}$ where g is dynamical field and f is auxiliary one.[3] Interestingly, without any fine-tuning, we get the double epsilon structure which is

[2] Note that any other approaches like extra-dimensions, non-locality can be rewritten equivalently as Einstein–Hilbert with additional degrees of freedom.

[3] By definition, if one has $g^{\beta\mu}f_{\mu\nu}=X^{\beta}_{\ \mu}X^{\mu}_{\ \nu}$, then $\sqrt{g^{\beta\mu}f_{\mu\nu}}=X^{\beta}_{\ \nu}$.

a hint for ghost-freeness of our model same as what happens in Lovelock theories, Galileon [17] and massive gravity [11]. It is worth to say few words why the double epsilon structure is a hint for ghost-freeness of our model. Assume a scalar field, π, with the following Lagrangian

$$\mathcal{L} \propto \epsilon^{\kappa\mu\rho\alpha}\epsilon_{\eta\nu\sigma\beta}\big(\partial_\kappa\partial^\eta\pi\big)\big(\partial_\mu\partial^\nu\pi\big)\big(\partial_\rho\partial^\sigma\pi\big)\big(\partial_\alpha\partial^\beta\pi\big). \tag{53}$$

For the first look the above Lagrangian will result in higher order derivative terms (i.e. more than second order) at the level of equations of motion. This means for solving the equation of motion one needs more than two initial conditions (i.e. $\pi(0)$ and $\dot{\pi}(0)$) which means the model contains more than one degree of freedom. Usually these unwanted degrees of freedom are ghost and make the model unstable by having a wrong sign of their kinetic term. However, the above Lagrangian has a very special form because of the double epsilon structure. A typical term of equation of motion can be written as

$$\mathcal{E} = \frac{\delta\mathcal{L}}{\delta\pi} \ni \partial_\gamma\partial^\zeta\frac{\partial\mathcal{L}}{\partial(\partial_\gamma\partial^\zeta\pi)} \ni \partial_\gamma\partial^\zeta\Big[\epsilon^{\gamma\mu\rho\alpha}\epsilon_{\zeta\nu\sigma\beta}\big(\partial_\mu\partial^\nu\pi\big)\big(\partial_\rho\partial^\sigma\pi\big)\big(\partial_\alpha\partial^\beta\pi\big)\Big] = \epsilon^{\gamma\mu\rho\alpha}\epsilon_{\zeta\nu\sigma\beta}\,\partial_\gamma\partial^\zeta\Big[\big(\partial_\mu\partial^\nu\pi\big)\big(\partial_\rho\partial^\sigma\pi\big)\big(\partial_\alpha\partial^\beta\pi\big)\Big], \tag{54}$$

where it is obvious from the last term that if we move a partial derivative (i.e. one of the ∂_γ or ∂^ζ) into the parentheses then the totally anti-symmetric structure of one of the epsilons (i.e. one of the $\epsilon^{\gamma\mu\rho\alpha}$ or $\epsilon_{\zeta\nu\sigma\beta}$) makes this term zero. This shows that the double epsilon structure can kill the new unwanted degrees of freedom. As we mentioned above exactly, this structure makes Lovelock theories ghost-free even when they are higher order.

We should emphasize that in our scenario we could get massive gravity terms up to $\mathcal{O}(X^2)$ and we need a kind of generalization if we want to have all orders of massive gravity terms e.g. working with another tetrad postulate or assuming higher order curvature in (48). The above Lagrangian has been deduced (more or less) in a bi-connection framework [13] where two connections satisfied Weyl nonmetricity with opposite signs.

Another issue in the above Lagrangian is the absence of kinetic term for the vector field κ_μ. This means our vector field is not propagating. However, one can add standard kinetic term for a vector i.e. $F_{\mu\nu}F^{\mu\nu}$ by hand where $F_{\mu\nu} = \nabla_\mu\kappa_\nu - \nabla_\nu\kappa_\mu$. A natural way to add the kinetic term for κ_μ is starting with higher order gravity models instead of (48) as it has been discussed in [10]. But we should notice that since our Lagrangian (52) is not $U(1)$ gauge invariant then it may suffer from a ghost. Another more natural scenario can be deduced by introducing a scalar field via $\partial_\mu\phi \equiv \sqrt{6}M\kappa_\mu$. In this case the Lagrangian becomes

$$\mathcal{L} = R - \partial^\mu\phi\,\partial_\mu\phi + b\,\epsilon^{\rho\sigma\mu\alpha}\,\epsilon_{\rho\sigma\nu\beta}\,\partial_\mu\phi\,\partial^\nu\phi\,X^\beta{}_\alpha - \frac{3}{2}b^2\,\epsilon^{\gamma\mu\rho\alpha}\,\epsilon_{\gamma\nu\sigma\beta}\,\partial_\mu\phi\,\partial^\nu\phi\,X^\sigma{}_\rho\,X^\beta{}_\alpha, \tag{55}$$

where $b \equiv A/3M$ is the only free parameter in this model. The above Lagrangian belongs to quintessence models where the scalar field is derivatively coupled to the metric via massive gravity building blocks. As we already mentioned, the additional terms can be written in double epsilon form. This means the above Lagrangian should be safe from any ghost in its dynamics. To show this we can write the most dangerous part of our model in decoupling limit by replacing $X^\beta{}_\alpha$ by $\Pi^\beta{}_\alpha = \partial^\beta\partial_\alpha\pi$ where π is the scalar degree of freedom of the metric field. Obviously the double epsilon structure does not allow the equations of motion have any term with more than two derivatives. This means our model is ghost free at least in decoupling limit. The above Lagrangian belongs to a family of massive gravity models with an additional scalar degree of freedom which has an extensive literature [6]. The more analysis of this model including its cosmology is beyond the scope of this paper and will be remained for future works.

5. Summary

In this paper we have studied the relation between the Lie groups and their geometric interpretation. Our interest in this subject is rooted in the relation between GR and gauge theories, which by a gauge theory we mean a theory that its action is invariant under a continuous group which form its symmetry. Our strategy is based on the building blocks of the Lie groups which show themselves in the Maurer–Cartan structure equations. These are indeed the objects such as metric, connection, torsion, curvature, which a geometry may be equipped with them. The role of the Maurer–Cartan equations is then to connect them to the generators and gauge fields of the corresponding Lie group.

The main steps we have taken to address this issue are as follows. The usual GR as a gauge theory is based on the Poincaré group. This group is a semidirect product of two Lie groups: the group of Lorentz transformations and the group of translations. The significance of this group becomes apparent when we want to bring spinors into GR. In order to do so, the usual GR whose connection is given by the Christoffel symbol compatible with the metric, should be formulated by the gauge transformation invariance such as Poincaré gauge group. For the Poincaré group we have shown that while a metric can be constructed by means of the group's gauge fields the Maurer–Cartan equations equip the resulting metric geometry with torsion and (Riemann) curvature. Assuming, a tetrad postulate, we have seen that the metric in this case satisfies the metricity condition, i.e. its covariant derivative vanishes. This means that the geometry beyond the Poincaré group is of the kind of the Riemann–Cartan geometry.

For the Weyl group, the story is almost the same except that here the metricity condition is replaced with a nonmetricity relation which is the characteristic of the Weyl geometry. This group is a larger group which includes the Poincaré group

as a subgroup. Here, in addition to the transformations of the Poincaré group, a new kind of curvature corresponding to the dilatation, which can resize the coordinates, has appeared. This means that in the corresponding gravity theory we have a setting in which nonmetricity, torsion and curvature are all in play.. Then, we have accomplished our main goal of constructing a geometrical structure by using the conformal group which in addition to all transformations of the Weyl group has an extra generator for the so-called co-translation. The co-solder forms σ, the gauge fields of this extra transformation together with the solder (tetrad) forms, θ, of the translations opened a new window for us to build a geometry with three metric. We saw that the Maurer–Cartan equations of the conformal group, which now included an additional curvature associated to the co-translations, do not change under replacements $\sigma \leftrightarrow \theta$ and $\kappa \rightarrow -\kappa$, where κ is the gauge field of the dilatation. This led us to find the most general nonmetricity conditions for the metrics from which we deduced the connection.

Finally, we have addressed the issue of whether a gravity theory can correspond to the geometric structure of the conformal group. Due to the more complicated structure of the conformal group compared to the Poincaré, its gravity theory is expected to have more degrees of freedom in comparison with the ordinary GR. In this regard, with the use of the resulting geometry we provided a gravity theory starting from an Einstein–Hilbert type action whose Lagrangian is the Ricci scalar of the metric constructed from the translation's gauge fields. We have shown that the proposed Lagrangian takes the form of the vector-tensor model with massive gravity building block. In addition, we rewrote the Lagrangian in a form which in the massive gravity terminology is called the double epsilon form. This helped us to show that the extra terms appeared in the Lagrangian do not include the ghosts, the existence of which must always be taken into consideration when dealing with such theories. To show this, we wrote the Lagrangian of a scalar field with double epsilon structure. In the first opinion, it seems that such a Lagrangian will result in unwanted degrees of freedom (ghosts) with wrong sing in the kinetic term which make the resulting model unstable. However, we explicitly showed because of the special form of the double epsilon structure these additional unwanted degrees of freedom will be removed due to the totally anti-symmetric property of one of the epsilons and thus, the proposed Lagrangian is safe from any ghost in its dynamics.

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