Journal of Geometry and Physics 145 (2019) 103482

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Geometric aspect of three-dimensional Chern–Simons-like gravity

Sen Hu [a], Zhi Hu [a,b,*]

[a] School of Mathematics, University of Science and Technology of China, Hefei, 230026, China
[b] Research Institute of Mathematical Science, Kyoto University, Kyoto, Japan

ARTICLE INFO

Article history:
Received 22 September 2018
Accepted 1 August 2019
Available online 7 August 2019

Keywords:
Chern–Simons action
Cotton tensor
Linear gravity
Supermanifold

ABSTRACT

In this paper, we detailed study the geometric aspect of Chern–Simons-like gravity in three dimensions. The vacua are provided by three dimensional conformally flat manifold, which admit a special configuration, a two dimensional system (M^2, h, Φ) consisting of metric h and scalar field Φ, by dimensional reduction. For this system we define the quasi-local mass. An interesting observation is that this system contains certain two dimensional dilaton gravity at the classical level. Via AdS/CFT, we check the Weyl anomaly and diffeomorphism anomaly for the boundary theory. We study the linearization of Chern–Simons-like gravity around the background with constant scalar curvature via linear Cotton tensor. If the background manifold is of positive constant scalar curvature, we show that there is no solutions for linearized vacuum equation. To find the solutions of the linear gravity with respect to the background with negative constant scalar curvature, we need to solve some Schrödinger-type equations. Related to supersymmetric quantum mechanics, one can find some exact solutions and some dualities between different components or modes of solutions. And it is also can be related to Seiberg–Witten theory via Picard–Fuchs equation in terms of WKB approximation. We discuss the ADM-type charge under the context of Chern–Simons-like gravity. Finally, we extend the Chern–Simons-like gravity to the supermanifolds by embedding the structure group of three-manifold into the body of othosymplectic supergroup.

1. Introduction

Since three dimensional Einstein gravity is locally trivial, one considers to add some higher-derivative terms that is responsible for the presence of a single propagating massive graviton. One possible choice is to supplement Chern–Simons action via Levi-Civita connection given by [6,11,12,22]

$$S[g] = \frac{1}{32\pi G_3 \zeta} \int_{M^3} d^3x \sqrt{|\det g|} \epsilon^{\lambda\mu\nu} \Gamma^{\rho}_{\lambda\sigma} (\partial_\mu \Gamma^{\sigma}_{\rho\nu} + \frac{2}{3} \Gamma^{\sigma}_{\mu\tau} \Gamma^{\tau}_{\nu\rho}), \tag{1.1}$$

where $\epsilon^{\lambda\mu\nu}$ is the 3-dimensional epsilon tensor, equal to $\varepsilon^{\lambda\mu\nu}/\sqrt{|\det g|}$, $\varepsilon^{\lambda\mu\nu}$ being the alternating symbol with $\varepsilon^{012} = 1$, and G_3 stands for the 3-dimensional gravitational constant and ζ is a constant with dimension of mass. This action can

* Corresponding author.
E-mail addresses: shu@ustc.edu.cn (S. Hu), halfask@mail.ustc.edu.cn (Z. Hu).

https://doi.org/10.1016/j.geomphys.2019.103482

be viewed as a boundary term of topological Hirzebruch–Pontryagin action on the bulk manifold

$$S_{\text{bulk}} = \int_{M^4} d^4x \epsilon^{abef} R_{abcd} R_{ef}{}^{cd}.$$

Another approach to the Chern–Simons action is produced with the spin connection based on the gauge group $SU(2)$ or $SL(2,\mathbb{R})$. Let $\omega = \frac{1}{2}\omega^{ij}\gamma_{ij}$ denote the 3-dimensional spin connection, and $e = e^i\gamma_i$ denote the dreibein 1-form, where γ stand for the 3-dimensional gamma matrices. Then the torsion 2-form and the curvature 2-form are given by $T^i = de^i + \omega^i_j \wedge e^j$ and $\mathcal{R}^i_j = d\omega^i_k \wedge \omega^k_j$ respectively. For the torsion-free connection, we have the following action

$$I[e] = \frac{1}{32\pi G_3 \zeta} \int_{M^3} \omega^i_j(e) d\omega^j_i(e) + \frac{2}{3}\omega^i_j(e)\omega^j_k(e)\omega^k_i(e). \tag{1.2}$$

We note that these two actions are not identical. However, by the relation between spin connection and the Levi-Civita connection $(\omega^i_j)_\mu = -\partial_\mu e^i_\nu e^\nu_j + \Gamma^\lambda_{\mu\nu} e^i_\lambda e^\nu_j$, one can connect them via the WZW action [5], up to a boundary term,

$$I[e] - S[g] = \frac{1}{96\pi G_3 \zeta} \int_{M^3} e^\mu_i de^j_\mu e^\mu_i de^j_\mu e^\nu_j de^k_\nu e^\lambda_j de^i_\lambda. \tag{1.3}$$

Einstein–Hilbert action plus Chern–Simons action describes so-called topologically massive gravity, although introducing Chern–Simons term leads to instability for general parameter ζ except a special value called chiral point [16]. In this paper, we focus on pure Chern–Simons-like gravity based on the action (1.1), whose geometric aspect will be studied detailed. The vacuum of Chern–Simons gravity is provided by three dimensional conformally flat manifold. In Section 2 we reduce a special vacuum to a two dimensional system (M^2, h, Φ) consisting of metric h and scalar field Φ. For this system we define the quasi-local mass. An interesting observation is that this system contains certain two dimensional dilaton gravity at the classical level. Chern–Simons-like gravity admits asymptotically AdS solutions so one can apply AdS/CFT correspondence. We check the Weyl anomaly and diffeomorphism anomaly for the boundary CFT. In Section 3, we study the linearization of Chern–Simons-like gravity around the background with constant scalar curvature via linear Cotton tensor. If the background manifold is of positive constant scalar curvature, we show that there is no solutions for linearized vacuum equation. To find the solutions of the linear gravity with respect to the background with negative constant scalar curvature, we need to solve some Schrödinger-type equations. Related to supersymmetric quantum mechanics, one can find some exact solutions and some dualities between different components or modes of solutions. And it is also can be related to Seiberg–Witten theory via Picard–Fuchs equation in terms of WKB approximation. We discuss the ADM-type charge under the context of Chern–Simons-like gravity. In the final section, we extend the Chern–Simons-like gravity to the supermanifolds by embedding the structure group of three-manifold into the body of othosymplectic supergroup.

2. The first-order variation

The first-order variation of the action (1.1) with respect to the metric g gives rise to the equations of motion plus the boundary terms

$$\delta S[g] = -\frac{1}{16\pi G_3 \zeta} \int_{M^3} d^3x \sqrt{|\det g|}\, \delta g_{\mu\nu} C^{\mu\nu} + \Sigma_1 + \Sigma_2, \tag{2.1}$$

where

- $C^{\mu\nu}$ is exactly the Cotton tensor defined by

$$C^{\mu\nu} = \frac{1}{2}(\epsilon^{\mu\alpha\beta}\nabla_\alpha R^\nu_\beta + \epsilon^{\nu\alpha\beta}\nabla_\alpha R^\mu_\beta) = \epsilon^{\mu\alpha\beta}\nabla_\alpha(R^\nu_\beta - \frac{1}{4}R\delta^\nu_\beta), \tag{2.2}$$

  which is symmetric, traceless, and covariantly conserved, and which vanishes if and only if the metric is locally conformally flat in three dimensions;
- the boundary term Σ_1 is given by

$$\Sigma_1 = \frac{1}{16\pi G_3 \zeta} \int_{\partial M^3} d^2x \sqrt{|\det h|}\, \delta g_{\mu\nu} \Theta^{\mu\nu} \tag{2.3}$$

  with the induced metric h on the boundary ∂M^3 (space-like or time-like surface), namely,

$$h_{\mu\nu} = \begin{cases} g_{\mu\nu} + n_\mu n_\nu, & \partial M^3 \text{ is space-like,} \\ g_{\mu\nu} - n_\mu n_\nu, & \partial M^3 \text{ is time-like,} \end{cases}$$

  for a unit normal vector n with respect to the boundary ∂M^3, whose direction depends on ∂M^3 being space-like or time-like; and the symmetric tensor

$$\Theta^{\mu\nu} = \frac{1}{2}(\epsilon^{\mu\alpha\beta} n_\alpha R^\nu_\beta + \epsilon^{\nu\alpha\beta} n_\alpha R^\mu_\beta); \tag{2.4}$$

- the boundary term Σ_2 is given by

$$\Sigma_2 = \frac{1}{64\pi G_3 \zeta} \int_{\partial M^3} \Gamma^{\sigma}_{\mu\rho} g^{\rho\lambda} (\nabla_\nu \delta g_{\lambda\sigma} + \nabla_\sigma \delta g_{\lambda\nu} - \nabla_\lambda \delta g_{\nu\sigma}) dx^\mu \wedge dx^\nu. \tag{2.5}$$

Therefore the equations of motion read as

$$C_{\mu\nu} = 0 \tag{2.6}$$

or equivalently,

$$C_{\mu\nu\gamma} := \nabla_\gamma R_{\mu\nu} - \nabla_\nu R_{\mu\gamma} - \frac{1}{4}(g_{\mu\nu}\nabla_\gamma R - g_{\mu\gamma}\nabla_\nu R) = 0. \tag{2.7}$$

The solutions of EOMs are called vacuum configurations of Chern–Simons-like gravity.

2.1. Newman–Penrose formalism

There are several approaches to express the Newman–Penrose formalism of EOMs.

Representation A: We introduce an orthonormal frame $\{e_0, e_1, e_2\}$, and then define a frame $\{e_0, e_+, e_-\}$ by $e_\pm = \frac{1}{\sqrt{2}}(e_0 + ie_1)$ such that the Lorentzian metric takes the matrix form $(g_{ij}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, where the indices i, j range over $0, +, -$. The complex Ricci rotation coefficients are defined by

$$\omega_{ijk} = \nabla_\nu (e_i)_\mu (e_j)^\mu (e_k)^\nu$$

with the anti-symmetries $\omega_{ijk} = -\omega_{jik}$. The following notations are employed to denote the independent components of Ricci rotation coefficients:

$$\rho = \omega_{+0-}, \sigma = \omega_{+0+}, \tau = \omega_{+--}, \kappa = \omega_{+00}, \epsilon = \omega_{+-0},$$

and the following operators are introduced:

$$D = (e_0)^\mu \partial_\mu, \delta = (e_+)^\mu \partial_\mu, \overline{\delta} = (e_-)^\mu \partial_\mu.$$

By some tedious calculations, we express the components of the Ricci tensor and the scalar curvature with the above notations as follows:

$$\begin{aligned}
R_{00} &= D\rho + D\overline{\rho} - \overline{\delta}\kappa - \delta\overline{\kappa} + \tau\kappa + \overline{\tau\kappa} + 2\kappa\overline{\kappa} - 2\sigma\overline{\sigma} - \rho^2 - \overline{\rho}^2 - \overline{\rho}^2, \\
R_{++} &= \delta\kappa - D\sigma + 2\epsilon\sigma + \overline{\tau}\kappa - \kappa^2 + \sigma\overline{\rho} + \rho\sigma, \\
R_{0+} &= -\overline{\delta}\sigma + \delta\rho + 2\tau\sigma - \kappa\rho + \kappa\overline{\rho}, \\
R_{0-} &= -\overline{\delta}\epsilon + D\tau - \kappa\overline{\sigma} + \rho\overline{\kappa} + \epsilon\tau + \epsilon\overline{\kappa} + \overline{\tau\sigma} - \tau\rho, \\
R_{+-} &= \overline{\delta}\kappa - D\rho + \delta\tau + \overline{\delta}\overline{\tau} - \epsilon\rho + \epsilon\overline{\rho} - \kappa\overline{\kappa} - \kappa\tau + \rho\overline{\rho} + \rho^2 - 2\tau\overline{\tau}, \\
\frac{1}{2}R &= 2\overline{\delta}\kappa - 2D\overline{\rho} + \delta\tau + \overline{\delta}\overline{\tau} - 2\kappa\overline{\kappa} - 2\overline{\kappa\tau} + 2\overline{\rho}^2 + \sigma\overline{\sigma} - \epsilon\rho + \epsilon\overline{\rho} + \rho\overline{\rho} - 2\tau\overline{\tau},
\end{aligned}$$

and we have the following identities that reflect the symmetries of the curvature tensor:

$$\begin{aligned}
-D\rho + \overline{\delta}\kappa - \kappa\tau + \rho^2 &= -D\overline{\rho} + \delta\overline{\kappa} - \overline{\kappa\tau} + \overline{\rho}^2, \\
\delta\overline{\sigma} - \overline{\delta}\overline{\rho} - \overline{\tau\sigma} + \overline{\kappa\rho} &= \overline{\delta}\epsilon - D\tau + \kappa\overline{\sigma} - \epsilon\tau - \epsilon\overline{\kappa} + \tau\rho.
\end{aligned}$$

Hence, we get the Newman–Penrose formalism as follows:

$$\begin{aligned}
&\mathrm{Im}[\overline{\delta}(-\overline{\delta}\sigma + \delta\rho + 2\tau\sigma - \kappa\rho + \kappa\overline{\rho}) + \rho(D\overline{\rho} - \delta\overline{\kappa} + \overline{\tau\kappa} + \kappa\overline{\kappa} - \sigma\overline{\sigma} - \overline{\rho}^2 + \delta\tau + \overline{\delta}\overline{\tau} - \epsilon\rho + \epsilon\overline{\rho} + \rho\overline{\rho} - 2\tau\overline{\tau}) \\
&- \tau(-\overline{\delta}\sigma + \delta\rho + 2\tau\sigma - \kappa\rho + \kappa\overline{\rho}) + \overline{\sigma}(\delta\kappa - D\sigma + 2\epsilon\sigma + \overline{\tau}\kappa - \kappa^2 + \sigma\overline{\rho})] = 0, \\
&\overline{\delta}(\delta\kappa - D\sigma + 2\epsilon\sigma + \overline{\tau}\kappa - \kappa^2 + \sigma\overline{\rho} + \rho\sigma) - \frac{1}{2}\delta(\delta\tau + \overline{\delta}\overline{\tau} - \epsilon\rho + \epsilon\overline{\rho} + \rho\overline{\rho} - 2\tau\overline{\tau} - \sigma\overline{\sigma}) \\
&+ 2\rho(-\overline{\delta}\sigma + \delta\rho + 2\tau\sigma - \kappa\rho + \kappa\overline{\rho}) - 2\tau(\delta\kappa - D\sigma + \epsilon\sigma + \overline{\tau}\kappa - \kappa^2 + 2\sigma\overline{\rho}) \\
&- \overline{\rho}(-\overline{\delta}\sigma + \delta\rho - \kappa\rho + \kappa\overline{\rho}) - \sigma(-\overline{\delta}\epsilon + D\tau - \kappa\overline{\sigma} + \rho\overline{\kappa} + \epsilon\tau + \epsilon\overline{\kappa} + \overline{\tau\sigma} + \tau\rho) = 0, \\
&\frac{1}{2}D(\delta\tau + \overline{\delta}\overline{\tau} - \epsilon\rho + \epsilon\overline{\rho} + \rho\overline{\rho} - 2\tau\overline{\tau} - \sigma\overline{\sigma}) - \delta(-\overline{\delta}\epsilon + D\tau - \kappa\overline{\sigma} + \rho\overline{\kappa} + \epsilon\tau + \epsilon\overline{\kappa} + \overline{\tau\sigma} - \tau\rho)
\end{aligned}$$

$$
\begin{aligned}
&- \overline{\rho}(D\overline{\rho} - \delta\overline{\kappa} + \overline{\tau\kappa} + \kappa\overline{\kappa} - \sigma\overline{\sigma} - \overline{\rho}^2 + \overline{\delta}\overline{\tau} - \epsilon\rho + \epsilon\overline{\rho} + \rho\overline{\rho} - 2\tau\overline{\tau}) - \rho(\overline{\delta}\overline{\kappa} - D\overline{\sigma} - 2\epsilon\overline{\sigma} - \overline{\kappa}^2 + \tau\overline{\kappa} + \overline{\sigma}\rho + \overline{\rho\sigma}) \\
&+ 2\mathrm{Re}[\kappa(-\overline{\delta}\epsilon + D\tau - \kappa\overline{\sigma} + \rho\overline{\kappa} + \epsilon\tau + \epsilon\overline{\kappa} + \overline{\tau\sigma} - \tau\rho)] + \overline{\tau}(-\overline{\delta}\epsilon + D\tau - \kappa\overline{\sigma} + \rho\overline{\kappa} + \epsilon\tau + \epsilon\overline{\kappa} + \overline{\tau\sigma} - \tau\rho) = 0, \\
&D(\delta\kappa - D\sigma + 2\epsilon\sigma - \overline{\tau}\kappa - \kappa^2 + \sigma\overline{\rho} + \rho\sigma) - \delta(-\overline{\delta}\sigma + \delta\rho + 2\tau\sigma - \kappa\rho + \kappa\overline{\rho}) \\
&+ (2\kappa - \overline{\tau})(-\overline{\delta}\sigma + \delta\rho + 2\tau\sigma - \kappa\rho + \kappa\overline{\rho}) - (2\epsilon + \overline{\rho})(\delta\kappa - D\sigma + 2\epsilon\sigma - \overline{\tau}\kappa - \kappa^2 + \sigma\overline{\rho} + \rho\sigma) \\
&- \sigma(D\overline{\rho} - \delta\overline{\kappa} + \overline{\tau\kappa} + \kappa\overline{\kappa} - \sigma\overline{\sigma} - \overline{\rho}^2 + \delta\tau + \overline{\delta\overline{\tau}} - \epsilon\rho + \epsilon\overline{\rho} + \rho\overline{\rho} - 2\tau\overline{\tau}) = 0.
\end{aligned}
$$

Representation B: One takes the frame $\{e_2, e_+, e_-\} = \{e_2, \frac{1}{\sqrt{2}}(\pm e_0 + e_1)\}$, which produces a real version of the Newman–Penrose formalism [10]. Compared with Representation A, the number of independent Ricci rotation coefficients is double by interchanging the null vectors $E_\pm$ but leaving the space-like vector e_2 invariant.

Representation C: Let (V, ε) be a two-dimensional symplectic real vector space with a standard symplectic form ε. The group of automorphisms preserving the symplectic form is isomorphic to $SL(2;\mathbb{R})$. Then the space $U = S^2V$, the symmetric tensor product of V, carries a natural Lorentzian metric g given by $g = \varepsilon^2$. The group of automorphisms preserving this metric is isomorphic to $O(1, 2)$. Thus V can be viewed as a space of spinors and U as a space of vectors. We take two normalized spinors o, ι, and define three vectors form a frame:

$$
e_o = o^2, e_1 = \frac{1}{\sqrt{2}}(o \otimes \iota + \iota \otimes o), e_2 = \iota^2.
$$

The metric takes the matrix form $(g_{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. The remaining things are similar [1].

2.2. Dimensional reduction

As a warm-up example, one assumes the spacetime $(M^3, g_{\mu\nu}dx^\mu dx^\nu)$ has the form of warped product.

Type I : $(M^3, g_{\mu\nu}dx^\mu dx^\nu) \simeq (M^2, h_{AB}dx^A dx^B) \times (\mathbb{R}^1, dt^2)$.

It is endowed with the static metric

$$
g_{\mu\nu}dx^\mu dx^\nu = -\Phi^2 dt^2 + h_{AB}dx^A dx^B, \tag{2.8}
$$

for a smooth function Φ on M^2. Then the non-trivial components of Ricci tensor and the scalar curvature are given by

$$
\begin{aligned}
R_{tt} &= \Phi \Delta^{(h)}\Phi, \\
R_{AB} &= \frac{1}{2}h_{AB}R^{(h)} - \frac{\nabla_A^{(h)}\nabla_B^{(h)}\Phi}{\Phi}, \\
R &= R^{(h)} - 2\frac{\Delta^{(h)}\Phi}{\Phi},
\end{aligned}
$$

where symbol (h) indicates that the corresponding calculation works on M^2 via the metric h, and the EOMs (2.7) reduce to

$$
\begin{aligned}
0 &= \nabla_{[A}R_{B]C} + \frac{1}{4}g_{C[A}\nabla_{B]}R \\
&= \frac{1}{4}h_{C[B}\partial_{A]}R^{(h)} + R^{(h)}\frac{h_{C[B}\partial_{A]}\Phi}{2\Phi} + \frac{h_{C[B}\partial_{A]}\Delta^{(h)}\Phi}{2\Phi} + \frac{\partial_{[A}\Phi\nabla_{B]}^{(h)}\partial_C\Phi}{\Phi^2} - \frac{\Delta^{(h)}\Phi h_{C[B}\partial_{A]}\Phi}{2\Phi^2},
\end{aligned}
$$

thus we have,

$$
\partial_A(\frac{1}{2}R^{(h)}\Phi^2 + \Phi\Delta^{(h)}\Phi - |d\Phi|_h^2) = 0. \tag{2.9}
$$

The scalar $\frac{1}{2}R^{(h)}\Phi^2 + \Phi\Delta^{(h)}\Phi - |d\Phi|_h^2$ and Gauss curvature $\frac{1}{2}R^{(h)}$ on two-dimensional manifold (M^2, h) are denoted by L and Q respectively. Hence the vacuum configuration of Chern–Simons-like gravity reduces to the two dimensional system (M^2, h, Φ) characterized by constant L.

Proposition 2.1. *Assume that M^2 is a closed manifold so that (M^2, h, Φ) forms a vacuum configuration of Chern–Simons-like gravity.*

(1) If M^2 is of negative (or positive) Euler number χ, and L is non-negative (or non-positive), Φ cannot be a sign-definite function.

(2) If Φ is a positive-definite (or negative-definite) function, we have the inequality

$$\frac{1}{5}\int_{M^2} d\mu_h Q\Phi\Delta^{(h)}\Phi^2 \leq (\text{or} \geq) \int_{M^2} d\mu_h |d\Phi|_h^2 \Delta^{(h)}\Phi.$$

(3) Assume Q is constant, if Φ is a positive-definite (or negative-definite) function, we have the inequality

$$\begin{aligned} &Q\int_{M^2} d\mu_h[4\Phi(\Delta^{(h)}\Phi)^2 + 3|d\Phi|_h^2\Delta^{(h)}\Phi] \\ \geq (\text{or} \leq) &\int_{M^2} d\mu_h[|d\Phi|_h^2(\Delta^{(h)})^2\Phi + \frac{1}{2}|Hess^{(h)}\Phi|_h^2\Delta^{(h)}\Phi + \Phi\langle Hess^{(h)}(\Delta^{(h)}\Phi), Hess^{(h)}\Phi\rangle_h \\ &- \frac{1}{2}(\Delta^{(h)}\Phi)^3 - |Hess^{(h)}\Phi|_h^3], \end{aligned}$$

where $|Hess^{(h)}\Phi|_h^3 = h^{AE}h^{BD}h^{CF}(Hess^{(h)}\Phi)_{AB}(Hess^{(h)}\Phi)_{CD}(Hess^{(h)}\Phi)_{EF}$.

Proof. (1) If Φ is sign-definite, we write $\Phi = \pm e^f, f \in C^\infty(M^2)$, then

$$L = e^{2f}(\Delta^{(h)}f + Q).$$

Therefore by compactness of M^2 we have

$$0 = \int_{M^2} d\mu_h \Delta^{(h)}f = \int_{M^2} d\mu_h(Le^{-2f} - Q),$$

thus

$$2\pi\chi = \int_{M^2} d\mu_h Le^{-2f},$$

which means that the signs of χ and L must be the same.

(2) Firstly, we calculate the following integral

$$\begin{aligned} &\int_{M^2} d\mu_h\Phi[|\mathrm{Hess}^{(h)}\Phi|^2 - (\Delta^{(h)}\Phi)^2] \\ =&\int_{M^2} d\mu_h\Phi(h^{AD}h^{BC} - h^{AB}h^{CD})\nabla_A^{(h)}\nabla_B^{(h)}\Phi\nabla_C^{(h)}\nabla_D^{(h)}\Phi \\ =&-\frac{1}{2}\int_{M^2} d\mu_h\Phi(h^{AD}h^{BC} - h^{AB}h^{CD})R^{(h)}{}_{ACDE}\nabla_B^{(h)}\Phi\nabla^{(h)E}\Phi \\ &-\int_{M^2} d\mu_h(h^{AD}h^{BC} - h^{AB}h^{CD})\nabla_A^{(h)}\Phi\nabla_B^{(h)}\Phi\nabla_C^{(h)}\nabla_D^{(h)}\Phi \\ =&-\frac{1}{2}\int_{M^2} d\mu_h Q\Phi(h^{AD}h^{BC} - h^{AB}h^{CD})(h_{AD}h_{CE} - h_{AE}h_{CD})\nabla_B^{(h)}\Phi\nabla^{(h)E}\Phi \\ &-\frac{1}{2}\int_{M^2} d\mu_h\langle d\Phi, d|d\Phi|_h^2\rangle_h + \int_{M^2} d\mu_h|d\Phi|_h^2\Delta^{(h)}\Phi \\ =&-\int_{M^2} d\mu_h Q\Phi|d\Phi|_h^2 + \frac{3}{2}\int_{M^2} d\mu_h(|d\Phi|_h^2\Delta^{(h)}\Phi), \end{aligned}$$

where $\mathrm{Hess}^{(h)}\Phi = \nabla^{(h)}(d\Phi)$. Thereby, it follows from (2.9) that

$$\begin{aligned} 0 &= \int_{M^2} d\mu_h(Q\Phi^2 + \Phi\Delta^{(h)}\Phi - |d\Phi|_h^2)\Delta^{(h)}\Phi \\ &= \int_{M^2} d\mu_h[Q\Phi(\Phi\Delta^{(h)}\Phi + |d\Phi|_h^2) + \Phi|\mathrm{Hess}^{(h)}\Phi|_h^2 - \frac{5}{2}|d\Phi|_h^2\Delta^{(h)}\Phi] \\ &= \int_{M^2} d\mu_h[\frac{1}{2}Q\Phi\Delta^{(h)}\Phi^2 + \Phi|\mathrm{Hess}^{(h)}\Phi|_h^2 - \frac{5}{2}|d\Phi|_h^2\Delta^{(h)}\Phi]. \end{aligned}$$

(3) The following identity is valid in two dimensions, which can be easily checked pointwisely under the normal coordinates

$$(\nabla_A^{(h)}\nabla_B^{(h)}\Phi - \frac{1}{2}h_{AB}\Delta^{(h)}\Phi)\Delta^{(h)}\Phi = (\nabla_A^{(h)}\nabla_C^{(h)}\Phi)(\nabla_B^{(h)}\nabla^{(h)C}\Phi) - \frac{1}{2}h_{AB}|\mathrm{Hess}^{(h)}\Phi|_h^2.$$

Then from (2.9) we obtain

$$\begin{aligned}0 =&\langle \mathrm{Hess}^{(h)}L, \mathrm{Hess}^{(h)}\Phi\rangle_h\\ =&(2Q\nabla^{(h)}_A\Phi\nabla^{(h)}_B\Phi+2\nabla^{(h)}_A\Phi\nabla^{(h)}_B\Delta^{(h)}\Phi)\nabla^{(h)A}\nabla^{(h)B}\Phi+\langle\Phi\mathrm{Hess}^{(h)}(\Delta^{(h)}\Phi)-\mathrm{Hess}^{(h)}(|d\Phi|^2_h), \mathrm{Hess}^{(h)}\Phi\rangle_h\\ &+2Q\Phi|\mathrm{Hess}^{(h)}\Phi|^2_h-\frac{1}{2}|\mathrm{Hess}^{(h)}\Phi|^2_h\Delta^{(h)}\Phi+\frac{1}{2}(\Delta^{(h)}\Phi)^3+|\mathrm{Hess}^{(h)}\Phi|^3_h.\end{aligned}$$

On the other hand, for any one form $\alpha\in\Lambda^1(M^2)$, Weitzenböck formula implies

$$\begin{aligned}\int_{M^2}d\mu_h\nabla^{(h)}_A\alpha_B\nabla^{(h)A}\nabla^{(h)B}\Phi &= \int_{M^2}d\mu_h\langle\alpha, \nabla^{(h)*}\nabla^{(h)}(d\Phi)\rangle_h\\ &=\int_{M^2}d\mu_h[\nabla^{(h)}_A\alpha^A\Delta^{(h)}\Phi-Q\alpha^A\nabla^{(h)}_A\Phi].\end{aligned}$$

Then for $\alpha=2Q\Phi d\Phi+2\Phi d\Delta^{(h)}\Phi-d|d\Phi|^2_h$, we have

$$\begin{aligned}&\int_{M^2}d\mu_h\nabla^{(h)}_A\alpha_B\nabla^{(h)A}\nabla^{(h)B}\Phi\\ =&\int_{M^2}d\mu_h[-2\Phi|d\Delta^{(h)}\Phi|^2_h-|d\Phi|^2_h(\Delta^{(h)})^2\Phi-2Q^2\Phi|d\Phi|^2_h+4Q\Phi(\Delta^{(h)}\Phi)^2+3Q|d\Phi|^2_h\Delta^{(h)}\Phi]\\ =&\int_{M^2}d\mu_h[\frac{1}{2}|\mathrm{Hess}^{(h)}\Phi|^2_h\Delta^{(h)}\Phi+\Phi\langle\mathrm{Hess}^{(h)}(\Delta^{(h)}\Phi), \mathrm{Hess}^{(h)}\Phi\rangle_h-\frac{1}{2}(\Delta^{(h)}\Phi)^3-|\mathrm{Hess}^{(h)}\Phi|^3_h].\end{aligned}$$

We complete the proof. □

Type II : $(M^3, g_{\mu\nu}dx^\mu dx^\nu)\simeq(M^2, h_{AB}dx^Adx^B)\times(S^1, d\theta^2)$.

The metric is given by

$$g_{\mu\nu}dx^\mu dx^\nu=h_{AB}dx^Adx^B+\Phi^2d\theta^2. \tag{2.10}$$

Now h is a two-dimensional Lorentzian metric, which can be locally expressed in terms of double-null coordinates as

$$h_{AB}dx^Adx^B=2F(u,v)dudv, \tag{2.11}$$

for two null vectors $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$. The EOMs then reduce to

$$\frac{1}{2}\partial_u(\frac{1}{F}\partial_u\partial_v\log F)-\frac{1}{F}\partial_v(\frac{\partial_u^2\Phi-\partial_u\log F\partial_u\Phi}{\Phi})=0, \tag{2.12}$$

$$\frac{1}{2}\partial_v(\frac{1}{F}\partial_u\partial_v\log F)-\frac{1}{F}\partial_u(\frac{\partial_v^2\Phi-\partial_v\log F\partial_v\Phi}{\Phi})=0. \tag{2.13}$$

Proposition 2.2. *Let (M^2,h,Φ) form a vacuum configuration. Assume $\nabla^{(h)}\Phi$ is a null vector, then the Gauss curvature of h is sign-definite.*

Proof. Since $\nabla^{(h)}\Phi$ is a null vector, thus $|d\Phi|^2_h=0$, one can choose $\Phi=\Phi(u)$ without loss of generality, then $\Delta^{(h)}\Phi=0$. Therefore EOMs are simplified to

$$\Phi\frac{dQ}{du}+2Q\frac{d\Phi}{du}=0.$$

Hence $Q\Phi^2$ is a constant, and Q is sign-definite. □

Including the contribution of the matter to the action, the EOMs become

$$C_{\mu\nu}=8\pi\zeta G_3T_{\mu\nu}, \tag{2.14}$$

where $T_{\mu\nu}$ is the energy–momentum tensor for matter fields. For the metric form (2.10) and the energy–momentum tensor $T_{\mu\nu}dx^\mu dx^\nu=2T_{A\theta}dx^Ad\theta$, we explicitly write down the EOMs

$$-\frac{\sqrt{|\det h|}}{2\Phi}\nabla^{(h)1}L=8\pi\zeta G_3T_{0\theta}, \tag{2.15}$$

$$\frac{\sqrt{|\det h|}}{2\Phi}\nabla^{(h)0}L=8\pi\zeta G_3T_{1\theta}. \tag{2.16}$$

The Kodama vector V associated with the metric (2.10) is defined as [14]

$$V=\epsilon^{AB}\nabla^{(h)}_B\Phi\frac{\partial}{\partial x^A}. \tag{2.17}$$

It follows immediately the following properties of Kodama vector from the definition: V and $\nabla^{(h)}\Phi$ are orthogonal to each other, i.e. $h(V, \nabla^{(h)}\Phi) = 0$, and V is a local conserved current, i.e. $\nabla_A^{(h)} V^A = 0$.

We define a vector J via

$$J = -V^A V^B T_{B\theta} \frac{\partial}{\partial x^A}, \tag{2.18}$$

which is a conserved current if and only if

$$\langle V \otimes \nabla^{(h)} L, \mathrm{Hess}\Phi \rangle_h = V^A \nabla_A^{(h)} \nabla_B^{(h)} \Phi \nabla^{(h)^B} L = 0. \tag{2.19}$$

Under the constraint (2.19), one has the conserved charge $\mathcal{Q}_\Sigma$ for a space-like surface Σ in (M^3, g) defined as

$$\mathcal{Q}_\Sigma = \int_\Sigma *_g J^\vee, \tag{2.20}$$

where $J^\vee$ is the dual 1-form with respect to the metric h, which is naturally viewed as a 1-form on M^3. By virtue of (2.15)–(2.20), we can define a function $\mathfrak{M}$ over M^2, called quasi-local mass associated to the system (M^2, h, Φ), as follows

$$\begin{aligned}\mathfrak{M} =& -2\pi \int \sqrt{\det g}[(V^0V^0T_{0\theta} + V^0V^1T_{1\theta})dx^1 - (V^1V^0T_{0\theta} + V^1V^1T_{1\theta})dx^0] \\ =& \frac{1}{8\zeta G_3} \int \langle d\Phi, dL \rangle_h d\Phi,\end{aligned} \tag{2.21}$$

hence,

$$d\mathfrak{M} = \frac{1}{16\pi\zeta G_3} \langle d \ln|\Phi|, dL \rangle_h dS, \tag{2.22}$$

where $S = \pi\Phi^2$ stands for the areal volume. It is immediately seen that (2.19) is exactly the compatibility condition of (2.22).

Example 2.3. For the Eddington–Finkelstein-like metric

$$g = -(1 - \frac{c}{r})du^2 + 2dudr + r^2 d\theta^2$$

with a positive constant c, we directly calculate

$$\begin{aligned}\mathfrak{M} =& \frac{1}{8\zeta G_3} \int (\frac{d}{dr}(\frac{2c}{r} - 1))(1 - \frac{c}{r})dr \\ =& \frac{1}{8\zeta G_3}(\frac{2c}{r} - \frac{c^2}{r^2}).\end{aligned}$$

Proposition 2.4. *Under the Ricci flow on* (M^2, h)

$$\frac{\partial h_{ij}}{\partial t} = -2R_{ij}^{(h)},$$

the quasi-local mass $\mathfrak{M}$ *associated to the system* (M^2, h, Φ) *evolves by*

$$8\zeta G_3 \frac{\partial \mathfrak{M}}{\partial t} = 4\int Q\langle d\Phi, dL \rangle_h d\Phi + 2\int L\langle d\Phi, dQ \rangle_h d\Phi + \int \langle d\Phi, d(\Phi^2 \Delta^{(h)} Q + 4Q|d\Phi|_h^2) \rangle_h d\Phi.$$

Proof. Under the Ricci flow, the scalar L obeys

$$\begin{aligned}\frac{\partial L}{\partial t} =& (\Delta^{(h)} R^{(h)} + R^{(h)^{ij}} R_{ij}^{(h)} - \nabla^{(h)^i} \nabla^{(h)^j} R_{ij}^{(h)})\Phi^2 \\ &+ \Phi(2R_{ij}^{(h)} \nabla^{(h)^i} \nabla^{(h)^j} \Phi + 2\nabla_i^{(h)} R^{(h)^{ik}} \nabla_k^{(h)} \Phi - \nabla_i^{(h)} R^{(h)} \nabla^{(h)^i} \Phi) + 2\nabla^{(h)^i} \Phi \nabla^{(h)^j} \Phi R_{ij}^{(h)} \\ =& R^{(h)} L + \frac{1}{2}\Phi^2 \Delta^{(h)} R^{(h)} + 2R^{(h)} |d\Phi|_h^2,\end{aligned}$$

which leads to the conclusion. □

2.3. Dilaton gravity in two dimensions

For the 2(= 1+1)-dimensional system (M^2, h, Φ), one constructs a model of dilaton gravity governed by the action [20]

$$S[h, \Phi] = \int_{M^2} d^2x \sqrt{|\det h|} (\frac{1}{3} R^{(h)} \Phi^3 - 2\Phi|d\Phi|_h^2 + c\Phi) \tag{2.23}$$

with a constant c. The variations with respect to the metric h and the dilaton field Φ yield EOMs respectively

$$\Phi(h_{\mu\nu}\Delta^{(h)} - \nabla^{(h)}_\mu\nabla^{(h)}_\nu)\Phi + 3|d\Phi|^2_h h_{\mu\nu} - 4\nabla^{(h)}_\mu\Phi\nabla^{(h)}_\nu\Phi - \frac{c}{2}h_{\mu\nu} = 0, \tag{2.24}$$

$$R^{(h)}\Phi^2 + 4\Phi\Delta^{(h)}\Phi + 2|d\Phi|^2_h + c = 0. \tag{2.25}$$

Taking the trace of the first equation leads to

$$\Phi\Delta^{(h)}\Phi + 2|d\Phi|^2_h = c,$$

substituting which into the second equation gives us

$$R^{(h)}\Phi^2 + 3\Phi\Delta^{(h)}\Phi = -2c.$$

Therefore the vacuum configurations of the 2-dimensional dilaton gravity can be embedded into those of the 3-dimensional Chern–Simons-like gravity. Then we immediately see that for the vacuum configuration (M^2, h, Φ), where M^2 is a closed manifold and Φ is sign-definite, the Euler number χ of M^2 and the constant c have the opposite signs.

Example 2.5. Assume (M^2, h, Φ) is a vacuum configuration with flat metric $h_{\mu\nu} = \eta_{\mu\nu}$, then the dilaton field Φ is subject to the following EOMs in terms of local light-cone coordinates $\xi^+ = \sigma^0 + \sigma^1$, $\xi^- = -\sigma^0 + \sigma^1$,

$$\begin{aligned}
&\Phi\partial_+^2\Phi + 4(\partial_+\Phi)^2 = 0,\\
&\Phi\partial_-^2\Phi + 4(\partial_-\Phi)^2 = 0,\\
&\Phi\partial_+\partial_-\Phi = -\frac{1}{6}c,\\
&\partial_+\Phi\partial_-\Phi = \frac{5}{24}c,
\end{aligned}$$

where $\partial_+ = \frac{\partial}{\partial\xi_+}$, $\partial_- = \frac{\partial}{\partial\xi_-}$.

- If M^2 is a torus, these equations only admit the solution of constant Φ with zero c.
- If $M^2 = \mathbb{R}^2$, one finds solutions $\Phi = a\xi_+^{\frac{1}{5}}, b\xi_-^{\frac{1}{5}}$ with zero c, where a, b are constants.

First-order formalism

The zweibein and the spin connection on M^2 are denoted by e^A, $\omega^A_B = \omega\epsilon^A_B = \omega_C\epsilon^A_B e^C$ respectively, where $\epsilon^A_B = \eta^{AC}\epsilon_{CB}$ ($\epsilon_{01} = -\epsilon^{01} = 1$). The torsion form and the curvature form are given by $T^A = de^A + \epsilon^A_B\omega\wedge e^B$ and $\mathcal{R}^A_B = \epsilon^A_B d\omega$, respectively. Introducing the auxiliary fields X^A and Y, let us consider the general first-order action constructed by Cartan variables e^A, ω_A and their first-order derivatives as follows [3]

$$S_{\text{f.o.}} = \int_{M^2} \eta_{AB}X^A T^B + Yd\omega + \epsilon V(X^A X_A, Y), \tag{2.26}$$

where the volume form $\epsilon = -\frac{1}{2}\epsilon_{AB}e^A\wedge e^B$. The EOMs from the variation of XA read

$$T^A + \epsilon\frac{\partial V}{\partial X_A} = 0,$$

thus

$$*T^A = -\frac{\partial V}{\partial X_A} = *de^A - \omega^A.$$

Reinserting it into the action (2.26) gives rise to

$$S_{\text{f.o.}} = \int_{M^2} -X_A\frac{\partial V}{\partial X_A}\epsilon + Yd\tilde{\omega} - dY - e_A\frac{\partial V}{\partial X_A} + \epsilon V, \tag{2.27}$$

where $\tilde{\omega} = e_A * de^A$ denotes the torsion-free part of the connection. The variation of X_A once again provides EOMs

$$(dY\wedge e_C + X_C\epsilon)\frac{\partial^2 V}{\partial X_C\partial X_A} = 0.$$

Assume $\det(\frac{\partial^2 V}{\partial X_C\partial X_A}) \neq 0$, then we arrive at

$$S_{\text{f.o.}} = \int_{M^2} Yd\tilde{\omega} + \epsilon V(X^A X_A, Y) = -\int_{M^2}\sqrt{|\det h|}\{\frac{Y}{2}R^{(h)} + V(-|dY^2|_h, Y)\}. \tag{2.28}$$

As a consequence, if we choose

$$Y = -\frac{2}{3}\Phi^3,\ V(X^AX_A, Y) = \frac{\eta_{AB}X^AX^B}{3Y} - c\sqrt[3]{-\frac{3}{2}Y},$$

the action (2.23) of dilaton gravity is recovered.

Liouville formalism

We introduce the following transformations of fields: $\Upsilon = a\Psi^2$, $\hat{h} = e^{-2\rho}h$ with $\Psi = \sqrt{\frac{1}{3}\Phi^3}$ and $\rho = \frac{a}{4}\Psi^2 + \frac{2}{3}\ln\Psi$, then

$$S[\hat{h}, \Upsilon] = \frac{1}{a}\int_{M^2} d^2x\sqrt{|\det\hat{h}|}(R^{(\hat{h})}\Upsilon + \frac{1}{2}|d\Upsilon|^2_{\hat{h}} + c'e^{\frac{\Upsilon}{2}}\Upsilon), \tag{2.29}$$

where $c' = \sqrt[3]{3}c$. It gives rise to the EOMs

$$2(\hat{h}_{\mu\nu}\Delta^{(\hat{h})} - \nabla^{(\hat{h})}_\mu\nabla^{(\hat{h})}_\nu)\Upsilon + \nabla^{(\hat{h})}_\mu\Upsilon\nabla^{(\hat{h})}_\nu\Upsilon - \hat{h}_{\mu\nu}(\frac{1}{2}|d\Upsilon|^2_{\hat{h}} + c'e^{\frac{\Upsilon}{2}}\Upsilon) = 0, \tag{2.30}$$

$$R^{(\hat{h})} - \Delta^{(\hat{h})}\Upsilon + c'e^{\frac{\Upsilon}{2}}(1 + \frac{\Upsilon}{2}) = 0. \tag{2.31}$$

It is easily seen that this formalism eliminates the vacuum configurations with constant scalar curvature unless $c = 0$. However, we let $\check{h} = e^{\frac{\Upsilon}{2}}\hat{h}$, $\Xi = \frac{1}{a}\Upsilon$, then the action (2.29) is further simplified to

$$S[\check{h}, \Xi] = \int_{M^2} d^2x\sqrt{|\det\check{h}|}(R^{(\check{h})}\Xi + c'\Xi), \tag{2.32}$$

whose vacuum configuration $(M^2, \check{h}, \Xi)$ consists of a surface with constant scalar curvature $-c'$, i.e. uniformization of surfaces, and a function Ξ governed by equation

$$\nabla^{(\check{h})}_\mu\nabla^{(\check{h})}_\nu\Xi = \frac{c'}{2}\check{h}_{\mu\nu}\Xi. \tag{2.33}$$

In particular, when M^2 is a closed manifold, one finds that

$$\text{if the vector field } \nabla^{(\check{h})}\Xi \text{ is } \begin{cases} \text{timelike,} \\ \text{spacelike,} \\ \text{lightlike,} \end{cases} \text{ then the scalar curvature of } \check{h} \text{ is } \begin{cases} \text{negative,} \\ \text{positive,} \\ \text{zero.} \end{cases}$$

In terms of locally conformally flat coordinates under which $\check{h}_{\mu\nu} = e^{2\rho}\eta_{\mu\nu}$, Eq. (2.33) is rewritten as

$$\partial_\mu\partial_\nu\Xi - \partial_\mu\rho\partial_\nu\Xi - \partial_\mu\Xi\partial_\nu\rho - \eta_{\mu\nu}(\frac{c'}{2}e^{2\rho}\Xi - \partial_\alpha\Xi\partial^\alpha\rho) = 0,$$

where the local function ρ satisfies Liouville equation

$$(-\partial_0^2 + \partial_1^2)\rho = \frac{c'}{2}e^{2\rho}.$$

Canonical formalism

It requires that M^2 has a $(1+1)$-decomposition $M^2 \simeq \Sigma\times\mathbb{R}$ topologically, and we write the metric in terms of ADM form

$$ds^2 = -N^2dt^2 + \sigma^2(dx + ndt)^2 \tag{2.34}$$

for the lapse function N and the shift function n. The Lagrangian density $\mathcal{L}$ of the action (2.23) is given by

$$\begin{aligned}\mathcal{L} =& -2\frac{\Phi^2\dot{\Phi}\dot{\sigma}}{N} - 2N(\frac{\Phi^2\Phi'}{\sigma})' - 2\frac{n\Phi^2(n\sigma)'\Phi'}{N} + 2\frac{n\Phi^2\dot{\sigma}\Phi'}{N} + 2\frac{n\Phi^2(n\sigma)'\dot{\Phi}}{N} \\ &+ 2\frac{\sigma\Phi(\dot{\Phi})^2}{N} - 4\frac{n\sigma\Phi(\Phi)'\dot{\Phi}}{N} + 2\frac{\Phi(\Phi')^2(\sigma^2n^2 - N^2)}{N\sigma} + c\sigma N\Phi,\end{aligned}$$

which shows N and n play the role of Lagrange multipliers. After introducing the canonical momenta for σ and Φ as

$$\begin{aligned}\Pi_\sigma &= \frac{\partial\mathcal{L}}{\partial\dot{\sigma}} = -2\frac{\Phi^2\dot{\Phi}}{N} + 2\frac{n\Phi^2\Phi'}{N}, \\ \Pi_\Phi &= \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = -2\frac{\Phi^2\dot{\sigma}}{N} + 2\frac{\Phi^2(n\sigma)'}{N} + 4\frac{\sigma\Phi\dot{\Phi}}{N} - 4\frac{n\sigma\Phi\Phi'}{N},\end{aligned}$$

up to a boundary term, the action (2.23) is reexpressed as

$$S = \int dt \int dx(\Pi_\sigma \dot{\sigma} + \Pi_\Phi \dot{\Phi} - N\mathcal{H}_0 - n\mathcal{H}_1), \tag{2.35}$$

which exhibits the secondary constraints in the Dirac sense given by

$$\begin{aligned}
\mathcal{H}_0 &= -\frac{1}{2\Phi^2}\Pi_\sigma \Pi_\Phi + 2(\frac{\Phi^2\Phi'}{\sigma})' + 2\frac{\Phi(\Phi')^2}{\sigma} - \frac{5\sigma}{2\Phi^3}\Pi_\sigma^2 - c\sigma\Phi \approx 0,\\
\mathcal{H}_1 &= \Pi_\Phi \Phi' - \sigma \Pi_\sigma' \approx 0.
\end{aligned}$$

More conveniently, working with the action (2.32), we have

$$\begin{aligned}
\mathcal{H}_0 &= -\frac{1}{2}\Pi_\sigma \Pi_\Xi + 2(\frac{\Xi'}{\sigma})' - c'\sigma\Xi,\\
\mathcal{H}_1 &= \Pi_\Xi \Xi' - \sigma\Pi_\sigma',
\end{aligned}$$

which form a closed Poisson bracket algebra

$$\begin{aligned}
\{\sigma(x)\mathcal{H}_0(x), \sigma(y)\mathcal{H}_0(y)\} &= -\delta'(x-y)(\mathcal{H}_1(x) + \mathcal{H}_1(y)),\\
\{\mathcal{H}_1(x), \sigma(y)\mathcal{H}_0(y)\} &= -\frac{3}{2}\delta'(x-y)(\sigma(x)\mathcal{H}_0(x) + \sigma(y)\mathcal{H}_0(y)),\\
\{\mathcal{H}_1(x), \mathcal{H}_1(y)\} &= -\delta'(x-y)(\mathcal{H}_1(x) + \mathcal{H}_1(y)).
\end{aligned}$$

2.4. Boundary terms

We consider a radial foliation of the spacetime M^3 such that metric near the boundary surface $\{r = \text{constant}\}$ is written as

$$g_{\mu\nu}dx^\mu dx^\nu = N^2(r)dr^2 + h_{ij}(r,x)dx^i dx^j \tag{2.36}$$

Then the unit normal vector n is given by $n = \frac{1}{N}\frac{\partial}{\partial r}$, and the extrinsic curvature of the boundary reads

$$K_{ij} = \frac{1}{2}\mathcal{L}_n h_{ij} = \frac{1}{2N}\frac{\partial h_{ij}}{\partial r} \tag{2.37}$$

By Gauss–Codazzi–Ricci equations

$$\begin{aligned}
R_{ij} &= \frac{1}{2}h_{ij}R^{(h)} - K_{ij}K + 2K_{im}K_j^m - \frac{1}{N}\frac{\partial K_{ij}}{\partial r},\\
R_{rr} &= -N\frac{\partial K}{\partial r} - N^2 K_i^j K_j^i,\\
R_{ir} &= N\nabla_j^{(h)}K_i^j - N\nabla_i^{(h)}K,
\end{aligned}$$

where $K_i^j = K_{il}h^{jl}$, $K = K_{ij}h^{ij}$, we have

$$\begin{aligned}
\Theta^{rr} &= 0,\\
\Theta^{ij} &= -\frac{1}{2}(\epsilon^i{}_k R^{jk} + \epsilon^j{}_k R^{ik})\\
&= \frac{1}{2}(\epsilon^i{}_k(K^{jk}K - 2K^{jm}K_m^k + \mathcal{L}_n K^{jk}) + \epsilon^j{}_k(K^{ik}K - 2K^{im}K_m^k + \mathcal{L}_n K^{ik})),\\
\Theta^{ri} &= -\frac{1}{2}\epsilon^i{}_k R^{rk} = -\frac{1}{2N}\epsilon^i{}_k(\nabla_j^{(h)}K^{kj} - \nabla^{(h)k}K),
\end{aligned}$$

where $\epsilon^{ij} = \frac{\varepsilon^{ij}}{\sqrt{|\det h|}}$. Therefore, we have

Proposition 2.6. *If the variation of the metric preserves the form in (2.36), the boundary terms are rewritten as*

$$\begin{aligned}
\Sigma_1 =& \frac{1}{16\pi G_3\mu}\int_{\partial M^3} d^2x\sqrt{|\det h|}\epsilon^i{}_k(K^{jk}K - 2K^{jm}K_m^k + \mathcal{L}_n K^{jk})\delta h_{ij},\\
\Sigma_2 =& \frac{1}{64\pi G_3\mu}\int_{\partial M^3} d^2x\sqrt{|\det h|}\epsilon^{ij}(\Gamma^{(h)})^m_{ik}h^{kl}(\nabla_j^{(h)}\delta h_{ml} + \nabla_m^{(h)}\delta h_{lj} - \nabla_l^{(h)}\delta h_{jm})\\
&- \frac{1}{16\pi G_3\mu}\int_{\partial M^3} d^2x\sqrt{|\det h|}\epsilon^{ij}K_i^k\delta K_{jk}
\end{aligned}$$

In particular, if M^3 is an asymptotically AdS_3 manifold, in a finite neighbourhood of ∂M^3, the metric takes the form

$$g_{\mu\nu}dx^\mu dx^\nu = d\rho^2 + h_{ij}(x,\rho)dx^i dx^j, \tag{2.38}$$

where h admits Fefferman–Graham expansion [7]

$$h_{ij} = e^{2\rho}h^{(0)}_{ij} + e^{\rho}h^{(1)}_{ij} + h^{(2)}_{ij} + e^{-\rho}h^{(3)}_{ij} + e^{-2\rho}h^{(4)}_{ij} + \mathcal{O}(e^{-3\rho}) \tag{2.39}$$

with the regular induced metric $h^{(0)}$ on the boundary ∂M^3 ($\rho \to \infty$). We calculate the Christoffel symbols, Ricci tensor and Ricci scalar in terms of Fefferman–Graham expansion

$$\begin{aligned}
\Gamma^i_{j\rho} =& \delta^i_j - \frac{1}{2}e^{-\rho}(h^{(1)})^i_j - e^{-2\rho}[(h^{(2)})^i_j - \frac{1}{2}((h^{(1)})^2)^i_j] + \mathcal{O}(e^{-3\rho}),\\
\Gamma^\rho_{ij} =& -e^{2\rho}h^{(0)}_{ij} - \frac{1}{2}e^{\rho}h^{(1)}_{ij} + \frac{1}{2}e^{-\rho}h^{(3)}_{ij} + e^{-2\rho}h^{(4)}_{ij} + \mathcal{O}(e^{-3\rho}),\\
\Gamma^k_{ij} =& (\Gamma^{(h^{(0)})})^k_{ij} + \frac{1}{2}e^{-\rho}(h^{(0)})^{kl}[\nabla^{(h^{(0)})}_i(h^{(1)})_{lj} + \nabla^{(h^{(0)})}_j(h^{(1)})_{il} - \nabla^{(h^{(0)})}_l(h^{(1)})_{ij}]\\
&+ \frac{1}{2}e^{-2\rho}[(h^{(0)})^{kl}(\nabla^{(h^{(0)})}_i(h^{(2)})_{lj} + \nabla^{(h^{(0)})}_j(h^{(2)})_{il} - \nabla^{(h^{(0)})}_l(h^{(2)})_{ij})\\
&- (h^{(1)})^{kl}(\nabla^{(h^{(0)})}_i(h^{(1)})_{lj} + \nabla^{(h^{(0)})}_j(h^{(1)})_{il} - \nabla^{(h^{(0)})}_l(h^{(1)})_{ij})] + \mathcal{O}(e^{-3\rho});\\
R^i_j =& -2\delta^i_j + \frac{1}{2}e^{-\rho}[(h^{(1)})^i_j + (\mathrm{Tr}h^{(1)})\delta^i_j] + e^{-2\rho}[\frac{1}{2}(R^{(h^{(0)})} + 2\mathrm{Tr}h^{(2)} - \mathrm{Tr}(h^{(1)})^2)\delta^i_j - \frac{1}{4}(\mathrm{Tr}h^{(1)})(h^{(1)})^i_j] + \mathcal{O}(e^{-3\rho}),\\
R^\rho_i =& \frac{1}{2}e^{-\rho}[\partial_i\mathrm{Tr}h^{(1)} - \nabla^{(h^{(0)})}_j(h^{(1)})^j_i] + e^{-2\rho}[\partial_i\mathrm{Tr}h^{(2)} - \nabla^{(h^{(0)})}_j(h^{(2)})^j_i + \frac{1}{2}(h^{(1)})_{ij}\nabla^{(h^{(0)})}_k(h^{(1)})^{jk}\\
&+ \frac{1}{2}(h^{(1)})^{jk}\nabla^{(h^{(0)})}_j(h^{(1)})_{ki} - \frac{3}{4}(h^{(1)})^{kl}\nabla^{(h^{(1)})}_i(h^{(1)})_{kl} - \frac{1}{4}(h^{(1)})^j_i\partial_j\mathrm{Tr}h^{(1)}] + \mathcal{O}(e^{-3\rho}),\\
R^i_\rho =& \frac{1}{2}e^{-3\rho}[\nabla^{(h^{(0)})^i}\mathrm{Tr}h^{(1)} - \nabla^{(h^{(0)})}_j(h^{(1)})^{ij}] + e^{-4\rho}[\nabla^{(h^{(0)})^i}\mathrm{Tr}h^{(2)} - \nabla^{(h^{(0)})}_j(h^{(2)})^{ij} + (h^{(1)})^i_j\nabla^{(h^{(0)})}_k(h^{(1)})^{jk}\\
&+ \frac{1}{2}(h^{(1)})^{jk}\nabla^{(h^{(0)})}_j(h^{(1)})^i_k - \frac{3}{4}(h^{(1)})^{kl}\nabla^{(h^{(1)})^i}(h^{(1)})_{kl} - \frac{3}{4}(h^{(1)})^{ij}\partial_j\mathrm{Tr}h^{(1)}] + \mathcal{O}(e^{-5\rho}),\\
R^\rho_\rho =& -2 + \frac{1}{2}e^{-\rho}\mathrm{Tr}h^{(1)} - \frac{1}{4}e^{-2\rho}\mathrm{Tr}(h^{(1)})^2 + \mathcal{O}(e^{-3\rho}),\\
R =& -6 + 2e^{-\rho}\mathrm{Tr}h^{(1)} + e^{-2\rho}[R^{(h^{(0)})} + 2\mathrm{Tr}h^{(2)} - \frac{5}{4}\mathrm{Tr}(h^{(1)})^2 - \frac{1}{4}(\mathrm{Tr}h^{(1)})^2] + \mathcal{O}(e^{-3\rho}),
\end{aligned}$$

where on the right hand side of the equalities all indices are raised or lowered by $h^{(0)}$ and Tr denotes the contraction with $h^{(0)}$. Let us assume M^3 be a vacuum configuration of Chern–Simons-like gravity, then substituting these expansions into EOMs $\partial_A R = 2\nabla_B R^B_A$ up to the second-order term leads to the constraint that the second-order component of the expansion of scalar curvature

$$R^{(2)} := R^{(h^{(0)})} + 2\mathrm{Tr}h^{(2)} - \frac{5}{4}\mathrm{Tr}(h^{(1)})^2 - \frac{1}{4}(\mathrm{Tr}h^{(1)})^2 \tag{2.40}$$

is a constant.[1] More generally, we propose the following conjecture.

Conjecture 2.7. *If (M^3, g) is a vacuum configuration of Chern–Simons-like gravity, then the even-order components $R^{(2k)}$ of the Fefferman–Graham expansion of scalar curvature R would be constant.*

Also due to Fefferman–Graham expansion, one finds that the value of the action vanishes tending to the boundary. Indeed, we have

$$\begin{aligned}
S[g] &\sim \int_{M^3} d^3x\varepsilon^{ij}(2\Gamma^k_{\rho l}\partial_i\Gamma^l_{jk} - 2\Gamma^k_{i\rho}\partial_\rho\Gamma^\rho_{kj} - \Gamma^k_{il}\partial_\rho\Gamma^l_{kj} + 2\Gamma^k_{\rho l}\Gamma^l_{im}\Gamma^m_{jk} + 2\Gamma^k_{\rho l}\Gamma^l_{i\rho}\Gamma^\rho_{jk})\\
&= \int_{M^3} d^3x\varepsilon^{ij}(-2\Gamma^k_{i\rho}\partial_\rho\Gamma^\rho_{kj} - \Gamma^k_{il}\partial_\rho\Gamma^l_{kj})\\
&\sim \int_{M^3} d^3x\mathcal{O}(e^{-\rho}).
\end{aligned}$$

[1] Here we assume that Witten–Yau conjecture that asserts that there exists a metric with non-negative curvature on the boundary is valid, which implies the connectedness of the boundary [23,24].

Only the boundary terms contribute to the on shell variation of the action, which gives rise to the Brown–York stress, namely we can define

$$(T^{\mathrm{BY}})^{ij} = \frac{2}{\sqrt{\det|h^{(0)}|}}\frac{\delta(\Sigma_1+\Sigma_2)}{\delta(h^{(0)})_{ij}}. \tag{2.41}$$

Omitting the term $\delta\Gamma^{(h)}$, we have the Fefferman–Graham expansion by virtue of Proposition 2.6

$$\begin{aligned}(T^{\mathrm{BY}})^{ij} &= \frac{1}{8\pi G_3\zeta}e^{2\rho}(\epsilon^{(h^{(0)})})^{ik}(K^j_kK - 2K^j_mK^m_k + \mathcal{L}_nK^j_k + K^j_k) + (i\leftrightarrow j)\\ &= \frac{1}{8\pi G_3\zeta}(\epsilon^{(h^{(0)})})^{ik}(e^{\rho}(h^{(1)})^j_k + \frac{1}{4}\mathrm{Tr}h^{(1)}(h^{(1)})^j_k + 3(h^{(2)})^j_k - 2((h^{(1)})^2)^j_k + (i\leftrightarrow j)) + \mathcal{O}(e^{-\rho}).\end{aligned}$$

We need to introduce the additional counterterm to kill the divergence. A suitable choice is given by

$$S_{\mathrm{ct}}[h] = \frac{1}{16\pi G_3\mu}\int_{\partial M^3}\sqrt{|\det h|}\epsilon^i{}_kK^l_i\mathcal{L}_nK^k_l,$$

which causes the stress tensor.

$$(T^{\mathrm{ct}})^{ij} = -\frac{1}{8\pi G_3\zeta}(\epsilon^{(h^{(0)})})^{ik}(e^{\rho}(h^{(1)})^j_k + 2(h^{(2)})^j_k - ((h^{(1)})^2)^j_k) + (i\leftrightarrow j) + \mathcal{O}(e^{-\rho}).$$

As a result, the total stress tensor given by

$$(T^{\mathrm{tot}})^{ij} = \frac{1}{8\pi G_3\zeta}(\epsilon^{(h^{(0)})})^{ik}(\frac{1}{4}\mathrm{Tr}h^{(1)}(h^{(1)})^j_k + (h^{(2)})^j_k - ((h^{(1)})^2)^j_k) + (i\leftrightarrow j) + \mathcal{O}(e^{-\rho})$$

is finite on the boundary. Obviously, it leads to no Weyl anomaly, but diffeomorphism anomaly for the boundary CFT by AdS/CFT correspondence:

$$\begin{aligned}&\langle T^i_i\rangle = 0,\\ &\nabla^{h^0}_i\langle T^{ij}\rangle = \frac{1}{8\pi G_3\zeta}\nabla^{h^0}_i[(\epsilon^{(h^{(0)})})^{ik}(\frac{1}{4}\mathrm{Tr}h^{(1)}(h^{(1)})^j_k + (h^{(2)})^j_k - ((h^{(1)})^2)^j_k) + (i\leftrightarrow j)].\end{aligned}$$

Define a (1, 1)-tensor $\mathcal{T}^j_k = \frac{1}{4}\mathrm{Tr}h^{(1)}(h^{(1)})^j_k + (h^{(2)})^j_k - ((h^{(1)})^2)^j_k$. The vanishing of diffeomorphism anomaly requires the equations

$$(\epsilon^{(h^{(0)})})^{ik}\nabla^{(h^{(0)})}_i\mathcal{T}^j_k + (\epsilon^{(h^{(0)})})^{jk}\nabla^{(h^{(0)})}_k\mathrm{Tr}\mathcal{T} = 0.$$

Obviously, the trace $\mathrm{Tr}\mathcal{T} = \frac{1}{4}(\mathrm{Tr}h^{(1)})^2 + \mathrm{Tr}h^{(2)} - \mathrm{Tr}(h^{(1)})^2$ is a constant if and only if $d_{\nabla^{(h^{(0)})}}\mathcal{T} = 0$. In particular, if $\mathcal{T}$ is trace-free and $R^{(2)}$ vanishes, $R^{(h^{(0)})} = \frac{3}{4}((\mathrm{Tr}h^{(1)})^2 - \mathrm{Tr}(h^{(1)})^2) = \frac{3}{2}f|\det h^{(1)}| > 0$ for a positive local function f when $h^{(1)}$ can be viewed as a metric on the boundary with the same signature as $h^{(0)}$, which agrees with the Witten–Yau conjecture.

3. The higher-order variation

3.1. Linear cotton operator

To consider the linear gravity, we should linearize the Cotton tensor with respect to a fixed background metric $\bar{g}$ around which there is a small perturbation $\mathfrak{h}$, thus we write $g_{\mu\nu} = \bar{g}_{\mu\nu} + \mathfrak{h}_{\mu\nu}$. For simplicity, the background metric is usually chosen to be a constant curvature space as the special vacuum configuration of Chern–Simons-like gravity.

Lemma 3.1. *Let $\mathcal{D}$ be the set of symmetric (0, 2)-tensors, and $\mathcal{D}'$ be a subset od $\mathcal{D}$ consisting of traceless and transverse tensors, i.e. $\mathcal{D}' := \{T \in \mathcal{D} : \mathrm{Tr}T = \bar{g}^{\mu\nu}T_{\mu\nu} = 0, \bar{\nabla}^{\mu}T_{\mu\nu} = 0\}$. We define three operators on $\mathcal{D}$: $\mathbb{L} = \bar{\Delta} - \frac{1}{3}\bar{R}$ and $(\Pi(T))_{\mu\nu} = \frac{1}{2}(\bar{\epsilon}_{\mu}{}^{\alpha\beta}\bar{\nabla}_{\alpha}T_{\beta\nu} + \bar{\epsilon}_{\nu}{}^{\alpha\beta}\bar{\nabla}_{\alpha}T_{\beta\mu})$ and the composition $\Xi = \mathbb{L}\circ\Pi$ called the linear Cotton operator with respect to the background metric of constant curvature, where all the barred quantities are taken with respect to the fixed background metric,*

(1) Ξ is a third-order self-adjoint differential operator on $\mathcal{D}'$ with respect to the global inner product induced by the background metric,

(2) if $T \in \mathcal{D}'$ is generated by a vector field X via $T_{\mu\nu} = \bar{\nabla}_{\mu}X_{\nu} + \bar{\nabla}_{\nu}X_{\mu}$, then $\Xi(T) = \frac{1}{3}\bar{R}\Pi(T)$,

(3) if $\mathfrak{h} \in \mathcal{D}'$, then $-\frac{1}{2}\Xi(\mathfrak{h})$ is exactly the first-order approximation of Cotton tensor.

Proof. (1) Firstly, we show that Ξ is an operator on $\mathcal{D}'$, namely $\Xi(T)$ is also traceless and transverse. The tracelessness is evident. We check the transversality, which is implied by the following identities

$$
\begin{aligned}
&\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\nabla_\alpha\bar\Delta T_{\beta\nu} = \bar\epsilon\ {}_{\mu}{}^{\alpha\beta}(\bar\Delta\bar\nabla_\alpha T_{\beta\nu} + \bar\nabla_{[\alpha}\bar\nabla_{\rho]}\bar\nabla^\rho T_{\beta\nu} + \bar\nabla^\rho\nabla_{[\alpha}\bar\nabla_{\rho]}\bar T_{\beta\nu})\\
=&\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}(\bar\Delta\bar\nabla_\alpha T_{\beta\nu} - \bar R\ {}_{\alpha\rho\lambda}{}^{\rho}\bar\nabla^\lambda T_{\beta\nu} + \bar R\ {}_{\alpha\rho\beta}{}^{\lambda}\bar\nabla^\rho T_{\lambda\nu} + \bar R\ {}_{\alpha\rho\nu}{}^{\lambda}\bar\nabla^\rho T_{\beta\lambda} + \bar\nabla^\rho(\bar R\ {}_{\alpha\rho\beta}{}^{\lambda}T_{\lambda\nu} + \bar R\ {}_{\alpha\rho\nu}{}^{\lambda}T_{\lambda\beta}))\\
=&\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\Delta\bar\nabla_\alpha T_{\beta\nu},
\end{aligned}
$$

and

$$
\begin{aligned}
&\bar\nabla^\mu(\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\nabla_\alpha T_{\beta\nu}) = \frac{1}{2}\bar\epsilon^{\mu\alpha\beta}(\bar R\ {}_{\mu\alpha\beta}{}^{\lambda}T_{\lambda\nu} + \bar R\ {}_{\mu\alpha\nu}{}^{\lambda}T_{\lambda\beta}) = 0,\\
&\bar\nabla^\mu(\bar\epsilon\ {}_{\nu}{}^{\alpha\beta}\bar\nabla_\alpha T_{\beta\mu}) = \frac{1}{2}\bar\epsilon\ {}_{\nu}{}^{\alpha\beta}(\bar R\ {}_{\mu\alpha\beta}{}^{\lambda}T^\mu_\lambda + \bar R_{\alpha\lambda}T^\lambda_\beta) = 0,
\end{aligned}
$$

where we have noted that the background metric is of constant curvature, thus

$$
\bar R_{\mu\nu\alpha\beta} = \frac{\bar R}{6}(\bar g_{\mu\alpha}\bar g_{\nu\beta} - \bar g_{\mu\beta}\bar g_{\nu\alpha}). \tag{3.1}
$$

Next we reformulate the operator Ξ. Let $(\Pi_0(T))_{\mu\nu} = \bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\nabla_\alpha T_{\beta\nu}$, then

$$
(\Xi(T))_{\mu\nu} = -\mathrm{sgn}(\det\bar g)\frac{1}{2}((\Pi_0^3(T))_{\mu\nu} + (\Pi_0^3(T))_{\nu\mu}) + \frac{1}{6}\bar R(\Pi(T))_{\mu\nu}. \tag{3.2}
$$

Indeed, by definitions, we have

$$
\begin{aligned}
-\mathrm{sgn}(\det\bar g)(\Pi_0^3(T))_{\mu\nu} =& -\mathrm{sgn}(\det\bar g)\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\epsilon\ {}_{\beta}{}^{\gamma\sigma}\bar\epsilon\ {}_{\sigma}{}^{\rho\lambda}\bar\nabla_\alpha\bar\nabla_\gamma\bar\nabla_\rho T_{\lambda\nu}\\
=& -\bar g_{\mu\eta}(\delta^\eta_\gamma\delta^\alpha_\sigma - \delta^\eta_\sigma\delta^\alpha_\gamma)\bar\epsilon^{\sigma\rho\lambda}\bar\nabla_\alpha\bar\nabla^\gamma\bar\nabla_\rho T_{\lambda\nu}\\
=&\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\Delta\bar\nabla_\alpha T_{\beta\nu} - \bar\epsilon^{\alpha\rho\lambda}\bar\nabla_\alpha\bar\nabla_{[\mu}\bar\nabla_{\rho]}T_{\lambda\nu} - \frac{1}{2}\bar\epsilon^{\alpha\rho\lambda}\bar\nabla_{[\alpha}\bar\nabla_{\rho]}\bar\nabla_\mu T_{\lambda\nu}\\
=&\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\Delta\bar\nabla_\alpha T_{\beta\nu} - \bar\epsilon^{\alpha\rho\lambda}\bar\nabla_\alpha(\bar R\ {}_{\mu\rho\lambda}{}^{\sigma}T_{\sigma\nu} + \bar R\ {}_{\mu\rho\nu}{}^{\sigma}T_{\sigma\lambda})\\
&-\frac{1}{2}\bar\epsilon^{\alpha\rho\lambda}(\bar R\ {}_{\alpha\rho\mu}{}^{\sigma}\bar\nabla_\sigma T_{\lambda\nu} + \bar R\ {}_{\alpha\rho\lambda}{}^{\sigma}\bar\nabla_\mu T_{\sigma\nu} + \bar R\ {}_{\alpha\rho\nu}{}^{\sigma}\bar\nabla_\mu T_{\lambda\sigma}).
\end{aligned}
$$

By means of (3.1), we arrive at

$$
-\mathrm{sgn}(\det\bar g)(\Pi_0^3(T))_{\mu\nu} = \bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\Delta\bar\nabla_\alpha T_{\beta\nu} - \frac{1}{3}\bar R\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\nabla_\alpha T_{\beta\nu} - \frac{1}{6}\bar R\bar\epsilon\ {}_{\nu}{}^{\alpha\beta}\bar\nabla_\alpha T_{\beta\mu}.
$$

Therefore, we only need to show Π_0 is self-adjoint, which easily follows from

$$
\int_{M^3} d\mu_{\bar g}\langle\Pi_0(T), S\rangle_{\bar g} = \int_{M^3} d\mu_{\bar g}\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\nabla_\alpha T_{\beta\nu}S^{\mu\nu} = \int_{M^3} d\mu_{\bar g}T^{\beta\nu}\bar\epsilon\ {}_{\beta}{}^{\alpha\mu}\bar\nabla_\alpha S_{\mu\nu} = \int_{M^3} d\mu_{\bar g}\langle T, \Pi_0(S)\rangle_{\bar g},
$$

for $S, T \in \mathcal{D}$.

(2) The vector field X should satisfy the conditions $\bar\Delta X = \bar\nabla\cdot X = 0$. Then

$$
\begin{aligned}
\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\nabla_\alpha\bar\Delta(\bar\nabla_\beta X_\nu + \bar\nabla_\nu X_\beta) &= \bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\nabla_\alpha(\bar\nabla^\rho\bar\nabla - [\rho\bar\nabla_{\beta]}X_\nu + \bar\nabla - [\rho\bar\nabla_{\beta]}\bar\nabla^\rho X_\nu + (\beta\leftrightarrow\nu))\\
&= \frac{2}{3}\bar R\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}(\bar\nabla_\alpha\bar\nabla_\nu X_\beta + \bar\nabla_\alpha\bar\nabla_\beta X_\nu).
\end{aligned}
$$

(3) We calculate the Cotton tensor up to the first-order of h

$$
\begin{aligned}
2C_{\mu\nu} =&\bar C_{\mu\nu} - \frac{1}{2}\mathrm{Tr}\mathfrak{h}\bar C_{\mu\nu} + \mathfrak{h}_{\mu\lambda}\bar\epsilon^{\lambda\alpha\beta}\bar\nabla_\alpha\bar R_{\beta\nu} + \frac{1}{2}\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\nabla_\alpha(\bar\nabla_\sigma\bar\nabla_\beta\mathfrak{h}^\sigma_\nu + \bar\nabla_\sigma\bar\nabla_\nu\mathfrak{h}^\sigma_\beta - \bar\Delta\mathfrak{h}_{\beta\nu} - \bar\nabla_\beta\bar\nabla_\nu\mathrm{Tr}\mathfrak{h})\\
&-\frac{1}{2}\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar R^\rho_\beta(\bar\nabla_\alpha\mathfrak{h}_{\rho\nu} + \bar\nabla_\nu\mathfrak{h}_{\rho\alpha} - \bar\nabla_\rho\mathfrak{h}_{\alpha\nu}) + (\mu\leftrightarrow\nu).
\end{aligned} \tag{3.3}
$$

Working in the transverse traceless gauge, we get

$$
\begin{aligned}
2C_{\mu\nu} =&\bar C_{\mu\nu} + \mathfrak{h}_{\mu\lambda}\bar\epsilon^{\lambda\alpha\beta}\bar\nabla_\alpha\bar R_{\beta\nu} + \frac{1}{2}\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\nabla_\alpha(3(\bar R_{\beta\lambda}\mathfrak{h}^\lambda_\nu + \bar R_{\nu\lambda}\mathfrak{h}^\lambda_\beta) - \bar R\mathfrak{h}_{\beta\nu} - 2\bar g_{\beta\nu}R_{\rho\lambda}\mathfrak{h}^{\rho\lambda} - \bar\Delta\mathfrak{h}_{\beta\nu})\\
&-\frac{1}{2}\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar R^\rho_\beta(\bar\nabla_\alpha\mathfrak{h}_{\rho\nu} + \bar\nabla_\nu\mathfrak{h}_{\rho\alpha} - \bar\nabla_\rho\mathfrak{h}_{\alpha\nu}) + (\mu\leftrightarrow\nu).
\end{aligned} \tag{3.4}
$$

Moreover for the background of constant curvature, we have

$$
C_{\mu\nu} = \frac{1}{4}\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}\bar\nabla_\alpha(\frac{1}{3}\bar R\mathfrak{h}_{\beta\nu} - \bar\Delta\mathfrak{h}_{\beta\nu}) + (\mu\leftrightarrow\nu) = \frac{1}{4}\bar\epsilon\ {}_{\mu}{}^{\alpha\beta}(\frac{1}{3}\bar R - \bar\Delta)\bar\nabla_\alpha\mathfrak{h}_{\beta\nu} + (\mu\leftrightarrow\nu). \tag{3.5}
$$

We complete the proof. □

Definition 3.2. For a perturbation $\mathfrak{h} \in \mathcal{D}'$, it is called the stable (unstable, marginal) perturbation if $\varXi(\mathfrak{h})$ is negative (positive, zero) everywhere. In particular, for the Lorentzian background manifold, a marginal perturbation is called the linear vacuum configuration of Chern–Simons-like gravity under the transverse-traceless gauge.

Proposition 3.3. *Assume M^3 is a closed manifold and $\bar{g}$ is a Riemannian metric of non-zero constant curvature. $\mathfrak{h}$ is a marginal perturbation on $(M^3, \bar{g})$ if and only if $d_{\bar{\nabla}}\mathfrak{h} = 0$, where $\mathfrak{h}$ is viewed as a 1-form valued in TM^3.*

Proof. Case I: $\bar{R}$ is positive. Since

$$\int_{M^3} d\mu_{\bar{g}} \langle \varXi(\mathfrak{h}), \varPi(\mathfrak{h})\rangle_{\bar{g}} = -\int_{M^3} d\mu_{\bar{g}} |\bar{\nabla}\varPi(\mathfrak{h})|^2_{\bar{g}} - \frac{\bar{R}}{3}\int_{M^3} d\mu_{\bar{g}} |\varPi(\mathfrak{h})|^2_{\bar{g}},$$

when $\bar{R} > 0$, $\mathfrak{h}$ is a marginal perturbation if and only if $\varPi(\mathfrak{h}) = 0$.

Case II: $\bar{R}$ is negative. We note that

$$(\varPi_0^2(\mathfrak{h}))_{\mu\nu} = (\varPi_0 \circ \varPi_0'(\mathfrak{h}))_{\mu\nu} = -\bar{\Delta}\mathfrak{h}_{\mu\nu} + \frac{\bar{R}}{2}\mathfrak{h}_{\mu\nu},$$

where $(\varPi_0'(\mathfrak{h}))_{\mu\nu} = \bar{\epsilon}\ _\nu{}^{\alpha\beta}\bar{\nabla}_\alpha T_{\beta\mu}$. Take advantage of the identity (3.2), we find that

$$\begin{aligned}\int_{M^3} d\mu_{\bar{g}} \langle \varXi(\mathfrak{h}), \varPi(\mathfrak{h})\rangle_{\bar{g}} &= -\int_{M^3} d\mu_{\bar{g}} \langle \varPi_0^3(\mathfrak{h}), \varPi(\mathfrak{h})\rangle_{\bar{g}} + \frac{\bar{R}}{6}\int_{M^3} d\mu_{\bar{g}} |\varPi(\mathfrak{h})|^2_{\bar{g}} \\ &= -2\int_{M^3} d\mu_{\bar{g}} |\varPi_0^2(\mathfrak{h})|^2_{\bar{g}} + \frac{\bar{R}}{6}\int_{M^3} d\mu_{\bar{g}} |\varPi(\mathfrak{h})|^2_{\bar{g}},\end{aligned}$$

from which, we see that $\varPi(\mathfrak{h})$ has to vanish for the marginal perturbation $\mathfrak{h}$ if $\bar{R} < 0$.

In either case, we have $\varPi(\mathfrak{h}) = 0$, thereby

$$\begin{aligned}\bar{\epsilon}^{\mu\rho\lambda}(\varPi(\mathfrak{h}))_{\mu\nu} &= \begin{vmatrix} \delta^\rho_\alpha & \delta^\lambda_\beta \\ \delta^\rho_\beta & \delta^\lambda_\alpha \end{vmatrix} \bar{\nabla}^\alpha \mathfrak{h}^\beta_\nu + \begin{vmatrix} \delta^\mu_\nu & \delta^\mu_\alpha & \delta^\mu_\beta \\ \delta^\rho_\nu & \delta^\rho_\alpha & \delta^\rho\beta \\ \delta^\lambda_\nu & \delta^\lambda_\alpha & \delta^\lambda_\beta \end{vmatrix} \bar{\nabla}^\alpha \mathfrak{h}^\beta_\mu \\ &= 2\bar{\nabla}^{[\rho}\mathfrak{h}^{\lambda]}_\nu = 0,\end{aligned}$$

where $\mathfrak{h}$ being traceless and transverse plays a role, thus $d_{\bar{\nabla}}\mathfrak{h} = 0$. □

Corollary 3.4 (*Rigidity Theorem*)**.** *If M^3 is simply connected, there is no non-trivial marginal perturbation.*

Proof. Firstly, we note that following identities

$$\int_{M^3} d\mu_{\bar{g}} |\varPi_0(\mathfrak{h})|^2_{\bar{g}} = \int_{M^3} d\mu_{\bar{g}} \langle \varPi_0(\mathfrak{h}), \varPi_0'(\mathfrak{h})\rangle_{\bar{g}} = \int_{M^3} d\mu_{\bar{g}} |\bar{\nabla}\mathfrak{h}|^2_{\bar{g}} + \frac{\bar{R}}{2}\int_{M^3} d\mu_{\bar{g}} |\mathfrak{h}|^2_{\bar{g}},$$

which means that there is no non-trivial marginal perturbation if $\bar{R}$ is positive. When M^3 is simply connected, any symmetric (0, 2)-tensor T whose covariant derivative is totally symmetric is generated by a function f via [17]

$$T = \mathrm{Hess} f + \frac{\bar{R}}{6} f\bar{g}.$$

If $T \in D'$, the function f is subject to the equation

$$\bar{\Delta} f + \frac{\bar{R}}{2} f = 0,$$

which implies $\bar{R}$ has to be positive if f is non-zero. However, we have shown that there is no non-trivial marginal perturbation if $\bar{R} > 0$. □

We consider the so-called linear Cotton flow, namely the small perturbation $\mathfrak{h}(t)$ evolves by equations

$$\frac{\partial \mathfrak{h}_{\mu\nu}}{\partial t} = (\varXi(\mathfrak{h}))_{\mu\nu}. \tag{3.6}$$

From the above corollary it follows that the flow has fixed points only when the background metric is of non-positive constant curvature (assuming the long time existence of solutions for initial data).

Proposition 3.5. *Assume the background manifold $(M^3, \bar{g})$ is a closed manifold of non-positive constant curvature, and we define the energy functional of perturbation $\mathfrak{h}$ as*

$$E(\mathfrak{h}) = \int_{M^3} d\mu_{\bar{g}} |\varXi(\mathfrak{h})|^2_{\bar{g}}. \tag{3.7}$$

Then under the linear Cotton flow, there is a positive constant C such that

$$\frac{dE}{dt} \leq C(\|\Xi(\mathfrak{h})\|_{2,2})^2.$$

where $\|\bullet\|_{2,2}$ *denotes the Sobolev norm.*

Proof. For any $T \in \mathcal{D}'$, we consider the integrals

$$\int_{M^3} d\mu_{\bar g}|\bar\nabla \circ \Pi(T) + \bar\nabla T|^2_{\bar g} = -\int_{M^3} d\mu_{\bar g}\langle 2\bar\Delta \circ \Pi(T) + \bar\Delta T, T\rangle_{\bar g} + \int_{M^3} d\mu_{\bar g}\bar\nabla_\alpha\bar\nabla_\beta T_{\rho\lambda}\bar\nabla^\alpha\bar\nabla^{[\beta}T^{\rho]\lambda},$$

and

$$\int_{M^3} d\mu_{\bar g}|\bar\epsilon_{\ \mu\alpha}{}^{\beta}T_{\beta\nu} + \bar\nabla_\alpha T_{\mu\nu}|^2_{\bar g} = -\int_{M^3} d\mu_{\bar g}\langle 2\Pi(T) + \bar\Delta T, T\rangle_{\bar g} + 2\int_{M^3} d\mu_{\bar g}|T|^2_{\bar g}$$

The following identity can be checked

$$\int_{M^3} d\mu_{\bar g}\bar\nabla_\alpha\bar\nabla_\beta T_{\rho\lambda}\bar\nabla^\alpha\bar\nabla^\rho T^{\beta\lambda} = \frac{5}{6}\bar R\int_{M^3} d\mu_{\bar g}\langle\bar\Delta T, T\rangle_{\bar g} + \frac{1}{3}\bar R^2\int_{M^3} d\mu_{\bar g}|T|^2_{\bar g}.$$

Therefore, if $\bar R \leq 0$, we obtain the inequality

$$\int_{M^3} d\mu_{\bar g}\langle\Xi(T), T\rangle_{\bar g} \leq \frac{1}{2}\int_{M^3} d\mu_{\bar g}[(1-\frac{\bar R}{3})|\bar\nabla T|^2_{\bar g} + |\bar\nabla \circ \bar\nabla T|^2_{\bar g} - \frac{2\bar R}{3}|T|^2_{\bar g}] \tag{3.8}$$

On the other hand, we have

$$\frac{dE}{dt} = 2\int_{M^3} d\mu_{\bar g}\langle\Xi(\mathfrak{h}), \Xi^2(\mathfrak{h})\rangle_{\bar g}.$$

Substituting $T = \Xi(\mathfrak{h})$ into (3.8) leads to the inequality

$$\frac{dE}{dt} \leq -\frac{2\bar R}{3}E + \int_{M^3} d\mu_{\bar g}[(1-\frac{\bar R}{3})|\bar\nabla\Xi(\mathfrak{h})|^2_{\bar g} + |\bar\nabla \circ \bar\nabla\Xi(\mathfrak{h})|^2_{\bar g}],$$

which yields the conclusion. □

Next we consider a family background metric $\bar g(t)$ with parameter $t \in [0, \infty)$, and let $\chi_{\mu\nu} = \frac{d\bar g_{\mu\nu}}{dt}$, $\chi^{\mu\nu} = \bar g^{\mu\alpha}\bar g^{\nu\beta}\frac{d\bar g_{\alpha\beta}}{dt} = -\frac{d\bar g^{\mu\nu}}{dt}$. Suppose $T \in \mathcal{D}$ is an eigentensor of the linear Cotton operator for the eigenvalue Λ, i.e. $(\Xi(T))_{\mu\nu} = \Lambda T_{\mu\nu}$, with the normalized condition $\int_{M^3}|T|^2_{\bar g} = 1$. Then by these set-ups, we reach

$$\begin{aligned}\frac{d\Lambda}{dt} &= \int_{M^3} d\mu_{\bar g}(\langle\frac{d\Xi}{dt}(T), T\rangle_{\bar g} + \langle\Xi(\frac{dT}{dt}), T\rangle_{\bar g} + \langle\Xi(T), \frac{dT}{dt}\rangle_{\bar g} \\ &\qquad - \langle\Xi(T), \chi\lrcorner T\rangle_{\bar g} - 2\Lambda\langle T, \frac{dT}{dt}\rangle + \Lambda\langle T, \chi\lrcorner T\rangle_{\bar g}) \\ &= \int_{M^3} d\mu_{\bar g}(\langle\frac{d\Xi}{dt}(T), T\rangle_{\bar g} + \langle\frac{dT}{dt}, \Upsilon(T)\rangle_{\bar g}),\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}(\chi\lrcorner T)_{\mu\nu} =& \chi_{\mu\alpha}T^\alpha_\nu + \chi_{\nu\alpha}T^\alpha_\mu, \\ (\Upsilon(T))_{\mu\nu} =& \frac{1}{2}\{\bar\epsilon_{\ \mu}{}^{\alpha\beta}(-\bar R^\lambda_\alpha\bar\nabla_\beta T_{\lambda\nu} + \frac{1}{2}\bar R\nabla_\beta T_{\alpha\nu} + \bar R_{\alpha\nu}\bar\nabla^\lambda T_{\beta\lambda} - \bar R^\lambda_\alpha\bar\nabla_\nu T_{\lambda\beta} \\ & - \bar\nabla_\beta(\bar R^\lambda_\alpha T_{\lambda\nu}) + \frac{1}{2}\nabla_\beta(\bar R T_{\alpha\nu}) + \bar\nabla^\lambda(\bar R_{\alpha\nu}T_{\beta\lambda}) - \bar\nabla_\nu(\bar R^\lambda_\alpha T_{\lambda\beta})) + (\mu \leftrightarrow \nu)\}.\end{aligned}$$

Therefore we need to determine the variation of linear Cotton operator. Indeed, the lengthy calculations show

$$\begin{aligned}(\frac{d(\bar\Delta \circ \Pi)}{dt}(T))_{\mu\nu} =& -\bar\nabla_\rho(\chi^{\rho\lambda}\bar\nabla_\lambda(\Pi(T))_{\mu\nu}) - \frac{1}{2}\mathrm{Tr}\chi(\bar\Delta \circ \Pi(T))_{\mu\nu} + \frac{1}{2}\bar\nabla^\sigma\mathrm{Tr}\chi\bar\nabla_\sigma(\Pi(T))_{\mu\nu} + \frac{1}{2}\{\chi_{\mu\lambda}(\bar\Delta \circ \Pi(T))^\lambda_\nu \\ & - \frac{1}{2}\bar\epsilon_{\ \mu}{}^{\alpha\beta}\bar\Delta((\bar\nabla_\nu\chi^\sigma_\alpha + \bar\nabla_\alpha\chi^\sigma_\nu - \bar\nabla^\sigma\chi_{\alpha\nu})T_{\beta\sigma}) - \frac{1}{2}\bar\epsilon_{\ \mu}{}^{\alpha\beta}\bar\nabla^\lambda(\bar\nabla_\lambda\chi^\sigma_\alpha + \bar\nabla_\alpha\chi^\sigma_\lambda - \bar\nabla^\sigma\chi_{\alpha\lambda})\bar\nabla_\sigma T_{\beta\nu} \\ & - \frac{1}{2}\bar\epsilon_{\ \mu}{}^{\alpha\beta}\bar\nabla^\lambda(\bar\nabla_\lambda\chi^\sigma_\beta + \bar\nabla_\beta\chi^\sigma_\lambda - \bar\nabla^\sigma\chi_{\beta\lambda})\bar\nabla_\alpha T_{\sigma\nu} - \frac{1}{2}\bar\epsilon_{\ \mu}{}^{\alpha\beta}\bar\nabla^\lambda(\bar\nabla_\lambda\chi^\sigma_\nu + \bar\nabla_\nu\chi^\sigma_\lambda - \bar\nabla^\sigma\chi_{\nu\lambda})\bar\nabla_\alpha T_{\beta\sigma} \\ & - \bar\epsilon_{\ \mu}{}^{\alpha\beta}(\bar\nabla_\lambda\chi^\sigma_\alpha + \bar\nabla_\alpha\chi^\sigma_\lambda - \bar\nabla^\sigma\chi_{\alpha\lambda})\bar\nabla^\lambda\bar\nabla_\sigma T_{\beta\nu} - \bar\epsilon_{\ \mu}{}^{\alpha\beta}(\bar\nabla_\lambda\chi^\sigma_\beta + \bar\nabla_\beta\chi^\sigma_\lambda - \bar\nabla^\sigma\chi_{\beta\lambda})\bar\nabla^\lambda\bar\nabla_\alpha T_{\sigma\nu} \\ & - \bar\epsilon_{\ \mu}{}^{\alpha\beta}(\bar\nabla_\lambda\chi^\sigma_\nu + \bar\nabla_\nu\chi^\sigma_\lambda - \bar\nabla^\sigma\chi_{\nu\lambda})\bar\nabla^\lambda\bar\nabla_\alpha T_{\beta\sigma} + (\mu \leftrightarrow \nu)\},\end{aligned}$$

and

$$(\frac{d(\bar{R}\circ\Pi)}{dt}(T))_{\mu\nu} = -(\frac{1}{2}\bar{R}\mathrm{Tr}\chi + \chi^{\rho\lambda}\bar{R}_{\rho\lambda} + \bar{\Delta}\mathrm{Tr}\chi - \bar{\nabla}^{\rho}\bar{\nabla}^{\lambda}\chi_{\rho\lambda})(\Pi(T))_{\mu\nu} + \frac{1}{2}\bar{R}\{\chi_{\mu\lambda}(\Pi(T))^{\lambda}_{\nu}$$
$$- \frac{1}{2}\bar{\epsilon}\ {}_{\mu}{}^{\alpha\beta}(\bar{\nabla}_{\alpha}\chi^{\sigma}_{\nu} + \bar{\nabla}_{\nu}\chi^{\sigma}_{\alpha} - \bar{\nabla}^{\sigma}\chi_{\alpha\nu})T_{\beta\sigma} + (\mu\leftrightarrow\nu)\}.$$

As a consequence, we have

Proposition 3.6. *If the initial data are taken as follows: $\bar{g}(0)$ is a metric of constant curvature and $T(0)$ is a marginal perturbation with respect to $\bar{g}(0)$, i.e. $(\Pi(T))(0) = 0$, then we have*

$$\frac{d\Lambda}{dt}|_{t=0} = \int_{M^3} d\mu_{\bar{g}}\langle\chi(0), (\mathcal{O}(T))(0)\rangle_{\bar{g}(0)}, \tag{3.10}$$

where

$$(\mathcal{O}(T))_{\alpha\sigma} = -\frac{1}{4}\{\bar{\epsilon}\ {}_{\alpha\mu}{}^{\beta}\bar{\nabla}_{\nu}T_{\beta\sigma}\bar{\Delta}T^{\mu\nu} - \bar{\epsilon}\ {}_{\alpha\mu}{}^{\beta}T^{\nu}_{\beta}\bar{\nabla}_{\nu}\bar{\Delta}T^{\mu}_{\sigma} - \bar{\epsilon}\ {}_{\alpha\mu}{}^{\beta}\bar{\Delta}(\bar{\nabla} - [\sigma T_{\beta}]_{\nu}T^{\mu\nu}) + \bar{\epsilon}\ {}_{\mu}{}^{\nu\beta}\bar{\nabla}_{\nu}\bar{\nabla}_{\alpha}(\bar{\nabla} - [\sigma T_{\beta}]_{\lambda}T^{\mu\lambda})$$
$$+ \bar{\epsilon}\ {}_{\alpha\mu}{}^{\beta}\bar{\nabla}^{\lambda}\bar{\nabla}_{\sigma}(\bar{\nabla}_{\lambda}T_{\beta\nu}T^{\mu\nu}) - \bar{\epsilon}\ {}_{\sigma\mu}{}^{\lambda}\bar{\nabla}_{\beta}\bar{\nabla}_{\alpha}(\bar{\nabla}_{\lambda}T^{\beta}_{\nu}T^{\mu\nu}) + 2\bar{\epsilon}\ {}_{\alpha\mu}{}^{\beta}\bar{\nabla}_{\lambda}(\bar{\nabla}^{\lambda}\bar{\nabla} - [\sigma T_{\beta}]_{\nu}T^{\mu\nu})$$
$$- 2\bar{\epsilon}\ {}_{\mu}{}^{\nu\beta}\bar{\nabla}_{\nu}(\bar{\nabla}_{\alpha}\bar{\nabla} - [\sigma T_{\beta}]_{\lambda}T^{\mu\lambda}) - 2\bar{\epsilon}\ {}_{\alpha\mu}{}^{\beta}\bar{\nabla}^{\lambda}(\bar{\nabla}_{\sigma}\bar{\nabla}_{\lambda}T_{\beta\nu}T^{\mu\nu}) + 2\bar{\epsilon}\ {}_{\sigma\mu}{}^{\lambda}\bar{\nabla}_{\beta}(\bar{\nabla}_{\alpha}\bar{\nabla}_{\lambda}T^{\beta}_{\nu}T^{\mu\nu})$$
$$- \frac{\bar{R}}{3}\bar{\epsilon}\ {}_{\alpha\mu}{}^{\beta}\bar{\nabla}_{\nu}T_{\beta\sigma}T^{\mu\nu} + \frac{\bar{R}}{3}\bar{\epsilon}\ {}_{\alpha\mu}{}^{\beta}T_{\beta\nu}\bar{\nabla}^{\nu}T^{\mu}_{\sigma} + (\alpha\leftrightarrow\sigma)\}.$$

3.2. ADM-type charges

In this section, the background manifold $(M^3, \bar{g})$ is still a closed Riemannian manifold of constant curvature.

Lemma 3.7.

(1) *Let $C^{(1)}$ stand for the first-order approximate of Cotton tensor around the background metric, then $C^{(1)}$ is also covariantly conserved and traceless.*

(2) *Let ξ be a Killing vector with respect to $\bar{g}$, and one introduces a new vector field $\check{\xi}^{\beta} = \bar{\epsilon}^{\alpha\nu\beta}\bar{\nabla}_{\alpha}\xi_{\nu}$ associated to ξ. Then $\check{\xi}$ is also a Killing vector field with respect to $\bar{g}$. Therefore we can define an operator κ on the space $\mathfrak{K}$ of Killing vector fields on $(M^3, \bar{g})$ by $\kappa(\xi) = \check{\xi}$ for any $\xi \in \mathfrak{K}$, then*

$$\kappa(\check{\xi}) = \kappa^2(\xi) = -\mathbb{L}(\xi),$$

hence $\bar{\Delta}\xi$ is also a Killing vector field.

Proof. (1) Let $\mathbb{G}$ denote the Einstein tensor defined by $\mathbb{G}^{\mu}_{\nu} = R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R$ and $\mathbb{G}^{(1)}$ denote the first-order approximate of $\mathbb{G}$ with respect to $\bar{g}$, which is given by

$$\mathbb{G}^{(1)\mu}{}_{\nu} = -\frac{\bar{R}}{3} + \frac{1}{2}(\bar{\nabla}^{\sigma}\bar{\nabla}^{\mu}\mathfrak{h}_{\sigma\nu} + \bar{\nabla}^{\sigma}\bar{\nabla}_{\nu}\mathfrak{h}^{\mu}_{\sigma} - \bar{\nabla}^{\mu}\bar{\nabla}_{\nu}\mathrm{Tr}\mathfrak{h}) + \frac{1}{2}\delta^{\mu}_{\nu}(\frac{\bar{R}}{3}\mathrm{Tr}\mathfrak{h} + \bar{\Delta}\mathrm{Tr}\mathfrak{h} - \bar{\nabla}^{\alpha}\bar{\nabla}^{\beta}\mathfrak{h}_{\alpha\beta}).$$

Then we calculate

$$\bar{\nabla}_{\mu}\bar{\nabla}^{\sigma}\bar{\nabla}^{\mu}\mathfrak{h}_{\sigma\nu} = \frac{\bar{R}}{6}\bar{\nabla}_{\nu}\mathrm{Tr}\mathfrak{h} - \frac{\bar{R}}{6}\bar{\nabla}^{\mu}\mathfrak{h}_{\mu\nu} + \bar{\nabla}_{\mu}\bar{\Delta}\mathfrak{h}^{\mu}_{\nu},$$
$$\bar{\nabla}_{\mu}\bar{\nabla}^{\sigma}\bar{\nabla}_{\nu}\mathfrak{h}^{\mu}_{\sigma} = -\frac{\bar{R}}{6}\bar{\nabla}_{\nu}\mathrm{Tr}\mathfrak{h} + \frac{5}{6}\bar{R}\bar{\nabla}_{\mu}\mathfrak{h}^{\mu}_{\nu} + \bar{\nabla}_{\nu}\bar{\nabla}^{\alpha}\bar{\nabla}^{\beta}\mathfrak{h}_{\alpha\beta},$$
$$\bar{\Delta}\bar{\nabla}_{\nu}\mathrm{Tr}\mathfrak{h} = \frac{\bar{R}}{3}\bar{\nabla}_{\nu}\mathrm{Tr}\mathfrak{h} + \bar{\nabla}_{\nu}\bar{\Delta}\mathrm{Tr}\mathfrak{h}.$$

Combining these identities leads to $\bar{\nabla}_{\mu}\mathbb{G}^{(1)\mu}{}_{\nu} = 0$. Then the conclusions can be easily derived from the definition of Cotton tensor and the above conclusion.

(2) Let $A_{\alpha\beta} = \bar{\nabla}_{\alpha}\check{\xi}_{\beta} + \bar{\nabla}_{\beta}\check{\xi}_{\alpha}$, then we have

$$\int_{M^3} d\mu_{\bar{g}}|A_{\alpha\beta}|^2_{\bar{g}} = 4\int_{M^3} d\mu_{\bar{g}}|\bar{\nabla}\circ\bar{\nabla}\xi|^2_{\bar{g}} - 4\int_{M^3} d\mu_{\bar{g}}|\bar{\Delta}\xi|^2_{\bar{g}}$$
$$= -4\int_{M^3} d\mu_{\bar{g}}\langle\bar{\nabla}\xi, \bar{\Delta}\bar{\nabla}\xi\rangle_{\bar{g}} - 4\int_{M^3} d\mu_{\bar{g}}|\bar{\Delta}\xi|^2_{\bar{g}}$$

$$= -\frac{2}{3}\bar{R}\int_{M^3} d\mu_{\bar{g}}\bar{\nabla}_\mu\xi_\nu\bar{\nabla}^\nu\xi^\mu - 4\bar{R}\int_{M^3} d\mu_{\bar{g}}\bar{\nabla}_\mu\xi_\nu(\frac{1}{3}\bar{\nabla}^\mu\xi^\nu + \frac{1}{6}\bar{\nabla}^\nu\xi^\mu)$$
$$= 0,$$

where the third equality follows from commuting the operators $\bar{\nabla}$ and $\bar{\Delta}$. Therefore $A_{\alpha\gamma} = 0$, which means that $\check{\xi}$ is a Killing vector field. The last claim follows from the following calculations

$$(\kappa(\check{\xi}))^\rho = \bar{\epsilon}^{\rho\gamma\delta}\bar{\epsilon}\ {}_\delta{}^{\alpha\beta}\bar{\nabla}_\gamma\bar{\nabla}_\alpha\xi_\beta = \bar{\nabla}_\alpha\bar{\nabla}^\rho\xi^\alpha - \bar{\Delta}\xi^\rho$$
$$= \frac{\bar{R}}{3}\xi^\rho - \bar{\Delta}\xi^\rho = -(\mathbb{L}(\xi))^\rho.$$

We complete the proof. $\square$

By ADM's approach [2] one can define a conserved charge $\mathcal{Q}_\xi(\mathfrak{h})$ associated to the fixed Killing vector field ξ as

$$\begin{aligned}(\mathcal{Q}_\xi(\mathfrak{h}))^\mu =& \frac{1}{8\pi\zeta G_3}\oint_\Sigma d^2x\sqrt{|\det\bar{h}|}n_\alpha(\xi_\nu\bar{\epsilon}^{\mu\alpha\beta}\mathbb{G}^{(1)\nu}{}_\beta + \xi_\nu\bar{\epsilon}^{\nu\alpha\beta}\mathbb{G}^{(1)\mu}{}_\beta + \xi_\nu\bar{\epsilon}^{\mu\nu\beta}\mathbb{G}^{(1)\alpha}{}_\beta \\ &- \check{\xi}_\nu\bar{\nabla}^\alpha\mathfrak{h}^{\mu\nu} + \check{\xi}_\nu\bar{\nabla}^\mu\mathfrak{h}^{\alpha\nu} - \check{\xi}^\alpha\bar{\nabla}^\mu\mathrm{Tr}\mathfrak{h} + \check{\xi}^\mu\bar{\nabla}^\alpha\mathrm{Tr}\mathfrak{h} + \check{\xi}^\alpha\bar{\nabla}_\nu\mathfrak{h}^{\mu\nu} - \check{\xi}^\mu\bar{\nabla}_\nu\mathfrak{h}^{\alpha\nu} \\ &+ \mathfrak{h}^{\mu\nu}\bar{\nabla}^\alpha\check{\xi}_\nu - \mathfrak{h}^{\alpha\nu}\bar{\nabla}^\mu\check{\xi}_\nu + \mathrm{Tr}\mathfrak{h}\bar{\nabla}^\mu\check{\xi}^\alpha),\end{aligned} \tag{3.11}$$

for a closed surface Σ in M^3 with the unit normal vector n and the induced metric $\bar{h}$.

Proposition 3.8. *The conserved charge $\mathcal{Q}_\xi(\mathfrak{h})$ is gauge invariant with respect to the diffeomorphisms.*

Proof. The diffeomorphism generated by a vector X causes the infinitesimal transformation of the perturbation $\mathfrak{h}$ given by $\delta_X\mathfrak{h}_{\mu\nu} = \bar{\nabla}_\mu X_\nu + \bar{\nabla}_\nu X_\mu$, then

$$\bar{\nabla}^\sigma\bar{\nabla}^\mu\delta_X\mathfrak{h}_{\sigma\nu} + \bar{\nabla}^\sigma\bar{\nabla}_\nu\delta_X\mathfrak{h}^\mu_\sigma = \frac{2}{3}\bar{R}(\bar{\nabla}^\mu X_\nu + \bar{\nabla}_\nu X^\mu) + \bar{\Delta}(\bar{\nabla}^\mu X_\nu + \bar{\nabla}_\nu X^\mu) + 2\bar{\nabla}^\mu\bar{\nabla}_\nu\bar{\nabla}\cdot X,$$
$$\frac{\bar{R}}{3}\mathrm{Tr}\delta_X\mathfrak{h} + \bar{\Delta}\mathrm{Tr}\delta_X\mathfrak{h} - \bar{\nabla}^\alpha\bar{\nabla}^\beta\delta_X\mathfrak{h}_{\alpha\beta} = 0.$$

Hence $\delta_X\mathbb{G}^{(1)} = 0$, which implies the gauge invariance. $\square$

We first deal with the case of asymptotically flat Riemannian manifold, namely (M^3, g) is diffeomorphic to the complement of a ball in Euclidean space outside a compact set, and in the coordinates given by this diffeomorphism, the metric satisfies the asymptotic conditions

$$g_{\mu\nu} = \delta_{\mu\nu} + \mathfrak{h}_{\mu\nu},$$

with appropriate decay of $\mathfrak{h}$ and its higher order derivatives as $|x| \to \infty$. One chooses a Killing vector ξ such that $\xi = (1, 0, 0)$ in terms of the Cartesian coordinates on the limit ends of the manifold, then we define the ADM-type mass $\mathcal{M}$ via the limit of $(\mathcal{Q}_\xi(\mathfrak{h}))^0$ as follows

$$\begin{aligned}\mathcal{M} &= \lim_{r\to\infty}\frac{1}{4\pi\zeta G_3}\oint_{S^2_r} d\mu\varepsilon^{ij}n_i\mathbb{G}^{(1)0}{}_j \\ &= \lim_{r\to\infty}\frac{1}{8\pi\zeta G_3}\oint_{S^2_r} d\mu[n_1(\partial_0\partial_2\mathfrak{h}_{21} + \partial_2\partial_1\mathfrak{h}_{02} - \partial_2\partial_2\mathfrak{h}_{01} - \partial_0\partial_1\mathfrak{h}_{22}) \\ &\qquad - n_2(\partial_0\partial_1\mathfrak{h}_{12} + \partial_1\partial_2\mathfrak{h}_{01} - \partial_1\partial_1\mathfrak{h}_{02} - \partial_0\partial_2\mathfrak{h}_{11})],\end{aligned} \tag{3.12}$$

where S^2_r is a Euclidean sphere with radius r, and $d\mu$ denotes the measure on S^2_r induced by the Euclidean metric, and the indices i, j runs over 1, 2.

Example 3.9. Let us consider 3-dimensional Schwarzschild-like manifold endowed with the metric with a parameter M

$$\begin{aligned}g &= \frac{1}{1-2Mr^{-1}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \\ &= (1 + \frac{M}{\sqrt{r^2-2Mr} + r - M})^4(dx_0^2 + dx_1^2 + dx_2^2).\end{aligned}$$

Under the asymptotical Cartesian coordinates, $g_{\mu\nu} = \delta_{\mu\nu}(1 + \frac{2M}{r})$. Then from (3.12) we find that

$$M \sim \lim_{r\to\infty}(\frac{1}{r} + O(\frac{1}{r})) = 0.$$

More complicatedly, we next consider the asymptotically hyperbolic Riemannian manifold (M^3, g), thus there exists a diffeomorphism φ from the complement of a compact set in M^3 into the complement of a ball $\mathbb{B}^3$ in 3-dimensional space $\mathbb{H}^3$ equipped with the background hyperbolic metric $\bar{g} = dr^2 + \sinh^2 r(d\theta^2 + \sin^2\theta d\varphi^2)$ in terms of geodesic polar coordinates such that $(\varphi^{-1})^* g$ and $\bar{g}$ are uniformly equivalent on $\mathbb{H} \setminus \mathbb{B}^3$ and some decay conditions should be imposed. With respect to background metric, the Killing vector fields on $\mathbb{H}^3$ are generated by

$$\begin{aligned}
\xi^{(1)} &= -\cos\theta\frac{\partial}{\partial r} + \coth r \sin\theta\frac{\partial}{\partial\theta},\\
\xi^{(2)} &= \sin\theta\cos\varphi\frac{\partial}{\partial r} + \coth r\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \coth r\csc\theta\sin\varphi\frac{\partial}{\partial\varphi},\\
\xi^{(3)} &= -\sin\theta\sin\varphi\frac{\partial}{\partial r} - \coth r\cos\theta\sin\varphi\frac{\partial}{\partial\theta} - \coth r\csc\theta\cos\varphi\frac{\partial}{\partial\varphi},\\
\xi^{(4)} &= \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi},\\
\xi^{(5)} &= \sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi},\\
\xi^{(6)} &= \frac{\partial}{\partial\varphi},
\end{aligned}$$

among which, there exist some duality relations via the operator κ, more precisely, we have

$$\begin{aligned}
&\kappa(\xi^{(1)}) = 2\xi^{(6)}, \quad \kappa(\xi^{(6)}) = -2\xi^{(1)},\\
&\kappa(\xi^{(2)}) = 2\xi^{(5)}, \quad \kappa(\xi^{(5)}) = -2\xi^{(2)},\\
&\kappa(\xi^{(3)}) = 2\xi^{(4)}, \quad \kappa(\xi^{(4)}) = -2\xi^{(3)},
\end{aligned}$$

which imply the Killing vector fields $\xi^{(1)}, \ldots, \xi^{(6)}$ are all eigenvectors of the Laplacian $\bar{\Delta}$ belong to the same eigenvalue 2. Then we consider the limits of ADM-type charges $(\mathcal{Q}_{\xi^{(i)}}(\mathfrak{h}))^\mu$, $i = 1, \ldots, 6$, for an asymptotically hyperbolic manifold as the same manner as the case of asymptotically flat manifold, where the integral domain is chosen the surface defined by the equation $r =$ constant. As an example, for Killing fields $\xi^{(1)}$, we express explicitly

$$\begin{aligned}
\mathcal{M}^{(1)\mu} &= \lim_{r\to\infty}((\mathcal{Q}_{\xi^{(1)}}(\mathfrak{h}))^\mu)\\
&= \begin{cases}
0, & \mu = 0,\\
\dfrac{1}{4\pi\zeta G_3}\lim_{r\to\infty}\displaystyle\int_0^\pi\int_0^{2\pi} d\theta d\varphi[\cos\theta\mathbb{G}^{(1)}{}_{02} - \coth r\sin\theta\mathbb{G}^{(1)}{}_{12} & \\
\qquad + \sin\theta(\bar{\nabla}_1\mathfrak{h}_{02} - \bar{\nabla}_0\mathfrak{h}_{12}) + \coth r\sin\theta\mathfrak{h}_{12} - \cos\theta\mathfrak{h}_{02}], & \mu = 1,\\
\dfrac{1}{8\pi\zeta G_3}\lim_{r\to\infty}\displaystyle\int_0^\pi\int_0^{2\pi} d\theta d\varphi[-2\cos\theta\mathbb{G}^{(1)}{}_{01} + \sinh r\cosh r\sin\theta(-\mathbb{G}^{(1)}{}^0_0 + \mathbb{G}^{(1)}{}^1_1 - \mathbb{G}^{(1)}{}^2_2) & \\
\qquad + 2\sin\theta(\bar{\nabla}_0\mathfrak{h}_{11} - \bar{\nabla}_1\mathfrak{h}_{01}) - 2\coth r\sin\theta\mathfrak{h}_{11} + 2\cos\theta\mathfrak{h}_{01}], & \mu = 2.
\end{cases}
\end{aligned}$$

Example 3.10. We modify the Schwarzschild-like metric by introducing a new parameter ℓ with the dimension of length as follows

$$\begin{aligned}
g &= \frac{1}{1 - 2Mr^{-1} + r^2\ell^{-2}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\\
&= \frac{1 + r^2\ell^{-2}}{1 - 2Mr^{-1} + r^2\ell^{-2}}\left[\frac{dr^2}{1 + r^2\ell^{-2}} + \frac{1 - 2Mr^{-1} + r^2\ell^{-2}}{1 + r^2\ell^{-2}}r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]\\
&\xlongequal{r\to\infty} \left(1 + \frac{2M\ell^2}{r(r^2 + \ell^2)}\right)\ell^2[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)],
\end{aligned}$$

where $r = \ell\sinh\chi$. Then by taking $\mathfrak{h}_{\mu\nu} = \frac{2M\ell^2}{r(r^2+\ell^2)}\bar{g}_{\mu\nu} = \frac{2M}{\ell\sinh\chi\cosh^2\chi}\bar{g}_{\mu\nu}$ in terms of the asymptotical geodesic polar coordinates $\{\chi, \theta, \varphi\}$, straightforward calculations show $\mathcal{M}^{(1)}$ vanishes.

Based on the above examples, we propose the following conjecture.

Conjecture 3.11. *If a conformally flat manifold (M^3, g) admits nontrivial ADM-type mass defined as above, then the Ricci curvature cannot be parallel with respect to g.*

3.3. Linear gravity and supersymmetric quantum mechanics

We return to the case with the metric signature $(-1, 1, 1)$ in this section. Around the background manifold $(M^3, \bar{g})$ of negative constant curvature, to find a marginal perturbation $\mathfrak{h} \in \mathcal{D}'$, one can consider the first-order equations

$$((\Pi_0 \pm \sqrt{\frac{-\bar{R}}{6}})\mathfrak{h})_{\mu\nu} = \bar{\epsilon}\ {}_{\mu}{}^{\alpha\beta}\bar{\nabla}_{\alpha}\mathfrak{h}_{\beta\nu} \pm \sqrt{\frac{-\bar{R}}{6}}\mathfrak{h}_{\mu\nu} = 0, \tag{3.13}$$

acting on both sides of which with the operators $\bar{\epsilon}\ {}_{\sigma}{}^{\rho\mu}\bar{\nabla}_{\rho} \mp \sqrt{\frac{-\bar{R}}{6}}\delta^{\mu}_{\nu}$ leads to

$$(\mathbb{L}\mathfrak{h})_{\mu\nu} = (\bar{\Delta} - \frac{\bar{R}}{3})\mathfrak{h}_{\mu\nu} = 0. \tag{3.14}$$

Definition 3.12.

(1) If a perturbation $\mathfrak{h} \in \mathcal{D}'$

- satisfies Eq. (3.13), it is called a type-I vacuum;
- satisfies Eq. (3.14), but not Eq. (3.13), it is called a type-II vacuum.

(2) If a marginal perturbation $\mathfrak{h} \in \mathcal{D}'$ can be expressed as the matrix form in terms of separated variables

$$\mathfrak{h} = \begin{pmatrix} H_{00}(r) & H_{01}(r) & H_{02}(r) \\ H_{01}(r) & H_{11}(r) & H_{12}(r) \\ H_{02}(r) & H_{12}(r) & H_{22}(r) \end{pmatrix} e^{i\omega t}e^{ik\varphi} \tag{3.15}$$

with additional conditions that

- $\omega, k \in \mathbb{R}$, it is called a real mode, and in particular, when $k = 0$, it is called an s-mode;
- the diagonal components all vanish, it is called an axial mode;
- the components $H_{02} = H_{12} = 0$, it is called a polar mode.

We will choose the following two common vacuum configurations of Chern–Simons-like gravity as the background manifolds.

- AdS$_3$ manifold endowed with metric

$$\bar{g}_{\text{AdS}} = -(1 + r^2)dt^2 + \frac{1}{1 + r^2}dr^2 + r^2 d\varphi^2. \tag{3.16}$$

The isometry group is $O(2, 2)$ whose identity component is exactly isomorphic to $(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))/\mathbb{Z}_2$ with the left and right moving sectors of generators

$$L_0 = \frac{i}{2}(\partial_t + \partial_\varphi), L_\pm = \frac{i}{2}e^{\mp i(t+\varphi)}(\frac{r}{\sqrt{1+r^2}}\partial_t \pm i\sqrt{1+r^2}\partial_r + \frac{\sqrt{1+r^2}}{r}\partial_\varphi),$$
$$\bar{L}_0 = \frac{i}{2}(\partial_t - \partial_\varphi), \bar{L}_\pm = \frac{i}{2}e^{\mp i(t-\varphi)}(\frac{r}{\sqrt{1+r^2}}\partial_t \pm i\sqrt{1+r^2}\partial_r - \frac{\sqrt{1+r^2}}{r}\partial_\varphi),$$

each of which forms Lie algebra $sl(2, \mathbb{R})$, thus

$$[L_0, L_\pm] = \pm L_\pm, [L_+, L_-] = 2L_0,$$
$$[\bar{L}_0, \bar{L}_\pm] = \pm\bar{L}_\pm, [\bar{L}_+, \bar{L}_-] = 2\bar{L}_0.$$

For a marginal perturbation $\mathfrak{h}$ with the form (3.15), we have $L_0(\mathfrak{h}) = -\frac{\omega+k}{2}\mathfrak{h}$, $\bar{L}_0(\mathfrak{h}) = -\frac{\omega-k}{2}\mathfrak{h}$, where $\omega \pm k$ is called the left(right)-weight of the mode, and the acting of $L_\pm$ on $\mathfrak{h}$ changes the left(right)-weight to $\omega + k \pm 2$ and leaves the right(left)-weight invariant.

- Non-rotating BTZ black hole endowed with the metric

$$\bar{g}_{\text{BTZ}} = -(-M + r^2)dt^2 + \frac{1}{-M + r^2}dr^2 + r^2 d\varphi^2, \tag{3.17}$$

with a positive parameter M that describes ADM mass of the black hole. The BTZ black hole can be viewed as the quotient space of AdS$_3$, i.e. locally AdS$_3$ space, and then the survived global Killing vector fields are only generated by $L_0, \bar{L}_0$.

Proposition 3.13. *Under the AdS_3 or BTZ background metric, if there exists a non-trivial axial mode such that Eq. (3.14) is satisfied and $r^{-\frac{1}{2}}H_{02}, r^{-\frac{1}{2}}H_{12}, r^{\frac{3}{2}}H_{12}$ are square-integrable over definition domains, then it must be a type-II vacuum and an s-mode.*

Proof. For AdS_3 background metric, Eq. (3.14) reduces to

$$\begin{aligned}
-2r^2(1+r^2)^2H_{00}'' - 2r(1-r^4)H_{00}' - (-2k^2+2r^2\omega^2-5r^2)H_{00} + 4r^4(1+r^2)^2H_{11} - 8i\omega r^3(1+r^2)H_{01} = 0,\\
-r^3(1+r^2)^3H_{01}'' - r^2(1+r^2)^2(1+3r^2)H_{01}' - r(1+r^2)(-1-3r^4-k^2(1+r^2)+r^2\omega^2)H_{01}\\
+2ik(1+r^2)^2H_{02} - 2i\omega r^4H_{00} - 2i\omega r^4(1+r^2)^2H_{11} = 0,\\
r^2(1+r^2)^2H_{02}'' - r(1+r^2)^2H_{02}' + (-k^2(1+r^2)+r^2\omega^2)H_{02} + 2ikr(1+r^2)^2H_{01} + 2i\omega r^3(1+r^2)H_{12} = 0,\\
-r^4(1+r^2)^4H_{11}'' - r^3(1+r^2)^3(1+7r^2)H_{11}' - r^2(1+r^2)^2(-2+2r^2+6r^4-k^2(1+r^2)+\omega^2r^2)H_{11}\\
+2r^6H_{00} - 4i\omega r^5(1+r^2)H_{01} - 2(1+r^2)^3H_{22} + 4ikr(1+r^2)^3H_{12} = 0,\\
-r^3(1+r^2)^3H_{12}'' - r^2(1+r^2)^2(-1+3r^2)H_{12}' - r(1+r^2)(-3(1+r^2)^2-k^2(1+r^2)+\omega^2r^2)H_{12}\\
-2ikr^2(1+r^2)^3H_{11} - 2i\omega r^4H_{02} + 2ik(1+r^2)^2H_{22} = 0,\\
r^2(1+r^2)^2H_{22}'' - r(1+r^2)(3+r^2)H_{22}' + (-(1+r^2)(k^2-2)+\omega^2r^2)H_{22} + 2r^2(1+r^2)^3H_{11}\\
+4ikr(1+r^2)^2H_{12} = 0,
\end{aligned}$$

and the transverse-traceless gauge conditions are given by

$$\begin{aligned}
r^2(1+r^2)^2H_{01}' + r(1+r^2)(1+3r^2)H_{01} + ik(1+r^2)H_{02} + i\omega r^2H_{00} = 0,\\
-r^3(1+r^2)^3H_{11}' - r^2(1+r^2)^2(1+4r^2)H_{11} - r^4H_{00} + (1+r^2)^2H_{22}\\
+i\omega r^3(1+r^2)H_{01} - ikr(1+r^2)^2H_{02} = 0,\\
-r^2(1+r^2)^2H_{12}' - r(1+r^2)(1+3r^2)H_{12} - ik(1+r^2)H_{22} + i\omega r^2H_{02} = 0,\\
r^2H_{00} - r^2(1+r^2)^2H_{11} - (1+r^2)H_{22} = 0.
\end{aligned}$$

One easily finds that $k=0$ for the non-trivial axial modes. Then the non-trivial H_{01}-component is subject to equations

$$\begin{aligned}
r^2(1+r^2)^2H_{01}'' + r(1+r^2)(1+3r^2)H_{01}' - (1+3r^4)H_{01} = 0,\\
r(1+r^2)H_{01}' + (1+3r^2)H_{01} = 0,
\end{aligned}$$

which are not compatible with each other. Therefore Eq. (3.14) and gauge conditions remain

$$\begin{aligned}
r(1+r^2)^2H_{02}'' - (1+r^2)^2H_{02}' + r\omega^2H_{02} + 2i\omega r^2(1+r^2)H_{12} = 0,\\
-r^2(1+r^2)^3H_{12}'' - r(1+r^2)^2(-1+3r^2)H_{12}' - (1+r^2)(-3(1+r^2)^2+\omega^2r^2)H_{12} - 2i\omega r^3H_{02} = 0,\\
-r(1+r^2)^2H_{12}' - (1+r^2)(1+3r^2)H_{12} + i\omega rH_{02} = 0,
\end{aligned}$$

thus the non-zero components respectively satisfy the equations

$$\begin{aligned}
H_{02}'' - \frac{1-r^2}{r(1+r^2)}H_{02}' - [\frac{4}{1+r^2} - \frac{\omega^2}{(1+r^2)^2}]H_{02} = 0,\\
H_{12}'' - \frac{1-5r^2}{r(1+r^2)}H_{12}' - [\frac{3}{r^2} - \frac{2(1+3r^2)}{(1+r^2)^2} - \frac{\omega^2}{(1+r^2)^2}]H_{12} = 0,
\end{aligned}$$

and they are connected by a compatibility relation

$$-i\omega H_{12} = \frac{2}{r}H_{02} - H_{02}'. \tag{3.18}$$

By introducing the new variable $\tilde{r}_{\mathrm{AdS}} = \arctan r$ and new fields

$$\begin{aligned}
\tilde{H}_{02}^{\mathrm{AdS}} = \frac{1}{\sqrt{r}}H_{02},\\
\tilde{H}_{12}^{\mathrm{AdS}} = \frac{1+r^2}{\sqrt{r}}H_{12},
\end{aligned}$$

we get the Schrödinger-type equations

$$\frac{d^2\tilde{H}_{02}^{\mathrm{AdS}}}{d\tilde{r}_{\mathrm{AdS}}^2} + [\omega^2 - V_{\mathrm{AdS}}]\tilde{H}_{02}^{AdS} = 0, \tag{3.19}$$

$$\frac{d^2\tilde{H}_{12}^{\mathrm{AdS}}}{d\tilde{r}_{\mathrm{AdS}}^2} + [\omega^2 - U_{\mathrm{AdS}}]\tilde{H}_{12}^{\mathrm{AdS}} = 0, \tag{3.20}$$

where the potential functions $V_{\mathrm{AdS}}, U_{\mathrm{AdS}}$ are given by

$$V_{\mathrm{AdS}} = \frac{3}{4}(1+\tan^2\tilde{r}_{\mathrm{AdS}})(5+\frac{1}{\tan^2\tilde{r}_{\mathrm{AdS}}}),$$
$$U_{\mathrm{AdS}} = \frac{3}{4}(1+\tan^2\tilde{r}_{\mathrm{AdS}})(1+\frac{5}{\tan^2\tilde{r}_{\mathrm{AdS}}}),$$

which are both positive.

Similarly, for the case of BTZ metric, we only need to substitute

$$\tilde{r}_{\mathrm{BTZ}} = \frac{1}{2\sqrt{M}}\ln|\frac{r-\sqrt{M}}{r+\sqrt{M}}|,$$

$$\tilde{H}_{02}^{\mathrm{BTZ}} = \tilde{H}_{02}^{\mathrm{AdS}}, \quad \tilde{H}_{12}^{\mathrm{BTZ}} = \frac{-M+r^2}{\sqrt{r}}H_{12},$$

$$V_{\mathrm{BTZ}} = \begin{cases} \frac{3}{4}M(\tanh^2(\sqrt{M}\tilde{r}_{\mathrm{BTZ}})-1)(5-\frac{1}{\tanh^2(\sqrt{M}\tilde{r}_{\mathrm{BTZ}})}), & 0<r<\sqrt{M}; \\ \frac{3}{4}M(\coth^2(\sqrt{M}\tilde{r}_{\mathrm{BTZ}})-1)(5-\frac{1}{\cosh^2(\sqrt{M}\tilde{r}_{\mathrm{BTZ}})}), & r>\sqrt{M}, \end{cases}$$

$$U_{\mathrm{BTZ}} = \begin{cases} \frac{3}{4}M(\tanh^2(\sqrt{M}\tilde{r}_{\mathrm{BTZ}})-1)(1-\frac{5}{\tanh^2(\sqrt{M}\tilde{r}_{\mathrm{BTZ}})}), & 0<r<\sqrt{M}; \\ \frac{3}{4}M(\coth^2(\sqrt{M}\tilde{r}_{\mathrm{BTZ}})-1)(1-\frac{5}{\cosh^2(\sqrt{M}\tilde{r}_{\mathrm{BTZ}})}), & r>\sqrt{M}, \end{cases}$$

within the final Schrödinger-type equations.

To confirm it is an s-mode, we still need to show ω is a real number. Generally, let $C_0^\infty(I)$, where $I=\{\tilde{r}:\tilde{r}_1<\tilde{r}<\tilde{r}_2\}$, denote the set of smooth real functions over I with compact support, acting on which we define the Schrödinger operators $\Delta_V = -\frac{d^2}{d\tilde{r}^2}+V$ and $\Delta_U = -\frac{d^2}{d\tilde{r}^2}+U$. Then these operators can be extended to self-adjoint operators on $L_2(I)$ by the Friedrichs extension. Hence since

$$\omega^2 = \frac{(\tilde{H}_{02},\Delta_V\tilde{H}_{02})}{(\tilde{H}_{02},\tilde{H}_{02})} = \frac{\tilde{H}_{12},\Delta_U\tilde{H}_{12}}{(\tilde{H}_{12},\tilde{H}_{12})},$$

where the pair $(\cdot,\cdot)$ is defined by $(A,B)=\int_{\tilde{r}_1}^{\tilde{r}_2} d\tilde{r}AB$ for $A,B\in L_2(I)$, we have

$$\omega^2 \geq \max\{\inf_{\phi\in C_0^\infty}\frac{(\phi,\Delta_V\phi)}{(\phi,\phi)}, \inf_{\phi\in C_0^\infty}\frac{(\phi,\Delta_U\phi)}{(\phi,\phi)}\}.$$

On the other hand, by Stokes theorem, we have

$$(\phi,\Delta_V\phi) = (\mathbb{D}_V\phi,\mathbb{D}_V\phi)+(\phi,\mathbb{V}\phi),$$

where

$$\mathbb{D}_V = \frac{d}{d\tilde{r}}+S$$

with S-deformed potential function [15]

$$\mathbb{V} = V+\frac{dS}{d\tilde{r}}-S^2$$

for any smooth function $S(\tilde{r})$ over I. This argument indicates that ω^2 cannot be negative under the assumptions in the proposition.

Finally, we should show that the axial s-mode cannot be made into a type-I vacuum. Indeed, we explicitly write the $(2,2)$-, $(0,0)$- and $(1,2)$-components of Eq. (3.13)

$$H'_{02}-\frac{1}{r}H_{02}-i\omega H_{12}=0,$$
$$H'_{02}-\frac{r}{1+r^2}H_{02}=0,$$
$$H_{12}\mp\frac{r^2}{1+r^2}H_{02}=0,$$

which only admit zero solution. □

Similarly, the type-II polar modes must have null k. The component H_{01} satisfies the equation

$$(r^2-2M-\omega^2)H''_{01}+\frac{5r^4-13Mr^2+2M^2-\omega^2(7r^2-M)}{r(-M+r^2)}H'_{01} + \frac{(3r^2-2M)(r^4-4Mr^2-M^2)+\omega^2(M^2-6r^4-\omega^2r^2)}{r^2(-M+r^2)^2}H_{01}=0,$$

where $M=-1$ correspondences to the AdS$_3$ metric, and $M>0$ corresponds to the BTZ metric. By defining

$$\tilde{H}_{01}=\frac{\sqrt{r}(-M+r^2)}{\sqrt{|r^2-2M-\omega^2|}}H_{01},$$

we once again obtain the Schrödinger-type equation

$$\frac{d^2\tilde{H}_{01}}{d\tilde{r}^2}+[\omega^2-W]\tilde{H}_{01}=0, \tag{3.21}$$

where $\tilde{r}$ is taken $\tilde{r}_{AdS}$ or $\tilde{r}_{BTZ}$ depending on the value of M, and the ω-dependent potential W is given by

$$W=(-M+r^2)[\frac{7}{4}-\frac{3M}{4r^2}+\frac{(-M+r^2)(-r^2+8M+4\omega^2)}{(r^2-2M-\omega^2)^2}].$$

Asymptotical behaviour

When $\tilde{r}$ tends to zero (i.e. $r\to 0$), we should consider the asymptotical Schrödinger-type equations with potentials $V_{\text{AdS}}=\frac{3}{4}\cot^2\tilde{r}$ or $U_{\text{AdS}}=\frac{15}{4}\cot^2\tilde{r}$. Let $\tilde{H}_{02}^{\text{AdS}}=\sin^{-\frac{1}{2}}\tilde{r}\Psi_0$, $\tilde{H}_{12}^{\text{AdS}}=\sin^{-\frac{3}{2}}\tilde{r}\psi_0$, and $\eta_0=\cos^2\tilde{r}$, then we obtain the following hypergeometric equations satisfied by Ψ_0 and ψ_0:

$$\eta_0(1-\eta_0)\frac{d^2\Psi_0}{d\eta_0^2}+\frac{1}{8}(2\omega^2+1)\Psi_0=0,$$
$$\eta_0(1-\eta_0)\frac{d^2\psi_0}{d\eta_0^2}+\frac{d\psi_0}{d\eta_0}+\frac{1}{8}(2\omega^2+3)\psi_0=0.$$

Similarly, when $\tilde{r}\to\frac{\pi}{2}$ (i.e. $r\to\infty$), one introduces $\tilde{H}_{02}^{\text{AdS}}=\cos^{-\frac{3}{2}}\tilde{r}\Psi_\infty$, $\tilde{H}_{12}^{\text{AdS}}=\cos^{-\frac{1}{2}}\tilde{r}\psi_\infty$, and $\eta_\infty=\sin^2\tilde{r}$, where Ψ_∞ and ψ_∞ satisfy the same equations control ψ_0 and Ψ_0 respectively. By virtue of the Gauss hypergeometric functions $F(\alpha,\beta,\gamma;z)$ and the function $\Upsilon(s)=\frac{\Gamma'(s)}{\Gamma(s)}$, we have

$$\Psi_0 \sim F(-\frac{1}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}},-\frac{1}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}},\frac{1}{2};\eta_0),\ \sqrt{\eta_0}F(\frac{1}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}},\frac{1}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}},\frac{3}{2};\eta_0),$$
$$\psi_0 \sim F(-\frac{3}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}},-\frac{3}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}},\frac{1}{2};\eta_0),\ \sqrt{\eta_0}F(-\frac{1}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}},-\frac{1}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}},\frac{3}{2};\eta_0),$$

$$\begin{aligned}\Psi_\infty \sim & F(-\frac{3}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}},-\frac{3}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}},-1,1-\eta_\infty),\\ & \ln(1-\eta_\infty)F(-\frac{3}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}},-\frac{3}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}},3,1-\eta_\infty)+\frac{8}{(17-2\omega^2)(1-\eta_\infty)}\\ & -\frac{64}{(17-2\omega^2)(53-2\omega^2)(1-\eta_\infty)^2}+\sum_{s=0}^{\infty}\frac{\Gamma(s-\frac{3}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}})\Gamma(s-\frac{3}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}})}{s!\Gamma(3+s)}(1-\eta_\infty)^s\\ & \times\{\Upsilon(s-\frac{3}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}})+\Upsilon(s-\frac{3}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}})-\Upsilon(3+s)-\Upsilon(1+s)-\Upsilon(-\frac{11}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}})\\ & -\Upsilon(-\frac{11}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{15}{4}})+\Upsilon(1)+\Upsilon(2)\},\\ \psi_\infty \sim & F(-\frac{1}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}},-\frac{1}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}},0;1-\eta_\infty),\\ & \ln(1-\eta_\infty)F(-\frac{1}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}},-\frac{1}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}},2,1-\eta_\infty)+\frac{8}{(11-2\omega^2)(1-\eta_\infty)}\end{aligned}$$

$$+\sum_{s=0}^{\infty}\frac{\Gamma(s-\frac{1}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}})\Gamma(s-\frac{1}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}})}{s!\Gamma(2+s)}(1-\eta_\infty)^s\{\Upsilon(s-\frac{1}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}})$$
$$+\Upsilon(s-\frac{1}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}})-\Upsilon(2+s)-\Upsilon(1+s)-\Upsilon(-\frac{5}{4}-\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}})-\Upsilon(-\frac{5}{4}+\frac{1}{2}\sqrt{\omega^2+\frac{3}{4}})+2\Upsilon(1)\}.$$

Related to supersymmetric quantum mechanics

We reformulate the potential functions V, U as the deformed potentials of constant ones, namely we write

$$V=S_V^2-\frac{dS_V}{d\tilde{r}}+\varpi_V^2,$$
$$U=S_U^2-\frac{dS_U}{d\tilde{r}}+\varpi_U^2,$$

where there are various choices of superpotentials

$$S_V=\begin{cases}\frac{5}{2}r+\frac{M\pm 2|M|}{2r},\\ -\frac{3}{2}r+\frac{M\pm 2|M|}{2r},\end{cases}\quad \varpi_V^2=\begin{cases}-10M\mp 6|M|,\\ -2M\pm 2|M|,\end{cases}$$

$$S_U=\begin{cases}\frac{3}{2}r+\frac{M\pm 4|M|}{2r},\\ -\frac{1}{2}r+\frac{M\pm 4|M|}{2r},\end{cases}\quad \varpi_U^2=\begin{cases}-8M\mp 8|M|,\\ -4M.\end{cases}$$

Then the corresponding supersymmetric partner potentials are given by

$$V^{\clubsuit}=S_V^2+\frac{dS_V}{d\tilde{r}}+\varpi_V^2=\begin{cases}(-M+r^2)(\frac{35}{4}-\frac{7M\pm 8|M|}{4r^2}),\\ (-M+r^2)(\frac{9}{4}-\frac{7M\pm 8|M|}{4r^2}),\end{cases}$$

$$U^{\clubsuit}=S_U^2+\frac{dS_U}{d\tilde{r}}+\varpi_U^2=\begin{cases}(-M+r^2)(\frac{15}{4}-\frac{19M\pm 16|M|}{4r^2}),\\ (-M+r^2)(\frac{1}{4}-\frac{19M\pm 16|M|}{4r^2}).\end{cases}$$

The original Schrödinger problems are translated into the following forms

$$(-\frac{d^2}{d\tilde{r}^2}+S^2\mp\frac{dS}{d\tilde{r}})\Psi=\begin{cases}A^\dagger A\Psi\\ AA^\dagger\Psi\end{cases}=(\omega^2-\varpi^2)\Psi$$

by introducing the operators

$$A=\frac{d}{d\tilde{r}}+S, A^\dagger=-\frac{d}{d\tilde{r}}+S$$

with the commutation relation $[A,A^\dagger]=2\frac{dS}{d\tilde{r}}$. In particular, one immediately obtains some type-II s-modes with special values of $\omega^2(=\varpi_V^2,\ \text{or}\ \varpi_U^2)$ via

$$H_{02}=\sqrt{r}\exp\{-\int dr\frac{S_V}{-M+r^2}\},$$

or

$$H_{02}=(-M+r^2)^2\frac{d}{dr}[\frac{\sqrt{r}}{-M+r^2}\exp\{-\int dr\frac{S_U}{-M+r^2}\}]+\frac{-M+3r^2}{\sqrt{r}}\exp\{-\int dr\frac{S_U}{-M+r^2}\},$$

for example,

$$H_{02}=\begin{cases}|-M+r^2|^{-\frac{3M\pm|M|}{2M}}r^{\frac{M\pm|M|}{M}}, & \omega^2=-10M\mp 6|M|;\\ |-M+r^2|^{\frac{M\mp|M|}{2M}}r^{\frac{M\pm|M|}{M}}, & \omega^2=-2M\pm 2|M|;\\ |-M+r^2|^{-\frac{M\pm|M|}{M}}r^{\pm 2\frac{|M|}{M}}, & \omega^2=-8M\mp 8|M|;\\ |-M+r^2|^{\mp\frac{|M|}{M}}r^{\pm 2\frac{|M|}{M}}(\frac{\pm 4|M|+2M}{M}r^2-2M\mp 2|M|), & \omega^2=-4M.\end{cases}$$

One can observe some duality relations by virtue of supersymmetric quantum mechanics.

- Choosing superpotential $S_U = \frac{3}{2}r - \frac{3M}{2r}$ whatever the sign of M is, since $V = U^{\clubsuit}$, if $\tilde{H}_{12}$ is a solution of the corresponding Schrödinger-type equation for axial mode, then $A_U\tilde{H}_{12}$ is a solution of the equation govern $\tilde{H}_{02}$. Conversely, the same thing occurs by acting on $\tilde{H}_{02}$ with the operator $A_U^{\dagger}$. This is nothing but a reinterpretation of the relation (3.18).
- Under the case of BTZ background metric, one is restricted to the region inside of the horizon, then when ω^2 is very large, there is an approximation $W \approx \frac{1}{5}V^{\clubsuit}$ by fixing $S_V = \frac{5}{2}r - \frac{3M}{2r}$. Indeed, one should resume a hidden parameter $\ell = \sqrt{-\frac{1}{\Lambda}}$ with dimension of length in AdS$_3$ or BTZ metric, where Λ stands for the cosmological constant, and is assumed to be -1 previously. Then after the following rescaling that does not affect the horizon
$$\ell \to \bar{\ell} = \sqrt[4]{5}\ell, M \to \bar{M} = \frac{1}{\sqrt{5}}M,$$
the $\tilde{H}_{02}$-component of the supersymmetric partner of axial mode is subject to the Schrödinger-type equation with potential $\frac{1}{5}V^{\clubsuit}_{(\ell,M)}$, so it corresponds to the $\tilde{H}_{01}$-component of the original polar mode, thus $A_{V_{(\bar{\ell},\bar{M})}}\tilde{H}_{02}^{(\bar{\ell},\bar{M})}$ produces a polar mode with large ω^2.
- We expand $\tilde{H}_{02}$ in terms of Laurent series as
$$\tilde{H}_{02} = \cdots + I_{-3}r^3 + I_{-2}r^2 + I_{-1}r + I_0 + \frac{I_1}{r} + \frac{I_2}{r^2} + \frac{I_3}{r^3} + \cdots.$$
At the spatial infinity $r \to \infty$, substituting it into the Schrödinger-type equation gives rise to the recurrence relations among the coefficients
$$(n(n-1) - \frac{15}{4})I_n + (\omega^2 + \frac{9M}{2})I_{n-2} - \frac{3M^2}{4}I_{n-4} = 0, n \in \mathbb{Z},$$
meanwhile, at $r \to 0$ the recurrence relations are given by
$$(n(n+1) - \frac{3}{4})I_n + M^{-2}(\omega^2 + \frac{9M}{2})I_{n+2} - \frac{15}{4M^2}I_{n+4} = 0, n \in \mathbb{Z}.$$
The similar asymptotic expansion can be done for the supersymmetric partner $\tilde{H}_{02}^{\mathrm{sp}}$ with coefficients J_n. Then the relations
$$(\frac{d}{d\tilde{r}} + S_V)\tilde{H}_{02} = (\omega - \varpi_V)\tilde{H}_{02}^{\mathrm{sp}},$$
$$(-\frac{d}{d\tilde{r}} + S_V)\tilde{H}_{02}^{\mathrm{sp}} = (\omega + \varpi_V)\tilde{H}_{02},$$
connect these coefficients. For example, adopting $S_V = \frac{5}{2}r + \frac{M+2|M|}{2r}$, we have
$$r \to \infty : \begin{cases} (n - \frac{5}{2})I_n - \frac{M+2|M|}{2}I_{n-2} = -(\omega + 10M + 6|M|)J_{n-1}, \\ (n + \frac{5}{2})J_n + \frac{M+2|M|}{2}J_{n-2} = (\omega - 10M - 6|M|)I_{n-1}, \end{cases}$$
$$r \to 0 : \begin{cases} (n + \frac{M+2|M|}{2M})I_n + \frac{5}{2M}I_{n+2} = \frac{\omega + 10M + 6|M|}{M}J_{n-1}, \\ (n - \frac{M+2|M|}{2M})J_n - \frac{5}{2M}J_{n+2} = -\frac{\omega - 10M - 6|M|}{M}I_{n-1}. \end{cases}$$
- The operator A can be reformulated as the Kac–Schwarz-type operator [13]
$$A = A^{(W,Q)} = \frac{1}{W'(r)}\frac{d}{dr} - c\frac{W''(r)}{W'(r)^2} + Q(r)$$
for $W(r) = \tilde{r}(r)$, $Q(r) = S(r) - 2cr$ and the constant c, whose dual operator is given by
$$\tilde{A} = A^{(Q,W)} = \frac{1}{Q'(r)}\frac{d}{dr} - c\frac{Q''(r)}{Q'(r)^2} + W(r).$$
The difference from the standard Kac–Schwarz operators is that our W, Q are not polynomials. Complexifying the parameter r as the coordinate on the complex plane removed zero point, we apply the theory of local Fourier transformations. In this theory, we need the category $\mathcal{C}$ of vector spaces with connections over the field $K = \mathbb{C}((r))$, where a connection ∇ on a d-dimensional vector space V over K is a $\mathbb{C}$-linear operator $\nabla : V \to V$ satisfying Leibniz

rule, and then ∇ can be written as $\nabla = \frac{d}{dr} + A$ for the connection matrix $A \in \mathfrak{gl}_d(K)$ with respect to a fixed basis of V. Let $K_p = \mathbb{C}((r^{\frac{1}{p}}))$ be the unique extension of K of degree $p \geq 1$ and f be an element in K_p, then one defines an object $E_{(f,p)} \in \mathcal{C}$ by $E_{(f,p)} = (K_p, \frac{d}{dr} + \frac{f}{r})$. The isomorphism class of $E_{(f,p)}$ depends only on the equivalent class of f under the equivalent relation $f \sim g$ if $f - g \in r^{\frac{1}{p}}\mathbb{C}[[r^{\frac{1}{p}}]] + \frac{1}{p}\mathbb{Z}$. Now for $A = \frac{d}{d\tilde{r}} + S$, near the point $\tilde{r} = 0$, since the order of $f(\tilde{r}) = \tilde{r}S$ is zero for $\forall p$, we get

$$A \sim \frac{d}{d\tilde{r}} + \frac{b}{\tilde{r}}$$

for a constant b, and hence

$$\tilde{A} \sim \frac{d}{d\hat{r}} - \frac{2c - b}{\hat{r}}$$

for $\hat{r} = \frac{b}{\tilde{r}}$. Then choosing $p = 1$, the local Fourier transformation of A is given by [8]

$$\mathcal{F}^{(0,\infty)}(A) = \frac{d}{d\mathring{r}} + \frac{f(\mathring{r})}{\mathring{r}} = A,$$

where the dual parameter $\mathring{r}$ is determined by $f = -\frac{\tilde{r}}{\mathring{r}}$.

Related to Seiberg--Witten theory

For the Schrödinger-type equations (3.19), (3.20), (3.21), we can consider the quantum correction to the Bohr–Sommerfeld integrals. Write $\tilde{H} = \exp(\frac{i}{\lambda}\int^{\tilde{r}} P(\tilde{r})d\tilde{r})$ with a parameter λ and expand P in terms of λ as $P = \sum_{n=0}(i\lambda)^n P_n$, then by considering the power of λ, we have the recurrence equations

$$\begin{aligned} P_0^2 &= w^2(\tilde{r}), \\ P_0' - 2P_1P_0 &= 0, \\ P_1' - P_1^2 - 2P_0P_2 &= 0, \\ &\ldots\ldots, \end{aligned}$$

thus, $P_1 = \frac{w'}{2w}$, $P_2 = \frac{w''}{4w^2} - \frac{3(w')^2}{8w^3}, \ldots$. Substituting the first order solution into the original equations gives rise to the equation satisfied by the function w:

$$w^2 = [\omega^2 - V] - \frac{1}{2}\{\int w d\tilde{r}; \tilde{r}\}_S,$$

where $\{\cdot;\cdot\}_S$ denotes the Schwartz derivative defined by

$$\{\int w d\tilde{r}; \tilde{r}\}_S = \frac{1}{w}\frac{d^2 w}{d\tilde{r}^2} - \frac{3}{2w^2}(\frac{dw}{d\tilde{r}})^2.$$

Hence the fewer lower order approximations are explicitly given by

$$\begin{aligned} P_0^{(0)} &= \sqrt{\omega^2 - V},\ P_0^{(1)} = \frac{\sqrt{16(\omega^2 - V)^3 - 4V''(\omega^2 - V) - 5(V')^2}}{4(\omega^2 - V)}, \\ P_1^{(0)} &= -\frac{V'}{4(\omega^2 - V)},\ P_1^{(1)} = -\frac{8(\omega^2 - V)^3 V' + 2(\omega^2 - V)^2 V''' + 7(\omega^2 - V)V'V'' + 5(V')^3}{2(\omega^2 - V)(16(\omega^2 - V)^3 - 4V''(\omega^2 - V) - 5(V')^2)} \\ P_2^{(0)} &= -\frac{4V''(\omega^2 - V) + 5(V')^2}{32(\omega^2 - V)^{\frac{5}{2}}}, \\ P_2^{(1)} &= -\frac{1}{2(\omega^2 - V)(16(\omega^2 - V)^3 - 4V''(\omega^2 - V) - 5(V')^2)^{\frac{5}{2}}}[2(\omega^2 - V)(16(\omega^2 - V)^3 - 4V''(\omega^2 - V) - 5(V')^2) \\ &\quad (-24(\omega^2 - V)^2(V')^2 + 8(\omega^2 - V)^3 V'' + 7(\omega^2 - V)(V'')^2 + 3(\omega^2 - V)V'V''' + 2(\omega^2 - V)^2 V'''' + 8(V')^2 V'') \\ &\quad - (8(\omega^2 - V)^3 V' + 7(\omega^2 - V)V'V'' + 2(\omega^2 - V)^2 V''' + 5(V')^3) \\ &\quad (-4V'(16(\omega^2 - V)^3 - 4V''(\omega^2 - V) - 5(V')^2) + (\omega^2 - V)(-48(\omega^2 - V)^2 V' - 4(\omega^2 - V)V''' - 6V'V'')) \\ &\quad + 3(8(\omega^2 - V)^3 V' + 7(\omega^2 - V)V'V'' + 2(\omega^2 - V)^2 V''' + 5(V')^3)^2], \\ &\quad \ldots\ldots. \end{aligned}$$

The "quantized" Bohr–Sommerfeld integral is given by

$$\begin{aligned}\Pi &= \int_a^b P(\tilde r)d\tilde r = \int_a^b P_0 d\tilde r + \lambda\int_a^b P_1 d\tilde r + \lambda^2\int_a^b P_2 d\tilde r + \cdots \\ &\approx \int_a^b \sqrt{\omega^2 - V}d\tilde r + \frac{\lambda}{4}(\log|\omega^2 - V|)_a^b - \frac{\lambda^2}{48}\int_a^b \frac{V''}{(\omega^2-V)^{\frac{3}{2}}}d\tilde r + \frac{5\lambda^2}{48}\frac{V'}{(\omega^2-V)^{\frac{3}{2}}}|_a^b + \cdots.\end{aligned}$$

For our cases $V = V_{\mathrm{AdS}}$ or U_{AdS}, we express the lowest order Bohr–Sommerfeld period $\Pi_0^{(0)}$ as

$$\Pi_0^{(0)} = \oint \sqrt{E - u\tan^2\tilde r - v\cot^2\tilde r}\,d\tilde r$$

with a parameter triple $(E, u, v) \in \mathbb{R}^3$, which satisfies the following series of Picard–Fuchs-type equations

$$\begin{aligned}[(E-2v)\partial_E^2 - E\partial_E\partial_u + 2v\partial_E\partial_v]\Pi_0^{(0)} &= 0,\\ [(E-4v)\partial_E^3 - (E+2u)\partial_E^2\partial_u + 4v\partial_E^2\partial_v - 2u\partial_E\partial_u^2]\Pi_0^{(0)} &= 0,\\ [(E-6v)\partial_E^4 - (E+4u)\partial_E^3\partial_u + 6v\partial_E^3\partial_v - 4u\partial_E^2\partial_u^2]\Pi_0^{(0)} &= 0,\\ &\cdots\cdots,\end{aligned}$$

due to the identities

$$\begin{aligned}d\frac{\tan\tilde r}{(E-u\tan^2\tilde r - v\cot^2\tilde r)^{\frac{n}{2}}} =& (-1)^{\frac{n+1}{2}}\frac{2^{\frac{n+3}{2}}}{\prod_{k=1}^{n-1}(2k-1)}[(E-(n+1)v)\partial_E^{\frac{n+3}{2}} - (E+(n-1)u)\partial_E^{\frac{n+1}{2}}\partial_u \\ &+ (n+1)v\partial_E^{\frac{n+3}{2}}\partial_v - (n-1)u\partial_E^{\frac{n-1}{2}}\partial_v^2]\sqrt{E-u\tan^2\tilde r - v\cot^2\tilde r}\,d\tilde r\end{aligned}$$

for an odd number n. We denote the parameters E, u, v by t_1, t_2, t_3 and write the Picard–Fuchs-type equations as the form

$$\frac{\partial^\alpha}{\partial t_{i_1}\cdots\partial t_{i_\alpha}}\Pi_0^{(0)} = \sum_{(j_1,\ldots,j_\alpha)\neq(i_1,\ldots,i_\alpha)} C_{i_1\cdots i_\alpha}^{j_1\cdots j_\alpha}(t_1,t_2,t_3)\frac{\partial^\alpha}{\partial t_{j_1}\cdots\partial t_{j_\alpha}}\Pi_0^{(0)},$$

where the coefficients C are all rational functions of t_1, t_2, t_3, and $\alpha = \frac{n+3}{2}$ is called the level. By these coefficients, one writes a 4×4-matrix $\mathbf{C}^{(\alpha)}$, called Picard–Fuchs matrix at level α,

$$\mathbf{C}^{(\alpha)} = \begin{array}{c} \mathbf{1}^\alpha \\ \mathbf{1}^{\alpha-1}\mathbf{2} \\ \mathbf{1}^{\alpha-1}\mathbf{3} \\ \mathbf{1}^{\alpha-2}\mathbf{2}^2 \end{array}\begin{pmatrix} 0 & \frac{t_1+(n-1)t_2}{t_1-(n+1)t_3} & -\frac{(n+1)t_3}{t_1-(n+1)t_3} & \frac{(n-1)t_2}{t_1-(n+1)t_3} \\ \frac{t_1-(n+1)t_3}{t_1+(n-1)t_2} & 0 & \frac{(n+1)t_3}{t_1+(n-1)t_2} & -\frac{(n-1)t_2}{t_1+(n-1)t_2} \\ -\frac{t_1-(n+1)t_3}{(n+1)t_3} & \frac{t_1+(n-1)t_2}{(n+1)t_3} & 0 & \frac{(n-1)t_2}{(n+1)t_3} \\ \frac{t_1-(n+1)t_3}{(n-1)t_2} & -\frac{t_1+(n-1)t_2}{(n-1)t_2} & \frac{(n+1)t_3}{(n-1)t_2} & 0 \end{pmatrix},$$

whose determinant is -3 independent of parameters and level. Then one can consider the characteristic polynomial of symmetric matrix $\mathfrak{C}^{(\alpha)} = \mathbf{C}^{(\alpha)}(\mathbf{C}^{(\alpha)})^\top$, it is given by

$$C^{(\alpha)}(x; t_1, t_2, t_3) := \det(\mathfrak{C}^{(\alpha)} - x) = x^4 + A(t)x^3 + B(t)x^2 + C(t)x + D(t),$$

with coefficients

$$\begin{aligned}A(t) =& -\frac{b^2+c^2+d^2}{a^2} - \frac{a^2+c^2+d^2}{b^2} - \frac{b^2+a^2+d^2}{c^2} - \frac{b^2+c^2+a^2}{d^2},\\ B(t) =& \frac{1}{a^2b^2c^2d^2}[c^2d^2(b^2+c^2+d^2)(a^2+c^2+d^2) + b^2d^2(b^2+c^2+d^2)(a^2+b^2+d^2)\\ &+ b^2c^2(b^2+c^2+d^2)(a^2+b^2+c^2) + a^2d^2(a^2+b^2+d^2)(a^2+c^2+d^2) + a^2c^2(a^2+b^2+c^2)(a^2+c^2+d^2)\\ &+ a^2b^2(a^2+b^2+d^2)(a^2+b^2+c^2) - c^2d^2(c^2+d^2)^2 - b^2d^2(b^2+d^2)^2 - b^2c^2(b^2+c^2)^2 - a^2c^2(a^2+c^2)^2\\ &- a^2d^2(a^2+d^2)^2 - a^2b^2(a^2+b^2)^2],\\ C(t) =& \frac{1}{a^2b^2c^2d^2}\{(a^2+c^2)^2[a^2(a^2+b^2+d^2)+c^2(b^2+c^2+d^2)] + (b^2+d^2)^2[b^2(a^2+b^2+c^2)+d^2(a^2+c^2+d^2)]\\ &+ (b^2+c^2)^2[c^2(a^2+c^2+d^2)+b^2(a^2+b^2+d^2)] + (a^2+d^2)^2[d^2(b^2+c^2+d^2)+a^2(a^2+b^2+c^2)]\\ &+ (a^2+b^2)^2[b^2(b^2+c^2+d^2)+a^2(a^2+c^2+d^2)] + (c^2+d^2)^2[d^2(a^2+b^2+d^2)+c^2(a^2+b^2+c^2)]\\ &- d^2(c^2+d^2)(a^2+d^2)(b^2+d^2) - a^2(a^2+c^2)(a^2+d^2)(a^2+b^2) - c^2(b^2+c^2)(c^2+d^2)(a^2+c^2)\end{aligned}$$

$$
\begin{aligned}
&- b^2(b^2+d^2)(a^2+b^2)(b^2+c^2) - d^2(b^2+d^2)(c^2+d^2)(a^2+d^2) - c^2(c^2+d^2)(a^2+c^2)(b^2+c^2)\\
&- b^2(b^2+c^2)(b^2+d^2)(a^2+b^2) - a^2(a^2+d^2)(a^2+c^2)(a^2+b^2)\\
&- d^2(b^2+c^2+d^2)(a^2+c^2+d^2)(a^2+b^2+d^2) - c^2(b^2+c^2+d^2)(a^2+c^2+d^2)(a^2+b^2+c^2)\\
&- b^2(b^2+c^2+d^2)(a^2+b^2+d^2)(a^2+b^2+c^2) - a^2(a^2+c^2+d^2)(a^2+b^2+d^2)(a^2+b^2+c^2)\},
\end{aligned}
$$

$$
\begin{aligned}
D(t) =&\frac{1}{a^2b^2c^2d^2}[(b^2+d^2)^2(a^2+c^2)^2 + (b^2+c^2)^2(a^2+d^2)^2 + (c^2+d^2)^2(a^2+b^2)^2\\
&- 2(c^2+d^2)(d^2+a^2)(a^2+b^2)(b^2+c^2) - 2(b^2+d^2)(d^2+c^2)(c^2+a^2)(a^2+b^2)\\
&- 2(b^2+d^2)(d^2+a^2)(a^2+c^2)(c^2+b^2) - (b^2+c^2+d^2)(a^2+b^2+d^2)(a^2+c^2)^2\\
&- (a^2+c^2+d^2)(a^2+b^2+c^2)(b^2+d^2)^2 - (a^2+c^2+d^2)(a^2+b^2+d^2)(b^2+c^2)^2\\
&- (b^2+c^2+d^2)(a^2+b^2+c^2)(a^2+d^2)^2 - (b^2+c^2+d^2)(a^2+c^2+d^2)(a^2+b^2)^2\\
&- (a^2+b^2+d^2)(a^2+b^2+c^2)(c^2+d^2)^2 + 2(c^2+d^2)(a^2+d^2)(b^2+d^2)(a^2+b^2+c^2)\\
&+ 2(a^2+c^2)(a^2+d^2)(a^2+b^2)(b^2+c^2+d^2) + 2(b^2+c^2)(a^2+c^2)(c^2+d^2)(a^2+b^2+d^2)\\
&+ 2(b^2+d^2)(a^2+b^2)(b^2+c^2)(a^2+c^2+d^2) + (b^2+c^2+d^2)(a^2+c^2+d^2)(a^2+b^2+d^2)(a^2+b^2+c^2)],
\end{aligned}
$$

where $a = t_1 - (n+1)t_3$, $b = t_1 + (n-1)t_2$, $c = (n+1)t_3$, $d = (n-1)t_2$. Obviously, $\mathcal{C}^{(\alpha)}(x; t_1, t_2, t_3)$ is invariant under the action of the product $S_2 \times S_3$ of 2-order and 3-order symmetry groups on the parameter space $\{t_1, t_2, t_3\}$. When $\alpha > 2$, for generic values of parameters, one associates $\mathcal{C}^{(\alpha)}(x; t_1, t_2, t_3)$ with a hyperelliptic curve viewed as a Seiberg–Witten curve defined by

$$
y^2 = \mathcal{C}^{(\alpha)}(x; t_1, t_2, t_3) = x^4 - \Lambda \prod_{r=1}^{3}(x - m_r)
$$

in $\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang–Mills theory with three hypermultiplets carrying masses m_r.

Finally, let us consider the extreme case $\upsilon \to 0$ corresponding to $r \to \infty$. By substituting the formal series [18]

$$
\Pi_0^{(0)}(\varepsilon) = \Pi_0^{(0)} + \varepsilon\widetilde{\Pi}_0^{(0)} + \mathcal{O}(\varepsilon^2) = E^{\frac{1}{2}+\varepsilon}(1 + \sum_{n>0} s_n(\frac{u}{E})^n)
$$

into the Picard–Fuchs equation of level 2, we have the recursion relation

$$
s_{n+1} = \frac{(2n-1-2\varepsilon)(2n+1-2\varepsilon)}{2(-2n-3+2\varepsilon)(n+1)} s_n = [-\frac{4n^2-1}{2(2n+3)(n+1)} + \frac{24n+1}{(2n+3)^2(n+1)}\varepsilon + \mathcal{O}(\varepsilon^2)]s_n,
$$

which yields the expansion

$$
\begin{aligned}
\Pi_0^{(0)}(\varepsilon) =&\sqrt{E}(1 + \frac{1}{6}\frac{u}{E} - \frac{1}{40}(\frac{u}{E})^2 + \cdots)\\
&+ \varepsilon[\sqrt{E}\log E(1 + \frac{1}{6}\frac{u}{E} - \frac{1}{40}(\frac{u}{E})^2 + \cdots) + \sqrt{E}(\frac{1}{9}\frac{u}{E} - \frac{1}{75}(\frac{u}{E})^2 + \cdots)].
\end{aligned}
$$

Let $\kappa = \Pi_0^{(0)}(0) = \Pi_0^{(0)} = \sqrt{E}(1 + \frac{1}{6}\frac{u}{E} - \frac{1}{40}(\frac{u}{E})^2 + \cdots)$, from which it follows that

$$
\sqrt{E} = \kappa(1 - \frac{1}{16}\frac{u}{\kappa^2} + \frac{27}{1280}(\frac{u}{\kappa^2})^2 + \cdots),
$$

hence we have

$$
\widetilde{\Pi}_0^{(0)} = \frac{d\Pi_0^{(0)}(\varepsilon)}{d\varepsilon}|_{\varepsilon=0} = 2\kappa\log\kappa - \frac{1}{8}\log k\frac{u}{\kappa} - \frac{5}{36}\frac{u}{\kappa} + \frac{27}{640}\log k\frac{u^2}{\kappa^3} + \frac{731}{28800}\frac{u^2}{\kappa^3} + \cdots,
$$

which produces the Seiberg–Witten prepotential $\mathcal{F}$ via $\frac{\partial\mathcal{F}}{\partial\kappa} = \widetilde{\Pi}_0^{(0)}$, thus

$$
\mathcal{F} = -\frac{\kappa^2}{2} + k^2\log\kappa - \frac{1}{16}u(\log\kappa)^2 - \frac{5}{36}u\log\kappa - \frac{27}{1280}\log\kappa\frac{u^2}{\kappa^2} - \frac{2677}{115200}\frac{u^2}{\kappa^2} + \cdots.
$$

4. Chern–Simons-like gravity on supermanifolds

Let us first briefly recall some basic materials for supermanifold [19]. Roughly speaking, a supermanifold X of dimension $p|q$ is a manifold with local coordinates $\{x^1, \ldots, x^p, \theta^1, \ldots, \theta^q\}$ where θ are Grassmannian variables. Mathematically, one defines a supermanifold of dimension $p|q$ as a ringed space $(X, \mathcal{O}_X)$, where X is a topological space, and $\mathcal{O}_X$ is a sheaf of supercommutative ring, locally isomorphic to $\mathbb{R}^{p|q} = (\mathbb{R}^p, C^\infty(\mathbb{R}^p)[\theta^1, \ldots, \theta^q])$. An $r|s$-vector bundle on a supermanifold $p|q$-supermanifold $(X, \mathcal{O}_X)$ is the quadruple $((E, \mathcal{O}_E), \pi, (X, \mathcal{O}_X), V)$ where $(E, \mathcal{O}_E) \to (X, \mathcal{O}_X)$ is a submersion of supermanifolds, V is a $r|s$-dimensional supervector space, and at each point $x \in X$ lies in a coordinate neighbourhood $U \subset X$, for which isomorphism Ψ_U, the following diagram commutes:

$$\begin{tikzcd}
(\pi^{-1}(U), \mathcal{O}_E|_{\pi^{-1}(U)}) \arrow[r, "\Psi_U"] \arrow[d, "\pi"'] & (U, \mathcal{O}_X|_U) \times V \arrow[d, "p_1"] \\
(U, \mathcal{O}_X|_U) \arrow[r, equal] & (U, \mathcal{O}_X|_U).
\end{tikzcd}$$

There is an open covering $\{U_\alpha\}$, a locally free sheaf $\mathcal{L}$ of $\mathcal{O}_X$-supermodule of rank $r|s$ and sheaf isomorphism $\mathcal{L}|_{U_\alpha} \mapsto (\mathcal{O}_X(U_\alpha))^r \oplus (\mathcal{O}_X(U_\alpha))^s$ such that on each $U_\alpha \cap U_\beta \neq \emptyset$, $g_{\alpha\beta} = g_\alpha g_\beta^{-1} : (\mathcal{O}_X(U_\alpha \cap U_\beta)_0)^r \oplus (\mathcal{O}_X(U_\alpha \cap U_\beta)_1)^s \to (\mathcal{O}_X(U_\alpha \cap U_\beta)_0)^r \oplus (\mathcal{O}_X(U_\alpha \cap U_\beta)_1)^s$ is an $\mathcal{O}_X(U_\alpha \cap U_\beta)$-supermodule isomorphism. Thus $g_{\alpha\beta}$ is of the form $g_{\alpha\beta} = \begin{pmatrix} A_{\alpha\beta} & B_{\alpha\beta} \\ C_{\alpha\beta} & D_{\alpha\beta} \end{pmatrix}$, where $A_{\alpha\beta} \in (\mathcal{O}_X(U_\alpha \cap U_\beta)_0)^{r\times r}$, $B_{\alpha\beta} \in (\mathcal{O}_X(U_\alpha \cap U_\beta)_1)^{r\times s}$, $C_{\alpha\beta} \in (\mathcal{O}_X(U_\alpha \cap U_\beta)_1)^{s\times r}$, $A_{\alpha\beta} \in (\mathcal{O}_X(U_\alpha \cap U_\beta)_0)^{s\times s}$, and $E \simeq E_r \oplus E_s$ where E_r and E_s are vector bundles over X of rank r and s given by cocycles $A_{\alpha\beta}$ and $D_{\alpha\beta}$ respectively. The supertangent bundle STX over $(X, \mathcal{O}_X)$ is a supervector bundle with $A_{\alpha\beta} = (\frac{\partial x^i_\beta}{\partial x^j_\alpha})$, $B_{\alpha\beta} = (\frac{\partial x^i_\beta}{\partial \theta^\lambda_\alpha})$, $C_{\alpha\beta} = (\frac{\partial \theta^\lambda_\beta}{\partial x^i_\alpha})$, $D_{\alpha\beta} = (\frac{\partial \theta^\lambda_\beta}{\partial \theta^\xi_\alpha})$. Locally, $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \theta^\alpha}\}$ forms a basis of $\mathcal{O}_X(U)$-module $STX(U)$. An even $\mathcal{O}_X$-bilinear map $G : STX \otimes STX \to \mathcal{O}_X$ is called a supermetric if $G(R, T) = (-1)^{|R||T|}G(T, R)$, $G(R, \cdot)$ defines an isomorphism between $\mathcal{O}_X$-modules STX and $(STX)^*$, and the restriction of G to $TX \otimes TX$ is a Riemannian metric. We can express the supermetric locally as $G = G_{ij}dx^i dx^j + G_{i\alpha}dx^i d\theta^\alpha + G_{\alpha\beta}d\theta^\alpha d\theta^\beta$, where $G_{ij} = G_{ji} \in \mathcal{O}_X(U)_0$, $G_{i\alpha} = G_{\alpha i} \in \mathcal{O}_X(U)_0$, $G_{\alpha\beta} = -G_{\beta\alpha} \in \mathcal{O}_X(U)_1$. One writes the supermetric as the form of block matrix as

$$G_{MN} = \begin{pmatrix} g_{ij} & \Lambda_{i\alpha} \\ \Lambda^\top_{\alpha i} & \Theta_{\alpha\beta} \end{pmatrix}, \tag{4.1}$$

then the inverse matrix is given by

$$G^{-1} = \begin{pmatrix} (1 - g^{-1}\Lambda\Theta^{-1}\Lambda^\top)^{-1}g^{-1} & -(1 - g^{-1}\Lambda\Theta^{-1}\Lambda^\top)^{-1}g^{-1}\Lambda\Theta^{-1} \\ -(1 - \Theta^{-1}\Lambda^\top g^{-1}\Lambda)^{-1}\Theta^{-1}\Lambda^\top g^{-1} & (1 - \Theta^{-1}\Lambda^\top g^{-1}\Lambda)^{-1}\Theta^{-1} \end{pmatrix}, \tag{4.2}$$

and the superdeterminant is given by

$$\mathrm{Sdet}(G) = \det(g - \Lambda\Theta^{-1}\Lambda^\top)\det(\Theta)^{-1}$$

The supermetric G is preserved by the othosymplectic supergroup $OSp(p|q)$ for even q whose underlying manifold is $SO(p) \times Sp(q)$. The notion of connection and covariant derivative have their corresponding supergeometric counterpart. Two conditions of vanishing of the covariant derivative of the supermetric vanishes and graded-symmetric of the lower indices of superconnection coefficients uniquely determine the superconnection given by the super Christoffel symbols

$$\Gamma^M_{NP} = \frac{1}{2}(-1)^{|Q|}G^{MQ}(G_{QN,P} + (-1)^{|N||P|}G_{QP,N} - (-1)^{|Q|(|N|+|P|)}G_{NP,Q}).$$

To define the integration over supermanifold X, the top form in usual manifold has to be replaced by the section of Berezinian line bundle $\mathrm{Ber}(X)$ [21].

We can naturally consider the following action on the supermanifold $X^{3|2}$

$$\mathbb{S} = \int_{X^{3|2}} d^3x d^2\theta (-1)^{|A|}\varepsilon^{ijk}(\Gamma^A_{iB}\partial_j\Gamma^B_{kA} + \frac{2}{3}\Gamma^A_{iB}\Gamma^B_{jC}\Gamma^C_{kA}). \tag{4.3}$$

Explicitly write the expansion of supermetric in terms of superfield formalism [9]

$$\begin{aligned}
g_{ij} &= h_{ij} + f_{ij\mu\nu}\theta^\mu\theta^\nu, \\
\Lambda_{i\mu} &= t_{i\mu\alpha}\theta^\alpha, \\
\Theta_{\mu\nu} &= y_{\mu\nu} + p_{\mu\nu\alpha\beta}\theta^\alpha\theta^\beta,
\end{aligned}$$

then the components of the inverse of supermetric are given by

$$\begin{aligned}
G^{ij} &= h^{ij} + h^{im}h^{jl}(t_{m\mu\alpha}t_{l\nu\beta}y^{\mu\nu} - f_{ml\alpha\beta})\theta^\alpha\theta^\beta = h^{ij} + (t^{i\mu}{}_\alpha t^j{}_{\mu\beta} - f^{ij}{}_{\alpha\beta})\theta^\alpha\theta^\beta, \\
G^{i\nu} &= h^{ij}t_{j\mu\alpha}y^{\mu\nu}\theta^\alpha = t^{i\nu}{}_\alpha\theta^\alpha, \\
G^{\mu\nu} &= y^{\mu\nu} + y^{\mu\alpha}y^{\beta\nu}(t_{i\alpha\rho}t_{j\beta\gamma}h^{ij} - p_{\alpha\beta\rho\gamma})\theta^\rho\theta^\gamma = y^{\mu\nu} - (t_i{}^\mu{}_\alpha t^{i\nu}{}_\beta - p^{\mu\nu}{}_{\alpha\beta})\theta^\alpha\theta^\beta,
\end{aligned}$$

where the indices in t, f denoted by latin letters i, j running over x^1, x^2, x^3 are raised or lowered by h and those in t, p denoted by Greek letters μ, ν running over θ^1, θ^2 are raised or lowered by y, and the super Christoffel symbols are decomposed as

$$\begin{aligned}
\Gamma^m_{in} =&(\Gamma^{(h)})^m_{in} + \frac{1}{2}[h^{ml}(\partial_n f_{li\alpha\beta} + \partial_i f_{ln\alpha\beta} - \partial_l f_{in\alpha\beta}) + (t^{m\mu}{}_\alpha t^l{}_{\mu\beta} - f^{ml}{}_{\alpha\beta})(\partial_n h_{li} + \partial_i h_{ln} - \partial_l h_{in})\\
&- t^{m\mu}{}_\alpha(\partial_n t_{i\mu\beta} + \partial_i t_{n\mu\beta} - 2f_{in\mu\beta})]\theta^\alpha\theta^\beta,\\
\Gamma^\mu_{in} =&\frac{1}{2}[t^{l\mu}{}_\alpha(\partial_n h_{li} + \partial_i h_{ln} - \partial_l h_{in}) - y^{\mu\nu}(\partial_n t_{i\nu\alpha} + \partial_i t_{n\nu\alpha} - 2f_{in\nu\alpha})]\theta^\alpha,\\
\Gamma^m_{i\mu} =&\frac{1}{2}[h^{mn}(2f_{ni\mu\alpha} + \partial_i t_{nu\alpha} - \partial_n t_{i\mu\alpha}) - t^{m\nu}{}_\alpha(t_{i\nu\mu} + \partial_i y_{\nu\mu} + t_{i\mu\nu})]\theta^\alpha,\\
\Gamma^\mu_{i\nu} =&-\frac{1}{2}y^{\mu\gamma}(t_{i\gamma\nu} + \partial_i y_{\gamma\nu} + t_{i\nu\gamma}) + \frac{1}{2}[t^{l\mu}{}_\alpha(2f_{li\nu\beta} + \partial_i t_{l\nu\beta} - \partial_l t_{i\nu\beta}) + (t_l{}^\mu{}_\alpha t^{l\gamma}{}_\beta - p^{\mu\gamma}{}_{\alpha\beta})\\
&(t_{i\gamma\nu} + \partial_i y_{\gamma\nu} + t_{i\nu\gamma}) - y^{\mu\gamma}\partial_i p_{\gamma\nu\alpha\beta}]\theta^\alpha\theta^\beta.
\end{aligned}$$

The integral over $d^2\theta$ picks up θ^2-terms, then the action (4.3) reduces to the usual integral over X. Apparently, if one fixes the gauge $y_{\mu\nu} = \varepsilon_{\mu\nu}$ and $p_{\mu\nu\alpha\beta} = \varepsilon_{\mu\nu}\varepsilon_{\alpha\beta}$, the bosonic degree of freedom coincides with the fermionic one. Then we explicitly express the action with superfields

$$\begin{aligned}
\mathbb{S} = \int_{X^3} & d^3x\varepsilon^{ijk}\{(\Gamma^{(h)})^n_{im}\partial_j[h^{ml}(\partial_n f_{lk\alpha}{}^\alpha + \partial_k f_{ln\alpha}{}^\alpha - \partial_l f_{kn\alpha}{}^\alpha) + (t^{m\mu}{}_\alpha t^l{}_\mu{}^\alpha - f^{ml}{}_\alpha{}^\alpha)(\partial_n h_{lk} + \partial_k h_{ln} - \partial_l h_{kn})\\
&- t^{m\mu}{}_\alpha(\partial_n t_{k\mu}{}^\alpha + \partial_k t_{n\mu}{}^\alpha - 2f_{kn\mu}{}^\alpha)] + \frac{1}{2}[h^{mn}(2f_{ni\mu\alpha} + \partial_i t_{nu\alpha} - \partial_n t_{i\mu\alpha}) - t^{m\nu}{}_\alpha(t_{i\nu\mu} + t_{i\mu\nu})]\\
&\partial_j[t^{l\mu\alpha}(\partial_m h_{lk} + \partial_k h_{lm} - \partial_l h_{km}) + (\partial_m t_k{}^{\mu\alpha} + \partial_k t_m{}^{\mu\alpha} - 2f_{km}{}^{\mu\alpha})] - \frac{1}{2}(t_i{}^\mu{}_\nu + t_{i\nu}{}^\mu)\\
&\partial_j[t^{l\nu}{}_\alpha(2f_{lk\mu}{}^\alpha + \partial_k t_{l\mu}{}^\alpha - \partial_l t_{k\mu}{}^\alpha) + (t_l{}^\nu{}_\alpha t^{l\gamma\alpha} - 2\varepsilon^{\nu\gamma})(t_{k\gamma\mu} + t_{k\mu\gamma})] + (\Gamma^{(h)})^n_{is}(\Gamma^{(h)})^s_{jm}\\
&[h^{ml}(\partial_n f_{lk\alpha}{}^\alpha + \partial_k f_{ln\alpha}{}^\alpha - \partial_l f_{kn\alpha}{}^\alpha) + (t^{m\mu}{}_\alpha t^l{}_{\mu\beta} - f^{ml}{}_{\alpha\beta})(\partial_n h_{lk} + \partial_k h_{ln} - \partial_l h_{kn})\\
&- t^{m\mu}{}_\alpha(\partial_n t_{k\mu}{}^\alpha + \partial_k t_{n\mu}{}^\alpha - 2f_{kn\mu}{}^\alpha)] + \frac{1}{2}(\Gamma^{(h)})^n_{im}[h^{ml}(2f_{lj\mu\alpha} + \partial_j t_{lu\alpha} - \partial_l t_{j\mu\alpha}) - t^{m\nu}{}_\alpha(t_{j\nu\mu} + t_{j\mu\nu})]\\
&[t^{s\mu\alpha}(\partial_n h_{sk} + \partial_k h_{sn} - \partial_s h_{kn}) + (\partial_n t_k{}^{\mu\alpha} + \partial_k t_n{}^{\mu\alpha} - 2f_{kn}{}^{\mu\alpha})] + \frac{1}{6}(t_i{}^\mu{}_\nu + t_{i\nu}{}^\mu)[t^{l\nu}{}_\alpha(\partial_n h_{lj} + \partial_j h_{ln} - \partial_l h_{jn})\\
&+ (\partial_n t_j{}^\nu{}_\alpha + \partial_j t_n{}^\nu{}_\alpha - 2f_{jn}{}^\nu{}_\alpha)][h^{mn}(2f_{mk\mu}{}^\alpha + \partial_k t_{mu}{}^\alpha - \partial_m t_{k\mu}{}^\alpha) - t^{n\nu\alpha}(t_{k\nu\mu} + t_{k\mu\nu})]\\
&- \frac{1}{4}(t_i{}^\mu{}_\nu + t_{i\nu}{}^\mu)(t_i{}^\nu{}_\lambda + t_{i\lambda}{}^\nu)[t^{l\lambda}{}_\alpha(2f_{lk\mu}{}^\alpha + \partial_k t_{l\mu}{}^\alpha - \partial_l t_{k\mu}{}^\alpha) + (t_l{}^\lambda{}_\alpha t^{l\gamma\alpha} - 2\varepsilon^{\lambda\gamma})(t_{k\gamma\mu} + t_{k\mu\gamma})]\}.
\end{aligned}$$

To include the contribution of the original bosonic action, one should consider the action

$$\begin{aligned}
\widetilde{\mathbb{S}} =& \int_{X^{3|2}} d^3x d^2\theta e^{\theta^1\theta^2}(-1)^{|A|}\varepsilon^{ijk}(\Gamma^A_{iB}\partial_j\Gamma^B_{kA} + \frac{2}{3}\Gamma^A_{iB}\Gamma^B_{jC}\Gamma^C_{kA})\\
=& S + \mathbb{S} - \frac{1}{4}\int_{X^3}\varepsilon^{ijk}(t_i{}^\mu{}_\nu + t_i{}^\nu{}_\mu)\partial_j(t_k{}^\nu{}_\mu + t_k{}^\mu{}_\nu) + \frac{1}{6}\int_{X^3}\varepsilon^{ijk}(t_i{}^\mu{}_\nu + t_i{}^\nu{}_\mu)(t_j{}^\nu{}_\gamma + t_j{}^\gamma{}_\nu)(t_k{}^\gamma{}_\mu + t_k{}^\mu{}_\gamma).
\end{aligned}$$

Another approach is to introduce the supervielbein $e^A(z) = dz^M e_M{}^A(z)$ on supermanifold $X^{3|2N}$ with coordinates $z^A = \{x^i, \theta^\alpha_I : i = 1, 2, 3; \alpha = 1, 2; I = 1, \cdots, N\}$ satisfying $e_A{}^M e_M{}^B = \delta^A_B$, $e_M{}^A e_A{}^N = \delta^N_M$ for the inverse supervielbein $e_A = e_A{}^M\frac{\partial}{\partial z^M}$, and introduce the covariant superderivatives on $N|2$-vector bundle on $X^{3|2N}$ with the structure group $OSp(N|2)$

$$\nabla_A = (\nabla_a, \nabla^I_\alpha) = e_A - \frac{1}{2}\Omega_A{}^{bc}\mathbb{M}_{bc} - \frac{1}{2}\Phi_A{}^{PQ}\mathbb{N}_{PQ} - \Psi^\alpha_{AI}\mathbb{Q}^I_\alpha,$$

where $\mathbb{M}$ are the Lorentz operators carrying two vector indices as $\mathbb{M}_{ab} = -\mathbb{M}_{ba}$, one vector indices as $\mathbb{M}_a = \frac{1}{2}\varepsilon^{abc}\mathbb{M}_{bc}$, and two spinor indices as $\mathbb{M}_{\alpha\beta} = (\gamma^a)_{\alpha\beta}\mathbb{M}_a = \mathbb{M}_{\beta\alpha}$ for $\gamma^a = (\mathrm{Id}, \sigma_1, \sigma_3)$, $\mathbb{N}$ are generators of $\mathfrak{so}(3)$, and $\mathbb{Q}$ are odd generators. The Lie superalgebra relations are given by

$$\begin{aligned}
[\mathbb{M}_{ab}, \mathbb{M}_{cd}] &= 2\eta_{ca}\mathbb{M}_{bd} + 2\eta_{db}\mathbb{M}_{ac} - 2\eta_{cb}\mathbb{M}_{ad} - 2\eta_{da}\mathbb{M}_{bc},\\
[\mathbb{N}_{IJ}, \mathbb{N}_{KL}] &= 2\delta_{KI}\mathbb{N}_{JL} + 2\delta_{LJ}\mathbb{N}_{IK} - 2\delta_{KJ}\mathbb{N}_{IL} - 2\delta_{LI}\mathbb{N}_{JK},\\
\{\mathbb{Q}^I_\alpha, \mathbb{Q}^J_\beta\} &= 2\delta^{IJ}\mathbb{M}_{\alpha\beta} - 2\varepsilon_{\alpha\beta}\mathbb{N}^{IJ},\\
[\mathbb{M}_{\beta\gamma}, \mathbb{Q}^I_\alpha] &= \varepsilon_{\alpha\beta}\mathbb{Q}^I_\gamma + \varepsilon_{\alpha\gamma}\mathbb{Q}^I_\beta,\\
[\mathbb{N}_{JK}, \mathbb{Q}^I_\alpha] &= 2\delta^I_J\mathbb{Q}_{\alpha K} - 2\delta^I_K\mathbb{Q}_{\alpha J},
\end{aligned}\tag{4.4}$$

and the (anti-)commutation relations of superderivatives are given by

$$[\nabla_A, \nabla_B] = -T_{AB}{}^C\nabla_C - \frac{1}{2}R(\mathbb{M})_{AB}{}^{cd}\mathbb{M}_{cd} - \frac{1}{2}R(\mathbb{N})_{AB}{}^{IJ}\mathbb{N}_{IJ} - R(\mathbb{Q})_{ABI}^{\ \ \ \alpha}\mathbb{Q}^I_\alpha,$$

where T, R denote torsion and curvature respectively

$$\begin{aligned}
T^a &:= \frac{1}{2}e^B \wedge e^A T_{AB}{}^a = de^a + e^b \wedge \Omega_b{}^a, \\
T_I^\alpha &:= \frac{1}{2}e^B \wedge e^A T_{ABI}^\alpha = de_I^\alpha + \frac{1}{2}e_I^\beta \wedge \Omega^c(\gamma_c)_\beta{}^\alpha + e_J^\alpha \wedge \Phi^J{}_I + e^a \wedge \Psi_I^\beta(\gamma_a)_\beta{}^\alpha, \\
(R(\mathbb{M}))^{ab} &:= \frac{1}{2}e^B \wedge e^A (R(\mathbb{M}))_{AB}{}^{cd} = d\Omega^{ab} + \Omega^{ac} \wedge \Omega_c{}^d + e_I^\alpha \wedge \Psi^{\beta I}(\gamma_c)_{\alpha\beta}\varepsilon^{cab}, \\
(R(\mathbb{N}))^{IJ} &:= \frac{1}{2}e^B \wedge e^A (R(\mathbb{N}))_{AB}{}^{IJ} = d\Phi^{IJ} + \Phi^{IK} \wedge \Phi_K{}^J + e^{\alpha I} \wedge \Psi_\alpha^J - e^{\alpha J} \wedge \Psi_\alpha^I, \\
(R(\mathbb{Q}))_I^\alpha &:= \frac{1}{2}e^B \wedge e^A (R(\mathbb{Q}))_{ABI}^\alpha = d\Psi_I^\alpha + \frac{1}{2}\Psi_I^\alpha \wedge \Omega^a(\gamma_a)_\beta{}^\alpha + \Psi^{\alpha J} \wedge \Phi_{JI},
\end{aligned} \tag{4.5}$$

one also chooses a Killing form on $\mathfrak{osp}(N|2)$

$$\begin{aligned}
&\mathrm{Tr}(\mathbb{M}_{ab}, \mathbb{M}_{cd}) = 2\eta_{ac}\eta_{bd} - 2\eta_{ad}\eta_{bc}, \\
&\mathrm{Tr}(\mathbb{N}_{IJ}, \mathbb{N}_{KL}) = 2\delta_{IK}\delta_{JL} - 2\delta_{IL}\delta_{JK}, \\
&\mathrm{Tr}(\mathbb{Q}_\alpha^I, \mathbb{Q}_\beta^J) = \delta^{IJ}\varepsilon_{\alpha\beta}.
\end{aligned} \tag{4.6}$$

Then we can define Chern–Simons-like action via (4.4)–(4.6) as follows [4]

$$\begin{aligned}
\mathcal{S} = &\int_{X^{3|2N}} \{\theta^6\delta^6(d\theta)\} \wedge \{\mathrm{Tr}(R(\mathbb{M}) \wedge \Omega + R(\mathbb{N}) \wedge \Phi + R(\mathbb{Q}) \wedge \Psi) \\
&- \frac{1}{6}\mathrm{Tr}([\Omega \wedge \Omega] \wedge (\Omega + \Psi) + [\Phi \wedge \Phi] \wedge (\Phi + \Psi) + ([\Psi \wedge \Psi] + 2[\Psi \wedge \Omega] + 2[\Psi \wedge \Phi]) \wedge (\Psi + \Omega + \Phi))\} \\
= &\int_{X^3} \{\mathrm{Tr}(R(\mathbb{M}) \wedge \Omega + R(\mathbb{N}) \wedge \Phi + R(\mathbb{Q}) \wedge \Psi) \\
&- \frac{1}{6}\mathrm{Tr}([\Omega \wedge \Omega] \wedge (\Omega + \Psi) + [\Phi \wedge \Phi] \wedge (\Phi + \Psi) + ([\Psi \wedge \Psi] + 2[\Psi \wedge \Omega] + 2[\Psi \wedge \Phi]) \wedge (\Psi + \Omega + \Phi))\}|_{\theta=0}.
\end{aligned} \tag{4.7}$$

Acknowledgments

The author Z. Hu would like to thank Dr. Ruiran Sun for his useful discussion.

References

[1] A. Aliev, Y. Nutku, Spinor formulation of topologically massive gravity, Classical Quantum Gravity 12 (1995) 2913.
[2] R. Arnowitt, S. Deser, C. Misner, Coordinate invariance and energy expressions in general relativity, Phys. Rev. 122 (2) (1961) 997–1006.
[3] L. Bergamin, D. Grumiller, W. Kummer, D. Vassilevich, Classical and quantum integrability of 2D dilaton gravities in Euclidean space, Classical Quantum Gravity 22 (2005) 1361–1381.
[4] L. Castellani, R. Catenacci, P. Grassi, Supergravity actions with integral forms, Nuclear Phys. B 889 (2014) 419.
[5] A. Chamseddine, J. Fröhlich, Two-dimensional Lorentz–Weyl anomaly and gravitational Chern–Simons theory, Comm. Math. Phys. 147 (3) (1992) 549–562.
[6] S. Deser, B. Tekin, Massive, topologically massive models, Classical Quantum Gravity 19 (2002) L97.
[7] C. Fefferman, C. Graham, Conformal Invariant, Elie Cartan et les Mathématiques D'aujour'Hui (Asteriaque 95), 1985.
[8] A. Graham-Squire, Calculation of local Fourier transforms, Ark. Mat. 51 (2013) 71–84.
[9] P. Grassi, M. Marescotti, Flus vacua and supermanifolds, J. High Energy Phys. 01 (2007) 068.
[10] G. Hall, T. Morgan, Z. Perjés, Three-dimensional space–times, Gen. Relativity Gravitation 19 (1987) 1137–1147.
[11] K. Hinterbichler, Theoretical aspects of massive gravity, Rev. Modern Phys. 84 (2012) 671.
[12] S. Hu, Z. Hu, On semi-classical topologically massive gravity in three dimensions, Eur. Phys. J. C 73 (2013) 2458.
[13] V. Kac, A. Schwarz, Geometric interpretation of the partition of 2D gravity, Phys. Lett. B 257 (1991) 329–334.
[14] H. Kodama, Conserved energy flux for the spherically symmetric system and the backreaction problem in the black hole evaporation, Progr. Theoret. Phys. 63 (1980) 1217–1228.
[15] H. Kodama, A. Ishibashi, Stability of generalised static black holes in higher dimensions, Progr. Theoret. Phys. 111 (2004) 29.
[16] W. Li, W. Song, A. Strominger, Chiral gravity in three dimensions, J. High Energy Phys. 04 (2008) 082.
[17] H. Liu, U. Simon, C. Wang, Higher order Codazzi tensors on conformally flat spaces, Contributions Algebra Geom. 39 (2) (1998) 329–348.
[18] A. Mironov, A. Morozov, Nekrasov functions and exact Bohr–Sommerfeld integrals, J. High Energy Phys. 04 (2010) 040.
[19] A. Roges, Supermanifolds: Theory and Applications, World Scientific, 2007.
[20] T. Strobl, Gravity in Two Spacetime Dimensions (Habilitation thesis). hep-th/0011240.
[21] E. Witten, Notes on supermanifolds and integration, arXiv:1209.2199.
[22] E. Witten, 2 + 1 dimensional gravity as an exactly soluble system, Nuclear Phys. B 311 (1988) 46.
[23] E. Witten, Anti-de sitter space and holography, Adv. Theor. Math. Phys. 2 (1998) 253–291.
[24] E. Witten, S.T. Yau, Connectedness of the boundary in the AdS/CFT correspondence, Adv. Theor. Math. Phys. 3 (1999) 1635, 1655.