Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Geometry and Physics 56 (2006) 1920–1939

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Orbifold conformal blocks and the stack of pointed G-covers

Matthew Szczesny*

Department of Mathematics and Statistics, Boston University, Boston, MA 02215, USA

Received 23 September 2004; received in revised form 26 October 2005; accepted 2 November 2005
Available online 27 December 2005

Abstract

Starting with a vertex algebra V, a finite group G of automorphisms of V, and a suitable collection of twisted V-modules, we construct (twisted) D-modules on the stack of pointed G-covers, introduced by Jarvis, Kaufmann, and Kimura. The fibers of these sheaves are spaces of orbifold conformal blocks defined in joint work with Edward Frenkel. The key ingredient is a G-equivariant version of the Virasoro uniformization theorem.

PACS: 14H30; 14H81; 22E67; 81T40

MSC: 17B69; 14H10; 32C38; 81T40

JGP SC: Moduli of algebraic curves; Vertex algebras; Conformal field theories

Keywords: Moduli stacks of algebraic curves; Vertex algebras; D-modules; Conformal blocks

1. Introduction

It is by now well-understood that conformal field theories (CFT's) in two dimensions give rise to sheaves with projectively flat connection over the moduli stack of n-pointed genus g curves, $\mathfrak{M}_{g,n}$. Mathematically, this process can be described as follows (see for instance the book [12] for more details). Given a vertex algebra V, which corresponds to a choice of CFT model, a collection of V-modules

$$\mathbb{M} = (M_1, \cdots, M_n),$$

* Tel.: +1 617 353 1487; fax: +1 617 3538100.
E-mail address: szczesny@math.bu.edu.

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2005.11.002

and an n-pointed curve $(X, \mathbf{p})$, $\mathbf{p} = (p_1, \cdots, p_n)$, $p_i \in X$, one obtains the vector space of conformal blocks, denoted $C_V(X, \mathbf{p}, \mathbb{M})$. Elements of this vector space can be used to construct chiral correlation functions in the CFT, which are sections of certain sheaves on powers of X with pairwise diagonals removed. As $(X, \mathbf{p})$ varies in $\mathfrak{M}_{g,n}$, one obtains a sheaf with a projectively flat connection, known as the Knizhnik-Zamolodchikov (KZ) connection. In certain cases, when the CFT is rational, the spaces $C_V(X, \mathbf{p}, \mathbb{M})$ are finite-dimensional. The sheaf is then a vector bundle that extends to the Deligne-Mumford compactification $\overline{\mathfrak{M}}_{g,n}$, and the connection to one with logarithmic singularities along the boundary divisor.

In several cases, the space of conformal blocks is related to a moduli problem. For instance, when $V = L_k(\mathfrak{g})$, the integrable basic $\hat{\mathfrak{g}}$-module of level $k \in \mathbb{Z}_+$, we have the isomorphism

$$C_{L_k(\mathfrak{g})}(X, p, L_k(\mathfrak{g})) \cong H^0(\mathrm{Bun}_G(X), \Theta^k)$$

where $\mathrm{Bun}_G(X)$ is the moduli stack of G-bundles on X, G is the simply-connected algebraic group with Lie algebra $\mathfrak{g}$, and Θ is the theta line bundle on $\mathrm{Bun}_G(X)$, In this case, the KZ connection yields a non-abelian analogue of the heat equation satisfied by abelian theta functions.

In recent years, much attention has been paid to the construction of CFT's by orbifolding. At the level of vertex algebras, this procedure can be described as follows. We are given a vertex algebra V, and a finite group G of automorphisms of V that also acts on the category of V-modules. To construct the orbifold model, one adjoins so called twisted modules $\{M^{[g]}\}$, for each conjugacy class $[g]$ in G. Geometrically, orbifold models possess the new feature of twist fields. These are objects that cause vertex operators to become multivalued, with monodromies given by elements of G.

In [13], the notion of conformal block was extended to include twisted modules. A key ingredient is the notion of a pointed G-cover. Let $(X, \mathbf{p})$ be a an n-pointed smooth projective curve. A pointed G-cover of $(X, \mathbf{p})$ is a set of data

$$(\pi : C \mapsto X, \mathbf{p}, \mathbf{q}), \quad \mathbf{q} = (q_1, \ldots, q_n)$$

where C is a smooth projective curve (not necessarily connected) with an effective action of G, such that $X = C/G$, the quotient map $\pi : C \mapsto X$ makes C in to a principal G-bundle over $X\backslash\{p_i\}$, and $q_i \in \pi^{-1}(p_i)$. The space of orbifold conformal blocks, denoted

$$C_V^G(C, X, \mathbf{p}, \mathbf{q}, \mathbb{M}) \tag{1.1}$$

is attached to a G-cover, and a collection of V-modules $\mathbb{M} = (M_1, \cdots, M_n)$, where M_i is twisted by the monodromy generator m_i at q_i. Orbifold conformal blocks can be used to construct chiral orbifold correlators, which are G-invariant sections of sheaves on powers of C with certain divisors removed. For a detailed treatment of this construction, see our forthcoming paper [22].

In [16], following earlier work by [1], the authors introduce a smooth Deligne-Mumford stack $\mathfrak{M}_{g,n}^G$ parameterizing smooth n-pointed G-covers. Given an appropriate collection $\mathbb{M}$ of (twisted) V-modules, it is a natural question how the spaces (1.1) vary as $(\pi : C \mapsto X, \mathbf{p}, \mathbf{q})$ moves in $\mathfrak{M}_{g,n}^G$. In this paper, we construct sheaves with a projectively flat connection over (certain components of) $\mathfrak{M}_{g,n}^G$, whose fibers at a G-cover $(\pi : C \mapsto X, \mathbf{p}, \mathbf{q})$ are the corresponding spaces (1.1). In other words, we construct a localization functor

$$\Delta : (V - \mathrm{mod}) \longrightarrow (\tilde{D}_{\mathfrak{M}_{g,n}^G} - \mathrm{mod})$$

where $\tilde{D}_{\mathfrak{M}_{g,n}^G}$ is a sheaf of twisted differential operators on $\mathfrak{M}_{g,n}^G$, depending on the Virasoro central charge. This yields an orbifold generalization of the KZ connection. The resulting sheaves should be useful in constructing G-modular functors (see [18]).

In the untwisted case, the key theorem in the construction is the so called Virasoro uniformization of $\mathfrak{M}_{g,n}$ (see [2,8,19,23]). Roughly, this theorem states that $\mathfrak{M}_{g,n}$ carries a transitive action of the Virasoro algebra. We extend the approach to G-covers.

Just as $\mathfrak{M}_{g,n}$ can be compactified to $\overline{\mathfrak{M}}_{g,n}$, so does $\mathfrak{M}^G_{g,n}$ have a compactification $\overline{\mathfrak{M}^G_{g,n}}$. In this paper we do not address the behavior of the D-modules on the boundary $\overline{\mathfrak{M}^G_{g,n}} \backslash \mathfrak{M}^G_{g,n}$. Just as in the untwisted case, the existence of a logarithmic extension requires the finite-dimensionality of conformal blocks. It is natural to conjecture that such an extension is possible for WZW models of positive integer level, when $V = L_k(\mathfrak{g})$, and where the twisted modules are realized as integrable representations of appropriate twisted affine algebras.

The structure of the paper is as follows. In Section 2 we quickly review the notion of vertex algebra and vertex algebra module/twisted module. Section 3 recalls some facts about the stack of pointed G-covers from [16]. In Sections 4 and 5 we review the construction of the space of orbifold conformal blocks $C^G_V(C, X, \mathbf{p}, \mathbf{q}, \mathbb{M})$ following [13]. Section 6 is devoted to reviewing Beilinson-Bernstein localization, which is the technique used to produce D-modules on $\mathfrak{M}^G_{g,n}$. Our treatment is essentially a condensed version of that in [12]. Section 7 contains a proof of the G-equivariant Virasoro uniformization of $\mathfrak{M}^G_{g,n}$ and the construction of the localization functor. Finally, Section 8 gives a formula for the orbifold KZ connection along the fibers of the projection

$$\mathfrak{M}^G_{g,n} \mapsto \mathfrak{M}_g$$

sending $(\pi : C \mapsto X, \mathbf{p}, \mathbf{q})$ to X, i.e. the locus of pointed G-covers with a fixed base curve.

2. Vertex algebras and twisted modules

In this paper we will use the language of vertex algebras, their modules, and twisted modules. For an introduction to vertex algebras and their modules see [15,17,12], and for background on twisted modules, see [14,10,11,7].

We recall that a conformal vertex algebra is a $\mathbb{Z}_+$-graded vector space

$$V = \bigoplus_{n=0}^{\infty} V_n,$$

together with a vacuum vector $|0\rangle \in V_0$, a translation operator T of degree 1, a conformal vector $\omega \in V_2$ and a map

$$\begin{aligned} Y : V &\to \operatorname{End} V[[z^{\pm 1}]], \\ A &\mapsto Y(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}. \end{aligned}$$

These data must satisfy certain axioms (see [15,17,12]). In what follows we will denote the collection of such data simply by V.

A vector space M is called a V-module if it is equipped with an operation

$$\begin{aligned} Y^M : V &\to \operatorname{End} M[[z^{\pm 1}]], \\ A &\mapsto Y^M(A, z) = \sum_{n \in \mathbb{Z}} A^M_{(n)} z^{-n-1} \end{aligned}$$

such that for any $v \in M$ we have $A^M_{(n)}v = 0$ for large enough n. This operation must satisfy the following axioms:

- $Y^M(|0\rangle, z) = \mathrm{Id}_M$;
- For any $v \in M$ there exists an element

  $$f_v \in M[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$$

  such that the formal power series

  $$Y^M(A, z)Y^M(B, w)v \quad \text{and} \quad Y^M(Y(A, z-w)B, w)v$$

  are expansions of f_v in $M((z))((w))$ and $M((w))((z-w))$, respectively.

The power series $Y^M(A, z)$ are called vertex operators. We write the vertex operator corresponding to ω as

$$Y^M(\omega, z) = \sum_{n\in\mathbb{Z}} L^M_n z^{-n-2},$$

where L^M_n are linear operators on V generating the Virasoro algebra. Following [10], we call M *admissible* if L^M_0 acts semi-simply with integral eigenvalues.

Now let σ_V be a conformal automorphism of V, i.e., an automorphism of the underlying vector space preserving all of the above structures (so in particular $\sigma_V(\omega) = \omega$). We will assume that σ_V has finite order $N > 1$. A vector space M^σ is called a σ_V-*twisted* V-module (or simply twisted module) if it is equipped with an operation

$$\begin{aligned} Y^{M^\sigma} &: V \to \operatorname{End} M^\sigma[[z^{\pm(1/N)}]], \\ &A \mapsto Y^{M^\sigma}(A, z^{1/N}) = \sum_{n\in(1/N)\mathbb{Z}} A^{M^\sigma}_{(n)} z^{-n-1} \end{aligned}$$

such that for any $v \in M^\sigma$ we have $A^{M^\sigma}_{(n)}v = 0$ for large enough n. Please note that we use the notation $Y^{M^\sigma}(A, z^{1/N})$ rather than $Y^{M^\sigma}(A, z)$ in the twisted setting. This operation must satisfy the following axioms (see [14,10,11,20]):

- $Y^{M^\sigma}(|0\rangle, z^{1/N}) = \mathrm{Id}_{M^\sigma}$;
- For any $v \in M^\sigma$, there exists an element

  $$f_v \in M^\sigma[[z^{1/N}, w^{1/N}]][z^{-(1/N)}, w^{-(1/N)}, (z-w)^{-1}]$$

  such that the formal power series

  $$Y^{M^\sigma}(A, z^{1/N})Y^{M^\sigma}(B, w^{1/N})v \quad \text{and} \quad Y^{M^\sigma}(Y(A, z-w)B, w^{1/N})v$$

  are expansions of f_v in $M^\sigma((z^{1/N}))((w^{1/N}))$ and $M^\sigma((w^{1/N}))((z-w))$, respectively.
- If $A \in V$ is such that $\sigma_V(A) = \mathrm{e}^{(2\pi i m/N)}A$, then $A^{M^\sigma}_{(n)} = 0$ unless $n \in \frac{m}{N} + \mathbb{Z}$.

The series $Y^{M^\sigma}(A, z)$ are called twisted vertex operators. In particular, the Fourier coefficients of the twisted vertex operator

$$Y^{M^\sigma}(\omega, z^{1/N}) = \sum_{n\in\mathbb{Z}} L^{M^\sigma}_n z^{-n-2},$$

generate an action of the Virasoro algebra on M^σ. The σ_V-twisted module M^σ is called *admissible* if $L_0^{M^\sigma}$ acts semi-simply with eigenvalues in $\frac{1}{N}\mathbb{Z}$.

Suppose that M^σ is an admissible module. Then we define a linear operator S_σ on M^σ as follows. It acts on the eigenvectors of $L_0^{M^\sigma}$ with eigenvalue $\frac{m}{N}$ by multiplication by $\mathrm{e}^{(2\pi i m/N)}$. Hence we obtain an action of the cyclic group of order N generated by σ on M^σ, $\sigma \mapsto S_\sigma$. According to the axioms of twisted module, we have the following identity:

$$S_\sigma^{-1} Y^{M^\sigma}(\sigma \cdot A, z^{1/N}) S_\sigma = Y^{M^\sigma}(A, z^{1/N}). \tag{2.1}$$

3. The stack of pointed G-covers

In this section we review the definition of the stack of pointed G-covers introduced in [16] following [1].

Let $\mathfrak{M}_{g,n}$ denote the stack of smooth n-pointed curves of genus g. The objects of $\mathfrak{M}_{g,n}$ consist of families $(\lambda : X \mapsto S, p_1, \cdots, p_n)$ where $\lambda : X \mapsto S$ is a smooth family of curves of genus g, and $p_i : S \mapsto X,\ i = 1, \cdots, n$ are pairwise disjoint sections of λ. Note that the p_i are ordered.

Definition 3.1. Let $(\lambda : X \mapsto S, p_1, \cdots, p_n) \in \mathfrak{M}_{g,n}$ be a smooth n-pointed curve, and G a finite group. A smooth G-cover of X consists of a morphism $\pi : C \mapsto X$ of smooth curves over S, satisfying the following properties:

- There is a right G-action on C preserving π.
- $C \backslash \cup_i \pi^{-1}(p_i)$ is a principal G-bundle over $X \backslash \cup_i p_i$

We denote the smooth G-cover by $(\pi : C \mapsto X, p_1, \ldots, p_n)$. A smooth n-pointed G-cover consists of a smooth G-cover $(\pi : C \mapsto S, p_1, \ldots, p_n)$ together with n sections $q_i : S \mapsto C$ such that $q_i \in \pi^{-1}(p_i)$. We denote it by $(\pi : C \mapsto X, p_1, \ldots, p_n, q_1, \ldots, q_n)$.

To avoid cumbersome notation, we will henceforth use $(X, \mathbf{p})$, $(C, X, \mathbf{p})$, and $(C, X, \mathbf{p}, \mathbf{q})$, to denote, respectively, an n-pointed curve, a G-cover, and a pointed G-cover, where $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$.

A morphism of pointed G-covers is a G-equivariant fibered diagram - that is, a morphism of the underlying curves X together with a G-equivariant morphism of the covers preserving the points q_i. Smooth n-pointed G-covers form a stack, denoted $\mathfrak{M}^G_{g,n}$. There is an obvious morphism

$$\begin{aligned} &p : \mathfrak{M}^G_{g,n} \mapsto \mathfrak{M}_{g,n} \\ &(C, X, \mathbf{p}, \mathbf{q}) \mapsto (X, \mathbf{p}) \end{aligned}$$

Let G_A denote the group G viewed as a right G-space under conjugation. There is a G-equivariant map

$$\mathbf{e} : \mathfrak{M}^G_{g,n} \mapsto G_A^n$$

sending $(C, X, \mathbf{p}, \mathbf{q})$ to the n-tuple $\mathbf{m} = (m_1, \ldots, m_n)$, where m_i is the monodromy of C around q_i. Let $\mathfrak{M}^G_{g,n}(\mathbf{m})$ denote the closed sub-stack $\mathbf{e}^{-1}(\mathbf{m})$. Note that $\mathfrak{M}^G_{g,n}(\mathbf{m})$ may be empty. We have

$$\mathfrak{M}^G_{g,n} = \bigcup_{\mathbf{m} \in G_A^n} \mathfrak{M}^G_{g,n}(\mathbf{m})$$

We have the following theorem:

Theorem 3.1. [16] *The stack* $\mathfrak{M}^G_{g,n}$ *and the stacks* $\mathfrak{M}^G_{g,n}(\mathbf{m})$ *are smooth Deligne-Mumford stacks, flat, proper, and quasi-finite over* $\mathfrak{M}_{g,n}$.

4. Groups and torsors

The purpose of this section is to review the structure of the group of formal coordinate changes on the "formal" disk $\mathrm{Spec}(R[[z]])$ and the "formal punctured disk" $\mathrm{Spec}(R((z)))$ for an arbitrary $\mathbb{C}$-algebra R. We will be primarily interested in the case when R is an Artin $\mathbb{C}$-algebra, such as $\mathbb{C}[\epsilon]/(\epsilon^2)$. For more on the structure of these groups, see Section 5.1 of [12], or the original source [21].

4.1. Groups

Let us denote $R[[z]]$ by $\mathcal{O}_R$, and $R((z))$ by $\mathcal{K}_R$. Let $\mathrm{Aut}(\mathcal{O}_R)$ denote the group of continuous R-algebra automorphisms of $R[[z]]$ preserving the ideal (z), and $\mathrm{Aut}(\mathcal{K}_R)$ the group of all continuous R-algebra automorphisms of $R((z))$. Since $R[[z]]$ is topologically generated by z, an automorphism ρ of $R[[z]]$ is completely determined by the image of z, which is a series of the form

$$\rho(z) = \sum_{n\in\mathbb{Z},n\geq 1} c_n z^n, \quad c_n \in R \tag{4.1}$$

where c_1 is a unit. Hence we identify $\mathrm{Aut}(\mathcal{O}_R)$ with the space of power series in z satisfying these conditions. Similarly, an element ρ of $\mathrm{Aut}(\mathcal{K}_R)$ can be identified with a formal Laurent series of the form

$$\rho(z) = \sum_{n\in\mathbb{Z},n\geq k} c_n z^n,$$

where c_n is nilpotent if $n \leq 0$, and c_1 is a unit. The functor

$$R \mapsto \mathrm{Aut}(\mathcal{O}_R)$$

is representable by a group scheme over $\mathbb{C}$ which we'll denote $\mathrm{Aut}\mathcal{O}$. The Lie algebra $\mathrm{Der}^{(o)}(\mathcal{O})$ of $\mathrm{Aut}(\mathcal{O})$ is topologically generated by elements

$$z^k\partial_z, \ k \geq 1$$

The functor

$$R \mapsto \mathrm{Aut}(\mathcal{K}_R)$$

is representable by an Ind-group scheme which we'll denote $\mathrm{Aut}(\mathcal{K})$. Denote by $\mathrm{Der}(\mathcal{K})$ the Lie algebra of $\mathrm{Aut}(\mathcal{K})$, generated by

$$z^k\partial_z, \quad k \in \mathbb{Z}$$

We now consider some groups arising in the study of ramified coverings of disks. Let $\mathrm{Aut}(R[[z^{1/N}]])$ denote the group of continuous R-algebra automorphisms of $R[[z^{1/N}]]$ preserving the ideal $(z^{1/N})$.

Definition 4.1. $\mathrm{Aut}_N(\mathcal{O}_R)$ is the subgroup of $\mathrm{Aut}(R[[z^{1/N}]])$ preserving the subalgebra $R[[z]] \subset R[[z^{1/N}]]$.

Thus, $\mathrm{Aut}_N\mathcal{O}$ consists of power series of the form

$$\rho(z^{1/N}) = \sum_{n\in\frac{1}{N}+\mathbb{Z}, n>0} c_n z^n, \quad c_n \in R \tag{4.2}$$

such that $c_{1/N}$ is a unit.

There is a homomorphism $\mu_N : \mathrm{Aut}_N(\mathcal{O}_R) \to \mathrm{Aut}(\mathcal{O}_R)$ which takes $\rho \in \mathrm{Aut}_N(\mathcal{O}_R)$ to the automorphism of $R[[z]]$ that it induces. At the level of power series, this is just the map $\mu_N : \rho(z^{1/N}) \mapsto \rho(z^{1/N})^N$. The kernel consists of the automorphisms of the form $z^{1/N} \mapsto \epsilon z^{1/N}$, where ϵ is an Nth root of unity, so we have the following exact sequence:

$$1 \to \mathbb{Z}/N\mathbb{Z} \to \mathrm{Aut}_N(\mathcal{O}_R) \to \mathrm{Aut}(\mathcal{O}_R) \to 1\,.$$

Thus $\mathrm{Aut}_N(\mathcal{O}_R)$ is a central extension of $\mathrm{Aut}(\mathcal{O}_R)$ by the cyclic group $\mathbb{Z}/N\mathbb{Z}$. One can define the group $\mathrm{Aut}_N(\mathcal{K}_R)$ in an obvious way. Denote by $\mathrm{Aut}_N(\mathcal{O})$ the group scheme representing the functor $R \mapsto \mathrm{Aut}_N(\mathcal{O}_R)$, and $\mathrm{Aut}_N(\mathcal{K})$ the Ind-group scheme representing $R \mapsto \mathrm{Aut}_N(\mathcal{K}_R)$. The Lie algebra $\mathrm{Der}_N^{(o)}(\mathcal{O})$ of $\mathrm{Aut}_N(\mathcal{O})$ can be identified with $z^{1/N}\mathbb{C}[[z]]\partial_{z^{1/N}}$. The homomorphism μ_N induces an isomorphism of the corresponding Lie algebras sending

$$z^{k+(1/N)}\partial_{z^{1/N}} \mapsto Nz^{k+1}\partial_z, \quad k \in \mathbb{Z}, k \geq 0. \tag{4.3}$$

Similarly, the Lie algebra $\mathrm{Der}_N(\mathcal{K})$of $\mathrm{Aut}_N(\mathcal{K})$ can be identified with $z^{1/N}\mathbb{C}((z))\partial_{z^{1/N}}$, and μ_N extends to a homomorphism $\mathrm{Der}_N(\mathcal{K}) \mapsto \mathrm{Der}(\mathcal{K})$.

Let X be a smooth curve over Spec$R$. If $x \in X$ is an R-valued point of X, denote by $\mathrm{Aut}(\hat{\mathcal{O}}_{x,R})$ the automorphisms of the formal neighborhood of x fixing x. Suppose now that C is a smooth curve over Spec$R$ with effective G-action, and that $X = C/G$. Denote the quotient map by $\pi : C \mapsto X$. Choosing a formal coordinate z at x yields an isomorphism $\mathrm{Aut}(\hat{\mathcal{O}}_{x,R}) \cong \mathrm{Aut}(\mathcal{O}_R)$ (For the definition of a formal coordinate, see the next section).

If $y \in C$, denote by $O(y)$ the G-orbit of y in C, and by $\widehat{O(y)}$ the formal neighborhood of $O(y)$. Let $\widehat{O(y)}^*$ denote the union of the formal punctured disks around points in $O(y)$. Note that $\widehat{O(y)}$ and $\widehat{O(y)}^*$ carry an action of G. Let

$$\mathrm{Aut}(\widehat{O(y)}_R)\, resp.\, \mathrm{Aut}(\widehat{O(y)}^*_R)$$

denote the automorphism group of $O(y)$ and $O(y)^*$, respectively, and

$$\mathrm{Aut}^G(\widehat{O(y)}_R)\, resp.\, \mathrm{Aut}^G(C, \widehat{O(y)}^*_R)$$

the corresponding subgroups of elements commuting with G. Finally, denote by

$$\mathrm{Aut}^G_e(\widehat{O(y)}_R)\, resp.\, \mathrm{Aut}^G_e(\widehat{O(y)}^*_R)$$

the identity components of the corresponding groups. For each $q \in O(y)$, by choosing a coordinate in which the action of the stabilizer G_q is linear (this is called a special coordinate, see the next section), we obtain isomorphisms

$$\psi_q : \mathrm{Aut}^G_e(\widehat{O(y)}_R) \mapsto \mathrm{Aut}_N(\mathcal{O}_R)$$

and

$$\psi^*_q : \mathrm{Aut}^G_e(\widehat{O(y)}^*_R) \mapsto \mathrm{Aut}_N(\mathcal{K}_R)$$

by restricting to the disk around q. The proof amounts to observing that any element of $\mathrm{Aut}_N(\mathcal{O}_R)$ has a unique G-equivariant extension to $\mathrm{Aut}^G_e(\widehat{O(y)}_R)$ (and the same with $\hat{O}$ replaced by $\hat{O}^*$).

4.2. Torsors

Let $\mathcal{D}_R = \mathrm{Spec}A$, where $A \cong R[[z]]$ is a "formal" R-disk, and let $x \in \mathcal{D}_R$ be an R-point. By a formal coordinate on $(\mathcal{D}_R, x)$ we mean a isomorphism $\mathcal{D}_R \cong \mathrm{Spec}(R[[z]])$ that identifies x with the origin (the origin being the R-point corresponding to the ideal $(z) \subset R[[z]]$). Let $\mathcal{A}ut(\mathcal{D}_{R,x})$ denote the set of formal coordinates on $(\mathcal{D}_R, x)$. It is an $\mathrm{Aut}(\mathcal{O}_R)$-torsor.

Suppose now that $(\mathcal{D}_R, x, \sigma_{\mathcal{D}})$ is a triple consisting of a formal disk $\mathcal{D}_R = \mathrm{Spec}B$, where $B \cong R[[z^{1/N}]]$, $x \in \mathcal{D}_R$, and $\sigma_{\mathcal{D}}$ is an automorphism of $\mathcal{D}$ (equivalently, of B) of order N fixing x. After a change of coordinate, $\sigma_{\mathcal{D}}$ is equivalent to the automorphism $z^{1/N} \mapsto \epsilon z^{1/N}$, where ϵ is a primitive Nth root of unity. We denote by $\bar{\mathcal{D}}$ the quotient of $\mathcal{D}$ by $\langle\sigma_{\mathcal{D}}\rangle$, i.e., the disk $\mathrm{Spec}B^{\sigma_{\mathcal{D}}}$, where $B^{\sigma_{\mathcal{D}}}$ is the subalgebra of $\sigma_{\mathcal{D}}$-invariant elements.

A formal coordinate t is called a *special coordinate* with respect to $\sigma_{\mathcal{D}}$ if $\sigma_{\mathcal{D}}(t) = \epsilon t$, where ϵ is an Nth root of unity, or equivalently, if t^N is a formal coordinate on $\bar{\mathcal{D}}$. We denote by $\mathcal{A}ut_N(\mathcal{D}_{R,x})$ the subset of $\mathcal{A}ut(\mathcal{D}_{R,x})$ consisting of special formal coordinates. The set $\mathcal{A}ut_N(\mathcal{D}_{R,x})$ carries a simply transitive right action of the group $\mathrm{Aut}_N(\mathcal{O}_R)$ given by $t \mapsto \rho(t)$, where ρ is the power series given in (4.2), i.e. $\mathcal{A}ut_N(\mathcal{D}_{R,x})$ is an $\mathrm{Aut}_N(\mathcal{O}_R)$-torsor.

Suppose now that $\pi : C \mapsto X = C/G$ as above, and that $y \in C$. Let N denote the order of the stabilizer G_y, which is cyclic, and generated by the monodromy around y, m_y. To this data we can associate the set $\mathcal{A}ut_N(\mathcal{D}_{R,y})$ of special coordinates on $(\mathrm{Spec}\hat{\mathcal{O}}_y, y)$ with respect to G_y, which is an $\mathrm{Aut}_N(\mathcal{O}_R)$-torsor.

Note that when $R = \mathbb{C}$, $\mathcal{D}_R$ has a unique R-point, and we will suppress it in the notation. Henceforth, we will also use the convention that $\mathcal{D}_{\mathbb{C}}$ is denoted $\mathcal{D}$, and suppress R when referring to groups, torsors, etc.

5. Orbifold conformal blocks

In this section we review the definition of orbifold conformal blocks introduced in [13].

5.1. Twisting modules by $\mathcal{A}ut_N(\mathcal{D})$

Let $\mathcal{D}$ be a disk with an Nth order automorphism as in the last section, and let M^σ be an admissible σ_V-twisted module over a conformal vertex algebra V. Define a representation r^{M^σ} of the Lie algebra $\mathrm{Der}_N^{(o)}(\mathcal{O})$ on M^σ by the formula

$$r^{M^\sigma} : z^{k+(1/N)}\partial_{z^{1/N}} \to -N \cdot L_k^{M^\sigma}.$$

It follows from the definition of a twisted module that the operators $L_k^{M^\sigma}$, $k > 0$, act locally nilpotently on M^σ and that the eigenvalues of $L_0^{M^\sigma}$ lie in $\frac{1}{N}\mathbb{Z}$, so that the operator $N \cdot L_0^{M^\sigma}$ has integer eigenvalues. This implies that the Lie algebra representation r^{M^σ} may be exponentiated to a representation R^{M^σ} of the group $\mathrm{Aut}_N(\mathcal{O})$. In particular, the subgroup $\mathbb{Z}/N\mathbb{Z}$ of $\mathrm{Aut}_N(\mathcal{O})$ acts on M^σ by the formula $i \mapsto S_\sigma^i$, where S_σ is the operator defined in Section 2.

We now twist the module M^σ by the action of $\mathrm{Aut}_N\mathcal{O}$ and define the vector space

$$\mathcal{M}^\sigma(\mathcal{D}) \stackrel{\text{def}}{=} \mathcal{A}ut_N(\mathcal{D}) \underset{\mathrm{Aut}_N\mathcal{O}}{\times} M^\sigma. \tag{5.1}$$

Thus, vectors in $\mathcal{M}^\sigma(\mathcal{D})$ are pairs (t, v), up to the equivalence relation

$$(\rho(t), v) \sim (t, R^{M^\sigma}(\rho) \cdot v), \quad t \in \mathcal{A}ut_N(\mathcal{D}), v \in M^\sigma.$$

When $\mathcal{D}=\mathcal{D}_x$, the formal neighborhood of a point x on an algebraic curve X, we will use the notation $\mathcal{M}_x^{\sigma}$.

5.2. The vector bundle $\mathcal{V}_X^G$

Let $(X, \mathbf{p})$ be a smooth n-pointed curve over Spec($\mathbb{C}$), and let $\pi : C \mapsto X$ be a smooth G-cover of $(X, \mathbf{p})$. Suppose furthermore that V is a conformal vertex algebra, and that G acts on V by conformal automorphisms. Let $\mathcal{A}ut_C$ be the Aut($\mathcal{O}$)-torsor over C whose fiber at $y \in C$ is $\mathcal{A}ut(\mathcal{D}_y)$the set of formal coordinates at y. As explained in [12], Aut($\mathcal{O}$) acts on V, and the action commutes with G. Let

$$\mathcal{V}_C = \mathcal{A}ut_C \underset{\mathrm{Aut}(\mathcal{O})}{\times} V$$

$\mathcal{V}_C$ has a flat connection ∇, given in the local coordinate z by the expression

$$d + L_{-1}^V \otimes \mathrm{d}z$$

The vector bundle $\mathcal{V}_C$ carries a G-equivariant structure lifting the action of G on C. It is given by

$$g \cdot (p, (A, z)) \overset{\mathrm{def}}{=} (g(p), (g(A), z \circ g^{-1})) \tag{5.2}$$

where $z \circ g^{-1}$ is the coordinate induced at $g(p)$ from z. Let $\overset{\circ}{C} \subset C$ denote the open set on which the G-action is free, and let $\overset{\circ}{X} \subset X = \pi(\overset{\circ}{C})$. Thus, $\overset{\circ}{C}$ is a G-principal bundle over $\overset{\circ}{X}$. $\mathcal{V}_{\overset{\circ}{C}}$ descends to a vector bundle $\mathcal{V}_{\overset{\circ}{X}}^G$ on $\overset{\circ}{X}$. More explicitly,

$$\mathcal{V}_{\overset{\circ}{X}}^G = \mathcal{A}ut_{\overset{\circ}{C}} \underset{\mathrm{Aut}(\mathcal{O}) \times G}{\times} V \tag{5.3}$$

Here, G acts on $\mathcal{A}ut_{\overset{\circ}{C}}$ by $g(p, z) = (g(p), z \circ g^{-1})$, and this action commutes with the action of Aut($\mathcal{O}$). The connection ∇ on $\mathcal{V}_C$ is G-invariant over $\overset{\circ}{C}$, and so descends to a connection ∇^G on $\mathcal{V}_{\overset{\circ}{X}}^G$.

5.3. Modules along G-orbits

Let $y \in C$. Then every point $r \in O(y)$ has a cyclic stabilizer of order N, which we denote G_r. Each G_r has a canonical generator h_r, which corresponds to the monodromy of a small loop around $p = \pi(y)$. For a point r in a generic orbit, $N = 1$, $G_r = \{e\}$ and we set $h_r = e$. Suppose that we are given the following data:

(1) A collection of admissible V-modules $\{M_r^{h_r}\}_{r \in O(y)}$, one for each point in the orbit, such that $M_r^{h_r}$ is h_r-twisted.

(2) A collection of maps $S_{g,r,g(r)} : M_r^{h_r} \mapsto M_{g(r)}^{h_{g(r)}}$, $g \in G$, $r \in \pi^{-1}(p)$, commuting with the action of $\mathrm{Aut}_N\mathcal{O}$ and satisfying

$$S_{gk,r,gk(r)} = S_{g,k(r),gk(r)} \circ S_{k,r,k(r)},$$

$$S_{g,r,g(r)}^{-1} = S_{g^{-1},g(r),r},$$

and

$$S_{g,r,g(r)}^{-1} Y^{M_{g(r)}^{h_{g(r)}}}(g \cdot A, z) S_{g,r,g(r)} = Y^{M_r^{h_r}}(A, z).$$

(3) If $g \in G_r$, then $S_{g,r,r} = S_g$, where S_g is the operator defined in Section 2.

Given a collection $\{M_r^{h_r}\}_{r \in O(y)}$, we can form the collection $\{\mathcal{M}_r^{h_r}(\mathcal{D}_r)\}_{r \in O(y)}$, where $\mathcal{M}_r^{h_r}(\mathcal{D}_r)$ is the $\mathrm{Aut}_N \mathcal{O}$-twist of $M_r^{h_r}$ by the torsor of special coordinates at r. Let

$$\overline{\mathcal{M}_{O(y)}} = \bigoplus_{r \in O(y)} \mathcal{M}_r^{h_r}(\mathcal{D}_r)$$

This is a representation of G, where G acts as follows. If $A \in M_r^{h_r}$, $z_r^{1/N}$ is a special coordinate at r, and $g \in G$, then

$$g \cdot (A, z_r^{1/N}) = (S_{g,r,g(r)} \cdot A, z_r^{1/N} \circ g^{-1})$$

Note that this action is well-defined since the S-operators commute with the action of $\mathrm{Aut}_N \mathcal{O}$. Now, let $\mathcal{M}_{O(y)} = (\overline{\mathcal{M}_{O(y)}})^G$, the space of G-invariants of $\overline{\mathcal{M}_{O(y)}}$. For every $r \in O(y)$ let

$$\phi_r : \mathcal{M}_{O(y)} \mapsto \mathcal{M}_r^{h_r}(\mathcal{D}_r)$$

be the isomorphism which is the composition of the inclusion $\mathcal{M}_{O(y)} \to \overline{\mathcal{M}_{O(y)}}$ and the projection $\overline{\mathcal{M}_{O(y)}} \to \mathcal{M}_r^{h_r}(\mathcal{D}_r)$.

For $v_r \in \mathcal{M}_r^{h_r}(\mathcal{D}_r)$, let $[v_r]$ denote $\phi_r^{-1}(v_r) \in \mathcal{M}_{O(y)}$. Note that for each $(A, z_r^{1/N})_r \in \mathcal{M}_r^{h_r}(\mathcal{D}_r)$, and $g \in G$, $[(A, z_r^{1/N})_r] = [(S_{g,r,g(r)} \cdot A, z_r^{1/N} \circ g^{-1})_{g(r)}]$ in $\mathcal{M}_p$.

Definition 5.1. We call $\mathcal{M}_{O(y)}$ a *V-module along* $O(y)$.

Henceforth, we will suppress the square brackets for elements of $\mathcal{M}_{O(y)}$ and refer to $[(A, z_r^{1/N})_r]$ simply as $(A, z_r^{1/N})$.

5.4. Construction of Modules along G-orbits

In this section we wish to describe an induction procedure which yields a module along $O(y)$ starting with a point $r \in O(y)$ and an h_r-twisted module $M_r^{h_r}$. Note that when G_r is trivial, this just an ordinary V-module M.

Thus, suppose we are given $r \in O(y)$, and an h_r-twisted module $M_r^{h_r}$. Observe that the monodromy generator at the point $g(r)$ is $h_{g(r)} = g h_r g^{-1}$, i.e. the monodromies are conjugate.

(1) For $g \in G$, define the module $M_{g(r)}^{g h_r g^{-1}}$ to be $M_r^{h_r}$ as a vector space, with the V-module structure given by the vertex operator

$$Y^{M_{g(r)}^{g h_r g^{-1}}}(A, z^{1/N}) = Y^{M_r^{h_r}}(g^{-1} \cdot A, z^{1/N}) \tag{5.4}$$

It is easily checked that this equips $M_{g(r)}^{g h_r g^{-1}}$ with the structure of a $g h_r g^{-1}$-twisted module. Furthermore, if $g \in G_r$, this construction results in an h_r-twisted module isomorphic to $M_r^{h_r}$.

(2) Recall that M^{h_s} is canonically isomorphic to $M_r^{h_r}$ as a vector space by the previous item. Thus, if $s \in O(y)$, and $g(s) \neq s$, define $S_{g,s,g(s)}$ to be the identity map.

(3) If $g \in G_s$, then g is conjugate to an element $g' \in G_r$. Define $S_{g,s,s} = S_{g',r,r}$ also using the canonical identification.

It is easy to check that this construction is well-defined, and satisfies the requirements of Definition 5.1.

Definition 5.2. We call a module along $O(y)$ obtained via this construction a *module along $O(y)$ induced from $M_r^{h_r}$*, and denote it $\mathrm{Ind}_r^{O(y)}(M_r^{h^r})$.

Remark 1. If G_r is trivial, and $M = V$, then for any $g \in G$, the new module structure (5.4) is isomorphic to the old one, and so the resulting module $\mathcal{M}_p$ along $O(y)$ is isomorphic to $\mathcal{V}_p^G$, the fiber of the sheaf $\mathcal{V}^G$ at p.

Remark 2. If $G = G_r$, then r is unique, and so any h_r-twisted module results in a module along $O(y)$.

Now, let $\mathbf{m} = (m_1, \ldots, m_n) \in G_A^n$ be a collection of monodromies. Let $(\pi : C \mapsto X, p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathbf{e^{-1}(m)}$ be a pointed G-cover with the prescribed monodromies. Given a collection $M_1, \ldots, M_n$ of V-modules, such that M_i is m_i-twisted, the above induction procedure yields a collection $\mathrm{Ind}_{q_1}^{O(q_1)}(M_1), \ldots, \mathrm{Ind}_{q_n}^{O(q_n)}(M_n)$, of modules along $O(q_i)$.

Note: We can label G-orbits on C by points of $p \in X$. We will use the notation $\mathcal{M}_p$ to denote a module along the G-orbit $\pi^{-1}(p)$.

5.5. Geometric Vertex Operators

Let $p \in X$, and $\mathcal{M}_p$ a V-module along $\pi^{-1}(p)$. Let $\mathcal{D}_p^\times$ denote the formal punctured disk $\mathrm{Spec}\mathcal{K}_p$, and Ω_X the sheaf of holomorphic 1-forms on X. It is shown in [13] that the vertex operator gives rise to a section

$$\mathcal{Y}^{\mathcal{M}_p,\vee} : \Gamma(\mathcal{D}_p^\times, \mathcal{V}^G_{\overset{\circ}{X}} \otimes \Omega_{\overset{\circ}{X}}) \to \mathrm{End}\mathcal{M}_p.$$

Moreover, this map factors through the quotient

$$U(\mathcal{V}_p^G) \overset{\mathrm{def}}{=} \Gamma(\mathcal{D}_p^\times, \mathcal{V}^G_{\overset{\circ}{X}} \otimes \Omega_{\overset{\circ}{X}})/\mathrm{Im}\nabla^G,$$

which has a natural Lie algebra structure. The corresponding map

$$y_p : U(\mathcal{V}_p^G) \to \mathrm{End}\mathcal{M}_p \tag{5.5}$$

is a homomorphism of Lie algebras. Note that p does not have to lie in $\overset{\circ}{X}$, but can be *any* point of X.

For each $r \in \pi^{-1}(p)$, composing the above maps with the isomorphism $\mathrm{End}\mathcal{M}_p \cong \mathrm{End}\mathcal{M}_r^{h_r}(\mathcal{D}_r)$ induced by ϕ_r yields a map:

$$\mathcal{Y}_r^{\mathcal{M}_p,\vee} : \Gamma(\mathcal{D}_p^\times, \mathcal{V}^G_{\overset{\circ}{X}} \otimes \Omega_{\overset{\circ}{X}}) \to \mathrm{End}\mathcal{M}_r^{h_r}(\mathcal{D}_r)$$

and a Lie algebra homomorphism

$$y_{p,r} : U(\mathcal{V}_p^G) \to \mathrm{End}\mathcal{M}_r^{h_r}(\mathcal{D}_r). \tag{5.6}$$

5.6. A sheaf of Lie algebras

Following Section 8.2.5 of [12], let us consider the following complex of sheaves (in the Zariski topology) on $\overset{\circ}{X}$:

$$0 \to \mathcal{V}^G_{\overset{\circ}{X}} \xrightarrow{\nabla} \mathcal{V}^G_{\overset{\circ}{X}} \otimes \Omega_{\overset{\circ}{X}} \to 0$$

where $\mathcal{V}^G_{\overset{\circ}{X}} \otimes \Omega_{\overset{\circ}{X}}$ is placed in cohomological degree 0 and $\mathcal{V}^G_{\overset{\circ}{X}}$ is placed in cohomological degree -1 (shifted de Rham complex). Let $h(\mathcal{V}^G_{\overset{\circ}{X}})$ denote the sheaf of the 0th cohomology, assigning to every Zariski open subset $\Sigma \subset \overset{\circ}{X}$ the vector space

$$U_\Sigma(\mathcal{V}^G_{\overset{\circ}{X}}) \overset{\text{def}}{=} \Gamma(\Sigma, \mathcal{V}^G_{\overset{\circ}{X}} \otimes \Omega_{\overset{\circ}{X}})/\mathrm{Im}\nabla^G$$

One can show as in Chapter 18 of [12] that this is a sheaf of Lie algebras.

For any $p \in \Sigma'$, where $\Sigma' \subset X$ is such that $\Sigma' \cap \overset{\circ}{X} = \Sigma$, restriction induces a Lie algebra homomorphism $\iota_p : U_\Sigma(\mathcal{V}^G_{\overset{\circ}{X}}) \to U(\mathcal{V}^G_p)$. We denote the image by $U_\Sigma(\mathcal{V}^G_p)$.

5.7. Conformal Blocks

Let $(\pi : C \mapsto X, \mathbf{p})$ be a G-cover, and $\mathbb{M} = (\mathcal{M}_{p_1}, \ldots, \mathcal{M}_{p_n})$ a collection of modules along $\pi^{-1}(p_1), \ldots, \pi^{-1}(p_n)$. Let

$$\mathbb{F} = \bigotimes \mathcal{M}_{p_i}$$

Composing the maps (5.5) with the map

$$U_{X\backslash\mathbf{p}}(\mathcal{V}^G_{\overset{\circ}{X}}) \xrightarrow{\sum \iota_{p_i}} \bigoplus_{i=1}^{n} U(\mathcal{V}^G_{p_i})$$

we obtain an action of the Lie algebra $U_{X\backslash\mathbf{p}}(\mathcal{V}^G_{\overset{\circ}{X}})$ on $\mathbb{F}$.

Definition 5.3. The space of *coinvariants* is the vector space

$$\mathcal{H}^G_V(C, X, \mathbf{p}, \mathbb{M}) = \mathbb{F}/U_{X\backslash\mathbf{p}}(\mathcal{V}^G_{\overset{\circ}{X}}) \cdot \mathbb{F}.$$

The space of *conformal blocks* is its dual: the vector space of $U_{X\backslash\mathbf{p}}(\mathcal{V}^G_{\overset{\circ}{X}})$-invariant functionals on $\mathbb{F}$

$$C^G_V(C, X, \mathbf{p}, \mathbb{M}) = \mathrm{Hom}_{U_{X\backslash\mathbf{p}}(\mathcal{V}^G_{\overset{\circ}{X}})}(\mathbb{F}, \mathbb{C}).$$

Suppose now that $(\pi : C \mapsto X, \mathbf{p}, \mathbf{q})$ is a pointed G-cover, that m_i denotes the monodromy at q_i, and that $M_1, \ldots, M_n$ is a collection of V-modules such that M_i is m_i-twisted. Then $\mathrm{Ind}^{O(q_i)}_{q_i}(M_i)$ is a module along $O(q_i)$, and we can apply our definition above with

$$\mathbb{M} = (\mathrm{Ind}^{O(q_1)}_{q_1}(M_1), \ldots, \mathrm{Ind}^{O(q_n)}_{q_n}(M_n))$$

In this case, to emphasize the dependence on the points q_i in the fiber, we use the notation $\mathcal{H}^G_V(C, X, \mathbf{p}, \mathbf{q}, \mathbb{M})$ and $C^G_V(C, X, \mathbf{p}, \mathbf{q}, \mathbb{M})$.

Denote by $\mathcal{M}_{q_i}$ the twist of M_i by the torsor of special coordinates at q_i. For each i, we have the following commutative diagram of Lie algebras:

$$\begin{array}{ccc} U(\mathcal{V}^G_{p_i}) & \xrightarrow{y_{p_i}} & \operatorname{End} \mathcal{M}_{p_i} \\ \| & & \phi_{q_i} \downarrow \\ U(\mathcal{V}^G_{p_i}) & \xrightarrow{y_{p_i,q_i}} & \operatorname{End} \mathcal{M}_{q_i} \end{array}$$

Letting $\tilde{\mathbb{F}} = \bigotimes \mathcal{M}_{q_i}$, the commutativity of the diagram implies that

$$\mathcal{H}^G_V(C, X, \mathbf{p}, \mathbf{q}, \mathbb{M}) \cong \tilde{\mathbb{F}}/U_{X\backslash\mathbf{p}}(\mathcal{V}^G_{\overset{\circ}{X}}) \cdot \tilde{\mathbb{F}}.$$

and

$$C^G_V(C, X, \mathbf{p}, \mathbf{q}, \mathbb{M}) = \operatorname{Hom}_{U_{X\backslash\mathbf{p}}(\mathcal{V}^G_{\overset{\circ}{X}})}(\tilde{\mathbb{F}}, \mathbb{C}).$$

6. Localization functors

The purpose of this section is to review the general yoga of Beilinson-Bernstein localization (see [3,4,8,12]) following [12].

Definition 6.1. A *Harish-Chandra* pair is a pair $(\mathfrak{g}, K)$ where $\mathfrak{g}$ is a Lie algebra and K is an algebraic group, equipped with the following data: an embedding $\mathfrak{k} \subset \mathfrak{g}$ of the Lie algebra $\mathfrak{k}$ of K into $\mathfrak{g}$, and an action Ad of K on $\mathfrak{g}$ compatible with the adjoint action of K on $\mathfrak{k}$ and the action of $\mathfrak{k}$ on $\mathfrak{g}$. A $(\mathfrak{g}, K)$-*module* is a vector space V carrying compatible actions of $\mathfrak{g}$ and K.

Definition 6.2. Let Z be a variety over $\mathbb{C}$. A $(\mathfrak{g}, K)$-action on Z is an action of K on Z, together with a Lie algebroid homomorphism $\alpha : \mathfrak{g} \otimes \mathcal{O}_Z \mapsto T_Z$ to the tangent sheaf of Z. The two actions must satisfy the following compatibility conditions:

(1) The restriction of α to $\mathfrak{k} \otimes \mathcal{O}_Z$ is the differential of the K-action.
(2) $\alpha(Ad_k(a)) = k\alpha(a)k^{-1}$

The action is said to be *transitive* if α is surjective, and *simply transitive* if α is an isomorphism.

These definitions extend naturally to the world of pro-algebraic groups, pro-varieties, and pro-stacks. (see [4,6]).

Suppose now that $Z \mapsto S$ is a principal K-bundle, and that Z carries a transitive $(\mathfrak{g}, K)$-action extending the fibrewise K-action. Let V be a $(\mathfrak{g}, K)$-module. The sheaf $V \otimes \mathcal{O}_Z$ carries an action of the algebroid $\mathfrak{g} \otimes \mathcal{O}_Z$, and it follows from the surjectivity of α that $V_{\text{stab}} = V \otimes \mathcal{O}_Z / ker(\alpha) \cdot (V \otimes \mathcal{O}_Z)$ is a module for the algebroid T_Z, and therefore a D_Z-module, where the latter denotes the sheaf of differential operators on Z. Moreover, the K-equivariance requirement in the definition of $(\mathfrak{g}, K)$-action and $(\mathfrak{g}, K)$-module ensures that V_{stab} is K-equivariant, and so descends to a D_S-module, which we denote $\Delta(V)$.

We will need a description of the fiber of $\Delta(V)$ at a point $s \in S$. Let Z_s denote the fiber of Z over s, and let $\mathfrak{g}^s$ denote the Lie algebra $Z_s \times_K \mathfrak{g}$. The ideals $ker(\alpha)_z$, $z \in Z_s$ give rise to a well-defined Lie ideal $\mathfrak{g}^s_{\text{stab}} \subset \mathfrak{g}^s$. Denote $Z_s \times_K V$ by $\mathcal{V}_s$. The action of $\mathfrak{g}$ on V induces an action of $\mathfrak{g}^s$ on $\mathcal{V}_s$. One can show (see [12]) that $\Delta(V)_s \cong \mathcal{V}_s/\mathfrak{g}^s_{\text{stab}} \cdot \mathcal{V}_s$.

Definition 6.3. The functor

$$\Delta : ((\mathfrak{g}, K) - \text{mod}) \longrightarrow (D_S - \text{mod})$$

sending V to $\Delta(V)$ is called the *localization functor* associated to the $(\mathfrak{g}, K)$-action on Z.

More generally, suppose that V is a module for a Lie algebra $\mathfrak{l}$, which contains $\mathfrak{g}$ as a Lie subalgebra and carries a compatible adjoint K-action. $V \otimes \mathcal{O}_Z$ is then a module over the Lie algebroid $\mathfrak{l} \otimes \mathcal{O}_Z$. Suppose also that we are given a subsheaf of Lie subalgebras $\tilde{\mathfrak{l}} \subset \mathfrak{l} \otimes \mathcal{O}_Z$ satisfying the following conditions:

(1) it is preserved by the action of K
(2) it is preserved by the action of the Lie algebroid $\mathfrak{g} \otimes \mathcal{O}_Z$
(3) it contains $ker(\alpha)$

Then, for the same reason as above, the sheaf $V \otimes \mathcal{O}_Z/\tilde{\mathfrak{l}} \cdot (V \otimes \mathcal{O}_Z)$ becomes a K-equivariant D_Z-module, and so descends to a D_S-module which we denote $\tilde{\Delta}(V)$. Let $s \in S$, and Z_s be as above. Denote by $\mathfrak{l}_s$ the Lie algebra $Z_s \times_K \mathfrak{l}$. The fibers $\tilde{\mathfrak{l}}_z$, $z \in Z_s$ give rise to a well-defined subalgebra $\tilde{\mathfrak{l}}_s \subset \mathfrak{l}_s$, and the fiber $\tilde{\Delta}(V)_s$ is isomorphic to $\mathcal{V}_s/\tilde{\mathfrak{l}}_s \cdot \mathcal{V}_s$.

6.1. *Localization of central extensions*

Suppose as above, that $(\mathfrak{g}, K)$ is a Harish-Chandra pair, and that $Z \mapsto S$ is a K-principal bundle with a transitive $(\mathfrak{g}, K)$-action extending the fibrewise K-action. Suppose also that $\mathfrak{g}$ has a central extension

$$0 \mapsto \mathbb{C} \mapsto \hat{\mathfrak{g}} \mapsto \mathfrak{g} \mapsto 0$$

which splits over $\mathfrak{k}$. Tensoring the above extension by $\mathcal{O}_Z$ we obtain an extension of Lie algebroids

$$0 \mapsto \mathcal{O}_Z \mapsto \hat{\mathfrak{g}} \otimes \mathcal{O}_Z \mapsto \mathfrak{g} \otimes \mathcal{O}_Z \mapsto 0 \tag{6.1}$$

We will assume henceforth that the extension (6.1) splits over $ker(\alpha)$, so that we get an embedding of the ideal $ker(\alpha)$ into $\hat{\mathfrak{g}} \otimes \mathcal{O}_Z$. The quotient $\mathcal{T}$ of $\hat{\mathfrak{g}} \otimes \mathcal{O}_Z$ by $ker(\alpha)$ now fits into an extension of Lie algebroids

$$0 \mapsto \mathcal{O}_Z \mapsto \mathcal{T} \mapsto T_Z \mapsto 0 \tag{6.2}$$

Let V be a $(\hat{\mathfrak{g}}, K)$-module. The sheaf $V \otimes \mathcal{O}_Z$ carries an action of the Lie algebroid $\hat{\mathfrak{g}} \otimes \mathcal{O}_Z$, and so the quotient $\hat{V}_{\text{stab}} = V \otimes \mathcal{O}_Z/ker(\alpha) \cdot (V \otimes \mathcal{O}_Z)$ carries an action of $\mathcal{T}$. Let $D'_Z = U(\mathcal{T})$, the enveloping algebroid of $\mathcal{T}$. This is a sheaf of twisted differential operators (a TDO), and the $\mathcal{T}$-action on $\hat{V}_{\text{stab}}$ extends naturally to an action of D'_Z. By the same argument as above, D'_Z and $\hat{V}_{\text{stab}}$ are K-equivariant, and so is the action of D'_Z on $\hat{V}_{\text{stab}}$. D'_Z therefore descends to a TDO D'_S on S, and $\hat{V}_{\text{stab}}$ to a D'_S-module which we denote $\Delta(V)$. Let $\hat{\mathfrak{g}}^s$ denote the Lie algebra $Z_s \times_K \hat{\mathfrak{g}}$. The ideals $ker(\alpha)_z$, $z \in Z_s$ give rise to a well-defined Lie ideal $\hat{\mathfrak{g}}^s_{\text{stab}} \subset \hat{\mathfrak{g}}^s$. Denote $Z_s \times_K V$ by $\mathcal{V}_s$. The action of $\hat{\mathfrak{g}}$ on V induces an action of $\hat{\mathfrak{g}}^s$ on $\mathcal{V}_s$. The fiber $\Delta(V)_s$ is isomorphic to $\mathcal{V}_s/\hat{\mathfrak{g}}^s_{\text{stab}} \cdot \mathcal{V}_s$. Thus, a central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ gives rise to a localization functor

$$\Delta : ((\hat{\mathfrak{g}}, K) - \text{mod}) \longrightarrow (D'_S - \text{mod})$$

sending V to $\Delta(V)$.

More generally, suppose $\hat{\mathfrak{g}}$ is a Lie subalgebra of $\hat{l}$, and we are given a subsheaf $\tilde{l} \subset \hat{l} \otimes \mathcal{O}_Z$ containing $ker(\alpha)$, preserved by the actions of K and $\hat{\mathfrak{g}} \otimes \mathcal{O}_Z$. Let V be a $\hat{l}$-module carrying a compatible K-action. The sheaf $V \otimes \mathcal{O}_Z/\tilde{l} \cdot (V \otimes \mathcal{O}_Z)$ is then a K-equivariant D'_Z-module, and descends to a D'_S-module on S, which we denote $\tilde{\Delta}(V)$. Let $\hat{l}_s = Z_s \times_K \hat{l}$, and denote by $\tilde{l}_s \subset \hat{l}_s$ the subalgebra arising from the stabilizers in Z_s. The fiber $\tilde{\Delta}(V)_s$ is isomorphic to $\mathcal{V}_s/\tilde{\mathfrak{l}}_s \cdot \mathcal{V}_s$.

7. *G*-equivariant Virasoro uniformization

Let $\widehat{\mathfrak{M}^G_{g,N}} = \{(\pi : C \mapsto X, \mathbf{p}, \mathbf{q}, \mathbf{z})\}$ where $(\pi : C \mapsto X, \mathbf{p}, \mathbf{q})$ is an n-pointed G-cover, and $\mathbf{z} = (z_1, \ldots, z_n)$, where z_i is a formal special coordinate at q_i. This is a projective limit of Deligne-Mumford stacks. Forgetting the coordinates yields a map

$$\begin{tikzcd} \widehat{\mathfrak{M}^G_{g,n}} \arrow[d, "\xi"'] \\ \mathfrak{M}^G_{g,n} \end{tikzcd}$$

Similarly, let $\widehat{\mathfrak{M}_{g,n}} = \{(X, \mathbf{p}, \mathbf{q}, \mathbf{z}')\}$ where $(X, \mathbf{p}) \in \mathfrak{M}_{g,n}$ and $\mathbf{z}' = (z'_1, \ldots, z'_n)$, where z'_i is a formal coordinate at p_i. Forgetting the coordinates yields a map

$$\begin{tikzcd} \widehat{\mathfrak{M}_{g,n}} \arrow[d, "\zeta"'] \\ \mathfrak{M}_{g,n} \end{tikzcd}$$

Moreover, there exists a map $\eta : \widehat{\mathfrak{M}^G_{g,n}} \mapsto \widehat{\mathfrak{M}_{g,n}}$ defined by

$$\eta : (\pi : C \mapsto X, \mathbf{p}, \mathbf{q}, \mathbf{z}) \mapsto (X, \mathbf{p}, \mathbf{z}')$$

where $z'_i = z_i^{N_i}$, and N_i is the order of G_{q_i} making the following diagram commute:

$$\begin{tikzcd} \widehat{\mathfrak{M}^G_{g,n}} \arrow[r, "\eta"] \arrow[d, "\xi"'] & \widehat{\mathfrak{M}_{g,n}} \arrow[d, "\zeta"] \\ \mathfrak{M}^G_{g,n} \arrow[r, "p"'] & \mathfrak{M}_{g,n} \end{tikzcd}$$

For a $\mathbb{C}$-algebra R, let

$$\mathrm{Aut}(C/X, \mathcal{O}_R) = \mathrm{Aut}_{N_1}(\mathcal{O}_R) \times \mathrm{Aut}_{N_2}(\mathcal{O}_R) \times \cdots \times \mathrm{Aut}_{N_n}(\mathcal{O}_R)$$

$$\mathrm{Der}^{(o)}(C/X, \mathcal{O}) = \mathrm{Der}^{(o)}_{N_1}(\mathcal{O}) \times \cdots \times \mathrm{Der}^{(o)}_{N_n}(\mathcal{O}).$$

Similarly, define $\mathrm{Aut}(C/X, \mathcal{K}_R)$ and $\mathrm{Der}(C/X, \mathcal{K})$ in the obvious way. We have that $\mathrm{Der}(C/X, \mathcal{O})$ (resp. $\mathrm{Der}(C/X, \mathcal{K})$) is the Lie algebra of the group scheme (resp. ind-scheme) representing $R \mapsto \mathrm{Aut}(C/X, \mathcal{O}_R)$ (resp. $R \mapsto \mathrm{Aut}(C/X, \mathcal{K}_R)$). We see that $\widehat{\mathfrak{M}^G_{g,n}}$ is an $\mathrm{Aut}(C/X)$-torsor over $\mathfrak{M}^G_{g,n}$. The following is a generalization of the Virasoro uniformization theorem (see [2,8,19,23]) to G-covers.

Theorem 7.1. $\widehat{\mathfrak{M}^G_{g,N}}$ *carries a transitive action of* $\mathrm{Der}(C/X, \mathcal{K})$ *extending the action of* $\mathrm{Aut}(C/X, \mathcal{O})$ *along the fibers of* ξ.

Proof. Let $R = \mathbb{C}[\epsilon]/(\epsilon^2)$. A family of G-covers and special coordinates over Spec$R$ is the same as a tangent vector to $\widehat{\mathfrak{M}^G_{g,N}}$. Thus to construct an action of Der$(C/X, \mathcal{K})$ on $\widehat{\mathfrak{M}^G_{g,N}}$ and prove transitivity, it suffices to do it over R. In fact, we construct an action of the corresponding group Aut$(C/X, \mathcal{K}_R)$. Suppose that $(\pi : C \mapsto X, \mathbf{p}, \mathbf{q}, \mathbf{z})$ is an R-point of $\widehat{\mathfrak{M}^G_{g,n}}$, and that $\rho \in$ Aut$(C/X, \mathcal{K}_R)$. Let

$$\mathrm{Aut}^G_e(C/X, \mathcal{K}_R) = \mathrm{Aut}^G_e(\widehat{O(q_1)}^*, \mathcal{K}_R) \times \cdots \times \mathrm{Aut}^G_e(\widehat{O(q_n)}^*, \mathcal{K}_R)$$

We have an isomorphism

$$\psi^*_{q_1} \times \cdots \times \psi^*_{q_n} : \mathrm{Aut}^G_e(C/X, \mathcal{K}_R) \mapsto \mathrm{Aut}(C/X, \mathcal{K}_R).$$

under which ρ corresponds to an element ρ^G of Aut$^G_e(C/X, \mathcal{K}_R)$.

At each point r of $O(q_i)$, there is a set of N_i special coordinates compatible with z_i (i.e. obtained from z_i by applying elements of G), and we choose one arbitrarily, and call it w_r. We now define the new ρ-twisted G-cover

$$(\pi : C_\rho \mapsto X_\rho, \mathbf{p}_\rho, \mathbf{q}_\rho, \mathbf{z}_\rho)$$

by the following: as a topological space, $C_\rho = C$ and $X_\rho = X$, but the structure sheaf of C_ρ is changed as follows. Let $U \in C$ be Zariski open. If $O(q_i) \cap U = \emptyset, \forall i$, define $\mathcal{O}_{C_\rho}(U) = \mathcal{O}_C(U)$. If U intersects the orbits $O(q_i)$ at the points $r_1, \ldots, r_m$, define $\mathcal{O}_{C_\rho}(U)$ to be the subring of $\mathcal{O}_C(U \backslash \{r_j\}_{j=1,\ldots,m})$ consisting of functions f whose expansion $f_{r_j}(w_{r_j}) \in R((w_{r_j}))$ at r_j in the coordinate w_{r_j} satisfies $f_{r_j}(\rho^{G,-1}(w_{r_j})) \in R[[w_{r_j}]]$. Since the gluing is G-equivariant, C_ρ is again a G-cover.

We now prove transitivity. The homomorphisms $\mu_N : \mathrm{Aut}_N(\mathcal{K}_R) \mapsto \mathrm{Aut}(\mathcal{K}_R)$ induce a homomorphism

$$\mu_{C/X} : \mathrm{Aut}(C/X, \mathcal{K}_R) \mapsto \mathrm{Aut}(X, \mathcal{K}_R) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\mathcal{K}_R) \times \cdots \times \mathrm{Aut}(\mathcal{K}_R)\,(n\mathrm{times})$$

The following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathfrak{M}^G_{g,n}} & \xrightarrow{\eta} & \widehat{\mathfrak{M}_{g,n}} \\ \rho \downarrow & & \downarrow \mu_{C/X}(\rho) \\ \widehat{\mathfrak{M}^G_{g,n}} & \xrightarrow[\eta]{} & \widehat{\mathfrak{M}_{g,n}} \end{array}$$

The transitivity follows from the fact that the action of Aut$(X, \mathcal{K}_R)$ on $\widehat{\mathfrak{M}_{g,n}}$ is transitive (by [2,8,19,23]), the fact that η is a quasi-finite map, and $\mu_{C/X}$ is surjective. □

Let $O(\mathbf{q}) = \bigcup_i O(q_i)$. The stabilizer of $(\pi : C \mapsto X, \mathbf{p}, \mathbf{q}, \mathbf{z})$ is the subgroup Aut$(C/X, \mathcal{K})_{out} \in$ Aut$(C/X, \mathcal{K})$ consisting of those elements that preserve $\mathcal{O}_C(C \backslash O(\mathbf{q}))$. The Lie algebra of Aut$(C/X, \mathcal{K})_{out}$ is the Lie algebra of G-invariant vector fields on $C \backslash O(\mathbf{q})$, denoted Vect$^G(C \backslash O(\mathbf{q}))$. Note that

$$\mathrm{Vect}^G(C \backslash O(\mathbf{q})) \cong \mathrm{Vect}(X \backslash \mathbf{p})$$

where $\mathrm{Vect}(X\backslash\mathbf{p})$ denotes the space of vector fields on $X\backslash\mathbf{p}$. We thus obtain a localization functor

$$\Delta : ((\mathrm{Der}(C/X), \mathrm{Aut}(C/X)) - \mathrm{mod}) \longrightarrow (D_{\mathfrak{M}^G_{g,n}} - \mathrm{mod})$$

$$\mathrm{M} \mapsto \Delta(\mathrm{M})$$

The fiber of $\Delta(\mathrm{M})$ at $(\pi : C \mapsto X, \mathbf{p}, \mathbf{q}, \mathbf{z})$ is $\mathcal{M}/\mathrm{Vect}^G(C\backslash O(\mathbf{q})\}) \cdot \mathcal{M}$, where $\mathcal{M}$ is the $\mathrm{Aut}(C/X)$-twist of M.

We want to construct a localization functor that sends modules along orbits to D-modules whose fibers are spaces of orbifold conformal blocks. We now proceed with this construction.

7.1. Construction of the Lie algebroid $\mathcal{U}^G(V)_{out}$

In this section we construct a Lie algebroid $\mathcal{U}^G(V)_{out}$ over $\widehat{\mathfrak{M}^G_{g,n}}$ whose fiber at $(\pi : C \mapsto X, \mathbf{p}, \mathbf{q}, \mathbf{z})$ is the Lie algebra $U_{X\backslash\mathbf{p}}(\mathcal{V}^G_{\underset{X}{\circ}})$ of Section 5.6.

Let $\mathcal{C}^G$ denote the universal G-cover over $\widehat{\mathfrak{M}^G_{g,n}}$. Thus $\mathcal{C}^G$ parametrizes tuples $\{(\pi : C \mapsto X, \mathbf{p}, \mathbf{q}, \mathbf{z}, y)\}$ where $y \in C$. Let $\widehat{\mathcal{C}^G}$ denote the $\mathrm{Aut}(\mathcal{O})$-bundle over $\mathcal{C}^G$ whose fiber over $\{(\pi : C \mapsto X, \mathbf{p}, \mathbf{q}, \mathbf{z}, y)\}$ is the set $\mathcal{A}ut(\mathcal{D}_y)$ of formal coordinates at y. The projection $\mathcal{C}^G \mapsto \mathfrak{M}^G_{g,n}$ comes with sections $q_i : \mathfrak{M}^G_{g,n} \mapsto \mathcal{C}^G$, and applying elements of the group G, we obtain sections passing through every point of $O(q_i)$.

Let

$$\mathcal{V}_{\mathcal{C}^G} = \widehat{\mathcal{C}^G} \underset{\mathrm{Aut}(\mathcal{O})}{\times} V$$

this is a sheaf over $\mathcal{C}^G$. $V_{\mathcal{C}^G}$ possesses a flat connection ∇ along the fibers of the projection $\mathcal{C}^G \mapsto \widehat{\mathfrak{M}^G_{g,n}}$, which along each fiber restricts to the connection ∇ of Section 5.2, i.e. we have a complex

$$0 \mapsto \mathcal{V}_{\mathcal{C}^G} \overset{\nabla}{\mapsto} \mathcal{V}_{\mathcal{C}^G} \otimes \Omega_{\mathcal{C}^G/\widehat{\mathfrak{M}^G_{g,n}}} \mapsto 0$$

Restricting this complex to $\mathcal{C}^G\backslash O(\mathbf{q})$, taking de Rham cohomology along the fibers, and then G-invariants, yields the Lie algebroid $\mathcal{U}^G(V)_{out}$.

7.2. Sheaves of Conformal Blocks

Let $\mathbf{m} = (m_1, \ldots, m_n)$ be a collection of monodromies, and $\mathbb{M} = (M_1, \ldots, M_n)$ a collection of V-modules such that M_i is m_i-twisted and admissible. Let $U(M_i)$ denote the Lie algebra of Fourier coefficients of (twisted) vertex operators acting on M_i, and let

$$U(\mathbb{M}) = U(M_1) \times \cdots \times U(M_n)$$

The data of the local coordinates in $\widehat{\mathfrak{M}^G_{g,n}}$ allows us to embed the Lie algebroid $\mathcal{U}^G(V)_{out}$ inside $U(\mathbb{M}) \otimes \mathcal{O}_{\widehat{\mathfrak{M}^G_{g,n}}}$ (by expanding vertex operators in the local coordinates), and the Virasoro operators acting on the modules yield the embedding

$$\mathrm{Der}(C/X, \mathcal{K}) \otimes \mathcal{O}_{\widehat{\mathfrak{M}^G_{g,n}}} \hookrightarrow U(\mathbb{M}) \otimes \mathcal{O}_{\widehat{\mathfrak{M}^G_{g,n}}}$$

Denote by $\mathrm{vect}^G(C\backslash O(\mathbf{q}))$ the Lie algebroid on $\widehat{\mathfrak{M}^G_{g,n}}$ whose fiber at $(\pi : C \mapsto X, \mathbf{p}, \mathbf{q}, \mathbf{z})$ is $\mathrm{Vect}^G(C\backslash O(\mathbf{q}))$. Again, the local coordinates allow us to identify it with a subalgebroid of $\mathrm{Der}(C/X, \mathcal{K}) \otimes \mathcal{O}_{\widehat{\mathfrak{M}^G_{g,n}}}$. Furthermore, under this identification,

$$\mathrm{vect}^G(C\backslash O(\mathbf{q})) \hookrightarrow \mathcal{U}^G(V)_{out}.$$

The same argument that is used in [12], chap. 16, shows that $\mathcal{U}^G(V)_{out}$ is preserved by the $\mathrm{Aut}(C/X)$-action on $\widehat{\mathfrak{M}^G_{g,n}}$

Let $\mathrm{M} = \bigotimes_i M_i$. This is a $U(\mathbb{M})$-module and a $(\mathfrak{g}, K)$-module for the Harish-Chandra pair $(\mathrm{Der}(C/X), \mathrm{Aut}(C/X))$. We are thus in the setup of Section 6, with $\mathfrak{l} = U(\mathbb{M}) \otimes \mathcal{O}_{\widehat{\mathfrak{M}^G_{g,n}}}$ and $\tilde{\mathfrak{l}} = \mathcal{U}^G(V)_{out}$. Suppose first that the Virasoro central charge c vanishes. Beilinson-Bernstein localization then yields a D-module $\mathcal{H}^G_V(\mathbb{M})$ on $\mathfrak{M}^G_{g,n}(\mathbf{m})$ whose fiber at $(\pi : C \mapsto X, \mathbf{p}, \mathbf{q})$ is

$$\tilde{\mathbb{F}}/U_{X\backslash\mathbf{p}}(\mathcal{V}^G_{\overset{\circ}{X}}) \cdot \tilde{\mathbb{F}} \tag{7.1}$$

where $\tilde{\mathbb{F}}$ denotes the $\mathrm{Aut}(C/X)$-twist of M by the torsor of special coordinates at the points q_i. By the comments at the end of Section 5.7, this space is exactly $\mathcal{H}^G_V(C, X, \mathbf{p}, \mathbf{q}, \mathbb{M})$.

Suppose now that c is non-zero, and let $\widehat{\mathrm{Der}}(C/X)$ denote the central extension of $\mathrm{Der}(C/X)$ obtained as the Baer sum of Virasoro cocycles of the individual factors. In this case M is a Harish-Chandra module for the pair $(\widehat{\mathrm{Der}}(C/X), \mathrm{Aut}(C/X))$. Moreover, the Lie algebroid extension

$$0 \mapsto \mathcal{O}_{\widehat{\mathfrak{M}^G_{g,n}}} \mapsto \widehat{\mathrm{Der}}(C/X) \otimes \mathcal{O}_{\widehat{\mathfrak{M}^G_{g,n}}} \mapsto \mathrm{Der}(C/X) \otimes \mathcal{O}_{\widehat{\mathfrak{M}^G_{g,n}}} \mapsto 0$$

splits over the kernel of the anchor map, as the Virasoro cocycle is trivial on $\mathrm{Vect}^G(C\backslash O(\mathbf{q}))$. Applying the machinery of Section 6 with $\widehat{l} = U(\mathbb{M}) \otimes \mathcal{O}_{\widehat{\mathfrak{M}^G_{g,n}}}$ and $\tilde{l} = \mathcal{U}^G(V)_{out}$., we obtain a module $\mathcal{H}^G_V(\mathbb{M})$ for a sheaf of twisted differential operators on $\mathfrak{M}^G_{g,n}(\mathbf{m})$, whose fiber at $(\pi : C \mapsto X, \mathbf{p}, \mathbf{q})$ is $\mathcal{H}^G_V(C, X, \mathbf{p}, \mathbf{q}, \mathbb{M})$.

Definition 7.1. Denote by ∇^{KZ} the connection on $\mathcal{H}^G_V(\mathbb{M})$ coming from the D-module structure.

8. The orbifold KZ connection in the direction of *X* fixed

In this section we give a formula for the orbifold KZ connection along the fibers of the projection

$$\kappa : \mathfrak{M}^G_{g,n}(\mathbf{m}) \mapsto \mathfrak{M}_g$$

sending $(C, X, \mathbf{p}, \mathbf{q})$ to X i.e., this amounts to fixing the base curve X. We begin by introducing some notation. Fix $\mathbf{m} = (m_1, \ldots, m_n)$ and $\mathbb{M} = (M_1, \ldots, M_n)$ as before, and let

$$\mathrm{M} = M_1 \otimes \cdots \otimes M_n$$

Let $\mathrm{Der}(\mathcal{O})$ denote the Lie algebra generated by $\{z^k \partial_z\}$, $k \geq 0$, and and $\mathrm{Der}_N(\mathcal{O})$ the one generated by $\{z^{k+\frac{1}{N}} \partial_{z^{1/N}}\}$, $k \geq -1$. Under the homomorphism (4.3), these two Lie algebras are isomorphic. We have

$$\mathrm{Der}^{(o)}(\mathcal{O}) \subset \mathrm{Der}(\mathcal{O}) \quad \mathrm{Der}^{(o)}_N(\mathcal{O}) \subset \mathrm{Der}_N(\mathcal{O})$$

Let

$$\mathrm{Der}(C/X, \mathcal{O}) = \mathrm{Der}_{N_1}(\mathcal{O}) \times \cdots \times \mathrm{Der}_{N_n}(\mathcal{O}).$$

We now wish to consider the two Harish-Chandra pairs $(\mathrm{Der}(\mathcal{O}), \mathrm{Aut}(\mathcal{O}))$ and $(\mathrm{Der}_N(\mathcal{O}), \mathrm{Aut}_N(\mathcal{O}))$. As explained in [12], the action of the pair $(\mathrm{Der}(\mathcal{O}), \mathrm{Aut}(\mathcal{O}))$ is simply transitive along the fibers of the projection $\widehat{\mathcal{M}_{g,1}} \mapsto \mathfrak{M}_g$ (and a similar statement applies in the case of multiple points). From the fact that the map

$$\eta : \widehat{\mathfrak{M}^G_{g,n}} \mapsto \widehat{\mathfrak{M}_{g,n}}$$

has finite fibers, and that the two Harish-Chandra pairs $(\mathrm{Der}(\mathcal{O}), \mathrm{Aut}(\mathcal{O}))$ and $(\mathrm{Der}_N(\mathcal{O}), \mathrm{Aut}_N(\mathcal{O}))$ are isogenous, we can deduce that the action of the pair $(\mathrm{Der}(C/X, \mathcal{O}), \mathrm{Aut}(C/X, \mathcal{O}))$ along the fibers of the map

$$\kappa \circ \xi : \widehat{\mathfrak{M}^G_{g,n}} \mapsto \mathfrak{M}_g$$

is also simply transitive. Let

$$\mathfrak{M}^G_{X,n}(\mathbf{m}) = \kappa^{-1}(X) \cap \mathfrak{M}^G_{g,n}(\mathbf{m})$$

and

$$\widehat{\mathfrak{M}^G_{X,n}}(\mathbf{m}) = (\kappa \circ \xi)^{-1}(X) \cap \widehat{\mathfrak{M}^G_{g,n}}(\mathbf{m})$$

and let

$$\tilde{\mathcal{H}}^G_V(\mathbb{M}, X) = \widehat{\mathfrak{M}^G_{X,n}}(\mathbf{m}) \underset{\mathrm{Aut}(C/X)}{\times} \mathrm{M}.$$

We deduce that along $\mathfrak{M}^G_{X,n}(\mathbf{m})$, the connection ∇^{KZ} on $\mathcal{H}^G_V(\mathbb{M})$ lifts to the sheaf $\tilde{\mathcal{H}}^G_V(\mathbb{M}, X)$, and an explicit formula in local coordinates can be obtained.

Choose coordinates z_i around p_i. Since the map p is quasi-finite, the z_i therefore define coordinates on $\kappa^{-1}(X)$. Choose compatible special coordinates z_i^{1/N_i} at q_i which are N_ith roots of the z_i. z_i induces coordinates $z_i - w_i$ at points near p_i, and likewise, the choice of N_ith root z_i^{1/N_i} at q_i induces a family of N_ith roots of $z_i - w_i$, which we denote $(z_i - w_i)^{1/N_i}$. We can thus trivialize $\tilde{\mathcal{H}}^G_V(\mathbb{M}, X)$ in a neighborhood W of $(C, X, \mathbf{p}, \mathbf{q})$:

$$\iota : \mathrm{M} \times W \mapsto \tilde{\mathcal{H}}^G_V(\mathbb{M}, X)$$

$$(A_1 \otimes \cdots \otimes A_n, w_1, \ldots, w_n) \mapsto [A_1, (z_1 - w_1)^{1/N_1}] \otimes \cdots \otimes [A_n, (z_n - w_n)^{1/N_n}]$$

and similarly, we obtain a trivialization ι^* of $\tilde{\mathcal{H}}^G_V(\mathbb{M}, X)^*$. We have the following theorem:

Theorem 8.1. *Along, $\mathfrak{M}^G_{X,n}(\mathbf{m})$, the orbifold KZ connection ∇^{KZ} on $\mathcal{H}^G_V(\mathbb{M})$ lifts to $\tilde{\mathcal{H}}^G_V(\mathbb{M}, X)$, and in the trivialization ι, is given in the local coordinates z_i by*

$$\nabla_{\partial_{z_i}} = \partial_{z_i} + L^{M_i}_{-1}$$

where $L^{M_i}_{-1}$ is the -1-st Virasoro operator acting on M_i.

Note: The vector field responsible for translations along the curve X is ∂_z. Observe that under the homomorphism (4.3), we have

$$\mu_N^{-1}(\partial_z) = \frac{1}{N} z^{-1+1/N} \partial_{z^{1/N}}$$

It follows that $\frac{1}{N} z^{-1+1/N} \partial_{z^{1/N}}$ is precisely the vector field responsible for moving the ramification points of the map $\pi : C \mapsto X$ along X. Note that in general, its action on $\widehat{\mathfrak{M}^G_{g,n}}$ will change the complex structure of the cover C.

Acknowledgments

I would like to thank Edward Frenkel and David Ben-Zvi for many valuable conversations. I would also like to gratefully acknowledge the hospitality of the Erwin Schrodinger Institute, where part of this work was completed. This research was partially supported by NSF grant DMS-0401619.

References

[1] D. Abramovich, A. Corti, A. Vistoli, *Twisted bundles and admissible covers*, preprint math.AG/0106211.
[2] E. Arbarello, C. De Concini, V.G. Kac, C. Procesi, Moduli spaces of curves and representation theory, Comm. Math. Phys. 117 (1) (1988) 1–36.
[3] A. Beilinson, J. Bernstein, Localisation de g-modules, (French) C. R. Acad. Sci. Paris Sr. I Math. 292 (1) (1981) 15–18.
[4] A. Beilinson, J. Bernstein, A proof of the Jantzen conjectures, Advances in Soviet Mathematics 16, Part 1, 1–50, AMS, 1993.
[6] A. Beilinson, B. Feigin, B. Mazur, it Introduction to algebraic field theory on curves, unpublished manuscript.
[7] B. Bakalov, V.G. Kac, Twisted modules over lattice vertex algebras, preprint, math.QA/0402315.
[8] A. Beilinson, V. Schechtman, Determinant bundles and Virasoro algebras Comm, Math. Phys. 118 (1988) 651–701.
[10] C. Dong, Twisted modules for vertex algebras associated with even lattices, J. Algebra 165 (1994) 91–112.
[11] C. Dong, H. Li, G. Mason, Twisted representations of vertex operator algebras, Math. Ann. 310 (1998) 571–600.
[12] E. Frenkel, D. Ben-Zvi, Vertex algebras and algebraic curves, Mathematical Surveys and Monographs 88, AMS, 2001.
[13] E. Frenkel, M. Szczesny, Twisted modules over vertex algebras on algebraic curves, to appear in Advances in Mathematics.
[14] A. Feingold, I. Frenkel, J. Reis, Spinor construction of vertex operator algebras, triality, and $E_8^{(1)}$, Contemp. Math. 121, AMS, 1991.
[15] I. Frenkel, J. Lepowsky, A. Meurman, Vertex operator algebras and the monster, Academic Press, 1988.
[16] T. Jarvis, R. Kaufmann, T. Kimura, Pointed Admissible G-Covers and G-equivariant Cohomological Field Theories, preprint math.AG/0302316.
[17] V. Kac, Vertex algebras for beginners, second ed., AMS, 1998.
[18] A. Kirillov Jr., On the modular functor associated with a finite group, preprint math.QA/0310087.
[19] M. Kontsevich, The Virasoro algebra and Teichmuller spaces, Funct. Anal. Appl. 21 (2) (1987) 156–157.
[20] H. Li, Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules, Contemp. Math 193 (1996) 203–236.
[21] J.M. Munoz Porras, F.J. Plaza Martin, Automorphism group of $k((z))$: applications to the Bosonic String, Comm. Math. Phys 216 (2001) 609–634.
[22] M. Szczesny, Orbifold chiral correlators from conformal blocks and KZ equations, in preparation.
[23] Tsuchiya, Akihiro; Ueno, Kenji; Yamada, Yasuhiko, Conformal field theory on universal family of stable curves with gauge symmetries. Integrable systems in quantum field theory and statistical mechanics, vol., 19 , Adv. Stud. Pure Math., Academic Press, Boston, MA, 1989, pp. 459–566.