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On isoperimetric surfaces in general relativity, II

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ABSTRACT

We determine the optimal way to enclose volume in a class of domains inside certain Friedmann–Robertson–Walker metrics. The method employed is an adaptation of the Bray–Morgan isoperimetric comparison procedure to the Lorentzian setting. We also make some remarks on isoperimetric comparison in the Riemannian setting, for rotationally-symmetric space-like slices in non-vacuum space-times.

1. Introduction

The classical isoperimetric problem in geometry is to determine how to enclose a given volume V with a hypersurface of optimal area. The condition for a (smooth) hypersurface to be critical for area with a volume constraint is that its mean curvature be constant. In a given explicit geometry, one may be able to determine a class of constant mean curvature (CMC) surfaces (for example, in case the surfaces are orbits of a subgroup of the isometry group) which stand as candidates for solutions to the optimization problem. It generally takes considerably more work to argue that the local condition of having constant mean curvature implies the hypersurface satisfies the global optimization problem. It is interesting, therefore, to be able to use a known isoperimetric profile to obtain the isoperimetric profile of a comparable space, as has been done in Bray–Morgan [1].

Our main results (Theorem 2 and Proposition 3) solve an isoperimetric problem in certain Friedmann–Robertson–Walker (FRW) space-times. Much is known about the class of space-like hypersurfaces with constant mean curvature in some special space-times, such as Minkowski space-time [2]. Moreover, the work of Bahn–Ehrlich settles an isoperimetric problem for a class of domains in Minkowski space-time [3]. We employ an adaptation of the Bray–Morgan approach to the Lorentzian setting, whereby we obtain the FRW isoperimetric profile (analogous to Bahn–Ehrlich) by comparing the FRW geometry to that of Minkowski space-time. In contrast to the Riemannian case, the unit sphere in a Lorentzian vector space is non-compact, and compact space-like hypersurfaces of Minkowski space-time necessarily have non-empty boundary. This impacts the domains we use for the isoperimetric problem, which in this case are tied to the causal structure. We remark that a similar amount of care in defining domains on which to measure volume is needed to achieve certain Lorentzian analogues of the classical volume comparison theorems, cf. [4,5].

Isoperimetric problems in the purely Riemannian context have been studied in connection with general relativity, on space-like slices of space-times. Consider a space-time $(\mathcal{S}, \bar{g})$ satisfying the Einstein equation $Ric(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} = 8\pi T$.

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If $M \subset \mathcal{S}$ is a space-like hypersurface with induced (Riemannian) metric g, unit (time-like) normal field N, and second fundamental form II (with trace H) of a space-like slice, then using the Einstein equation along with the Gauss equation, we obtain the Hamiltonian constraint on M: $R(g) - \|\mathit{II}\|^2 + H^2 = 16\pi\mu$, where $R(g)$ is the scalar curvature and $\mu = T(N, N)$ is the local energy density of the matter fields. In the totally geodesic ($\mathit{II} = 0$) case (also known as the *time-symmetric* case), these constraints reduce to the condition $R(g) = 16\pi\mu$, which indicates a role for the scalar curvature in the study of the energy content of isolated systems (cf. the Positive Mass Theorem and the Penrose Inequality). However, Bray notes in [6] that the "potential energy contributions between matter (to the extent this is well-defined in certain examples) tends to make a negative contribution to the total mass" of the isolated system. In Section 4 we will see that negative scalar curvature (energy density) provides a stabilizing effect for constant mean curvature hypersurfaces in certain geometries. To be precise, we will construct examples of rotationally-symmetric, time-symmetric initial data for the Einstein constraint equations with scalar curvature of fixed sign, which will illustrate that non-positive energy density has a stabilizing effect on the rotationally-symmetric spheres. It might seem strange from a physical point of view that a natural variational problem seems to prefer exotic matter, but it is quite natural from a geometric point of view. Indeed, recall the well-known expansion for the volume of small geodesic balls $B_r(p)$ about p in a Riemannian three-manifold [7]:

$$V(B_r(p)) = \frac{4\pi r^3}{3}\left(1 - \frac{R(p)}{30}r^2 + O(r^3)\right). \tag{1}$$

We see that the scalar curvature measures the top-order deviation of the volumes of small geodesic balls from that of their Euclidean counterparts, and that negative scalar curvature implies that small geodesic balls contain more volume than their Euclidean counterparts. Of course, volume is just one part of the isoperimetric problem, and this volume behavior needs to be compared with the area profile of the rotationally-symmetric spheres S_r.

Bray [8] established a connection between isoperimetric profiles and the Riemannian Penrose inequality from general relativity, using a Hawking mass quantity associated to the profile. In particular, he established the isoperimetric profile for the standard space-like slice of the Schwarzschild space-time, an argument which was codified by Bray and Morgan [1], and further explored in [9]. The main idea of the argument, as we will recall in more detail below, is that under certain special circumstances, one can glean the isoperimetric profile of a space by comparison to a space where the isoperimetric profile is already established. In the relativistic context, appropriate comparison to the Euclidean space establishes that the rotationally-symmetric spheres in the standard space-like slice of the (positive mass) Schwarzschild space-time minimize area for given volume enclosed with the horizon [1]. The analogous result holds for the Reissner–Nördstrom space-time, while comparison to the hyperbolic space proves the analogous result for Schwarzschild–anti-DeSitter space-time [9]. The relation between isoperimetric problems and the mass-energy of space-times has recently been developed by Huisken [10], in part motivated by (1).

2. Preliminaries

Let (M, g) be a Riemannian or Lorentzian space, with Levi–Civita connection ∇. In local coordinates x^i we have $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma^k_{ij}\frac{\partial}{\partial x^k}$, with $\Gamma^k_{ij} = \frac{1}{2}g^{km}(g_{im,j} + g_{mj,i} - g_{ij,m})$. We use the Einstein summation convention, and the comma denotes partial differentiation. We use the curvature convention $R(X, Y, Z) = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z$, and $R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = R^{\ell}_{ijk}\frac{\partial}{\partial x^\ell}$. The Ricci curvature $Ric(g)$ has components $R_{ik} = R^{\ell}_{i\ell k}$, and the scalar curvature is the metric trace of Ric: $R(g) = g^{ik}R_{ik}$.

2.1. The Bahn–Ehrlich theorem

Let $\mathbb{L}^{n+1} = (\mathbb{R}^{n+1}, \eta)$ be the $(n+1)$-dimensional Minkowski space-time, with metric signature $(-, +, \ldots, +)$. Recall that a set A in a time-oriented Lorentzian space-time is *achronal* if no future-pointing time-like curve intersects A in more than one point. Let S be a compact, simply connected, achronal hypersurface contained in the causal future $I^+(O)$ of the origin of Minkowski space-time. Let $d(O, \cdot)$ be the Minkowski distance function on $I^+(O)$: $d(O, x) = \sqrt{-\eta(x, x)} =: \|x\|$. We assume without further remark that S is (piecewise) smooth with (piecewise) smooth boundary. $I^+(O)$ is foliated by umbilic space-like hypersurfaces $\mathbb{H}(r) = \{x \in I^+(O) : d(O, x) = r\}$, $r > 0$, each of which is isometric to a hyperbolic space (of curvature $K = -\frac{1}{r^2}$). We can rescale a hypersurface S to $\mu(S) = \left\{\frac{x}{\|x\|} : x \in S\right\} \subset \mathbb{H}(1)$. We let the closed *cone* of S be given by $C(S) = \{\lambda x : x \in S, \lambda \in [0, 1]\}$. We now recall the isoperimetric inequality of Bahn and Ehrlich.

Theorem 1 (*Bahn–Ehrlich [3]*). *Let $S \subset I^+(O) \subset \mathbb{L}^{n+1}$ be a compact, simply connected, achronal, space-like hypersurface. Let $t^* = d(O, S)$. Let A_0 be Lorentzian area, let V_0 be Lorentzian volume, and let $\omega_S = V_0(C(\mu(S)))$. Then*

$$(A_0(S))^{n+1} \le (n+1)^{n+1}\omega_S(V_0(C(S)))^n.$$

Equality holds if and only if $S \subset \mathbb{H}(t^*)$.

It follows that for hypersurfaces with a given $\mu(S)$, only subsets of the hyperbolic leaves *maximize* area for given volume enclosed by the cone $C(S)$.

We now turn to spaces for which we want to generalize the above result. Let $I = (0, b) \subseteq (0, +\infty)$ be an open interval. Note that the metric η on $I^+(O)$ can be written as a warped product metric on $(0, +\infty) \times \mathbb{H}^n$ as $\eta = -dt^2 + t^2 g_{\mathbb{H}^n}$. On $I \times \mathbb{H}^n$ we consider other warped product metrics $g = -dt^2 + (a(t))^2 g_{\mathbb{H}^n}$. Such metrics form a class of Friedmann–Robertson–Walker (FRW) metrics, which more generally take the form $g = -dt^2 + (a(t))^2 g_\kappa$ on $M = I \times \Sigma$, where g_κ is a Riemannian metric on Σ of constant sectional curvature κ. Such metrics are used to model homogeneous and isotropic cosmologies, and the warping factor $a(t)$ encodes the dynamics of the space-time.

We will consider smooth maps $F : I \times \mathbb{H}^n \to I^+(O) \approx (0, +\infty) \times \mathbb{H}^n$ of the form $F(t, \omega) = (\psi(t), \omega)$, for some suitably chosen ψ. We take ψ to strictly increasing, so that F is a diffeomorphism onto its image.

We now define subsets of FRW space-times analogous to those used in the Bahn–Ehrlich Theorem and examine how they behave under F. First, note that the curves $\alpha(s) = (s, \omega)$ are time-like geodesics: since $\alpha'(s) = \frac{\partial}{\partial t}|_{\alpha(s)}$, the acceleration vector is given by $\alpha''(s) := D_{\alpha'(s)}\alpha'(s) = \Gamma^k_{00} \frac{\partial}{\partial x^k}$, where $x^0 = t$ and $(x^1, \dots, x^n)$ are coordinates on the hyperbolic factor, and $\Gamma^k_{00} = \frac{1}{2} g^{km}(2g_{m0,0} - g_{00,m}) = 0$. Now consider a compact, space-like, achronal hypersurface S. We define a cone-like set $\Gamma(S)$ on S as follows: $\Gamma(S) = \{(t, \omega) : (\tau, \omega) \in S \text{ and } 0 < t < \tau\}$. We also want to define the analogous *shadow set* $\hat{\mu}_t(S) \subset \{t\} \times \mathbb{H}^n$ by $\hat{\mu}_t(S) = \{(t, \omega) : \text{for some } \tau > 0,\ (\tau, \omega) \in S\}$. Note that since ψ is increasing, $F(\hat{\mu}_t(S)) = \mu_{\psi(t)}(F(S))$. The sets $\Gamma(S)$ and $\hat{\mu}_t(S)$ are defined by using time-like geodesics passing through S, and so are tied to the causal structure. In the Minkowski case, the geodesics emanate from an origin O; in the general FRW case, if $\lim_{t\to 0^+} a(t) = 0$, we see that the diameter of the shadow $\hat{\mu}_t(S)$ of any such surface S (compact) shrinks to zero as $t \to 0^+$. The geodesics might thus be interpreted as emanating from a single point; in case $\lim_{t\to 0^+} a'(t) = +\infty$, one may interpret this as a *big-bang* singularity.

2.2. Second variation of area under a volume constraint

Let Σ be either a smooth domain contained in some $\{t_0\} \times \mathbb{H}^n$ inside an FRW space $(M = I \times \mathbb{H}^n, g)$ as above, or $\Sigma = S_r = \{r\} \times \mathbb{S}^2$ in a rotationally-symmetric Riemannian metric g on $M = I \times \mathbb{S}^2$ (see Section 4). In either case, Σ has constant mean curvature in M. In fact Σ is totally umbilic, since in the case $\Sigma = S_r$, the ambient metric is rotationally invariant, and similarly in the case $\Sigma = \{t_0\} \times \mathbb{H}^n$ the ambient FRW metric is preserved by hyperbolic isometries. In this latter case we compute the second fundamental form with respect to the unit normal $\frac{\partial}{\partial t}$ to Σ: $\mathit{II}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g\left(\frac{\partial}{\partial t}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right) = -\Gamma^0_{ij} = -\frac{a'(t_0)}{a(t_0)} g_{ij} =: -h(t_0) g_{ij}$, so that the mean curvature of $\{t_0\} \times \mathbb{H}^n$ in FRW is just $H = g^{ij}\mathit{II}_{ij} = -nh(t_0)$. In either case, let N be a unit normal vector field to Σ, let $\epsilon = g(N, N) = \pm 1$, and let $V = V^T + V^N$ be the decomposition of a vector $V \in T_pM$, $p \in \Sigma$, into tangential and normal parts. The mean curvature vector field of Σ is then $\vec{H} = \sum_{i=1}^n (\nabla_{E_i} E_i)^N$, so that for any normal vector field W on Σ, $div_\Sigma W = \sum_{i=1}^n g(\nabla_{E_i} W, E_i) = -g(W, \vec{H})$, and $\vec{H} = \epsilon H N$.

We now recall the relevant formulas for the first and second variation of area of Σ (in either case posed above). Consider a variation $\Phi : (-\epsilon_0, \epsilon_0) \times \Sigma \to M$ with variation field $X = (\Phi_\tau)|_{\tau=0}$, where $\Phi_\tau := \Phi_*(\frac{\partial}{\partial \tau})$. Let $\gamma(\tau)$ be the induced metric on $\Phi(\tau, \Sigma)$, and let the area of the embedding $\Phi(\tau, \Sigma)$ be $A(\tau)$. Let $\{E_i\}$ be a local trivialization of $T\Phi(\tau, \Sigma)$ which commutes with the variation field Φ_τ along Φ, and which is a local orthonormal frame field on Σ, and let ν be the outward-pointing co-normal on $\partial\Sigma$ (in the FRW case). From the identity $\frac{d}{d\tau}\sqrt{\det\gamma} = \frac{1}{2}\sqrt{\det\gamma}\left(\gamma^{ij}\frac{d}{d\tau}\gamma_{ij}\right) = (div_{\Sigma_\tau}\Phi_\tau)\sqrt{\det\gamma}$, we can easily derive the first variation of area formula (assuming $X \perp \nu$ on $\partial\Sigma$): $A'(0) = -\int_\Sigma g(X, \vec{H})\, dA_\Sigma$.

We consider variation fields normal to Σ (at $\tau = 0$). Indeed in the Riemannian case, we consider $\Phi_\tau = f(\tau, x)N(\tau, x)$ to be normal to the embedding. In the FRW (Lorentzian) case, we let the variation $\Phi_\tau = f(\tau, x)\frac{\partial}{\partial t}|_{\Phi(\tau,x)}$ be in the t-direction. Moreover, we also assume the volume $V(\tau)$ enclosed by $\Phi(\tau, \Sigma)$ (either the cone volume, or the homological volume, depending on the case) is constant. We let $D(\Sigma) \subset \mathbb{H}^n$ be the projection of Σ onto the second factor of $I \times \mathbb{H}^n$; there is a function $t(\tau, p)$ so that $V(\tau) = \int_{p\in D(\Sigma)} \int_0^{t(\tau,p)} a(s)^n\ ds\ dA_{\mathbb{H}^n}$. In either case, $V'(0) = \int_\Sigma f\ dA_\Sigma$, and by applying the calculation for the first variation of area, we obtain $V''(0) = \int_\Sigma \left(\frac{\partial f}{\partial \tau} - Hf^2\right)\ dA_\Sigma = 0$.

The derivation of the second variation of area follows the argument in [11] (up to any sign change from the Lorentzian signature); see also [12] (note that their time-like Ricci curvature is negative of ours), or [13] for the FRW case. We apply the volume constraint $V'(0) = 0 = V''(0)$ to arrive at the second variation of area for volume-preserving normal variations, and then use $X = fN$, $\nabla^N_{E_i} X = (\nabla_{E_i} X)^N = E_i[f]\, N$, and $\frac{\partial}{\partial t} = N$ on $\{t_0\} \times \mathbb{H}^n$ to obtain (2):

$$\begin{aligned} A''(0) &= \int_\Sigma \left(\sum_{i=1}^n g(\nabla^N_{E_i} X, \nabla^N_{E_i} X) - \sum_{i,j=1}^n (g(\nabla_{E_i} E_j, X))^2 - g(R(X, E_i)X, E_i) \right)\ dA_\Sigma \\ &= \int_\Sigma \left(\epsilon |\nabla^\Sigma f|^2 - f^2 \|\mathit{II}\|^2 - f^2 Ric(g)(N, N) \right)\ dA_\Sigma. \end{aligned} \quad (2)$$

In FRW space-times satisfying the *time-like convergence condition Ric*$(g)(N, N) \geq 0$ (equivalently $a''(t) \leq 0$ by (4)), we have $A''(0) \leq 0$. More generally, $\mathit{II} = -\frac{a'(t)}{a(t)} g$ along with (4) also implies $A''(0) \leq 0$ in case $(a(t)a''(t) - (a'(t))^2) \leq 0$.

3. Lorentzian isoperimetric comparison

We will now discuss an adaptation of the Bray–Morgan isoperimetric comparison technique to our setting. We let $(M, g) = (I \times \mathbb{H}^n, -dt^2 + (a(t))^2 g_{\mathbb{H}^n})$ be an FRW space-time as above, and let $(I^+(O), \eta)$ be the future of O in the Minkowski space-time, which we identify with $((0, +\infty) \times \mathbb{H}^n, -dt^2 + t^2 g_{\mathbb{H}^n})$. We consider a diffeomorphism $F : M \to I^+(O)$ which has the form $F(t, \omega) = (\psi(t), \omega)$ with respect to the indicated identifications, with ψ increasing in t. We define the area stretch factor α_Σ for a space-like hypersurface $\Sigma \subset M$ by $F^*(\mathrm{d}A_{F(\Sigma)}) = \alpha_\Sigma \mathrm{d}A_\Sigma$, where $\mathrm{d}A_\Sigma$ and $\mathrm{d}A_{F(\Sigma)}$ are the induced area forms on the surfaces, and where we assume $F(\Sigma)$ is also space-like; a similar definition holds for time-like hypersurfaces. The volume stretch factor β is given by $F^*(dV_0) = \beta dV_M$, where dV_M and dV_0 denote the respective volume forms.

To estimate the stretch factors, we first compute them for two types of surfaces. We let $D \subset \mathbb{H}^n$ be any regular domain, and let $D_t = \{t\} \times D \subset M$. Let $\alpha_1(t) = \alpha_{D_t}$, so that

$$\alpha_1(t) = \frac{\int_D \psi^n(t)\ \mathrm{d}A_{\mathbb{H}^n}}{\int_D a^n(t)\ \mathrm{d}A_{\mathbb{H}^n}} = \frac{\psi^n(t)}{a^n(t)}.$$

We define the stretch factor $\alpha_0(t)$ for the annular time-like surface determined by flowing an $(n-1)$-dimensional submanifold $\Gamma \subset \mathbb{H}^n$ along the time-like direction field $\frac{\partial}{\partial t}$, and we let $\mathrm{d}\sigma_\Gamma$ be the $g_{\mathbb{H}^n}$-induced area element along Γ:

$$\alpha_0(t) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\psi(t_0)}^{\psi(t)}\int_\Gamma \tau^{n-1}\ \mathrm{d}\sigma_\Gamma\ \mathrm{d}\tau}{\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t}\int_\Gamma a^{n-1}(\tau)\ \mathrm{d}\sigma_\Gamma\ \mathrm{d}\tau} = \frac{\psi^{n-1}(t)\psi'(t)}{a^{n-1}(t)}.$$

Finally the volume stretch is given by

$$\beta(t) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\psi(t_0)}^{\psi(t)}\int_D \tau^{n}\ \mathrm{d}A_{\mathbb{H}^n}\ \mathrm{d}\tau}{\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t}\int_D a^{n}(\tau)\ \mathrm{d}A_{\mathbb{H}^n}\ \mathrm{d}\tau} = \frac{\psi^n(t)\psi'(t)}{a^n(t)} = \sqrt[n]{\alpha_1(t)}\ \alpha_0(t).$$

Let $\Sigma \subset M$ be a space-like hypersurface; at a (smooth) point $p \in \Sigma \cap (\{t\} \times \mathbb{H}^n)$, $T_p\Sigma \cap T_p(\{t\} \times \mathbb{H}^n)$ is at least $(n-1)$-dimensional. Let $E_2, \dots, E_n$ be an orthonormal set in this intersection. Let E_1 complete these to form an orthonormal basis for $T_p(\{t\} \times \mathbb{H}^n)$, and let $E_0 = \frac{\partial}{\partial t}$. There exist $\xi_0, \xi_1 \in \mathbb{R}$ with $\xi_1^2 - \xi_0^2 = 1$ so that $\xi_0 E_0 + \xi_1 E_1 \in T_p\Sigma$ is a unit vector orthogonal to E_j, $j = 2, \dots, n$. Therefore we see

$$\begin{aligned}\alpha_\Sigma &= |\mathrm{d}A_{F(\Sigma)}(F_*(E_2), \dots, F_*(E_n), F_*(\xi_0 E_0 + \xi_1 E_1))| \\ &= \|F_*(E_2)\| \cdots \|F_*(E_n)\| \|\xi_0 F_*(E_0) + \xi_1 F_*(E_1)\| \\ &= \frac{\psi^{n-1}(t)}{a^{n-1}(t)}\sqrt{\xi_1^2\frac{\psi^2(t)}{a^2(t)} - \xi_0^2(\psi'(t))^2} = \sqrt{\xi_1^2\alpha_1^2 - \xi_0^2\alpha_0^2}.\end{aligned} \tag{3}$$

Therefore, $\alpha_\Sigma^2 = \xi_1^2\alpha_1^2 - \xi_0^2\alpha_0^2 = \alpha_1^2 + \xi_0^2(\alpha_1^2 - \alpha_0^2)$.

Theorem 2. *Let S be a compact, simply-connected, space-like, achronal hypersurface in an FRW space-time $(M, g) = (I \times \mathbb{H}^n, -dt^2 + (a(t))^2 g_{\mathbb{H}^n})$, with $V(\Gamma(S)) < +\infty$. Let $T(S) = \{t : S \cap (\{t\} \times \mathbb{H}^n) \neq \emptyset\}$, and let $t^* > 0$ be such that the volume $V(\Gamma(\hat{\mu}_{t^*}(S)))$ from the shadow $\hat{\mu}_{t^*}(S)$ is precisely $V(\Gamma(S))$. Furthermore, suppose a map F as above can be constructed so that $\beta(t) \le \beta(t')$ for $t, t' \in T(S)$ with $t \ge t^* \ge t'$ (e.g. in case $\beta(t)$ is non-increasing for $t \in T(S)$), so that the average value of α_S over S is at least $\alpha_1(t^*) = \frac{A_0(F(\hat{\mu}_{t^*}(S)))}{A(\hat{\mu}_{t^*}(S))}$, and so that $F(S)$ is space-like and achronal. Then the area $A(S)$ of S is at most the area of the shadow $\hat{\mu}_{t^*}(S)$, with equality if and only if $S = \hat{\mu}_{t^*}(S)$.*

Proof. Suppose S and $t^* > 0$ are as in the theorem, and moreover that $A(S) \ge A(\hat{\mu}_{t^*}(S))$. Then we have by change of variables and the fact that $\frac{1}{A(S)}\int_S \alpha_S\ \mathrm{d}A_S \ge \alpha_1(t^*)$,

$$A_0(F(S)) = \int_{F(S)} \mathrm{d}A_{F(S)} = \int_S \alpha_S\ \mathrm{d}A_S \ge A_0(F(\hat{\mu}_{t^*}(S))) = A_0(\mu_{\psi(t^*)}(F(S))).$$

Since $\beta(t)$ is decreasing in t, the volume $V_0\,(C(F(S)) \cap \{(\tau, \omega) : \tau > \psi(t^*)\})$ is less than the volume $V_0(C(F(S)) \cap \{(\tau, \omega) : \tau < \psi(t^*)\})$. We may thus conclude that $V_0(C(F(S))) \le V_0(C(\mu_{\psi(t^*)}(F(S))))$. By assumption on the map F, $F(S)$ is achronal and space-like, so that we may apply the Bahn–Ehrlich theorem to prove $(A_0(F(S)))^{n+1} = (n+1)^{n+1}\omega_{F(S)}(V_0(C(F(S))))^n$, and thus conclude $F(S) \subset \{\psi(t^*)\} \times \mathbb{H}^n$. Pulling back to M, we can conclude the theorem. □

Remark. We remark that the assumption $V(\Gamma(S)) < +\infty$ will be satisfied in our applications below. In any case, we could instead choose $t_0 > 0$ and replace $V(\Gamma(S))$ with $V(\Gamma(S), t_0) = V(\Gamma(S) \cap \{(t, \omega) \in M : t > t_0\})$, which is always finite for compact S, and state an analogous theorem.

3.1. Applications

It remains to be shown to what extent the theorem can be applied, in other words, under what restrictions on the FRW metric we can find ψ so that F satisfies the conditions of the theorem. We begin with the following proposition.

Proposition 3. *Consider an FRW metric $g = -\mathrm{d}t^2 + (a(t))^2 g_{\mathbb{H}^n}$ on $I \times \mathbb{H}^n$ with $0 < a'(t) \leq 1$, and $a''(t) \leq 0$. For any compact, simply connected, space-like, achronal hypersurface S, we let $t^* > 0$ be chosen so that $V(\Gamma(\hat{\mu}_{t^*}(S))) = V(\Gamma(S))$. Then the area $A(S)$ of S is at most the area $A(\hat{\mu}_{t^*}(S))$ of the shadow, with equality if and only if $S = \hat{\mu}_{t^*}(S)$.*

Proof. We verify the conditions of Theorem 2. We define F by taking $\psi(t) = a(t)$. Then the area stretch $\alpha_1(t) = 1$, while $\alpha_0(t) = a'(t)$. Thus $\alpha_1(t) \geq \alpha_0(t)$ if and only if $a'(t) \leq 1$, in which case by (3), $\alpha_S \geq \alpha_1$. The volume stretch is $\beta(t) = a'(t)$, so β is non-increasing precisely for $a''(t) \leq 0$.

We now consider how F affects the causal nature of vectors. Let $v = \frac{\partial}{\partial t} + \lambda u$, where u is a $g_{\mathbb{H}^n}$-unit vector. Then $g(v, v) = -1 + \lambda^2 a^2(t)$, and $F_*(v) = \psi'(t)\frac{\partial}{\partial t} + \lambda u$, so that

$$\eta(F_*(v), F_*(v)) = -(\psi'(t))^2 + \lambda^2\psi^2(t) = -(a'(t))^2 + \lambda^2 a^2(t).$$

Under the condition $0 < a'(t) \leq 1$, we obtain

$$\eta(F_*(v), F_*(v)) \geq g(v, v) = -1 + \lambda^2 a^2(t).$$

Thus we see F_* maps space-like vectors to space-like vectors, so that $F(S)$ is space-like. This inequality also shows that $F(S)$ is achronal: if there were a future-pointing time-like curve $\gamma = F \circ \alpha$ from $F(p)$ to $F(q)$ on $F(S)$, then α is a future-pointing time-like curve from p to q on S, which contradicts achronality of S. □

Let $N = \frac{\partial}{\partial t}$, and let V and W be tangent to $\{t\} \times \mathbb{H}^n$. The Ricci tensor and scalar curvature of an FRW metric are given by the following [14]: $Ric(g)(N, V) = 0$, $Ric(g)(V, W) = \big(a(t)a''(t) + (n-1)((a'(t))^2 - 1)\big)$,

$$Ric(g)(N, N) = -\frac{na''(t)}{a(t)} \tag{4}$$

and thus $R(g) = (a(t))^{-2}\left[2na(t)a''(t) + n(n-1)\left((a'(t))^2 - 1\right)\right]$. The metrics covered by the proposition, then, have non-negative Ricci curvature in the time direction (orthogonal to the hyperbolic space-like slices), and non-positive Ricci curvature in the spatial directions.

Let us interpret the metrics covered by this proposition as solutions of Einstein's equation in the case $n = 3$. Using the above curvature formulas, one easily obtains the equation $Ric(g) - \frac{1}{2}R(g)g + \Lambda g = 8\pi T$, where Λ is a constant (cosmological constant), and the tensor T has the form $T_{ab} = \rho N_a N_b + P(g_{ab} + N_a N_b)$ of the stress-energy tensor of a perfect fluid with velocity N, density ρ and pressure P, for functions $\rho(t)$ and $P(t)$ which will be specified below. The density and pressure are related to the metric function $a(t)$ as follows, where $h(t) = \frac{a'(t)}{a(t)}$ is called the *Hubble parameter* [14]:

$$(h(t))^2 := \frac{(a'(t))^2}{(a(t))^2} = \frac{1}{3}\left(8\pi\rho + \frac{3}{a^2(t)} + \Lambda\right), \qquad \frac{a''(t)}{a(t)} = -\frac{4\pi}{3}(\rho + 3P) + \frac{1}{3}\Lambda.$$

We note the condition that $\rho \geq 0$ is equivalent to $h^2 \geq a^{-2} + \Lambda/3$, i.e. $(a'(t))^2 \geq 1 + \frac{1}{3}\Lambda(a(t))^2$. Observe that in the classical case of vanishing cosmological constant, the metrics covered by the preceding proposition have $\rho \leq 0$; for the proposition to accommodate models with positive matter density, one must use an Einstein equation with $\Lambda < 0$.

We remark that in the proof of the proposition, we constructed a specific comparison map F, and that it might be possible to find a different comparison map that may capture a different regime of FRW metrics, and in particular, may allow models with $\rho \geq 0$ in case $\Lambda = 0$. For example, consider (3) and impose the condition $\alpha_0(t) = \alpha_1(t)$; by the above calculations, this condition is equivalent to $\frac{\psi'(t)}{\psi(t)} = \frac{1}{a(t)}$. The solutions to this equation are determined up to a constant factor. We note that with this choice of ψ, F preserves the causal nature of vectors, and hence $F(S)$ is space-like and achronal if S is: as above, for $v = \frac{\partial}{\partial t} + \lambda u$, we have $\eta(F_*(v), F_*(v)) = -(\psi'(t))^2 + \lambda^2\psi^2(t) = \frac{\psi^2(t)}{a^2(t)}g(v, v)$. By design, $\alpha_S = \alpha_1 = \alpha_0$. Furthermore, this choice of ψ yields a monotone volume stretch $\beta(t) = \frac{\psi^{n+1}(t)}{a^{n+1}(t)}$. Indeed we find

$$\beta'(t) = (n+1)\beta^{\frac{n}{n+1}}(t)\frac{\psi'(t)a(t) - \psi(t)a'(t)}{a^2(t)} = (n+1)\beta^{\frac{n}{n+1}}(t)\frac{\psi(t)(1 - a'(t))}{a^2(t)}.$$

Thus, β is non-increasing precisely if $a'(t) \geq 1$, which corresponds to non-negative density $\rho \geq 0$ in case $\Lambda = 0$.

However there is a catch: the last item to check is the area stretch factor for S. We note that since $\alpha_1(t) = \beta^{\frac{n}{n+1}}(t)$, the sign of $\alpha_1'(t)$ is the same as the sign of $\beta'(t)$. This is problematic, since one way to satisfy the area stretch condition in Theorem 2 is to arrange $\alpha_S \geq \alpha_1(t^*)$, which would be satisfied if α_1 has a minimum at t^*. This simple statement will not hold in this case, but at least the monotone behavior of $\alpha_1(t)$ does give us a way to interpret the condition on the area of S used in Theorem 2: in trying to minimize volume and maximize area (for a given shadow set), the only possible way to improve upon a hyperboloidal hypersurface is to try to arrange for a majority of the surface to lie in $\{t > t^*\}$, while at the same time adhering to the volume constraint. We conjecture this cannot improve on the hyperboloidal hypersurface.

4. Remarks on energy density and stability of CMC surfaces

The Lorentzian isoperimetric comparison in Proposition 3 can be interpreted to hold in certain FRW space-times with $\Lambda = 0$ and $\rho \leq 0$. In this section we consider the Riemannian setting, and in the context of the time-symmetric Einstein constraint equation $R(g) = 16\pi\mu$, we remark that negative energy density tends to promote stability of CMC spheres in rotationally-symmetric perturbations of the standard space-like slices in Schwarzschild space-times.

In particular, we will consider metrics on domains $E_R = \{x \in \mathbb{R}^3 : |x| > R\}$ of the form $g(x) = u^4(r)g_E(x)$, where g_E is the Euclidean metric, and $u(r) > 0$, $(r = |x|)$. An important family of such metrics is given by the Schwarzschild metrics: $g^S(x) = \left(1 + \frac{m}{2r}\right)^4 g_E$. The parameter m is called the *mass*; it measures the deviation of the metrics from the Euclidean metric, and it has an interpretation in terms of the energy of isolated gravitational systems [15]. For $m > 0$, there is a unique minimal (in fact totally geodesic) sphere $r = m/2$, which in the context of general relativity is called the (apparent) horizon; in this case the metric is complete with two asymptotically flat ends, as the map $r \mapsto \frac{m^2}{4r}$ is an isometric inversion in the horizon sphere. We note that we can do a radial change of coordinates so that half of the Schwarzschild metric can be written $\left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2$, where $\mathrm{d}\Omega^2$ is the standard metric on the sphere $\mathbb{S}^2$, and $r > 2m$; in these coordinates $r = 2m$ corresponds to the horizon. When the mass m is negative, the metric $(1 - \frac{2m}{r})^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$ is defined on $r > 0$, and is an inextendible metric with no minimal sphere. The metric is incomplete: as $r \to 0^+$ along radial geodesics, the arclength remains bounded but the Ricci tensor blows up.

Of present interest for us is the fact that the stability of the area functional at the constant mean curvature spheres S_r of constant r depends on the sign of the mass m. Let N be a unit normal field to S_r, and let $A(\tau)$ be the area function induced from a variation of S_r with normal variation field $X = fN$; we assume that the volume is preserved (at least through the second order). We use coordinates for which the Schwarzschild metric takes the form $g^S = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$, so that the area of S_r is $4\pi r^2$, and the first non-zero eigenvalue of the Laplacian is $\lambda_1 = \frac{2}{r^2}$. By the volume constraint we have $\int_{S_r} f \, \mathrm{d}A_{S_r} = 0$, so we can apply the Poincaré inequality $\lambda_1 \int_\Sigma f^2 \, \mathrm{d}A_\Sigma \leq \int_\Sigma |\nabla^\Sigma f|^2 \, \mathrm{d}A_\Sigma$ on $\Sigma = S_r$. From the second variation (2) we then obtain $A''(0) \geq 6mr^{-3} \int_{S_r} f^2 \, \mathrm{d}A_{S_r}$, with equality if and only if f is in the λ_1-eigenspace (which is the span of the restrictions of Cartesian coordinate functions to the sphere). We see from this that in the *positive* mass Schwarzschild case, the second variation must be positive for (non-trivial) volume-preserving deformations; note that by applying Theorem 4 below, one can show the spheres are in fact *isoperimetric* for $m > 0$. In case $m < 0$, however, we see that for f a λ_1-eigenfunction, $A''(0)$ is negative, so the spheres are *unstable*. Of course in the Euclidean $(m = 0)$ borderline case, the spheres are stable, but not strictly stable, as the translations are isometries.

The Schwarzschild metrics have zero scalar curvature. A natural follow-up to the above observations is to determine geometries that are in some sense perturbations of Schwarzschild geometries, for the which the scalar curvature is non-zero and the rotationally-symmetric spheres enjoy some stability or isoperimetry. We will in fact deform the area profiles of Schwarzschild geometries to obtain rotationally-symmetric geometries in which the spheres S_r (for some range of r values) have a desired stability property, while keeping the sign of the scalar curvature fixed. The examples will point out that non-positive energy density has a stabilizing effect on the rotationally-symmetric spheres.

Before we state our results, we recall a simple form of the Bray–Morgan comparison theorem we will use for our setting.

Theorem 4 (*Bray–Morgan [1]*). *Consider a Riemannian three-manifold $M = I \times \mathbb{S}^2$ with a rotationally-symmetric metric (so the metric can be written in the form $\mathrm{d}r^2 + f^2(r)\mathrm{d}\Omega^2$, or equivalently $h^2(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$), and let $S_r = \{r\} \times \mathbb{S}^2$ be a radially-symmetric sphere. Suppose M has non-positive radial Ricci curvature, and that M has non-negative tangential sectional curvature (with respect to the spheres S_r). Suppose furthermore that S_r has non-negative mean curvature in the $-\frac{\partial}{\partial r}$ direction, for all $r \in I$. Then the radially-symmetric spheres S_r minimize surface area among smooth surfaces enclosing the same volume (against S_{r_0}, say).*

We now compute the quantities of interest for the metric $g = u(r)^4 g_E = u^4(r)(\mathrm{d}r^2 + r^2(\mathrm{d}\varphi^2 + \sin^2\varphi \, \mathrm{d}\theta^2))$. For this purpose, we index (r, φ, θ) as $(1, 2, 3)$ respectively, and we let $\partial_r, \partial_\varphi, \partial_\theta$ be the coordinate vector fields. The area profile is just $A(r) = A(S_r) = 4\pi r^2 u^4$, and the non-vanishing Christoffel symbols are as follows.

$$\Gamma^1_{11} = \frac{2u'}{u}, \quad \Gamma^1_{33} = -\sin^2\varphi\left(\frac{2u'}{u}r^2 + r\right) = \sin^2\varphi\,\Gamma^1_{22}, \quad \Gamma^2_{33} = -\sin\varphi \, \cos\varphi,$$

$$\Gamma^3_{23} = \cot\varphi, \quad \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{2u'}{u} + \frac{1}{r}.$$

We find $R^k_{232}\partial_k = R(\partial_\varphi, \partial_\theta, \partial_\varphi) = \nabla_{\partial_\theta}\nabla_{\partial_\varphi}\partial_\varphi - \nabla_{\partial_\varphi}\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\theta}(\Gamma^1_{22}\partial_r) - \nabla_{\partial_\varphi}(\Gamma^3_{23}\partial_\theta) = \left(1 - r^2(\frac{1}{r} + \frac{2u'}{u})^2\right)\partial_\theta$. The tangential sectional curvature is then $K^M(r) = K^M(T_pS_r) = \frac{g(R(\partial_\varphi,\partial_\theta,\partial_\varphi),\partial_\theta)}{\|\partial_\theta\|^2\|\partial_\varphi\|^2} = \frac{-4}{u^6}\left((u')^2 + \frac{uu'}{r}\right)$. We also note $R^2_{121} = R^3_{131} = \frac{-2u''}{u} + \frac{2u'^2}{u^2} - \frac{2u'}{ru}$, so that the radial Ricci curvature is given by $\mathrm{Ric}^M(\partial_r, \partial_r) = R^1_{111} + R^2_{121} + R^3_{131} = \frac{-4u''}{u} + \frac{4u'^2}{u^2} - \frac{4u'}{ru}$. The scalar curvature transforms as

$$R(g) = -8u^{-5}\Delta_{g_E}u = -8u^{-5}\left(u'' + \frac{2}{r}u'\right). \tag{5}$$

We consider the unit normal $N = -u^{-2}\partial_r$ to S_r, and we compute the second fundamental form and mean curvature of S_r with respect to N. We find $\mathit{\Pi}_{\theta\theta} = \mathit{\Pi}(\partial_\theta, \partial_\theta) = g(-\nabla_{\partial_\theta} N, \partial_\theta) = g(u^{-2}\Gamma^k_{31}\partial_k, \partial_\theta) = u^{-2}(\frac{2u'}{u} + \frac{1}{r})g_{\theta\theta} = u^2r^2\sin^2\varphi(\frac{2u'}{u} + \frac{1}{r}) = \sin^2\varphi\ \ \mathit{\Pi}_{\varphi\varphi}$. From here it is easy to find $\|\mathit{\Pi}\|^2 = g^{ik}_{S_r}g^{jl}_{S_r}\mathit{\Pi}_{ij}\mathit{\Pi}_{kl} = g^{\theta\theta}g^{\theta\theta}\mathit{\Pi}^2_{\theta\theta} + g^{\varphi\varphi}g^{\varphi\varphi}\mathit{\Pi}^2_{\varphi\varphi} = \frac{2}{u^4}(\frac{2u'}{u} + \frac{1}{r})^2$ and $H = g^{\varphi\varphi}\mathit{\Pi}_{\varphi\varphi} + g^{\theta\theta}\mathit{\Pi}_{\theta\theta} = \frac{2}{u^2}\left(\frac{2u'}{u} + \frac{1}{r}\right)$. Finally we apply the second variation formula (2) to S_r with $X = fN$, and use the Poincaré inequality with $\lambda_1 = \frac{1}{u^4r^2}$ to obtain

$$\begin{aligned} A''(0) &\geq \int_{S_r} f^2\left(\frac{2}{u^4r^2} - \frac{2}{u^4}\left(\frac{2u'}{u} + \frac{1}{r}\right)^2 - u^{-4}\left(\frac{-4u''}{u} + \frac{4u'^2}{u^2} - \frac{4u'}{ur}\right)\right)\,\mathrm{d}A_{S_r} \\ &= 4u^{-6}\left(uu'' - \frac{uu'}{r} - 3(u')^2\right)\int_{S_r} f^2\,\mathrm{d}A_{S_r}. \end{aligned}$$

Let $\Xi(r) = uu'' - \frac{uu'}{r} - 3(u')^2$, so that stability is equivalent to $\Xi(r) \geq 0$.

We will produce some examples by picking appropriate functions u. We first note $u_0(r) = 1 + \frac{m}{2r}$ gives the general solutions (up to a constant factor) of $u'' + \frac{2}{r}u' = 0$; in particular u_0 are the rotationally invariant harmonic functions (up to scale). With an eye on (5), we will look for solutions of the equation

$$u'' + \frac{2}{r}u' = -\frac{C}{r^p}, \tag{6}$$

where C is a constant whose sign gives the sign of μ, and where p gives the decay rate of μ as $r \to +\infty$. Solutions of this equation are given by

$$u(r) = u_0(r) + \frac{C}{(2-p)(p-3)}\frac{1}{r^{p-2}} = 1 + \frac{m}{2r} + \frac{C}{(2-p)(p-3)}\frac{1}{r^{p-2}},$$

where again u_0 is determined up to a constant factor. We take $p > 3$ so that u_0 gives the top-order part of u. We now proceed by specifying the parameters m, C and p, and then checking whether and where the corresponding metric $g = u^4(r)g_E$, defined on some $E_R = \{u > 0\}$, satisfies the Bray–Morgan conditions for isoperimetry and/or the condition $\Xi(r) \geq 0$ for stability.

We let $p = 4$, and for reference we collect here the following identities:

$$A(r) = 4\pi r^2\left(1 + \frac{m}{2r} - \frac{C}{2r^2}\right)^4 \tag{7}$$

$$Ric^M\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = -\frac{4(2mr^2 - C(8r+m))}{r(r(2r+m) - C)^2} \tag{8}$$

$$K^M(r) = \frac{64r^6(mr - 2C)(2r^2 + C)}{(r(2r+m) - C)^6} \tag{9}$$

$$H(r) = \frac{8r^3(r(2r-m) + 3C)}{(r(2r+m) - C)^3} \tag{10}$$

$$\Xi(r) = \frac{6mr^3 - Cr(16r - m) - 4C^2}{4r^6}. \tag{11}$$

Case $m = 0$. We take $m = 0$ and $p = 4$ in the definition of u, so $u = 1 - \frac{C}{2r^2}$, and so for large r, then, the metric $g = u^4g_E$ approaches the flat metric, with top-order deviation $O(r^{-2})$.

If we let $C < 0$, then the metric is defined on $\{r > 0\}$, and from $H(r) = \frac{8r^3(2r^2+3C)}{(2r^2-C)^3}$, we see there is a unique minimal sphere at $r_0 = \sqrt{\frac{-3C}{2}}$: note that the spheres for $r > r_0$ are convex. The scalar curvature $R(g)$ is negative in this case, and $R(g) \in L^1(\{r > r_0\}, d\mu_g)$, where $d\mu_g$ is the metric volume measure, which is u^6 times the Euclidean volume measure. We note that $N = \pm u^{-2}\frac{\partial}{\partial r}$ is a unit normal to S_r, so that substituting $m = 0$ into (7)–(9) above, we obtain $\lim_{r\to 0^+} A(r) = +\infty$, and $\lim_{r\to 0^+} Ric^M(N, N) = 0 = \lim_{r\to 0^+} K^M(T_pS_r)$. Indeed, the metric is asymptotically flat with *two* ends; however as $r \to 0^+$, the difference of the metric from the flat metric is only $O(s^{-2/3})$, where s is the intrinsic distance along radial geodesics approaching $r = 0$; in fact, $R(g) \notin L^1(\{r_0 > r > 0\}, d\mu_g)$. In any case, using the above identities, we see the Bray–Morgan comparison proves that the spheres S_r for $r > r_0$ are isoperimetric. Moreover, these isoperimetric spheres are *strictly* stable, unlike in the Euclidean case. In fact, $\Xi(r) = -\frac{C(4r^2+C)}{r^6} > 0$ if and only if $r > r_c = \frac{\sqrt{-C}}{2}$, and $r_0 > r_c$.

The Penrose Inequality ([6,15,16]) states that in an asymptotically flat metric with *non-negative* scalar curvature, the total mass measured at infinity in an asymptotically flat end is always at least $\sqrt{\frac{A}{16\pi}}$, where A is the area of an *outermost* minimal surface. If we let $(M, g) = (\{r \geq r_0\}, u^4 g_E)$ in our example above, then we see the Penrose Inequality does not hold for g; of course, the scalar curvature is negative in this example.

If we now let $C > 0$, so that the scalar curvature is positive, we note that the metric is defined on $\left\{r > \sqrt{\frac{C}{2}}\right\}$. There are no minimal spheres: this is no surprise, in light of the Penrose inequality. The hypotheses of the Bray–Morgan theorem above do not hold; in fact $\Xi(r) < 0$, so the spheres S_r are *unstable*.

Case $m < 0$. We let $m = -2$ and $p = 4$, so that $g = u^4 g_E$ with $u(r) = 1 - \frac{1}{r} - \frac{C}{2r^2}$.

Suppose we try to stabilize spheres S_r by increasing C from zero to positive (thus by increasing the scalar curvature from zero to positive); the metric in this case is defined for $r > \frac{1+\sqrt{1+2C}}{2} = r_c$. Then by (11), $\Xi(r) = -\frac{6r^3 + Cr(8r+1) + 2C^2}{2r^6} < 0$, and so S_r is unstable.

Suppose instead we seek to stabilize some range of spheres S_r by making the scalar curvature negative. Note from (11), that no matter what C is, $\Xi(r) < 0$ for large r; this makes sense, since the spheres S_r are unstable in the negative mass case, and the mass term dominates the scalar curvature term for large r. From the form of r_c above, we see that for $C < -\frac{1}{2}$, the metric is defined on $\{r > 0\}$. By (10), $H = \frac{8r^3(2r^2+2r+3C)}{(2r^2-2r-C)^3}$, so there is a unique minimal sphere at $r_0 = \frac{1}{2}(-1 + \sqrt{1-6C})$. For C just below $-\frac{1}{2}$, the radius r_0 of the minimal sphere is just above $\frac{1}{2}$, and there is a small range of r values near r_0 about which the spheres are stable; the range becomes smaller as $C \to -\frac{1}{2}^-$. For example, if let $C = -1$, we produce an example of a metric g on $\{r > 0\}$ with negative scalar curvature, and with a unique minimal sphere S_{r_0} with $r_0 = \frac{\sqrt{7}-1}{2} \approx 0.823$. We note that $\Xi(r)$ is positive on an interval $I \supset (0.557, 1.254)$ around r_0, so that the spheres S_r are stable in this range. Moreover, on the interval $J = (r_0, 1)$, we have negative radial Ricci curvature, positive (inward) mean curvature, and positive tangential sectional curvature. Hence by Bray–Morgan, the spheres S_r are isoperimetric in $(J \times \mathbb{S}^2, g)$.

Case $m > 0$. We finally consider the case where $m > 0$, say $m = 2$. We let $p = 4$ again, so that $g = u^4 g_E$ with $u(r) = 1 + \frac{1}{r} - \frac{C}{2r^2}$.

When $C < 0$ (negative scalar curvature), the metric is defined on $\{r > 0\}$, and by (10), $H = \frac{8r^3(2r^2-2r+3C)}{(2r^2+2r-C)^3}$, we see there is a unique minimal sphere S_{r_0} with $r_0 = \frac{1}{2}(1 + \sqrt{1-6C})$. Since $2r_0^2 + C > 0$, a glance at (8)-(10) shows that the Bray–Morgan comparison can be applied to show that each S_r for $r > r_0$ is isoperimetric.

When $C > \frac{1}{6}$, however, the geometry does not resemble the Schwarzschild geometry of mass $m = 2$ for small r. Indeed, the metric is defined for $r > r_c = \frac{1}{2}(-1 + \sqrt{1+2C})$, in which range there are no minimal spheres. As $r \to r_c^+$, the metric becomes singular (consider the intrinsic curvature quantities in (8) and (9)), and the spheres S_r can be unstable: for example if $C = 4$, then $r_c = 1$ and $\Xi(r_c) < 0$.

In conclusion, the above examples illustrate the stabilizing effect of negative scalar curvature. The borderline $(m = 0)$ case may illustrate this the best: the effect of the scalar curvature is shown no matter how small C is, and the metric $u^4 g_E$ converges uniformly on compact subsets of $\{r > r_c\}$ (where r_c may be 0) to the Euclidean metric as C tends to 0. We also emphasize that in the negative mass cases, positive scalar curvature promotes further instability, while in the positive mass case, negative scalar curvature promotes further stability. When one tries to change the stability properties, one must add enough scalar curvature of the appropriate sign to make a large enough deviation in the geometry to change the stability properties of the spheres in some interval; though the geometry may change a lot, we maintain control on the sign of the scalar curvature.

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