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Review

Conformal superspace σ-models[☆]

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ABSTRACT

We review recent developments in the context of two-dimensional conformally invariant σ-models. These quantum field theories play a prominent role in the covariant superstring quantization in flux backgrounds and in the analysis of disordered systems.

We present supergroup WZW models as primary examples of logarithmic conformal field theories, whose structure is almost entirely determined by the underlying supergeometry. In particular, we discuss the harmonic analysis on supergroups and supercosets and point out the subtleties of Lie superalgebra representation theory that are responsible for the emergence of logarithmic representations. Furthermore, special types of marginal deformations of supergroup WZW models are studied which only exist if the Killing form is vanishing. We show how exact expressions for anomalous dimensions of boundary fields can be derived using quasi-abelian perturbation theory. Finally, the knowledge of the exact spectrum is used to motivate a duality between the OSP(4|2) symmetric Gross-Neveu model and the $S^{3|2}$ supersphere σ-model.

Contents

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1. Introduction

The development of exactly solvable two-dimensional quantum field theories provides one of the nicest and most fruitful examples of a co-evolution of physics and mathematics. Due to the fundamental importance of such models in string theory, a promising candidate for a theory of quantum gravity, the subject naturally connects geometrical, topological and algebraic questions. On the algebraic side, it is naturally related to quantum groups as well as infinite dimensional Lie algebras such as affine Kac–Moody algebras and the Virasoro algebra. On the other hand, simple algebraic manipulations, such as applying an automorphism of a chiral vertex algebra, are intimately related to deep geometric phenomena such as mirror symmetry between Calabi–Yau manifolds.

In this review, we discuss the mathematical structures that arise in a special class of two-dimensional quantum field theories, namely conformally invariant superspace σ-models. The sole inclusion of the word "super" leads to a number of subtleties and features which do not have counterparts in comparable bosonic models. One of the issues addressed in this article is the harmonic analysis on supergroups and supercosets which determines the σ-model spectrum at large volume. In particular, we exhibit the existence of logarithmic representations as an immediate consequence of the underlying supergeometry. We then discuss two different types of deformations of supergroup WZW models, which require the supergroup to have a vanishing Killing form. In both cases, we are able to determine certain open string partition functions exactly for all values of the coupling. We finally use these results to argue for a duality between the $\mathrm{OSP}(4|2)$ Gross–Neveu model and the $S^{3|2}$ supersphere σ-model.

The types of σ-models discussed in this review are an essential ingredient in the covariant quantization of superstrings in flux backgrounds, specifically in various types of anti-de Sitter spaces, see Table 1. Further applications exist in condensed matter theory (see Table 2), where Efetov's supersymmetry trick allows us to express physical observables in disordered systems (such as quantum Hall systems) in terms of correlation functions in a superspace σ-model [1] (see also [2,3]). Also, the universality classes of certain loop ensembles can be described by this class of models.

2. Nonlinear σ-models on superspaces

In this section, we provide a brief introduction to superspace σ-models. We then discuss issues related to conformal invariance and discuss the moduli space of such theories. Circle theories are used as an illustrative example where exact results can be obtained.

2.1. Superspace σ-models

Nonlinear σ-models are quantum field theories describing the embedding of a two-dimensional surface Σ, the world-sheet, into some (pseudo-)Riemannian (super)manifold $\mathcal{M}$. The latter may be equipped with some additional structure besides the metric. Examples include vector bundles and gerbes. In the case Σ is a cylinder, the model describes the propagation of a closed string in the "universe" $\mathcal{M}$, in the case of a strip the propagation of an open string. More complicated world sheets with multiple holes and boundary components have to be used to describe the interaction between different strings.

The dynamics of the world sheet is governed by an action functional $\mathcal{S}$ in which the metric and the other structures of the manifold $\mathcal{M}$ enter as parameters. In the simplest case, the functional $\mathcal{S}$ reads

$$\mathcal{S}[X] = \frac{1}{2\pi}\int \mathrm{d}^2 z\Big[G_{\mu\nu}(X) + B_{\mu\nu}(X)\Big]\partial X^\mu \bar{\partial} X^\nu. \tag{1}$$

It assigns a real number to each embedding $X : \Sigma \to \mathcal{M}$. The model can be regarded as a two-dimensional quantum field theory with background fields $G(X)$ (the metric) and $B(X)$ (a two-form) playing the role of coupling constants.

Besides their physical importance for the description of strings, nonlinear σ-models are also interesting from a mathematical point of view since they probe the symmetries and the topology of $\mathcal{M}$. Indeed, the isometries of the space $\mathcal{M}$ reflect themselves as internal symmetries of the quantum field theory. Moreover, both open and closed strings as well as D-branes can wind around non-trivial cycles in $\mathcal{M}$. The nature and size of these cycles are encoded in the spectrum of string excitations.

Table 1
Supercosets and their applications in string theory. The supercosets and supergroups in the lower row describe a supersymmetrized version of the geometries in the first line.

Minkowski	$AdS_5 \times S^5$	$AdS_4 \times \mathbb{CP}^3$	$AdS_3 \times S^3$	$AdS_3 \times S^3 \times S^3$
$\frac{\text{super-Poincaré}}{\text{Lorentz}}$	$\frac{PSU(2,2\|4)}{SO(1,4)\times SO(5)}$	$\frac{OSP(6\|2,2)}{U(3)\times SO(1,3)}$	$PSU(1,1\|2)$	$D(2,1;\alpha)$

Table 2
Supercosets and their applications in condensed matter theory and statistical physics.

IQHE	Dilute polymers (SAW)	Dense polymers
(non-conformal) $\frac{U(1,1\|2)}{U(1\|1)\times U(1\|1)}$	$S^{2S+1\|2S}$ $\frac{OSP(2S+2\|2S)}{OSP(2S+1\|2S)}$	$\mathbb{CP}^{S-1\|S}$ $\frac{U(S\|S)}{U(1)\times U(S-1\|S)}$

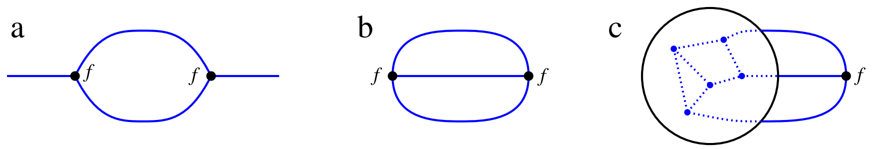

Fig. 1. (a) The Killing form. (b) & (c) Vanishing contributions to the β-function.

While the geometric interpretation of the σ-model is clear at large scales, it becomes less obvious on smaller scales where quantum effects become important. In a sense, nonlinear σ-models provide a way of defining a notion of "quantum geometry". Some of the surprises of quantum geometry are sketched below, e.g. the fact that a small circle cannot be distinguished from a small three-sphere if the sizes are chosen appropriately.

2.2. Conformal invariance

In general, nonlinear σ-models on arbitrary spaces $\mathcal{M}$ are not solvable. They are complicated interacting quantum field theories whose couplings obey intricate perturbative renormalization group equations. For this reason, we wish to restrict our attention to a special class of σ-models whose extended symmetry considerably reduces the complexity of the problem. More precisely, we wish to consider two specific classes of conformally invariant σ-models. In two dimensions, the conformal invariance implies the existence of an infinite dimensional spectrum generating algebra, diminishing the actual number of independent degrees of freedom.

The two classes of target spaces that are considered in this review are supergroups $\mathcal{M} = G$ and supercosets $\mathcal{M} = G/H$.[2] In the latter case, the identification is given by $g \sim gh$, with $g \in G$ and $h \in H$. The spaces G and G/H have isometries $G \times G$ and G, respectively, which act as left (and right in the first case) multiplication. In order to guarantee conformal invariance, some extra conditions on the metric and on the B-field have to be met. Since we impose G-invariance, we only have one parameter at our disposal for the metric in case G is a simple supergroup. Indeed, the metric is uniquely determined up to a scalar in that case. In most of the examples, conformal invariance uniquely determines B in terms of the metric. In these cases, the volume of G will hence be the only modulus, cf. Fig. 2. Exceptions are the deformations of supergroup WZW models discussed in Section 4 where two moduli are available.

In the case of supercosets, it is custom to assume that H is the fixed point set under some finite order automorphism of G. To first order in perturbation theory, the associated nonlinear σ-models are conformally invariant if the Killing form of G is vanishing [4]. After fixing a basis T^a of the Lie superalgebra $\mathfrak{g}$ underlying G, this statement is equivalent to

$$K^{ab} = \text{str}(\text{ad}_{T^a} \circ \text{ad}_{T^b}) = -(-1)^{\deg(d)} f^{ac}{}_d f^{bd}{}_c = 0. \tag{2}$$

For many supercosets relevant in physical applications (see Tables 1 and 2), this condition is even sufficient to all orders (cf. the tables in [5]). While for a purely bosonic Lie group Eq. (2) can never be satisfied if G is simple, it is well possible for certain families of Lie supergroups. These series are $PSL(N|N)$, $OSP(2S+S|2S)$ and $D(2,1;\alpha)$.[3] The interesting physical applications of these supergroups are displayed in the Tables 1 and 2.

The condition (2) for conformal invariance has a nice diagrammatic interpretation, see part (a) of Fig. 1 for an illustration of the Killing form. Intuitively, it is now clear why the β-function does not receive corrections. As an invariant with respect to the global G-symmetry, it is defined as a sum over Feynman diagrams without external legs. All these diagrams are constructed from straight lines and trivalent vertices, corresponding to the metric and the structure constants $f^{ab}{}_c$ on $\mathfrak{g}$,

[2] Note that a supergroup G can be realized as a diagonal supercoset $G \times G/G$. For reasons that become clear below, it is nevertheless useful to distinguish these two cases.

[3] There are further series which, however, are not interesting from a physical point of view since they do not admit a non-degenerate metric.

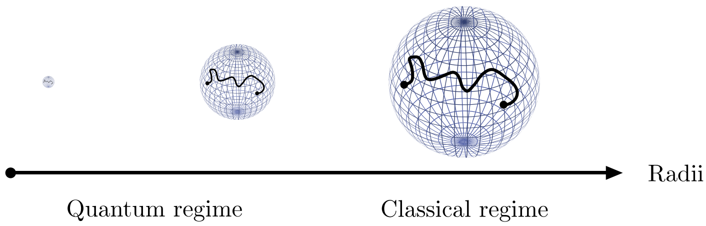

Fig. 2. The moduli space of generic supercoset σ-models.

respectively. Considering a potential contribution involving at least one vertex, the latter has to be connected to the rest of the diagram through three legs. The contribution in part (b) of Fig. 1 clearly vanishes. Let us therefore consider a general diagram as in part (c) of the same figure. For the cases at hand, there is just one invariant three-tensor, and hence the "blob" will be proportional to the structure constants f. The diagram is then essentially equivalent to the one in part (b) and hence vanishes, provided that Eq. (2) is satisfied [6]. The remaining diagrams without trivalent vertices can also be found in an abelian theory for which the β-function is known to be zero.

2.3. Spectra and partition functions

All information about a conformal field theory is contained in its correlation functions. The quantity we are aiming for in this review is a boundary partition function, or in other words the vacuum correlation function on an annulus. This kind of partition function encodes the spectrum of open strings starting and ending on a given D-brane. The energies of the excitations, the conformal dimensions h of the states, will depend on the moduli of $\mathcal{M}$. In particular, they will change if the volume of the manifold is increased or decreased. Below the moduli will be collectively referred to as the "radius" R.

The D-branes we will consider in this review preserve a global isometry G in addition to the conformal symmetry (represented by the Virasoro algebra Vir). The string excitations can therefore be organized by their transformation behavior with respect to the total symmetry $\mathfrak{g} \oplus \mathrm{Vir}$. The partition function can thus be written as

$$Z(q, z|R) = \mathrm{tr}_{\mathcal{H}}\left[z^H q^{L_0 - \frac{c}{24}}\right] = \sum_{\Lambda} \underbrace{\psi_\Lambda(q, R)}_{\text{Dynamics}} \; \underbrace{\chi_\Lambda(z)}_{\text{Symmetry}} , \tag{3}$$

where $\chi_\Lambda(z)$ are characters of $\mathfrak{g}$. In this expression, L_0 and c refer to the energy operator and the central charge of the Virasoro algebra. The second insertion $z^H = \prod z_i^{H_i}$ keeps track of the eigenvalues of the Cartan generators H_i of $\mathfrak{g}$. On the left-hand side and the right-hand side, we made explicit that the partition function depends on the moduli of $\mathcal{M}$. In the middle, this is implicit since the state space $\mathcal{H}$ will depend on R.

At large values of R, the partition function (3) is entirely determined by the geometric and topological properties of $\mathcal{M}$. In particular, the energy of string excitations can be obtained by performing a harmonic analysis on $\mathcal{M}$. For small values of R, however, the σ-model becomes strongly coupled and geometry starts to loose its meaning (see the example below). In this regime, calculating the partition function is a highly non-trivial task. While the character $\chi_\Lambda(z)$ only reflect the symmetries of the model, the dynamical information is contained in the branching functions $\psi_\Lambda(q, R)$. These functions describe which multiplets are located at which energy level for a given value of the moduli R. An example of a spectrum is sketched below in Fig. 6.

2.4. An example: the circle S^1

The simplest non-trivial target space is the circle S^1. Since the circle does not support a B-field, the nonlinear σ-model only has one parameter R, the circle radius. Since moreover the circle is flat (i.e. it has no intrinsic curvature), the theory is described by a *free* boson. This theory can be solved exactly for all values of the radius R using the underlying $\widehat{U}(1)$ current algebra

$$J(z)J(w) = \frac{1}{(z-w)^2}. \tag{4}$$

The partition functions of an open string with free boundary conditions on both ends turns out to be

$$Z(q, z|R) = \mathrm{tr}_{\mathcal{H}}\left[z^P q^{L_0 - \frac{c}{24}}\right] = \frac{1}{\eta(q)} \sum_{w \in \mathbb{Z}} z^w q^{\frac{w^2}{2R^2}}. \tag{5}$$

In this formula, w denotes the eigenvalues of the quantized momentum quantum operator P on a circle and the factor $\eta(q)$ keeps track of the energies of string oscillation modes. Note that the existence of P is related to the U(1) isometry of the circle.

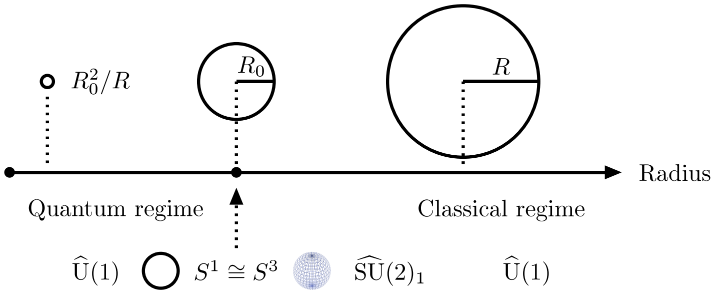

Fig. 3. The moduli space of circle theories and the corresponding symmetries.

Even though the circle theory is free and therefore trivial to solve, it nevertheless exhibits a few surprises. The first one concerns an exact equivalence under the replacement $R \leftrightarrow R_0^2/R$ (and a simultaneous exchange of momentum and winding modes as well as free and fixed boundary conditions), also known as T-duality. In this case, it implies the unexpected statement that the classical regime (very large R) is equivalent to the quantum regime (very small R). The second one concerns a symmetry enhancement from $\widehat{\mathrm{U}}(1)$ to $\widehat{\mathrm{SU}}(2)_1$ at the self-dual radius $R = R_0$. At this special point, the circle theory can be identified with a $\widehat{\mathrm{SU}}(2)_1$ WZW model which in turn can be thought of as describing a special point in the quantum regime of σ-models on S^3. A sketch of the free boson moduli space can be found in Fig. 3. We will later use the analogy with the free boson to argue for the existence of a much more complicated equivalence between two seemingly different conformal field theories, one geometric, one non-geometric.

3. Harmonic analysis on supergroups and supercosets

This section focuses on mathematical aspects of superspace σ-models. We review the representation theory of Lie superalgebras as well as the harmonic analysis on supergroups and supercosets, specifically superspheres. The results of this section have direct implications for the large volume spectra of superspace σ-models but they are also interesting from a purely mathematical point of view.

3.1. Representation theory of Lie superalgebras

The finite dimensional simple modules of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ fall into two classes, typical and atypical representations.[4] The typical sector is characterized by the property that its simple modules are projective (in the sense of category theory) while for the atypical representations this is not the case. The atypical modules can be thought of as arising from degenerate limits of typical modules where the typical representation becomes non-semisimple.[5] Alternative characterizations in terms of highest weights exist [7], but they will not be needed here.

For the sake of completeness let us also provide some further details on the notion of a projective module. A $\mathfrak{g}$-module $\mathcal{P}$ is called projective if and only if for every surjective $\mathfrak{g}$-homomorphism $f : M \twoheadrightarrow \mathcal{P}$ there exists a $\mathfrak{g}$-homomorphism $h : \mathcal{P} \to M$ such that $f \circ h = \mathrm{id}$. In other words, in case M is a cover of $\mathcal{P}$ then it contains $\mathcal{P}$ as a direct summand and the map f can be thought of as the projection onto $\mathcal{P}$. In the present context, i.e. restricting all considerations to the category of finite-dimensional $\mathfrak{g}$-modules, projective modules also satisfy the dual property of being injective. One could then replace our previous definition by the requirement that any projective submodule $\mathcal{P}$ of an arbitrary module M always appears as a direct summand.

Let $\mathrm{Rep}(\mathfrak{g})$ denote the set of (equivalence classes of) all finite dimensional simple modules. Elements from this set will be denoted by $\mathcal{L}_\mu$, with μ running through some set of weights. We split the weights into typical ones and atypical ones, $\mathrm{Typ}(\mathfrak{g})$ and $\mathrm{Atyp}(\mathfrak{g})$. To each weight μ, one can associate precisely one simple module $\mathcal{L}_\mu$ and further indecomposable modules which contain $\mathcal{L}_\mu$ is a simple quotient. The most important of these modules is the projective cover $\mathcal{P}_\mu$. A representation $\mathcal{L}_\mu$ is typical if and only if $\mathcal{L}_\mu \cong \mathcal{P}_\mu$. A further example would be the Kac module $\mathcal{K}_\mu$.

Another important module, even though not indecomposable, is the module $\mathcal{B}_\mu = \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(V_\mu)$ which is induced from a finite dimensional simple $\mathfrak{g}_0$-module V_μ. $\mathcal{B}_\mu$ is projective since V_μ is projective and hence it possesses a decomposition

$$\mathcal{B}_\mu = \bigoplus_{\nu \in \mathrm{Rep}(\mathfrak{g})} m_{\mu\nu} \mathcal{P}_\nu \tag{6}$$

[4] In physics terminology, one would call them non-BPS and BPS representations or long and (semi-)short representations, respectively.

[5] In physics terminology one would say that the long multiplet splits into short multiplets at the BPS bound, even though this seems to suggest a decomposition into irreducible components which certainly is not the case.

into projective covers. Due to the relation $\dim \mathrm{Hom}_{\mathfrak{g}}\big(\mathcal{P}_\mu, \mathcal{L}_\nu\big) = \delta_{\mu\nu}$, the multiplicities can be obtained as the dimension

$$m_{\mu\nu} = \dim \mathrm{Hom}_{\mathfrak{g}}\big(\mathcal{B}_\mu, \mathcal{L}_\nu\big) \tag{7}$$

of a suitable space of $\mathfrak{g}$-homomorphisms.

Finally, we introduce an equivalence relation on the set $\mathrm{Rep}(\mathfrak{g})$. Two weights μ and ν are said to be in the same block if there exists a non-split extension

$$0 \to \mathcal{L}_\mu \to \mathcal{A} \to \mathcal{L}_\nu \to 0. \tag{8}$$

In other words, if $\mathcal{L}_\mu$ and $\mathcal{L}_\nu$ can be obtained as a submodule and a quotient of $\mathcal{A}$, respectively, but nevertheless $\mathcal{A}$ is different from $\mathcal{L}_\mu \oplus \mathcal{L}_\nu$. The division of weights into blocks defines an equivalence relation. We will use the symbol $[\sigma]$ to denote the block σ belongs to. A weight is typical if and only if the corresponding block contains precisely one element. The symbol $\mathrm{AtypBlocks}(\mathfrak{g})$ will be reserved for the set of blocks obtained from *atypical* modules σ.

3.2. Harmonic analysis on supergroups

Roughly speaking, a Lie supergroup G is a fermionic extension of an ordinary Lie group G_0 by fermionic coordinates which transform in a suitable representation of G_0. All the properties of a supergroup are inherited from these data. For this reason, we will review the well-known harmonic analysis on ordinary compact Lie groups first. The goal of harmonic analysis is to learn about the structure of a manifold from studying the action of differential operators – Lie derivatives and Laplace operators – on its algebra of functions.

Let us consider a compact, simple, simply-connected Lie group G_0. According to the Peter–Weyl theorem, the algebra $\mathcal{F}(G_0)$ of square integrable functions on G_0 (with respect to the Haar measure) decomposes as

$$\mathcal{F}(G_0) \cong \bigoplus_{\mu\in\mathrm{Rep}(G_0)} V_\mu \otimes V_\mu^* \tag{9}$$

under the left–right regular action $l\times r\cdot f : g \mapsto f(l^{-1}gr)$ of $G_0 \times G_0$, where the sum is over all finite dimensional irreducible representations of G_0. Each individual term in the decomposition can be thought of as being associated with representation matrices $\rho_\mu(g) \in \mathrm{End}(V_\mu)$ with $g \in G_0$. The statement of the Peter–Weyl theorem is that these matrix elements can be used to approximate any function on G_0 with arbitrary precision.

The extension to supergroups is straightforward. Compared to an ordinary group, a supergroup comes with additional Grassmann algebra valued coordinates which generate the exterior algebra $\bigwedge(\mathfrak{g}_1^*)$. This space admits an obvious action of $\mathfrak{g}_0$ by the Lie bracket or, equivalently, by conjugation with elements from G_0. The algebra of functions on the supergroup G is the induced module (with respect to the right action of G_0)

$$\mathcal{F}(G) = \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathcal{F}(G_0) = \mathcal{F}(G_0) \otimes \bigwedge(\mathfrak{g}_1^*). \tag{10}$$

This definition has a natural interpretation arising from formally expanding functions on G in a Taylor series in the odd coordinates. Apart from the right action of G, $\mathcal{F}(G)$ also admits a left action, just as in the bosonic case. Our goal is to understand the decomposition of this algebra as a $\mathfrak{g}\oplus\mathfrak{g}$-module (with respect to the left and right regular action). The result will provide a super-analogue of the Peter–Weyl theorem.

Since all finite dimensional representations of a reductive Lie algebra $\mathfrak{g}_0$ are projective, the same will be true for the induced module $\mathcal{F}(G)$. Hence, as a right $\mathfrak{g}$-module, $\mathcal{F}(G)$ has the decomposition

$$\mathcal{F}(G) = \bigoplus_{\mu\in\mathrm{Rep}(G)} L_\mu \otimes \mathcal{P}_\mu^*, \tag{11}$$

where the sum is over all projective covers of $\mathfrak{g}$ and the L_μ are some multiplicity spaces. As a left $\mathfrak{g}$-module, $\mathcal{F}(G)$ has precisely the same decomposition. Indeed, the algebra of functions has to be isomorphic with respect to the left and the right regular action due to the existence of the isomorphism $\Omega : \mathcal{F}(G) \to \mathcal{F}(G)$ which acts as $\Omega(f) : g \mapsto f(g^{-1})$ and which intertwines the left and right regular actions.

In the typical sector, $\mathcal{P}_\mu$ agrees with $\mathcal{L}_\mu$. Given the symmetry between the left and the right action, it is then obvious that $L_\mu \cong \mathcal{L}_\mu$ as vector spaces. We will now show that this is indeed always the case, not only in the typical sector but also in the atypical sector. First of all we notice that, by definition and Eq. (6), the algebra of functions on G has the form

$$\mathcal{F}(G) = \bigoplus_{\mu\in\mathrm{Rep}(\mathfrak{g}_0)} V_\mu \otimes \mathcal{B}_\mu^* = \bigoplus_{\mu\in\mathrm{Rep}(\mathfrak{g}_0)} m_{\mu\nu} V_\mu \otimes \mathcal{P}_\mu^* \tag{12}$$

as a $\mathfrak{g}_0 \oplus \mathfrak{g}$-module. We then employ Frobenius reciprocity to rewrite Eq. (7) as

$$m_{\mu\nu} = \dim \mathrm{Hom}_{\mathfrak{g}_0}\big(V_\mu, \mathcal{L}_\nu\big), \tag{13}$$

which proves our assertion, given that all $\mathfrak{g}_0$-modules are fully reducible.

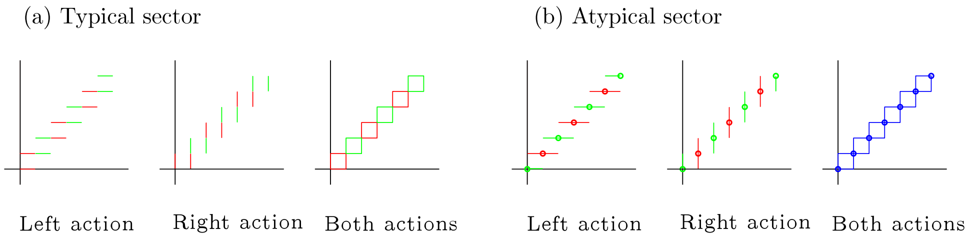

Fig. 4. Sketch of the harmonic analysis on supergroups using the example of GL(1|1). It is shown how the space of functions organizes itself with respect to the left and the right action of $\mathfrak{g}$ (the axes correspond to eigenvalues of the respective Cartan generators) and with respect to the simultaneous action of both. In the typical sector, (a) we observe a factorization of representation (green and red) while in the atypical sector (b) the functions organize themselves in infinite dimensional non-factorizing representations (blue) due to the extension of simple modules into projective covers. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The result just obtained suggests that we have a factorization $\mathcal{L}_\mu \otimes \mathcal{L}_\mu^*$ of the individual contributions in the typical sector, just as in the case of $\mathcal{F}(\mathrm{G}_0)$. In the atypical sector, however, such a factorization is not possible since the projective covers $\mathcal{P}_\mu$ are strictly larger than the simple modules $\mathcal{L}_\mu$. For this reason, the left and right modules in the atypical sector are entangled in a complicated way, arranging themselves in infinite dimensional non-chiral indecomposable $\mathfrak{g} \oplus \mathfrak{g}$-modules $\mathcal{I}_{[\sigma]}$ for each individual *block* $[\sigma]$. We finally find[6]

$$\mathcal{F}(\mathrm{G}) = \bigoplus_{\mu \in \mathrm{Typ}(\mathfrak{g})} \mathcal{L}_\mu \otimes \mathcal{L}_\mu^* \oplus \bigoplus_{[\sigma] \in \mathrm{AtypBlocks}(\mathfrak{g})} \mathcal{I}_{[\sigma]} \tag{14}$$

for the decomposition of $\mathcal{F}(\mathrm{G})$ as a $\mathfrak{g} \oplus \mathfrak{g}$-module. For Lie supergroups of type I, this has been worked out in great detail in [8].

It should be emphasized that the Laplace operator is not diagonalizable on the atypical projective covers $\mathcal{P}_\mu$ and on the non-chiral modules $\mathcal{I}_{[\sigma]}$. For σ-models on supergroups, this implies that they are generally logarithmic conformal field theories. Exceptions may occur for small volumes where the spectrum can be truncated in such a way that the modules $\mathcal{I}_{[\sigma]}$ no longer contribute.

3.3. *Example:* GL(1|1)

Since our previous result has been very abstract, let us explain it in more detail using the example of GL(1|1) (first considered in [9]). The corresponding Lie superalgebra gl(1|1) has two Cartan elements E and N. Consequently, states in a module are labeled by two numbers (e, n), the eigenvalues of E and N. Since E is central, the value e merely plays the role of a spectator. For $e = 0$, the state sits in an atypical representation, otherwise in a typical one. Typical representations $\mathcal{L}(e, n)$ (with $e \neq 0$) are two-dimensional while atypical simple modules $\mathcal{L}(n)$ are one-dimensional. On the other hand, the projective cover $\mathcal{P}(n)$ of the atypical simple module $\mathcal{L}(n)$ is four-dimensional, having a composition series $\mathcal{P}(n) : \mathcal{L}(n) \rightarrow \mathcal{L}(n+1) \oplus \mathcal{L}(n-1) \rightarrow \mathcal{L}(n)$.

The most important features of the harmonic analysis on GL(1|1) are sketched in Fig. 4. In the typical sector, the states organize themselves into two-dimensional representations, both with respect to the left and with respect to the right action (red and green lines). Under the combined action, they combine into four-dimensional representations (red and green boxes) which correspond to the tensor product of two representations. In the atypical sector, however, the picture is very different. Here the states organize themselves in four-dimensional projective covers if only one of the two actions is considered (red and green lines). One should imagine a diamond which is perpendicular to the plane, with two of the vertices being located in the plane. In this case, the states cannot be organized in a tensor product under the combined action for obvious reasons. Instead they have to combine into infinite dimensional indecomposable multiplets (blue), one for each value of n mod1.

3.4. *Harmonic analysis on supercosets*

The harmonic analysis on a supercoset G/H where the supergroup elements are identified according to the rule $g \sim gh$ with $h \in \mathrm{H}$ can be immediately deduced from that of the supergroup case. Indeed, the algebra of functions on G/H can be thought of as the space of H-invariant functions on G,

$$\mathcal{F}(\mathrm{G/H}) = \mathrm{Inv}_H \mathcal{F}(\mathrm{G}), \tag{15}$$

where an element h of the supergroup H acts on $f \in \mathcal{F}(\mathrm{G})$ according to

$$h \cdot f(g) = f(gh). \tag{16}$$

[6] Versions of this result appear to be known among mathematicians specialized on Lie superalgebras, even though no specific reference seems to exist. In the physics literature, the result was first noted in [8].

It is obvious that the space G/H and hence also the algebra of functions $\mathcal{F}(G/H)$ still admits an action of G. The isometry supergroup of G/H might be bigger than G but for simplicity we will only consider the symmetry G.

Writing the invariant subspace (15) explicitly as a direct sum over indecomposable G-modules turns out to be rather involved in the general case. The main reason is that the modules over Lie superalgebras are not fully decomposable. On the one hand, such modules already appear in $\mathcal{F}(G)$, as G-modules with respect to the right regular action, see Eq. (11). On the other hand, they may also arise when decomposing simple G-modules after restricting the action to the supergroup H. Finally, the invariants that need to be extracted when restricting from $\mathcal{F}(G)$ to $\mathcal{F}(G/H)$ can be either true H-invariants of $\mathcal{F}(G)$ (i.e. simple H-modules) or they can sit in a larger indecomposable H-module. A general solution to this intricate problem is currently beyond reach. Nevertheless, we can state one general lesson: in case H is purely bosonic the algebra of functions $\mathcal{F}(G/H)$ will necessarily involve projective covers of simple G-modules. In particular, this observation is relevant for the supercosets describing AdS backgrounds in string theory, see Table 1.

The problem sketched in the previous paragraph can be circumvented when working on the level of characters since these are not sensitive to the indecomposable structure of modules. For our purposes, this will be sufficient.

3.5. Example: the supersphere $S^{3|2}$

Instead of treating the general case, we discuss one example in some detail, namely the supersphere $S^{3|2}$. This supersphere can be thought of as being embedded into the flat superspace $\mathbb{R}^{4|2}$, i.e. we have four bosonic coordinates x^i and two fermionic coordinates η_1, η_2 subject to the constraint $\vec{x}^2+\eta_1\eta_2 = R^2$. For our purposes, we can identify the algebra of functions $\mathcal{F}(S^{3|2})$ with the polynomial algebra in the six coordinates $X^a \in \mathbb{R}^{4|2}$ modulo the ideal $\vec{X}^2 = R^2$.

These coordinates transform in the vector representation of SO(4) and the defining representation of SP(2), respectively. Moreover, these transformations leave the constraint invariant. In addition, one can consider transformations which mix bosons and fermions. The resulting supergroup of isometries is OSP(4|2). Since the stabilizer of an arbitrary point on $S^{3|2}$ is OSP(3|2), this confirms that the supersphere possesses a representation as a supercoset OSP(4|2)/OSP(3|2) (cf. Table 2).

In order to determine the character for the OSP(4|2)-module $\mathcal{F}(S^{3|2})$, we proceed as follows. We first identify the Cartan subalgebra of OSP(4|2) with the Cartan subalgebra of su(2) × su(2) × su(2), where we use the identification so(4) ≅ su(2) × su(2) and sp(2) ≅ su(2). We then choose linear combinations of the six coordinates such that we can assign the weights $(\epsilon, \eta, 0)$ and $(0, 0, \epsilon)$ to them, with $\epsilon, \eta = \pm 1$. In addition we introduce a quantum number for the polynomial grading. Each of the six coordinates X^a then contributes a term $z_1^{m_1} z_2^{m_2} z_3^{m_3} t$ to the character, where $m_i \in \{0, \pm 1\}$ have to be chosen according to the respective weights and t keeps track of the polynomial grade. For a product of coordinates, these individual contributions have to be multiplied with each other since the quantum numbers and the polynomial degree add up.

After these preparations, it is very simple to write down the character of all polynomial functions in the coordinates X^a. Dividing out the ideal $\vec{X}^2 = R^2$ is taken into account by multiplying the previous character with $1 - t^2$ since the constraint relates singlets at polynomial degree n to singlets at polynomial degree $n - 2$. At the end, we take the limit $t \to 1$ since the grade is not a good quantum number once we impose the constraint. The total character is thus given by

$$Z_{\mathcal{F}(S^{3|2})}(z_1, z_2, z_3) = \lim_{t\to 1} \frac{(1-t^2)(1+tz_3)(1+t/z_3)}{(1-tz_1z_2)(1-tz_1/z_2)(1-tz_2/z_1)(1-t/z_1z_2)}. \tag{17}$$

This expression can be expanded and represented as a linear combination of characters of OSP(4|2). Without going into the details we just write down the result

$$Z_{\mathcal{F}(S^{3|2})}(z_1, z_2, z_3) = \chi_{[0,0,0]}(z_1, z_2, z_3) + \sum_{k=0}^{\infty} \chi_{[1/2,k/2,k/2]}(z_1, z_2, z_3), \tag{18}$$

where the labels $[j_1, j_2, j_3]$ refer to simple modules (see [10]). Since $[0, 0, 0]$ denotes the trivial representation and all the other labels correspond to typical representations, this character decomposition at the same time yields the result for the harmonic analysis on $S^{3|2}$. The considerations of this section will be used in Section 5.3 when we discuss the spectrum of freely moving open strings on $S^{3|2}$.

4. Supergroup WZW models and their deformations

A special class of conformal superspace σ-models are WZW models. They are distinguished by the existence of an infinite dimensional affine Kac–Moody superalgebra symmetry. We review supergroup WZW models and discuss two deformations which are special for supergroups with vanishing Killing form.

4.1. WZW models

Let us fix a supergroup G and a non-degenerate invariant bilinear form $\langle\cdot,\cdot\rangle$. We assume the supergroup to be simple and simply connected and the invariant form to be normalized in the standard way (see below). The supergroup WZW model is

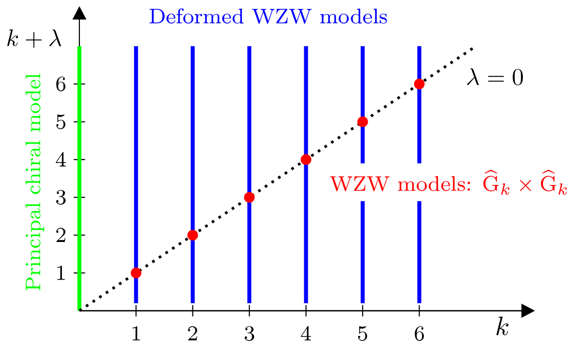

Fig. 5. The moduli space of G × G preserving deformations of WZW models on supergroups with vanishing Killing form. The WZW models and the principal chiral models are associated with the special loci $\lambda = 0$ and $k = 0$, respectively.

a two-dimensional σ-model describing the propagation of strings on G. The action functional is given by

$$\mathcal{S}^{\mathrm{WZW}}[g] = -\frac{k}{2\pi}\int_\Sigma \mathrm{d}^2 z \langle g^{-1}\partial g, g^{-1}\bar{\partial} g\rangle - \frac{\mathrm{i}k}{24\pi}\int_B \langle g^{-1}\mathrm{d}g, [g^{-1}\mathrm{d}g, g^{-1}\mathrm{d}g]\rangle, \tag{19}$$

where Σ is a closed Riemann surface and B is a three-dimensional extension of this surface such that $\partial B = \Sigma$. The form $\langle\cdot,\cdot\rangle$ is supposed to be normalized such that the topological Wess–Zumino term is well-defined up to multiples of $2\pi\mathrm{i}$ as long as k is an integer. The level k is thus the only parameter of the model.[7]

By construction, every WZW model has a global symmetry G × G corresponding to multiplying the field $g(z,\bar{z})$ by arbitrary group elements from the left and from the right. In fact, this symmetry is elevated to an affine Kac–Moody superalgebra symmetry $\widehat{G}_k$

$$J^a(z)J^b(w) = \frac{k\kappa^{ab}}{(z-w)^2} + \frac{\mathrm{i}f^{ab}{}_c J^c(w)}{z-w} + \text{non-singular} \tag{20}$$

and a corresponding anti-holomorphic symmetry if one allows these elements to depend holomorphically and antiholomorphically on z, respectively. In the last formula, the currents are defined by $J = -k\partial g g^{-1}$ and $\bar{J} = kg^{-1}\bar{\partial}g$. The equations of motion guarantee that they are holomorphic and antiholomorphic, respectively.

4.2. G × G *preserving deformations*

WZW models possess an obvious deformation which amounts to allowing for a different normalization of the two contributions to the action (19). Since the coefficient of the topological term is quantized (at least for supergroups having a compact simple part), the only freedom is in fact to change the coefficient of the kinetic term. In most cases, especially for all simple bosonic groups, such a deformation would spoil conformal invariance. The only exception occurs if G is a supergroup with vanishing Killing form as we will now review.

The implementation of the G × G-preserving deformation in the operator language is somewhat cumbersome. We have to find a field $\Phi(z,\bar{z})$ that is invariant under G × G and possesses conformal dimension $(h,\bar{h}) = (1,1)$. The deformed model would then formally be described by the deformed action $\mathcal{S}_{\mathrm{def}} = \mathcal{S}^{\mathrm{WZW}} + \lambda/2\pi\int \mathrm{d}^2 z\Phi(z,\bar{z})$. The two currents J and $\bar{J}$ transform non-trivially under either the left or the right action of G on itself. For this reason, an invariant can only be built by conjugating one of the two currents. In algebraic terms, we have to consider the normal ordered operator

$$\Phi_1(z,\bar{z}) = :\big\langle J(z), \mathrm{Ad}_g\big(\bar{J}(\bar{z})\big)\big\rangle: = :\ J^a\phi_{ab}\bar{J}^b\ :(z,\bar{z}) \tag{21}$$

involving the non-chiral affine primary field $\phi_{ab}(z,\bar{z})$ which transforms in the representation ad × ad with respect to the symmetry G × G. At lowest order in perturbation theory, the field $\Phi_1(z,\bar{z})$ is marginal if and only if the field ϕ_{ab} has conformal dimensions $(h,\bar{h}) = (0,0)$. Since the conformal dimensions of the affine primary field $\phi_{\mu\nu}$ are proportional to the quadratic Casimir in the adjoint representation, this condition is equivalent to the vanishing of the Killing form.

A sketch of the moduli spaces for conformal supergroup σ-models and the symmetries at each individual point can be found in Fig. 5. In case G is bosonic (but not abelian) or its Killing form is not vanishing, conformal invariance only holds for WZW models. The moduli space of G × G symmetric models is discrete in that case. WZW models possess a special kind of D-branes preserving the diagonal G symmetry. The simplest of these is a point-like D-brane which is localized at the identity element of G. The spectrum of anomalous dimensions on such a D-brane has been determined in [11],

$$\delta h_\Lambda = -\frac{k\lambda}{1+k\lambda}\frac{C_\Lambda}{k}. \tag{22}$$

7 For simple Lie supergroups, all invariant forms are unique up to rescaling.

Here the label refers to a field which transforms in the representation Λ with respect to the global G-symmetry and C_Λ is the corresponding eigenvalue of the quadratic Casimir.

The analysis above is directly relevant for the study of strings on $\mathrm{AdS}_3 \times S^3$ [12]. The coefficients λ and k are in one-to-one correspondence to the number of RR and NS fluxes in the background. Our formula (22) provides the first exact string spectrum on $\mathrm{AdS}_3 \times S^3$ with non-trivial RR flux.

4.3. G *preserving deformations*

Supergroups with vanishing Killing form admit a second type of deformation. This deformation is easier to describe since it avoids the use of the cumbersome field $\phi_{\mu\nu}$. On the other hand, it is more difficult to interpret geometrically since the group structure is broken. In this case, the deformation is implemented by the perturbing field

$$\Phi_2(z,\bar{z}) = \left\langle J(z), \bar{J}(\bar{z})\right\rangle = J_a\bar{J}^a(z,\bar{z}). \tag{23}$$

This field transforms non-trivially under the full G $\times$ G-symmetry but trivially under the diagonal subgroup G. As a consequence, the original global G $\times$ G-symmetry is broken to the diagonal G-symmetry as soon as the deformation is switched on. In the present case, it is less obvious than for $\Phi_1(z,\bar{z})$ but again we need to impose the vanishing of the Killing form since otherwise the field $\Phi_2(z,\bar{z})$ would break conformal invariance at higher orders in perturbation theory.

The deformed model admits *D*-branes which preserve the full global G-symmetry. They are obtained from symmetry preserving *D*-branes in the WZW model that preserve one copy of the current algebra $\widehat{\mathrm{G}}_k$. For open strings ending on such *D*-branes, the anomalous dimensions can be determined explicitly using conformal perturbation theory [10]. We postpone the explanation to Section 6 and just state the result

$$\delta h_\Lambda = -\frac{k\lambda}{1+k\lambda}\frac{C_\Lambda}{k}. \tag{24}$$

The agreement of this expression with (22) is no coincidence since both have their origin in the same kind of combinatorics. Again, C_Λ refers to the eigenvalue of the quadratic Casimir acting on the representation Λ.

5. A duality between Gross–Neveu models and supersphere σ-models

In this section, we present arguments which support a conjectured duality between $\mathrm{OSP}(2S+2|2S)$ Gross–Neveu models and σ-models on the superspheres $S^{2S+1|2S}$. We first point out the formulation of the Gross–Neveu model as a deformed WZW model and how boundary spectra can be calculated along the lines of Section 4. For simplicity, we restrict ourselves to the case of $S = 1$.

5.1. *The* $\mathrm{OSP}(2S+2|2S)$ *Gross–Neveu model as a deformed WZW model*

The results of the previous section may be used to present arguments in favor of a duality between non-linear σ-models on superspheres $S^{2S+1|2S}$ and $\mathrm{OSP}(2S+2|2S)$ Gross–Neveu models [10]. In the case $S = 0$, this duality reduces to the well-known correspondence between the massless Thirring model (also known as Luttinger liquid in the condensed matter community) and the free compactified boson. All cases $S \geq 1$ can be thought of as non-abelian generalizations of this equivalence.

The $\mathrm{OSP}(4|2)$ Gross–Neveu model is a non-geometric theory defined by the following Lagrangian:

$$\mathcal{S}^{\mathrm{GN}}[\Psi] = \frac{1}{2\pi}\int \mathrm{d}^2z\Big[\langle\Psi,\bar{\partial}\Psi\rangle + \langle\bar{\Psi},\partial\bar{\Psi}\rangle + g^2\langle\Psi,\bar{\Psi}\rangle^2\Big]. \tag{25}$$

Here, $\Psi = (\psi_1,\ldots,\psi_4,\beta,\gamma)$ is a fundamental $\mathrm{OSP}(4|2)$ multiplet with four fermions and two bosons, all having conformal dimension $h = 1/2$. The theory has a single coupling constant g which determines the strength of the quadratic potential. Instead of working with the $\mathrm{OSP}(4|2)$ Gross–Neveu model, it is convenient to use the components of the free fields Ψ^i to construct the currents of an $\widehat{\mathrm{OSP}}(4|2)$ affine Lie superalgebra at level $k = 1$. Using these currents, the previous Lagrangian can be written as a deformed WZW model:

$$\mathcal{S}^{\mathrm{GN}} = \mathcal{S}^{\mathrm{WZW}} + g^2\mathcal{S}_{\mathrm{def}} \quad \text{with } \mathcal{S}_{\mathrm{def}} = \frac{1}{2\pi}\int \mathrm{d}^2z\big\langle J, \omega(\bar{J})\big\rangle, \tag{26}$$

ω being induced from the exchange automorphism of $\widehat{\mathrm{SU}}(2)_1\times\widehat{\mathrm{SU}}(2)_1$. This kind of deformation is covered by our discussion in Section 4.3.

The solution of the $\mathrm{OSP}(4|2)$ WZW model is relatively straightforward, since it can simply be formulated as an orbifold

$$\widehat{\mathrm{OSP}}(4|2)_1 = \Big(\widehat{\mathrm{SU}}(2)_{-\frac{1}{2}}\times\widehat{\mathrm{SU}}(2)_1\times\widehat{\mathrm{SU}}(2)_1\Big)\Big/\mathbb{Z}_2 \tag{27}$$

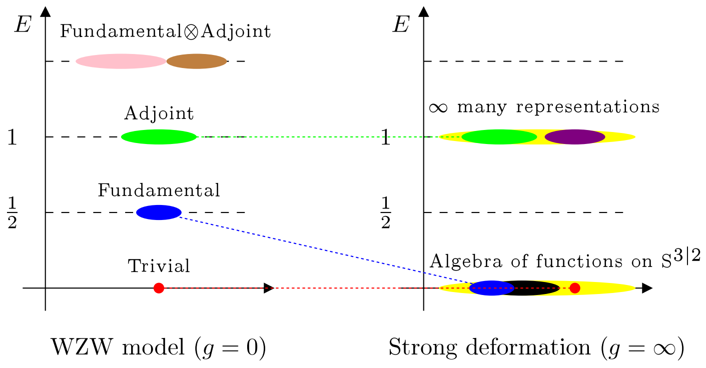

Fig. 6. A distinguished boundary spectrum of the OSP(4|2) Gross–Neveu model at zero and infinite coupling. The interpolation between these two spectra for other values of the coupling g is described by formula (29).

of purely bosonic WZW models.[8] The two copies of $\mathrm{SU}(2)_1$ arise from the two pairs of fermions, while the $\widehat{\mathrm{SU}}(2)_{-\frac{1}{2}}$ arises from the bosonic $\beta\gamma$ system. Both theories are well-understood, even though the $\beta\gamma$ system exhibits some rather unexpected features [13]. It is worth noting that despite the subtleties, the WZW model on OSP(4|2) at level $k = 1$ is not a logarithmic CFT, in contrast to WZW models at higher levels [8,9]. This is due to the realization in terms of free fields.[9]

The D-brane we wish to study corresponds to trivial gluing conditions in the $\widehat{\mathrm{SU}}(2)_{-\frac{1}{2}}$ part and permutation gluing conditions in the $\widehat{\mathrm{SU}}(2)_1 \times \widehat{\mathrm{SU}}(2)_1$ part. Its spectrum can easily be determined to be [10]

$$Z_{\mathrm{GN}}(q, z|0) = \frac{\eta(q)}{\theta_4(z_1)}\left[\frac{\theta_2(q^2, z_2^2)\theta_2(q^2, z_3^2)}{\eta(q)^2} + \frac{\theta_3(q^2, z_2^2)\theta_3(q^2, z_3^2)}{\eta(q)^2}\right]. \tag{28}$$

It is just the sum of the affine $\widehat{\mathrm{OSP}}(4|2)_1$ characters based on the trivial and the fundamental representation of OSP(4|2).

5.2. Deformed boundary spectrum

Once the WZW model has been solved, it is straightforward to determine the deformed boundary spectrum using the results of Section 4.3. According to formula (24), the anomalous dimensions of a boundary field only depend on the transformation properties with respect to the global OSP(4|2)-symmetry. Taking care of the proper normalizations, the partition function then reads[10]

$$Z_{\mathrm{GN}}(q, z|g^2) = \sum_{\Lambda} q^{-\frac{1}{2}\frac{g^2}{1+g^2}C_\Lambda}\, \psi_\Lambda^{\mathrm{WZW}}(q)\chi_\Lambda(z), \tag{29}$$

where $\psi_\Lambda^{\mathrm{WZW}}(q)$ denotes the branching functions at zero coupling. Under the present circumstances, the decomposition of the WZW spectrum (28) into representations of OSP(4|2) can actually be performed explicitly [10], resulting in

$$\psi_{[j_1,j_2,j_3]}^{\mathrm{WZW}}(q) = \frac{1}{\eta(q)^4}\sum_{n,m=0}^{\infty}(-1)^{n+m}q^{\frac{m}{2}(m+4j_1+2n+1)+j_1+\frac{n}{2}-\frac{1}{8}}\left[q^{(j_2-\frac{n}{2})^2} - q^{(j_2+\frac{n}{2}+1)^2}\right]\left[q^{(j_3-\frac{n}{2})^2} - q^{(j_3+\frac{n}{2}+1)^2}\right]. \tag{30}$$

We recognize from Eq. (29) that the deformed branching functions have a very simple dependence on the coupling g.

Let us discuss the consequences of formula (29) in more detail, see also Fig. 6. At zero coupling, the spectrum is characterized by the following features: all states have either integer or half-integer energy and at each energy level there is only a finite number of states. As mentioned above, these states are accounted for by the two affine $\widehat{\mathrm{OSP}}(4|2)_1$ representations built on top of the vacuum (with $h = 0$) and the fundamental representation (with $h = 1/2$), respectively. Once the deformation is switched on, the affine symmetry is broken and the states will receive an anomalous dimension depending on their transformation behavior under global OSP(4|2) transformations (the zero-modes of the current algebra). In particular, multiplets belonging to a representation with vanishing Casimir do not receive any correction. These are all

8 The orbifold corresponds to the action of the simple current (1/2, 1/2, 1/2). It implements a target space GSO projection.

9 However, in close analogy to [14], it is possible to construct a logarithmic lift of this theory by including fermionic zero-modes.

10 We use a different normalization of the Casimir here.

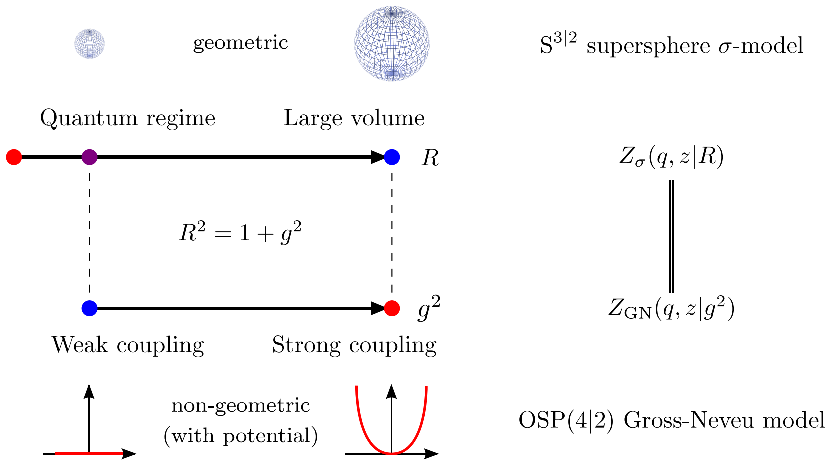

Fig. 7. The duality between the OSP(4|2) Gross–Neveu model and the $S^{3|2}$ supersphere σ-model in pictures.

protected BPS representations.[11] This applies in particular to the adjoint representation and ensures that the currents stay at conformal dimension $h = 1$.

At intermediate coupling, the spectrum is very complicated, exhibiting almost no sign of an underlying organizing principle, except for the preserved global G and the Virasoro symmetry. However, at infinite coupling we again recover a special situation. The energy of a multiplet Λ is shifted by $-C_\Lambda/2$ in this case. It can be shown that despite this shift all conformal dimensions remain non-negative. Even more surprising, the spectrum is very regular again, exhibiting an integer level spacing (as opposed to the half-integer spacing at $g = 0$). Nevertheless, the spectrum now has entirely different characteristics than at zero coupling. Indeed, at infinite coupling we find an infinite number of states on each energy level, see again Fig. 6.

5.3. *Identification with the large volume spectrum of a supersphere σ-model*

We now wish to argue that the spectrum of the Gross–Neveu model discussed in the previous section coincides with the large volume partition function of the σ-model on the supersphere $S^{3|2}$ when we send the coupling g to infinity. At infinite volume the partition function is easy to write down since the fields $\vec{X}$ become free. The most general field is obtained by considering the normal ordered products $\prod \partial^{n_i}\bar{\partial}^{m_i}X^{a_i}$ of the fields X^a and their derivatives and the energy (scaling dimension) of such a field is just given by the number of derivatives.

We assume Neumann boundary conditions, i.e. a freely moving open string. In this case, we are only left with one type of derivatives. In close analogy to the harmonic analysis on $S^{3|2}$, we can write down the open string partition function

$$Z_{S^{3|2}}(q, z|R = \infty) = \lim_{t\to 1}\prod_{n=1}^{\infty}\frac{(1 - t^2q^n)(1 + tq^nz_3)(1 + tq^n/z_3)}{(1 - tq^nz_1z_2)(1 - tq^nz_1/z_2)(1 - tq^nz_2/z_1)(1 - tq^n/z_1z_2)}. \tag{31}$$

The only difference compared to Eq. (17) are the additional terms involving powers q^n. These correspond to counting derivatives $\partial^n\vec{X}$ instead of plain coordinates $\vec{X}$. Since the constraint $\vec{X}^2 = R^2$ also leads to constraints on derivatives of $\vec{X}$ also the first term in the numerator had to be extended to an infinite product.

Even though it is by no means obvious, the decomposition of the partition function (31) into irreducible characters of OSP(4|2) precisely agrees with the limit $g \to \infty$ of the expression (29) [10]. This suggests that the moduli spaces of the two theories indeed overlap – and employing their common symmetry – actually coincide, see Fig. 7 for an illustration. Complementary calculations based on either lattice models [15], background field methods or a cohomological reduction [5] confirm this picture and predict that the couplings should in fact be related as $R^2 = 1 + g^2$.

6. Quasi-abelian perturbation theory

In this section, we will explain the origin of the formulas (23) and (29) for the distinguished boundary spectra of deformed WZW models. For simplicity, we restrict our presentation to the case of the G-preserving deformation discussed in Section 4.3. The G × G preserving deformation can be discussed along similar lines but the analysis is slightly more complicated due to the presence of the additional field $\phi_{\mu\nu}(z, \bar{z})$ [11]. Our presentation here will be based on a simple analogy with the free boson.

[11] It should be noted, however, that there are short/BPS representations for OSP(4|2) which are not protected in this sense.

6.1. Anomalous dimensions

In order to determine the anomalous dimensions of boundary operators $\psi_\mu(x)$, we need to calculate the two-point correlation functions

$$\left\langle \psi_\mu(x)\psi_\nu(y)\right\rangle_\lambda = \frac{C_{\mu\nu}(\lambda)}{(x-y)^{\Delta_\mu(\lambda)+\Delta_\nu(\lambda)}}. \tag{32}$$

It should be emphasized that $\Delta(\lambda)$ will in general have a diagonal contribution $h(\lambda)$ and a nilpotent part $\delta(\lambda)$. The two-point function is determined perturbatively through the formula

$$\begin{aligned}\left\langle \psi_\mu(x)\psi_\nu(y)\right\rangle_\lambda &= \left\langle \psi_\mu(x)\psi_\nu(y)\mathrm{e}^{-\lambda \mathcal{S}_{\mathrm{def}}}\right\rangle_{\mathrm{WZW}} \\ &= \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\left\langle \psi_\mu(x)\psi_\nu(w)\prod_{i=1}^{n}\int \mathrm{d}^2 w_i \langle J(w_i),\bar{J}(\bar{w}_i)\rangle\right\rangle_{\mathrm{WZW}}.\end{aligned} \tag{33}$$

As usual, the expression in the second line needs regularization and renormalization to be well defined. We wish to avoid entering the technical details here but rather to make an analogy with the free boson.

In general, it is impossible to make precise statements about correlations function beyond a few orders in perturbation theory. The same applies here with regard to the structure functions $C_{\mu\nu}(\lambda)$ and the nilpotent part of $\Delta(\lambda)$. On the other hand, the diagonal part $h(\lambda)$ of $\Delta(\lambda)$ is accessible since it is a G-invariant. In other words, for the calculation of $h(\lambda)$ all terms involving structure constants can be dropped, following the reasoning of Section 2.2 (cf. (20) with (4)). The combinatorics (and also the regularization and renormalization) hence effectively reduce to those of the radius perturbation of a multi-component free boson. For the latter, the conformal dimensions are known explicitly for all values of the radius even without making use of perturbation theory. We are thus in the comfortable position to avoid any tedious combinatorics.

Let us now fix the self-dual radius R_0 as a reference radius. The free boson at all other radii $R = R_0\sqrt{1+\lambda}$ can be considered as a current–current deformation of the type $\langle J,\bar{J}\rangle$ of the reference theory. Looking at the concrete expression (5) for the partition function, we find the anomalous dimension of a field with momentum quantum number w,

$$\delta h_w(\lambda) = \left[\frac{1}{1+\lambda}-1\right]\frac{w^2}{2R_0^2} = -\frac{\lambda}{1+\lambda}\frac{w^2}{2R_0^2}. \tag{34}$$

The last term should be interpreted as the operator J_0^2 acting on a boundary field. In the WZW context, this would give rise to the quadratic Casimir C_Λ. Taking into account the different normalization of the currents and the Lagrangians we finally end up with Eqs. (24) and (29). A more detailed discussion of these issues can be found in [10,11].

7. Conclusions and outlook

We presented a few of the main peculiarities of conformal σ-models on supergroups G and supercosets G/H. In these models, conformal invariance is closely tied to the condition that G has vanishing Killing form, rendering the model quasi-abelian. We discussed the harmonic analysis on G and G/H in order to get a hold on the large volume regime of the σ-models. In particular, the emergence of non-chiral indecomposable representations was pointed out as a consequence of supergeometry. Afterwards we introduced supergroup WZW and their deformations. As a concrete example, we presented the OSP(4|2) Gross–Neveu model. Based on an exact perturbative expression for a boundary spectrum, we finally argued for a duality between the non-geometric Gross–Neveu model and the $S^{3|2}$ supersphere σ-model. Throughout the paper we used analogies with the free boson to motivate observations and part of the formulas.

The content of this review can be extended into several directions. A more complicated application of supercoset σ-models concerns projective superspaces (cf. Table 2) which, in contrast to superspheres, admit a topological term. In that case an additional idea, namely cohomological reduction to a symplectic fermion theory, has to be employed in order to derive exact boundary partition functions [5,16]. A different direction concerns the study of the current algebras associated with deformed WZW models [17–19]. Away from the WZW point, there is still a current algebra but it becomes non-chiral and current three-point functions start exhibiting logarithmic singularities. One may assume that a better understanding of the interplay between this local current algebra and additional non-local conserved charges will provide the key to an exact solution of the models beyond the results that have been presented here.

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