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Symplectic structures on the integration of exact Courant algebroids

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ABSTRACT

We construct an infinite-dimensional symplectic 2-groupoid as the integration of an exact Courant algebroid. We show that every integrable Dirac structure integrates to a "Lagrangian" sub-2-groupoid of this symplectic 2-groupoid. As a corollary, we recover a result of Bursztyn–Crainic–Weinstein–Zhu that every integrable Dirac structure integrates to a presymplectic groupoid.

1. Introduction

In the late 80s, T. Courant and A. Weinstein [1] introduced the notion of *Dirac structure* as a way of unifying Poisson, symplectic, and presymplectic structures. An important ingredient in the definition of Dirac structure is a bracket, now called the *Courant bracket*, defined on the direct sum of the spaces of vector fields and 1-forms on a manifold. In the 90s, Z. Liu, Weinstein, and P. Xu [2] formalized the properties of the Courant bracket in the definition of a *Courant algebroid*.

A *Dirac structure* is a subbundle of a Courant algebroid that is maximally isotropic and whose sections are closed under the Courant bracket. The restriction of a Courant bracket to the sections of a Dirac structure is a Lie bracket, making the Dirac structure into a Lie algebroid. In the case of the standard Courant algebroid $TM \oplus T^*M$ (as well as its twisted versions), H. Bursztyn, M. Crainic, Weinstein, and C. Zhu [3] showed that, if a Dirac structure is integrable in the sense of [4], then the Lie groupoid integrating it carries a natural closed (or H-closed, in the twisted case), multiplicative 2-form, making it a *presymplectic groupoid*.

In this article, we study the integration problem for Courant algebroids, which was one of open problems raised by Z. Liu, A. Weinstein, and P. Xu [2]:

> "**Open Problem 5.** *What is the global, groupoid-like object corresponding to a Courant algebroid? In particular, what is the double of a Poisson groupoid?*"

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In [5], generalizing the idea of rational homotopy theory in the language of NQ-manifolds, P. Ševera outlined a formal proof that the solution to this problem is a *symplectic* 2-*groupoid*. A key missing piece in [5] is a proof of the smoothness of the involved constructions. Recently, the authors [6], D. Li-Bland and P. Ševera [7], and Y. Sheng and C. Zhu [8] independently constructed (local) Lie 2-groupoids integrating certain subclasses of Courant algebroids, with the standard Courant algebroid being the common element of all three subclasses. For these constructions, the term "integration" can be justified by showing that the Courant algebroid structure can be recovered via Ševera's 1-jet construction [9].

An important question that has remained unaddressed is how a symplectic 2-groupoid integrating a Courant algebroid is related to the presymplectic groupoids integrating the Dirac structures that sit inside the Courant algebroid. In fact, one can easily find examples showing that the symplectic 2-groupoids of [6,7], and [8] are not large enough to contain all the presymplectic groupoids arising from Dirac structures. In this article, we show that this problem can be resolved, at the cost of working with infinite-dimensional manifolds.

Inspired by the works of A. Cattaneo and G. Felder [10] and M. Crainic and R. L. Fernandes [4] on integration of Poisson manifolds and Lie algebroids, we construct, for any manifold M, an infinite-dimensional Lie 2-groupoid, i.e. a Kan simplicial (Banach) manifold $\{X_\bullet\}$ for which the horn fillings are unique in degrees greater than 2. For any closed $H \in \Omega^3(M)$, we obtain a natural multiplicative symplectic 2-form ω_2^H on X_2, making $\{X_\bullet\}$ into a symplectic 2-groupoid, which we call the *Liu–Weinstein–Xu* 2-*groupoid*, or $\mathrm{LWX}(M)$ for short.

A brief description of $\mathrm{LWX}(M)$ in low degrees is as follows. The space of "0-simplices" is $\mathrm{LWX}_0(M) = M$. The space $\mathrm{LWX}_1(M)$ of "1-simplices" consists of bundle maps from the tangent bundle of the standard 1-simplex to T^*M. An element of the space $\mathrm{LWX}_2(M)$ of "2-simplices" is given by a quadruplet $([f], \psi_0, \psi_1, \psi_2)$, where $[f]$ is an equivalence class of maps from the standard 2-simplex to M, modulo boundary-fixing homotopies, and where each ψ_i is an element of $\mathrm{LWX}_1(M)$ whose base map is the ith edge of f. This construction can be seen as a concrete implementation of the proposal by Ševera [5] to integrate Courant algebroids via equivalence classes of maps. In order to endow $\mathrm{LWX}(M)$ with a smooth structure, we require the maps to have certain fixed orders of differentiability; the details are in Section 3.

Our most significant results arise from the observation that $\mathrm{LWX}(M)$ has a natural 2-form ω_1^H on $\mathrm{LWX}_1(M)$ for which

$$d\omega_1^H = \delta H, \qquad \delta\omega_1^H = \omega_2^H, \tag{1.1}$$

where $\delta : \Omega^\bullet(\mathrm{LWX}_k(M)) \to \Omega^\bullet(\mathrm{LWX}_{k+1}(M))$ is the simplicial coboundary map. In other words, ω_2^H is the coboundary of $\omega_1^H + H$ in the Bott–Shulman–Stasheff complex of $\mathrm{LWX}(M)$. The existence of ω_1^H seems to be specific to the case of exact Courant algebroids, so it does not appear in the general construction of [5].

We associate to any integrable Dirac structure a sub-2-groupoid of $\mathrm{LWX}(M)$ whose 1-truncation can be identified with the Lie groupoid integrating the Dirac structure. We prove that the pullback of ω_2^H vanishes on this sub-2-groupoid; as a result, we can deduce that the pullback of ω_1^H descends to the 1-truncation, inducing an H-closed, multiplicative 2-form on the Lie groupoid integrating the Dirac structure. This 2-form precisely coincides with the one constructed by H. Bursztyn, M. Crainic, A. Weinstein, and C. Zhu [3]. We can thus view the Liu–Weinstein–Xu 2-groupoid as being the geometric origin of presymplectic groupoids.

We prove that the sub-2-groupoid associated to a Dirac structure is in fact Lagrangian at the "units" of the 2-groupoid. We conjecture that it is Lagrangian everywhere, and we prove the conjecture in a special case. We believe that the Lagrangian property is the origin of the nondegeneracy condition in [3] and therefore deserves further study.

Another issue that we do not address here is that of the relationship between $\mathrm{LWX}(M)$ and the finite-dimensional symplectic 2-groupoids of [6–8]. There are natural maps between these different symplectic 2-groupoids, e.g. [7, Remark 7]. Clearly, there should be a notion of equivalence between symplectic 2-groupoids (see [11] for discussion about Morita equivalence of symplectic groupoids) but the precise nature of the equivalence remains an open question.

Organization of the paper. In Section 2, we briefly review some background material on Courant algebroids and simplicial manifolds. In Section 3, we construct an infinite-dimensional simplicial manifold $\{\mathfrak{C}_\bullet(M)\}$ associated to any manifold M. We show that $\{\mathfrak{C}_\bullet(M)\}$ can be truncated to an infinite-dimensional Lie 2-groupoid, which we denote $\mathrm{LWX}(M)$. In Section 4, we construct canonical symplectic forms ω_i on $\mathrm{LWX}_i(M)$ for $i = 1, 2$, as well as twisted versions ω_i^H associated to any closed 3-form H on M. In particular, we show that the relations (1.1) are satisfied. In Section 5, we construct a simplicial manifold $\{\mathfrak{G}_\bullet(\mathcal{D})\}$ associated to any Dirac structure $\mathcal{D}$, whose 1-truncation is the Lie groupoid G integrating the Dirac structure. There is a natural inclusion map $\mathfrak{G}_\bullet(\mathcal{D}) \hookrightarrow \mathfrak{C}_\bullet(M)$, and we show that the pullback of ω_2 vanishes, implying that ω_1 induces a presymplectic structure on G. Finally, in Section 6, we show that the image of $\mathfrak{G}_2(\mathcal{D})$ in $\mathrm{LWX}(M)$ is Lagrangian at the units and conjecture that it is Lagrangian everywhere.

2. Background

In this section, we briefly review the definitions and constructions about Courant algebroids and simplicial manifolds.

2.1. Courant algebroids

We begin with the definition of Courant algebroid.

Definition 2.1. A *Courant algebroid* is a vector bundle $E \to M$ equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a bundle map $\rho : E \to TM$ (called the *anchor*), and a bracket $[\![\cdot, \cdot]\!]$ (called the *Courant bracket*) on $\Gamma(E)$ such that

(1) $[\![e_1, fe_2]\!] = \rho(e_1)(f)e_2 + f[\![e_1, e_2]\!]$,
(2) $\rho(e_1)(\langle e_2, e_3\rangle) = \langle [\![e_1, e_2]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle$,
(3) $[\![[\![e_1, e_2]\!], e_3]\!] = [\![e_1, [\![e_2, e_3]\!]]\!] - [\![e_2, [\![e_1, e_3]\!]]\!]$,
(4) $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathcal{D}\langle e_1, e_2\rangle$,

for all $f \in C^\infty(M)$ and $e_i \in \Gamma(E)$, where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined by

$$\langle \mathcal{D}f, e\rangle = \rho(e)(f).$$

Note that, in Definition 2.1, the bracket is generally not skew-symmetric, but a Jacobi identity (condition (3)) holds. This definition of Courant algebroid appeared in [2], and it was shown to be equivalent to the original definition of Liu–Weinstein–Xu in [12] (see also [13] and [14]).

For any manifold M, the *standard Courant algebroid* over M is the bundle $E = TM \oplus T^*M$, where ρ is the projection onto the TM component, the bilinear form is given by

$$\langle X_1 + \xi_1, X_2 + \xi_2\rangle = \xi_2(X_1) + \xi_1(X_2),$$

and the bracket is given by

$$[\![X_1 + \xi_1, X_2 + \xi_2]\!] = [X_1, X_2] + L_{X_1}\xi_2 - \iota_{X_2} d\xi_1$$

for $X_i \in \mathfrak{X}(M)$, $\xi_i \in \Omega^1(M)$. Given a closed 3-form H on M, one can define the H-twisted Courant bracket by

$$[\![X_1 + \xi_1, X_2 + \xi_2]\!]_H = [X_1, X_2] + L_{X_1}\xi_2 - \iota_{X_2} d\xi_1 + \iota_{X_1}\iota_{X_2} H.$$

Definition 2.2. A Courant algebroid $E \to M$ is called *exact* if the sequence

$$0 \to T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \to 0$$

is exact.

Ševera [15] showed that every exact Courant algebroid is isomorphic to an H-twisted version of $TM \oplus T^*M$ for some closed $H \in \Omega^3(M)$.

Dirac structures in $TM \oplus T^*M$ were introduced by Courant [16] as a bridge between Poisson geometry and presymplectic geometry. The twisted versions were studied in [14].

Definition 2.3. An (H-twisted) *Dirac structure* on M is a maximally isotropic subbundle $D \subset TM \oplus T^*M$ such that $\Gamma(D)$ is closed under the (H-twisted) Courant bracket.

2.2. Simplicial manifolds

Definition 2.4. A *simplicial manifold* is a sequence $\{X_\bullet\} = \{X_m : m \geq 0\}$, of manifolds equipped with surjective submersions $d_i^m : X_m \to X_{m-1}$ (called *face maps*), $i = 0, \ldots, m$, and embeddings $s_i^m : X_m \to X_{m+1}$ (called *degeneracy maps*), $i = 0, \ldots, m$, such that

$$d_i^{m-1} d_j^m = d_{j-1}^{m-1} d_i^m, \quad i < j,$$
$$s_i^{m+1} s_j^m = s_{j+1}^{m+1} s_i^m, \quad i < j,$$
$$d_i^{m+1} s_j^m = \begin{cases} s_{j-1}^{m-1} d_i^m, & i < j, \\ id, & i = j,\ i = j+1, \\ s_j^{m-1} f_{i-1}^m, & i > j+1. \end{cases}$$

When the domain of a face or degeneracy map is clear, we will generally omit the superscript index.

For $m \geq 1$ and $0 \leq \ell \leq m$, an (m, ℓ)-*horn* of X consists of an m-tuple $(x_0, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots x_m)$, where $x_i \in X_{m-1}$, such that

$$d_i x_j = d_{j-1} x_i \tag{2.1}$$

for $i < j$. The space of all (m, ℓ)-horns is denoted $\Lambda_{m,\ell}(X)$. The *horn maps* $\lambda_{m,\ell} : X_m \to \Lambda_{m,\ell}(X)$ are defined as

$$\lambda_{m,\ell}(x) = (d_0 x, \ldots, \widehat{d_k x}, \ldots d_m x)$$

for $x \in X_m$.

A simplicial manifold $\{X_\bullet\}$ is said to satisfy the *Kan condition* if all of its horn spaces $\Lambda_{m,\ell}(X)$ are smooth and all of the horn maps $\lambda_{m,\ell}$ are surjective submersions. The following is a more general notion.

Definition 2.5. A morphism $f : \{X_\bullet\} \to \{Y_\bullet\}$ of simplicial manifolds is a *Kan fibration* if all of the fiber products $\Lambda_{m,\ell}(X) \times_{\Lambda_{m,\ell}(Y)} Y$ are smooth and all of the maps $X_m \to Y \times_{\Lambda_{m,\ell}(Y)} \Lambda_{m,\ell}(X)$ given by $x \mapsto (f(x), \lambda_{m,\ell}(x))$ are surjective submersions.

Clearly, $\{X_\bullet\}$ satisfies the Kan condition if and only if the map $\{X_\bullet\} \to \{*\}$ is a Kan fibration.

The following statement is a special case of [17, Lemma 2.9]. Although that paper mainly deals with reduced simplicial manifolds, i.e. those for which $X_0 = \{*\}$, the proof given there does not rely on that assumption.

Lemma 2.6. *If $\{Y_\bullet\}$ is a Kan simplicial manifold and if $f : \{X_\bullet\} \to \{Y_\bullet\}$ is a Kan fibration, then $\{X_\bullet\}$ is Kan.*

2.3. Lie n-groupoids

Definition 2.7. A *Lie n-groupoid* is a Kan simplicial manifold whose horn maps $\lambda_{m,\ell}$ are diffeomorphisms for $m > n$.

Duskin [18] introduced a *truncation* functor $\tau_{\le n}$ which may be applied to a Kan simplicial set $\{X_\bullet\}$ to produce an n-groupoid. It is defined as follows:

- $(\tau_{\le n}X)_m = X_m$ for $m < n$;
- $(\tau_{\le n}X)_n = X_n/\sim$, where $x \sim y$ if and only if there exists $z \in X_{n+1}$ such that $d_n z = x$, $d_{n+1}z = y$, and $d_i z \in \mathrm{im}(s_{n-1})$ for $0 \le i < n$;
- $(\tau_{\le n}X)_m = X_m/\sim$ for $m > n$, where $x \sim y$ if and only if the n-skeletons of x and y are equivalent with respect to the equivalence relation on X_n.

The following result can be found in [19, Section 2].

Lemma 2.8. *If $\{X_\bullet\}$ is a Kan simplicial (Banach) manifold, then $\tau_{\le n}X$ is an n-groupoid. If, furthermore, $(\tau_{\le n}X)_n = X_n/\sim$ is a (Banach) manifold, then $\{(\tau_{\le n}X)_\bullet\}$ is a Lie n-groupoid.*

2.4. Symplectic 2-groupoids

The notion of *symplectic* 2-*groupoid* was defined in [6,7]. We somewhat imprecisely state the definition as follows:

Definition 2.9. A *symplectic* 2-*groupoid* is a Lie 2-groupoid $\{X_\bullet\}$ that is equipped with a closed, multiplicative 2-form $\omega \in \Omega^2(X_2)$, i.e. $d\omega = \delta\omega = 0$ satisfying a nondegeneracy condition.

In Definition 2.9, we have intentionally left the content of the nondegeneracy condition ambiguous. Li-Bland and Ševera [7] required the 2-form to be genuinely nondegenerate, so that (X_2, ω) is a symplectic manifold. In [6], a weaker condition was stated, where the kernel of ω is required to be controlled in a certain way by the simplicial structure. We refer the reader to [20] for ideas from shifted symplectic geometry about the nondegeneracy condition, and to [21] for more detailed discussion about the nondegeneracy condition.

In this paper, we will use the genuine nondegeneracy condition; however, since we are dealing with Banach manifolds, we only require weak nondegeneracy, which is recalled in the following definition.

Definition 2.10. Let X be a Banach manifold. A 2-form $\omega \in \Omega^2(X)$ is *weakly nondegenerate* if for every $x \in X$, the map $\omega_x^\flat : T_xX \to T_x^*X$, $v \mapsto \omega_x(v, \cdot)$, is injective, i.e. $\omega_x(v, w) = 0$ for all $w \in T_xX$ if and only if $v = 0$.

3. Construction of LWX(M)

In this section, we describe the construction of the Liu–Weinstein–Xu 2-groupoid, which involves first constructing an infinite-dimensional simplicial manifold and then truncating it to obtain a Lie 2-groupoid.

The idea of integrating infinitesimal structures by spaces of maps has a long history, going back to Lie's Third Theorem (see [22]). A general construction was given by D. Sullivan [23] in the context of rational homotopy theory. The application of this idea to the integration of symplectic NQ-manifolds was described in [5,15] and is closely related to AKSZ theories [24]. In particular, the integration of Poisson manifolds and Lie algebroids [4,10] utilizes this approach.

3.1. A simplicial manifold

Let M be a manifold. Recall that, if X is a manifold and $E \xrightarrow{\pi} M$ is a vector bundle, then a map $\phi : X \to E$ is said to be of class $C^{p,q}$ if ϕ is C^q and $\pi \circ \phi$ is C^p. Clearly, if this is the case, then it is necessary that $p \ge q$.

Fix $p \ge q$. For each integer $m \ge 0$, let $\mathfrak{C}_m(M)$ denote the set of $C^{p,q}$ bundle maps from $T\Delta^m$ to T^*M, where Δ^m is the standard m-dimensional simplex in $\mathbb{R}^m$.

Lemma 3.1. *The space $\mathfrak{C}_m(M)$ is a Banach manifold.*

Proof. Let $T^*_m M$ be the m-fold direct sum of T^*M. That is,

$$T^*_m M := \underbrace{T^*M \oplus \cdots \oplus T^*M}_{m}.$$

Then we may use the standard trivialization $T\Delta^m = \Delta^m \times \mathbb{R}^m$ to obtain a one-to-one correspondence between bundle maps $\varphi : T\Delta^m \to T^*M$ and maps $\tilde{\varphi} : \Delta^m \to T^*_m M$. Specifically, given such a φ, we obtain $\tilde{\varphi}$ by evaluating φ on the standard basis vectors of $\mathbb{R}^m$. This correspondence preserves order of differentiability, in that φ is $C^{p,q}$ if and only if $\tilde{\varphi}$ is $C^{p,q}$.

The statement immediately follows from the fact that the space of $C^{p,q}$-maps from Δ^m to $T^*_m M$ is a Banach manifold. □

There is a cosimplicial manifold structure on $\{T\Delta^\bullet\}$, obtained by applying the tangent functor to the standard cosimplicial manifold $\{\Delta^\bullet\}$. Thus there is an induced simplicial manifold structure on $\{\mathfrak{C}_\bullet(M)\}$. We note that $\mathfrak{C}_0(M) = M$, and that $\mathfrak{C}_1(M) = \{C^{p,q}$ bundle maps $T[0, 1] \to T^*M\}$ can be identified with the space of $C^{p,q}$ paths on T^*M.

Lemma 3.2. *The horn space $\Lambda_{m,\ell}(\mathfrak{C}(M))$ is smooth for all m, ℓ.*

Proof. We will prove the statement by constructing a coordinate chart on a neighborhood $\mathcal{U}$ of any element in $\Lambda_{m,\ell}(\mathfrak{C}(M))$. Since horns are contractible, we may assume without loss of generality that $\mathcal{U}$ is contained in the space of horns whose images are in $T^*U \cong T^*\mathbb{R}^n$ for some coordinate chart $U \subseteq M$. Thus, it is sufficient to show that $\Lambda_{m,\ell}(\mathfrak{C}(\mathbb{R}^n))$ can be given the structure of a Banach space.

Since $T^*_m\mathbb{R}^n$ can be identified with $\mathbb{R}^{n(m+1)}$, we can identify $\mathfrak{C}_m(\mathbb{R}^n)$ with the Banach space of $C^{p,q}$ maps from Δ^n to $\mathbb{R}^{n(m+1)}$. Under this identification, the face maps $d_i^m : \mathfrak{C}_m(\mathbb{R}^n) \to \mathfrak{C}_{m-1}(\mathbb{R}^n)$ are bounded linear maps between Banach spaces. To construct the horn space $\Lambda_{m,\ell}(\mathfrak{C}(\mathbb{R}^n))$, we first consider the Banach space

$$P_{m-1} := \underbrace{\mathfrak{C}_{m-1}(\mathbb{R}^n) \oplus \cdots \oplus \mathfrak{C}_{m-1}(\mathbb{R}^n)}_{m}$$

consisting of m-tuples of $(m-1)$-simplices. We then observe that the horn compatibility conditions (2.1) can be written in the form $d_i x_j - d_{j-1} x_i$, so that $\Lambda_{m,\ell}(\mathfrak{C}(\mathbb{R}^n))$ can be seen as the intersection of the kernels of a set of bounded linear maps and is therefore a Banach space. □

Proposition 3.3. *The simplicial manifold $\{\mathfrak{C}_\bullet(M)\}$ satisfies the Kan condition.*

Proof. For each m, let $S_m(M)$ denote the set of C^p maps from Δ^m to M. It is known (see, for example, [17, Lemma 5.7]) that $\{S_\bullet(M)\}$ is a Kan simplicial manifold. There is a natural projection map $\{\mathfrak{C}_\bullet(M)\} \to \{S_\bullet(M)\}$, so, by Lemma 2.6, it suffices to show that this map is a Kan fibration.

The fact that $\{S_\bullet(M)\}$ satisfies the Kan condition, together with Lemma 3.2, implies that the fiber product $S_m(M) \times_{\Lambda_{m,\ell}(S(M))} \Lambda_{m,\ell}(\mathfrak{C}(M))$ is smooth. It remains to show that the map $\mathfrak{C}_m(M) \to S_m(M) \times_{\Lambda_{m,\ell}(S(M))} \Lambda_{m,\ell}(\mathfrak{C}(M))$, taking $\varphi \in \mathfrak{C}_m(M)$ to $(\bar{\varphi}, \lambda_{m,\ell}(\varphi))$, where $\bar{\varphi}$ is the underlying base map of φ, is a surjective submersion for all ℓ and $m \geq 1$. We will prove this using a method similar to that used in [17, Lemma 5.7].

Let $f : \Delta^m \to M$ be a C^p map, and let $\psi_i : T\Delta^{m-1} \to T^*M$, $i \neq \ell$ be a collection of $C^{p,q}$ maps forming a horn in $\Lambda_{m,\ell}(\mathfrak{C}(M))$ that is compatible with f. For each nonempty $I \subset \{0, \ldots, m\} \smallsetminus \{\ell\}$, let $F_I \subset \Delta^m$ denote the $(m - |I|)$-dimensional subface whose vertices are $\{0, \ldots, m\} \smallsetminus I$. As a result of the horn compatibility conditions, the maps ψ_i induce well-defined $C^{p,q}$ maps $\psi_I : TF_I \to T^*M$.

For each I, let $p_I : \Delta^m \to F_I$ be the affine projection map collapsing the vertices in I onto ℓ. Fix a Riemannian metric on M. Then, for each $t \in \Delta^m$, we may use parallel transport along the image of the line from t to $p_I(t)$ to identify $T^*_{f(t)}M$ with $T^*_{f(p_I(t))}M$.

We now define a map $\varphi : T\Delta^m \to T^*M$ with base map f, given by

$$\varphi = \sum_{I \subset \{0,\ldots,m\} \smallsetminus \{\ell\}} (-1)^{|I|+1} \psi_I \circ Tp_I.$$

It is clear by construction that φ is $C^{p,q}$, and it follows from the identities satisfied by the projection and face maps that $d_i\varphi = \psi_i$ for each $i \neq \ell$. This proves surjectivity.

To show that the map is a submersion, we observe that, under a sufficiently small change in f, we can use parallel transport to accordingly change any horn filling φ, and under a change in $\{\psi_i\}$, we can apply the above construction to the difference to accordingly change φ. This process gives a local section through any $\varphi \in \mathfrak{C}_m(M)$. □

3.2. 2-groupoid truncation

By applying Lemma 2.8 to $\{\mathfrak{C}_\bullet(M)\}$ for $n = 2$, we obtain the *Liu–Weinstein–Xu* 2-*groupoid* $\mathrm{LWX}(M) := \tau_{\leq 2}\mathfrak{C}(M)$. The main result of this section is the following:

Theorem 3.4. *The quotient $\mathfrak{C}_2(M)/\sim$ is a Banach manifold, and therefore* $\mathrm{LWX}(M)$ *is a Lie* 2*-groupoid.*

Theorem 3.4 follows directly from Lemmas 3.6–3.8. In these lemmas we will assume, without loss of generality, that M is connected.

Lemma 3.5. *The map $\mathfrak{C}_1(M) \to M \times M$ given by $\psi \mapsto (d_0\psi, d_1\psi)$ is a surjective submersion.*

Proof. Surjectivity follows from the assumption that M is connected. To prove that the map is a submersion, we will describe a way to construct local sections.

Recall that $\mathfrak{C}_1(M)$ can be identified with the space of $C^{p,q}$ paths on T^*M. Let ψ be such a path. Choose a Riemannian metric on M, and, for $i = 1, 2$, let U_i be a neighborhood of $d_i\psi$ for which the exponential map is a diffeomorphism. Using the exponential map and parallel transport along the base path $\bar{\psi} : [0, 1] \to M$, we may then identify the neighborhood $U_1 \times U_2$ of $(d_0\psi, d_1\psi)$ with a neighborhood of $(0, 0)$ in $T_{d_0\psi}M \times T_{d_0\psi}M$.

For any $(v_0, v_1) \in T_{d_0\psi}M \times T_{d_0\psi}M$, we may (again using parallel transport) view $v(t) := (1 - t)v_0 + tv_1$ as a vector field along $\bar{\psi}$. By exponentiating $\bar{\psi}$ in the direction of $v(t)$ and parallel transporting the cotangent vectors of ψ, we obtain a path $\psi' \in \mathfrak{C}_1(M)$ for which $(d_0\psi', d_1\psi') = (v_0, v_1)$. This process provides a well-defined local section through ψ. □

Let $\mathfrak{B}(M)$ be defined as the space of "triangles" of paths in $\mathfrak{C}_1(M)$. More precisely, $\mathfrak{B}(M)$ is the space of triples (ψ_0, ψ_1, ψ_2), $\psi_i \in \mathfrak{C}_1(M)$, such that

$$d_0\psi_2 = d_1\psi_0, \qquad d_0\psi_1 = d_0\psi_0, \qquad d_1\psi_1 = d_1\psi_2. \tag{3.1}$$

Lemma 3.6. *$\mathfrak{B}(M)$ is a Banach manifold.*

Proof. Recall (see, for example, [17, Lemma 4.4]) that Banach manifolds are closed under fiber products where one of the maps is a surjective submersion. Since the face maps of a simplicial manifold are surjective submersions, the horn space $\Lambda_{2,1}(\mathfrak{C}(M))$, consisting of pairs $(\psi_0, \psi_2) \in \mathfrak{C}_1(M) \times \mathfrak{C}_1(M)$ satisfying the first equation in (3.1), is a Banach manifold.

We may view $\mathfrak{B}(M)$ as the fiber product over $M \times M$ of $\mathfrak{C}_1(M)$ and $\Lambda_{2,1}(\mathfrak{C}(M))$, where the fiber product imposes the latter two equations in (3.1). Since this fiber product involves the surjective submersion $\mathfrak{C}_1(M) \to M \times M$ from Lemma 3.5, it follows that $\mathfrak{B}(M)$ is a Banach manifold. □

Let $\pi_{\mathfrak{B}} : \mathfrak{B}(M) \to C^p(\partial\Delta^2; M)$ be the map taking $(\psi_0, \psi_1, \psi_2) \in \mathfrak{B}(M)$ to its base map $(\bar{\psi}_0, \bar{\psi}_1, \bar{\psi}_2)$.

Lemma 3.7. *The map $\pi_{\mathfrak{B}}$ is a surjective submersion.*

Proof. Surjectivity is clear, since the edges of any map from $\partial\Delta^2$ to M can be lifted to zero maps $T\Delta^1 \to T^*M$.

Choose a Riemannian metric on M. For any map $f : \partial\Delta^2 \to M$, we can use the exponential map to identify any sufficiently close maps with lifts $\tilde{f} : \partial\Delta^2 \to TM$. Given such a lift, we can use parallel transport along the exponential paths to translate any element of $\mathfrak{B}(M)$ whose base map is f. This process gives a local section of $\pi_{\mathfrak{B}}$. □

There is a natural "1-skeleton" map $\nu : \mathfrak{C}_2(M) \to \mathfrak{B}(M)$, given by $\nu(\varphi) = (d_0\varphi, d_1\varphi, d_2\varphi)$. The map ν is invariant under the equivalence relation that defines $\mathrm{LWX}_2(M) = \mathfrak{C}_2(M)/\sim$.

Let $\mathfrak{B}_0(M)$ be the connected component of $\mathfrak{B}(M)$ consisting of elements β for which $\pi_{\mathfrak{B}}(\beta)$ is contractible. Clearly, the image of ν is contained in $\mathfrak{B}_0(M)$. Thus, we see that ν induces a map $\hat{\nu} : \mathrm{LWX}_2(M) \to \mathfrak{B}_0(M)$.

There is another map $\pi_{\mathfrak{C}} : \mathfrak{C}_2(M) \to S_2(M) := C^p(\Delta^2; M)$, taking φ to its base map $\bar{\varphi}$. An equivalence between elements $\varphi, \varphi' \in \mathfrak{C}(M)$ induces a boundary-fixing homotopy between $\bar{\varphi}$ and $\bar{\varphi}'$, so $\pi_{\mathfrak{C}}$ descends to a map from $\mathrm{LWX}_2(M)$ to $S_2(M)/\sim$, where the equivalence relation is boundary-fixing homotopy. We observe that $S_2(M)/\sim$ is a covering of the component of contractible maps in $C^p(\partial\Delta^2; M)$ and is therefore a Banach manifold.[1] In particular, if $\pi_2(M) = 0$, then $S_2(M)/\sim = C^p(\partial\Delta^2; M)$.

Lemma 3.8. *The map $(\hat{\nu}, \pi_{\mathfrak{C}})$ is a bijection from* $\mathrm{LWX}_2(M)$ *to the fiber product (over $C^p(\partial\Delta^2; M)$) of $\mathfrak{B}_0(M)$ with $S_2(M)/\sim$.*

Proof. Throughout this proof, we will assume that a choice of Riemannian metric on M has been fixed, and we will implicitly use parallel transport to identify cotangent spaces at different points along paths in M.

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[1] This statement is a higher-dimensional analogue of the fact that the fundamental groupoid of a manifold M is a cover of $M \times M$, and the proof is similar. We leave the details to the reader.

We will first show that $(\hat{\nu}, \pi_{\mathfrak{C}})$ is surjective. Let (ψ_0, ψ_1, ψ_2) be in $\mathfrak{B}_0$, and let f be a compatible map in $C^p(\Delta^2; M)$. Using the standard trivializations of Δ^1 and Δ^2, we can identify each ψ_i with a path in T^*M, and we can identify $\mathfrak{C}_2(M)$ with the space of $C^{p,q}$ maps from Δ^2 to $T_2^*M := T^*M \oplus T^*M$.

For $i = 0, 1, 2$, let β_i be the path in T_2^*M with the same base path as ψ_i, given by

$$\beta_0(t) = (\psi_0(t), (1-t)(\psi_1(0) - \psi_0(0)) + t\psi_2(0)),$$
$$\beta_1(t) = ((1-t)\psi_0(0) + t(\psi_1(t) - \psi_2(1)), (1-t)(\psi_1(t) - \psi_0(0)) + t\psi_2(1)),$$
$$\beta_2(t) = ((1-t)\psi_0(1) + t(\psi_1(1) - \psi_2(1)), \psi_2(t)).$$

These paths agree at the endpoints, in that $\beta_2(0) = \beta_0(1)$, $\beta_0(0) = \beta_1(0)$, and $\beta_1(1) = \beta_2(1)$, so they form a well-defined map $\beta : \partial\Delta^2 \to T_2^*M$ that can be extended to a C^q map $\varphi : \Delta^2 \to T_2^*M$ for which the base map is f. By construction, we have that $\hat{\nu}(\varphi) = (\psi_0, \psi_1, \psi_2)$ and $\pi_{\mathfrak{C}}(\varphi) = f$.

Next, we will show that $(\hat{\nu}, \pi_{\mathfrak{C}})$ is one-to-one. Suppose that φ, φ' are elements of $\mathfrak{C}_2(M)$ for which $\nu(\varphi) = \nu(\varphi')$ and $\pi_{\mathfrak{C}}(\varphi) \sim \pi_{\mathfrak{C}}(\varphi')$. By a process similar to the proof of surjectivity, one can construct a map from $\partial\Delta^3$ to T_3^*M which, if it could be extended to a map $\zeta : \Delta^3 \to T_3^*M$, would satisfy the conditions of an equivalence between φ and φ'. The assumption that $\pi_{\mathfrak{C}}(\varphi)$ and $\pi_{\mathfrak{C}}(\varphi')$ are in the same boundary-fixing homotopy class guarantees that such an extension ζ does exist (and can be chosen to have the same order of differentiability as the boundary), proving that φ and φ' represent the same element of $\mathrm{LWX}_2(M)$. □

Remark 3.9. If $\pi_2(M) = 0$, then Lemma 3.8 implies that $\mathrm{LWX}_2(M)$ is naturally diffeomorphic to $\mathfrak{B}_0(M)$. If $\pi_1(M) = 0$, then $\mathfrak{B}_0(M) = \mathfrak{B}(M)$. Thus, if M is 2-connected, then $\hat{\nu}$ is a diffeomorphism from $\mathrm{LWX}_2(M)$ to $\mathfrak{B}(M)$. This fact provides a simple description (in the 2-connected case) of elements of $\mathrm{LWX}_2(M)$ as triangles of paths in $\mathfrak{C}_1(M)$. For general M, an appropriate modification is as follows: an element of $\mathrm{LWX}_2(M)$ corresponds to a quadruplet $([f], \psi_0, \psi_1, \psi_2)$, where

- $[f]$ is a class of C^p maps from Δ^2 to M, modulo boundary-fixing homotopy, and
- each ψ_i is a C^q lift of the ith edge of f to T^*M.

4. Symplectic structures

In this section, we describe how the canonical symplectic form on T^*M induces symplectic structures on $\mathrm{LWX}_1(M)$ and $\mathrm{LWX}_2(M)$.

4.1. Multiplicative forms and truncation

Let $\{X_\bullet\}$ be a Kan simplicial manifold, and let α be a k-form on X_m for some m. Recall that the simplicial coboundary of $\alpha \in \Omega^k(X_m)$ is defined as

$$\delta\alpha := \sum_{i=0}^{m+1} (-1)^i d_i^* \alpha \in \Omega^k(X_{m+1}).$$

There is also the de Rham differential

$$d : \Omega^k(X_m) \to \Omega^{k+1}(X_m).$$

The two differentials d and δ commute and hence define a bicomplex structure on the total space of differential forms $\bigoplus_{k,m} \Omega^k(X_m)$. The associated total complex is called the *Bott–Shulman–Stasheff complex* [25].

Definition 4.1. We say that $\alpha \in \Omega(X_m)$ is *multiplicative* if $\delta\alpha = 0$.

Proposition 4.2. *If $\alpha \in \Omega(X_n)$ is multiplicative, then α is basic with respect to the quotient map $X_n \to (\tau_{\leq n} X)_n$.*

Proof. Recall that the quotient is defined by the equivalence relation where $x \sim y$ if and only if there exists $z \in X_{n+1}$ such that $d_n z = x$, $d_{n+1} z = y$, and $d_i z \in \mathrm{im}(s_{n-1})$ for $0 \leq i < n$. In this case, it follows that $d_i z = s_{n-1} d_i x = s_{n-1} d_i y$ for $0 \leq i < n$.

From the definition of the equivalence relation, we can see that a vector $v \in TX_n$ is tangent to a fiber of the quotient map if and only if there exists a vector $\tilde{v} \in TX_{n+1}$ such that $(d_{n+1})_* \tilde{v} = v$ and $(d_i)_* \tilde{v} = 0$ for $0 \leq i \leq n$. If this is the case, then, for any $\alpha \in \Omega^k(X_n)$,

$$\alpha(v, \cdot, \dots, \cdot) = \pm(\delta\alpha)(\tilde{v}, \cdot, \dots, \cdot).$$

Therefore, if α is multiplicative, then any vector tangent to a fiber of the quotient map is in $\ker \alpha$.

If α is multiplicative, then $d\alpha$ is also multiplicative, and any vector tangent to a fiber of the quotient map is also in $\ker d\alpha$. Since both α and $d\alpha$ annihilate vectors tangent to the fibers, we conclude that α is basic. □

4.2. Lifting differential forms

Recall that, for each m, the space $\mathfrak{C}_m(M)$ consists of $C^{p,q}$ bundle maps from $T\Delta^m$ to T^*M. For $\varphi \in \mathfrak{C}_m(M)$, a tangent vector at φ is given by a $C^{p,q}$ lift $X : T\Delta^m \to TT^*M$ that is linear over TM:

$$\begin{tikzcd}
 & TT^*M = T^*TM \arrow[dr] \arrow[dd] & \\
T\Delta^m \arrow[ur, "X"] \arrow[rr, "\varphi"] \arrow[dd] & & T^*M \arrow[dd] \\
 & TM \arrow[dr] & \\
\Delta^m \arrow[ur, "X_0"] \arrow[rr, "f"] & & M
\end{tikzcd} \tag{4.1}$$

For each pair of tangent vectors $X, Y \in T_\varphi \mathfrak{C}_m(M)$, we can use the canonical symplectic form ω_{can} on T^*M to obtain a function $\eta^m_{X,Y}$ on $T\Delta^m$, given by

$$\eta^m_{X,Y}(v) = \omega_{\mathrm{can}}(X(v), Y(v))$$

for $v \in T\Delta^m$.

Proposition 4.3. *The function $\eta^m_{X,Y}$ is linear and can therefore be identified with a 1-form on Δ^m.*

Proof. The result is a direct consequence of the linearity property of ω_{can} with respect to the bundle structure of $T^*M \to M$. $\square$

For $m = 1$, the operation $(X, Y) \mapsto \int_{\Delta^1} \eta^1_{X,Y}$ is bilinear and skew-symmetric, and so it determines a 2-form $\omega_1 \in \Omega^2(\mathfrak{C}_1(M))$. For $m = 2$, we can also define a 2-form $\omega_2 \in \Omega^2(\mathfrak{C}_2(M))$ by the formula

$$\omega_2(X, Y) = \int_{\Delta^2} d\eta^2_{X,Y} = \int_{\partial\Delta^2} \eta^2_{X,Y}. \tag{4.2}$$

Proposition 4.4. *ω_2 is the simplicial coboundary of ω_1.*

Proof. For $i = 0, 1, 2$, let $\sigma_i : \Delta^1 \to \Delta^2$ be the ith coface map (which is essentially dual to the face map d_i). For any $X, Y \in T_\varphi \mathfrak{C}_2(M)$ and $v \in T\Delta^1$, we have that

$$\begin{aligned}
\eta^1_{Td_i(X),Td_i(Y)}(v) &= \omega_{\mathrm{can}}(Td_i(X)(v), Td_i(Y)(v)) \\
&= \omega_{\mathrm{can}}(X(T\sigma_i(v)), Y(T\sigma_i(v))) \\
&= \eta^2_{X,Y}(T\sigma_i(v)).
\end{aligned} \tag{4.3}$$

Using (4.3), we see that

$$\begin{aligned}
(d_i^*\omega_1)(X, Y) &= \omega_1(Td_i(X), Td_i(Y)) \\
&= \int_{\Delta^1} \eta^1_{Td_i(X),Td_i(Y)} \\
&= \int_{\Delta^1} \sigma_i^* \eta^2_{X,Y}.
\end{aligned} \tag{4.4}$$

The result then follows from (4.2) and (4.4). $\square$

Proposition 4.5. *The 2-forms $\omega_1 \in \Omega^2(\mathfrak{C}_1(M))$ and $\omega_2 \in \Omega^2(\mathfrak{C}_2(M))$ are exact.*

Proof. Let λ_{can} denote the tautological 1-form on T^*M, satisfying the property $\omega_{\mathrm{can}} = -d\lambda_{\mathrm{can}}$. We can use λ_{can} to induce forms on the mapping spaces in a manner similar to the construction of ω_1 and ω_2. Specifically, for $X \in T_\varphi \mathfrak{C}_n(M)$, let θ^n_X be the function on $T\Delta^n$ given by

$$\theta^n_X(v) = \lambda_{\mathrm{can}}(X(v)).$$

Because of the linearity property of λ_{can}, we have that θ^n_X is a linear function and can therefore be identified with a 1-form on Δ^n.

Then, let $\lambda_1 \in \Omega^1(\mathfrak{C}_1(M))$ and $\lambda_2 \in \Omega^1(\mathfrak{C}_2(M))$ be defined by

$$\lambda_1(X) = \int_{\Delta^1} \theta^1_X, \qquad \lambda_2(X) = \int_{\Delta^2} d\theta^2_X = \int_{\partial\Delta^2} \theta^2_X.$$

The proof of Proposition 4.4, with appropriate modification, can be used to show that $\lambda_2 = \delta\lambda_1$.

We claim that $\omega_1 = -d\lambda_1$ (and, since d commutes with δ, therefore $\omega_2 = -d\lambda_2$). We can check it locally in M, as follows.

Let (x^i, p_i) be canonical coordinates on a neighborhood in T^*M. Any $C^{p,q}$ bundle map $\varphi : T\Delta^1 \to T^*M$ is locally described by the pullbacks $f^i := \varphi^*(x^i)$ and $\xi_i := \varphi^*(p_i)$, where $f^i \in C^p(\Delta^1)$ and $\xi_i \in C^q_{\text{linear}}(T\Delta^1)$ can be identified with 1-forms on Δ^1. A tangent vector $X \in T_\varphi\mathfrak{C}_1(M)$ is locally given by (v^i, χ_i), where the v^i and χ_i are functions and 1-forms, respectively, on Δ^1.

We can locally describe λ_1 by the formula

$$\lambda_1|_{(f^i,\xi_i)}(v^i, \chi_i) = \int_{\Delta^1} v^i\xi_i. \tag{4.5}$$

The directional derivatives of λ_1 are given by

$$D_{(v^i,\chi_i)}\lambda_1|_{(f^i,\xi_i)}(v'^i, \chi'_i) = \int_{\Delta^1} v'^i\chi_i,$$

from which we obtain the result

$$d\lambda_1((v^i, \chi_i), (v'^i, \chi'_i)) = \int_{\Delta^1} v'^i\chi_i - v^i\chi'_i = -\omega_1((v^i, \chi_i), (v'^i, \chi'_i)). \quad \square$$

4.3. A symplectic 2-groupoid

Since $\omega_2 \in \Omega^2(\mathfrak{C}_2(M))$ is multiplicative, it descends (by Proposition 4.2) to $\mathrm{LWX}_2(M) = \mathfrak{C}_2(M)/\!\sim$.

Theorem 4.6. LWX(M)*, equipped with the* 2*-form* ω_2*, is a symplectic* 2*-groupoid.*

Proof. The fact that ω_2 descends to a closed, multiplicative 2-form on the truncation is an immediate consequence of Propositions 4.2, 4.4, and 4.5.

It remains to check the nondegeneracy condition. For this, we first observe that $\omega_1 \in \Omega^2(\mathfrak{C}_1(M))$ is (weakly) nondegenerate. Then, using the description of $\mathrm{LWX}_2(M)$ obtained in Lemma 3.8, one can see that a tangent vector in $\mathrm{LWX}_2(M)$ is given by a compatible triplet of tangent vectors in $\mathfrak{C}_1(M)$. If such a triplet (b_0, b_1, b_2) does not vanish everywhere, it is a straightforward exercise to construct another triplet (b'_0, b'_1, b'_2) for which the pairing

$$\omega_2((b_0, b_1, b_2), (b'_0, b'_1, b'_2)) = \omega_1(b_0, b'_0) - \omega_1(b_1, b'_1) + \omega_1(b_2, b'_2)$$

does not vanish, thereby proving the (weak) nondegeneracy of ω_2 on $\mathrm{LWX}_2(M)$. $\square$

Remark 4.7.

(1) We recommend the reader to compare Theorem 4.6 with [5, Lemma 3] for a formal proof of a more general statement.
(2) In [26], a symplectic form is constructed by the AKSZ method for the Courant-sigma model. It would be interesting to see if the symplectic form ω_2 on $\mathrm{LWX}_2(M)$ could be interpreted as a Hamiltonian reduction of the symplectic form for the Courant-sigma model, in analogy to the result of [10].

4.4. 3-form twisting

Let H be a 3-form on M. In this section, we describe how H can be used to twist the 2-forms ω_1 and ω_2. We suggest the reader to compare these constructions with the ones in [7, Section 4.1]. In [7], a metric on M is chosen and only geodesic paths are considered, whereas in this article we consider all paths.

For $\varphi \in \mathfrak{C}_m(M)$, let $f : \Delta^m \to M$ be the base map underlying $\varphi : T\Delta^m \to T^*M$. For $X, Y \in T_\varphi\mathfrak{C}_m(M)$, let $X_0, Y_0 : \Delta^m \to TM$ be the respective base maps (see (4.1)). We can use H to obtain a 1-form $H^m_{X,Y}$ on Δ^m, given by

$$H^m_{X,Y}(s) = f^*(H(X_0(s), Y_0(s), \cdot))$$

for $s \in \Delta^m$. We then define 2-forms $\phi^H_1 \in \Omega^2(\mathfrak{C}_1(M))$ and $\phi^H_2 \in \Omega^2(\mathfrak{C}_2(M))$ by

$$\phi^H_1(X, Y) = \int_{\Delta^1} H^1_{X,Y},$$
$$\phi^H_2(X, Y) = \int_{\Delta^2} dH^1_{X,Y}.$$

Remark 4.8. The forms ϕ_i^H, $i = 1, 2$, only depend on the information about the underlying base maps, so are actually pullbacks of forms on $S_i(M) := C^p(\Delta^i, M)$. The construction of ϕ_1^H is a special case of a more general transgression procedure taking any $\beta \in \Omega^k(M)$ to $\phi_i^\beta \in \Omega^{k-1}(S_i(M))$.

We define the twisted versions of ω_1 and ω_2 as follows:

$$\omega_i^H := \omega_i + \phi_i^H.$$

Proposition 4.9. *ϕ_2^H is the simplicial coboundary of ϕ_1^H. Therefore, $\omega_2^H = \delta\omega_1^H$.*

Proof. The result follows from an argument similar to the proof of Proposition 4.4. □

Proposition 4.10. *If H is closed, then $d\phi_1^H = \delta H$, and therefore $d\omega_1^H = \delta H$.*

Proof. In local coordinates on M, write $H = \frac{1}{6}H_{ijk}dx^i \wedge dx^j \wedge dx^k$. Then, in the local neighborhood, for $f = (f^i) \in C^p(\Delta^1; M)$ and any C^p sections $X_0 = (X_0^i)$, $Y_0 = (Y_0^i)$ of $f^*(TM)$, we have

$$\phi_1^H|_f(X_0, Y_0) = \int_{\Delta^1} f^*(H_{ijk})X_0^i Y_0^j df^k.$$

The differential of ϕ_1^H is then given by

$$d\phi_1^H|_f(X_0, Y_0, Z_0) = \int_{\Delta^1} X_0^*(dH_{ijk})Y_0^i Z_0^j df^k + f^*(H_{ijk})Y_0^i Z_0^j dX_0^k + \{\text{cycl.}\}. \tag{4.6}$$

The integral of the first term on the right side of (4.6), together with its cyclic permutations, is equal to

$$\phi_1^{dH}|_f(X_0, Y_0, Z_0) + \int_{\Delta^1} f^*(dH_{ijk})X_0^i Y_0^j Z_0^k. \tag{4.7}$$

The integral of the second term on the right side of (4.6), together with its cyclic permutations, is

$$\int_{\Delta^1} f^*(H_{ijk})d(X_0^i Y_0^j Z_0^k). \tag{4.8}$$

Putting (4.7) and (4.8) together, we have

$$\begin{aligned} d\phi_1^H|_f(X_0, Y_0, Z_0) &= \phi_1^{dH}|_f(X_0, Y_0, Z_0) + \int_{\Delta^1} d\left(f^*(H_{ijk})X_0^i Y_0^j Z_0^k\right) \\ &= \phi_1^{dH}|_f(X_0, Y_0, Z_0) + \left[f^*(H_{ijk})X_0^i Y_0^j Z_0^k\right]_0^1, \end{aligned}$$

or, in other words,

$$d\phi_1^H = \phi_1^{dH} + \delta H. \tag{4.9}$$

In particular, if H is closed, then $d\phi_1^H = \delta H$. □

Since $\omega_2^H \in \Omega^2(\mathfrak{C}_2(M))$ is multiplicative, it descends to $\mathrm{LWX}_2(M)$. We now arrive at the main result of this section.

Theorem 4.11. *Let M be a manifold, and let H be a closed 3-form on M. Then $\mathrm{LWX}_2(M)$, equipped with the 2-form ω_2^H, is a symplectic 2-groupoid.*

Proof. It follows from Propositions 4.9 and 4.10 that ω_2^H is closed and multiplicative.

As in the proof of Theorem 4.6, the nondegeneracy of ω_2^H follows from the observation that ω_1^H is nondegenerate. This can be seen by showing that, for any $X \in T_\varphi\mathfrak{C}_1(M)$, one can construct $Y \in T_\varphi\mathfrak{C}_1(M)$ for which $\phi_1^H(X, Y) = 0$ and $\omega_1(X, Y) \neq 0$. We leave the details as an exercise. □

Remark 4.12.

(1) The geometric intuition behind the construction of ω_1^H is as follows. The space of 1-simplices $\mathfrak{C}_1(M) = \mathrm{LWX}_1(M)$ can be naturally identified with the cotangent bundle of $S_1(M)$, and ω_1 can then be seen as the canonical symplectic form on $T^*(S_1(M))$. Since ϕ_1^H is the pullback of a 2-form on $S_1(M)$, we see that ω_1^H is the modification of ω_1 by the magnetic term ϕ_1^H and is therefore "automatically" nondegenerate.

(2) The results of Propositions 4.9 and 4.10 can be concisely stated by saying that ϕ_2^H is the coboundary of $\phi_1^H + H$ in the Bott–Shulman–Stasheff complex of $\mathrm{LWX}(M)$. Putting this together with Propositions 4.4 and 4.5, we see that ω_2^H is the coboundary of $\omega_1^H + H$ in the Bott–Shulman–Stasheff complex.

5. Integration of Dirac structures

In this section, we study the geometry of integration of Dirac structures in relation to the Liu–Weinstein–Xu 2-groupoid.

5.1. A-path integration

Let $\mathcal{D}$ be a Dirac structure in an exact Courant algebroid $(TM \oplus T^*M, H)$. Letting ρ be the canonical projection from $TM \oplus T^*M$ to TM, we have that $(\mathcal{D}, \rho)$ forms a Lie algebroid. The integration of $(\mathcal{D}, \rho)$ via "A-paths" was studied in [3,4].

In this section, we will construct a simplicial manifold $\{\mathfrak{G}_\bullet(\mathcal{D})\}$ that connects the A-path integration of a Dirac structure to $\{\mathfrak{C}_\bullet(M)\}$. First, we briefly review the A-path construction.

Define $P(\mathcal{D})$ to be the space of $C^{2,1}$ paths $\alpha : \Delta^1 \to T^*M$ satisfying

$$\left(\frac{d}{dt}(\pi \circ \alpha)(t), \alpha(t)\right) \in \mathcal{D}, \ \forall t \in \Delta^1.$$

It is proved in [4, Lemma 4.6] that $P(\mathcal{D})$, which is called the space of *A-paths*, is a Banach manifold.

Recall that $\mathfrak{C}_1(M)$ can be identified with the space of all $C^{p,q}$ maps from $\Delta^1 \to T^*M$. Taking $p = 2$ and $q = 1$, we have a natural, smooth inclusion map $\iota_1 : P(\mathcal{D}) \to \mathfrak{C}_1(M)$.

To construct a groupoid integrating $\mathcal{D}$, one needs to impose a homotopy relation on A-paths; we refer to [4, Definition 1.4] for the precise definition. Crainic and Fernandes proved [4, Theorem 2.1] that the quotient $\mathsf{G} := P(\mathcal{D})/\sim$ is a source-simply-connected topological groupoid. In general, G could fail to be smooth, and necessary and sufficient conditions for $\mathcal{D}$ to be integrable to a Lie groupoid were obtained in [4, Theorem 4.1].

We will now describe a simplicial manifold associated to a Dirac structure $\mathcal{D}$. For simplicity, we will assume that $\mathcal{D}$ is integrable to a Lie groupoid, although many of the results will carry through in the general case.[2]

For each $m \geq 0$, let $\mathfrak{G}_m(\mathcal{D})$ denote the set of C^2 groupoid morphisms from the pair groupoid $\Delta^m \times \Delta^m$ to G. There is a natural simplicial structure on $\{\mathfrak{G}_\bullet(\mathcal{D})\}$, induced by the cosimplicial structure of $\{\Delta^\bullet\}$.

Lemma 5.1. *The space $\mathfrak{G}_m(\mathcal{D})$ is a Banach manifold.*

Proof. Given a C^2 groupoid morphism $\Sigma : \Delta^m \times \Delta^m \to \mathsf{G}$, we define a C^2 map $\sigma : \Delta^m \to \mathsf{G}$ by the equation $\sigma(w) = \Sigma(w, 0)$. We observe that σ satisfies the following two properties:

- $\sigma(0)$ is a unit of G,
- $\sigma(w)$ is in the same source-fiber as $\sigma(0)$ for all $w \in \Delta^m$.

Conversely, given any C^2 map $\sigma : \Delta^m \to \mathsf{G}$ satisfying the above properties, we may obtain $\Sigma \in \mathfrak{G}_m(\mathcal{D})$ by setting $\Sigma(w_1, w_2) = \sigma(w_1)\sigma(w_2)^{-1}$, so we have a one-to-one correspondence.

We will now show that the space of all σ satisfying the above properties (and hence $\mathfrak{G}_m(\mathcal{D})$) is a Banach manifold. Let $s : \mathsf{G} \to M$ denote the source map. Since s is a submersion, we have that $s^{-1}(x)$ is a submanifold of G for each $x \in M$, so $C^2(\Delta^m; s^{-1}(x))$ is a Banach manifold.

Consider the evaluation map $\mathrm{ev}_0 : C^2(\Delta^m; s^{-1}(x)) \to s^{-1}(x)$, defined by $\mathrm{ev}_0(f) := f(0)$ for $f \in C^2(\Delta^m; s^{-1}(x))$. It is not difficult to check that ev_0 is a surjective submersion between Banach manifolds. Therefore, $C^2(\Delta^m; s^{-1}(x))_0 := \mathrm{ev}_0^{-1}(x)$ is a Banach manifold.

As $\sigma(\Delta^m)$ is compact and the map $s : \mathsf{G} \to M$ is submersive, we have that $\mathfrak{G}(\mathcal{D})_m$ near σ is locally a product of a neighborhood of σ in $C^2(\Delta^m, s^{-1}(x))_0$ and $\mathbb{R}^{\dim(M)}$. This shows that $\mathfrak{G}_m(\mathcal{D})$ is a Banach manifold. $\square$

Remark 5.2. We point out that, since G might not be a Hausdorff manifold [4], it is possible for $\mathfrak{G}_m(\mathcal{D})$ to be non-Hausdorff as well.

Following the approach of the proof of Lemma 5.1, one can show that the horn spaces of $\{\mathfrak{G}_\bullet(\mathcal{D})\}$ are smooth and the horn maps are surjective submersions, thereby completing the proof of the following statement.

Proposition 5.3. *$\{\mathfrak{G}_\bullet(\mathcal{D})\}$ is a Kan simplicial manifold.*

For any $\Sigma \in \mathfrak{G}_m(\mathcal{D})$, we may apply the Lie functor to obtain a $C^{2,1}$ Lie algebroid morphism $\bar{\Sigma}$ from $T\Delta^m$ to $\mathcal{D}$; in fact, this process gives a one-to-one correspondence, since G and the pair groupoid $\Delta^m \times \Delta^m$ are both source-simply-connected.

When $m = 1$, this correspondence allows us to identify $\mathfrak{G}_1(\mathcal{D})$ with the A-path space $P(\mathcal{D})$. Furthermore, when we consider the truncation $\tau_{\leq 1}\mathfrak{G}(\mathcal{D})$, the equivalence imposed by the truncation corresponds to homotopy equivalence of A-paths (see [4, Propositions 1.1, 1.3]). In other words, $\left(\tau_{\leq 1}\mathfrak{G}(\mathcal{D})\right)_1$ can be identified with $\mathsf{G} = P(\mathcal{D})/\sim$. Thus we see that G can be recovered from $\{\mathfrak{G}_\bullet(\mathcal{D})\}$ by truncation.

2 It has been communicated to us by Rui Fernandes and Ionut Marcut [27] and Pavol Ševera and Michal Siran [28,29] that Lemma 5.1 holds even without the assumption that $\mathcal{D}$ is integrable.

To connect $\{\mathfrak{G}_\bullet(\mathcal{D})\}$ to $\{\mathfrak{C}_\bullet(M)\}$, let π_T and π_{T^*} be the canonical projections from $TM \oplus T^*M$ to TM and T^*M, respectively. The map taking $\Sigma \in \mathfrak{G}_m(\mathcal{D})$ to the bundle map $\pi_{T^*} \circ \bar{\Sigma} : T\Delta^m \to T^*M$ defines a natural map of simplicial manifolds

$$F_\bullet : \mathfrak{G}_\bullet(\mathcal{D}) \to \mathfrak{C}_\bullet(M). \tag{5.1}$$

It may seem that $F_\bullet$ discards the information about the TM-component of $\bar{\Sigma}$; however, the fact that $\bar{\Sigma}$ is a Lie algebroid morphism implies that the TM-component can be recovered by applying the tangent functor to the underlying map $\Delta^m \to M$. To be more explicit, suppose that $\bar{\Sigma}$ is a Lie algebroid morphism from $T\Delta^m$ to $\mathcal{D}$ with underlying map $f : \Delta^m \to M$. The anchor map $\mathcal{D} \to TM$ is the restriction of π_T to $\mathcal{D}$, so compatibility of $\bar{\Sigma}$ with the anchor maps requires that $\pi_T \circ \bar{\Sigma} = Tf$. Therefore, the map $F_\bullet$ is injective.

The following diagram of Kan simplicial manifolds summarizes the various relationships we have described.

$$\begin{tikzcd} \mathfrak{G}_\bullet(\mathcal{D}) \arrow[r, hook, "F_\bullet"] \arrow[d, "\tau_{\leq 1}"'] & \mathfrak{C}_\bullet(M) \arrow[d, "\tau_{\leq 2}"] \\ \mathsf{G} & \mathsf{LWX}(M) \end{tikzcd} \tag{5.2}$$

Example 5.4. To help illustrate the relationships in (5.2), consider the case where $\mathcal{D} = TM$. Then $\mathfrak{G}_m(TM)$ is space of C^2 groupoid morphisms from the pair groupoid $\Delta^m \times \Delta^m$ to the fundamental groupoid $\Pi_1(M)$, which is the source-simply-connected Lie groupoid integrating TM. Such groupoid morphisms are in one-to-one correspondence with $C^{2,1}$ Lie algebroid morphisms from $T\Delta^m$ to TM, which can be identified with C^2 maps from Δ^m to M. The map F_m takes $f : \Delta^m \to M$ to the trivial bundle map $\tilde{f} : T\Delta^m \to T^*M$ which has f as its underlying base map and is the zero map on each fiber.

5.2. *(Twisted) Presymplectic 2-form*

In Section 4.2, we introduced 2-forms ω_i, as well as their twisted versions ω_i^H, on $\mathfrak{C}_i(M)$ for $i = 1, 2$. In this subsection, we study the relationship of these 2-forms to the simplicial manifold $\{\mathfrak{G}_\bullet(\mathcal{D})\}$ associated to a Dirac structure $\mathcal{D}$. Our main results are as follows.

Theorem 5.5. *The 2-form $F_2^*\omega_2^H \in \Omega^2(\mathfrak{G}_2(\mathcal{D}))$ vanishes.*

Together with the results of Section 4 (specifically, Propositions 4.2, 4.9, and 4.10), Theorem 4.11 implies the following result.

Corollary 5.6. *The 2-form $F_1^*\omega_1^H$ is multiplicative and therefore descends to a multiplicative 2-form on the Lie groupoid* G *integrating $\mathcal{D}$. Additionally, $F_1^*\omega_1^H$ is H-closed, in the sense that $d(F_1^*\omega_1^H) = \delta H$.*

Remark 5.7.

(1) In Corollary 5.6, we recover one of the main results of [3]. However, [3] showed that the 2-form on G satisfies an additional property that controls the extent to which it is degenerate. It remains unclear how this property arises from the inclusion (5.1), but we expect that it is related to the Lagrangian property discussed in Section 6.

(2) From Corollary 5.6, $F_1^*\omega_1^H$ is an H-closed 2-form on $\mathfrak{G}_1(\mathcal{D})$, and therefore defines a δH-twisted Dirac structure on $\mathfrak{G}_1(\mathcal{D})$. From the property that $F_1^*\omega_1^H$ is multiplicative, it is not hard to check the truncation map $\tau_{\leq 1}$ is a forward Dirac map. It is natural to expect that the groupoid G could be obtained from $\mathfrak{G}_1(\mathcal{D})$ via Hamiltonian reduction as in [10] and [4]. We will not pursue this direction in this paper.

The proof of Theorem 5.5 is somewhat technical, so we will first give an intuitive explanation for why one might expect it to hold. By [13] and [5], there is a correspondence between Courant algebroids and degree 2 differential graded (dg)-symplectic manifolds. In particular, the Courant algebroid $TM \oplus T^*M$ corresponds to the dg-symplectic manifold $T^*[2]T[1]M$. We can thus view the symplectic 2-groupoid $\mathsf{LWX}(M)$ as the integration of $T^*[2]T[1]M$.

It is known [5] that Dirac structures $\mathcal{D} \subset TM \oplus T^*M$ correspond to a certain dg-Lagrangian submanifolds $\mathfrak{L}_{\mathcal{D}}$ of $T^*[2]T[1]M$. It seems reasonable to expect that dg-Lagrangian submanifolds should integrate to Lagrangian sub-2-groupoids. We point out that the analogous result in the $n = 1$ case is known to be true [30].

In the remainder of this section, we will prove Theorem 5.5. For simplicity, we assume $H = 0$. The extension to the general case is straightforward and left to the reader.

It suffices to check the statement locally on a coordinate chart of M. On such a chart, let (x^i, q^i, p_i) be coordinates on $TM \oplus T^*M$. A $C^{2,1}$ bundle map $\varphi : T\Delta^2 \to T^*M$ is locally given by $\varphi = (f^i, \xi_i)$, where $f^i := \varphi^*(x^i)$ is a function on Δ^2 and $\xi_i := \varphi^*(p_i)$ is a 1-form on Δ^2 for each i. Together, the f^i form a C^2 map $f : \Delta^2 \to M$, and the ξ_i form a C^1 element of $\Omega^1\left(\Delta^2, f^*(T^*M)\right)$.

A tangent vector on $\mathfrak{C}_2(M)$ at φ is locally given by (v^i, χ_i), where the v^i describe a section of $f^*(TM)$ and the χ_i describe an element of $\Omega^1\left(\Delta^2, f^*(T^*M)\right)$. In the above coordinates, η_2 has the form

$$\eta_2\left((v^i, \chi_i), (v'^i, \chi'_i)\right) = v^i\chi'_i - v'^i\chi_i,$$

so the 2-form ω_2 on $\mathfrak{C}_2(M)$ is given by

$$\omega_2((v^i, \chi_i), (v'^i, \chi'_i)) = \int_{\Delta^2} d(v^i\chi'_i - v'^i\chi_i). \tag{5.3}$$

Let $n = \dim M$. Since the rank of the Dirac structure $\mathcal{D}$ is n, we can locally find linearly independent sections $\Theta_\alpha = Q_\alpha + P_\alpha = q^i_\alpha\partial_i + p_{i\alpha}dx^i$, for $\alpha = 1, \dots, n$, that span $\mathcal{D}$. The following properties hold by definition of Dirac structures:

$$\langle\Theta_\alpha, \Theta_\beta\rangle = q^i_\alpha p_{i\beta} + q^i_\beta p_{i\alpha} = 0, \tag{5.4}$$

$$\left[\Theta_\alpha, \Theta_\beta\right] = C^\gamma_{\alpha\beta}\Theta_\gamma, \tag{5.5}$$

where $C^\gamma_{\alpha\beta}$ is a smooth function on M, and $[\cdot,\cdot]$ is the Courant bracket. Using the definition of the Courant bracket, (5.5) implies that

$$C^\gamma_{\alpha\beta}P_\gamma = L_{Q_\alpha}P_\beta - \iota_{Q_\beta}dP_\alpha, \tag{5.6}$$

which in coordinates becomes

$$C^\gamma_{\alpha\beta}p_{i\gamma} = q^j_\alpha\partial_j(p_{i\beta}) - q^j_\beta\partial_j(p_{i\alpha}) + p_{j\beta}\partial_i(q^j_\alpha) + q^j_\beta\partial_i(p_{j\alpha}). \tag{5.7}$$

Recall from Section 5.1 that a point Ψ of $\mathfrak{G}_2(\mathcal{D})$ can be identified with a $C^{2,1}$ Lie algebroid morphism from $T\Delta^2$ to $\mathcal{D}$. A $C^{2,1}$ bundle map from $T\Delta^2$ to $\mathcal{D}$ can be locally described by a C^2 map $f : \Delta^2 \to M$ and C^1 elements $\psi^\alpha \in \Omega^1(\Delta^2)$ where, for $v \in T\Delta^2$,

$$\Psi(v) = \psi^\alpha(v)\Theta_\alpha.$$

Using the characterization of Lie algebroid morphisms in terms of differentials, we have that a bundle map $\Psi : T\Delta^2 \to \mathcal{D}$ is a Lie algebroid morphism if and only if

$$df^i = f^*(q^i_\alpha)\psi^\alpha, \tag{5.8}$$

$$d\psi^\gamma = -\frac{1}{2}f^*(C^\gamma_{\alpha\beta})\psi^\alpha \wedge \psi^\beta. \tag{5.9}$$

Given $\Psi \in \mathfrak{G}_2(\mathcal{D})$, the induced bundle map $\hat{\Psi} := F_2(\Psi)$ from $T\Delta^2$ to T^*M is given by

$$\hat{\Psi} = (f^i, f^*(p_{i\alpha})\psi^\alpha).$$

A tangent vector on $\mathfrak{G}_2(\mathcal{D})$ at Ψ is given by a collection of C^2 functions v^i on Δ^2, representing a vector field along f, and C^1 1-forms μ^α, satisfying the following equations, which are obtained by differentiating (5.8) and (5.9):

$$dv^i = v^jf^*(\partial_jq^i_\alpha)\psi^\alpha + f^*(q^i_\alpha)\mu^\alpha, \tag{5.10}$$

$$d\mu^\gamma = -\frac{1}{2}v^jf^*(\partial_jC^\gamma_{\alpha\beta})\psi^\alpha \wedge \psi^\beta - f^*(C^\gamma_{\alpha\beta})\mu^\alpha \wedge \psi^\beta. \tag{5.11}$$

The induced tangent vector on $\mathfrak{C}_2(M)$ is given by (v^i, χ_i), where

$$\chi_i = v^kf^*(\partial_kp_{i\alpha})\psi^\alpha + f^*(p_{i\alpha})\mu^\alpha. \tag{5.12}$$

Putting this into (5.3), we obtain the formula

$$(F_2^*\omega_2)\left((v^i, \mu^\alpha), (v'^i, \mu'^\alpha)\right) = \int_{\Delta^2} d\Xi,$$

where

$$\Xi = f^*(\partial_kp_{i\alpha})\psi^\alpha(v^iv'^k - v'^iv^k) + f^*(p_{i\alpha})(v^i\mu'^\alpha - v'^i\mu^\alpha) \tag{5.13}$$

is a 1-form on Δ^2. We claim that $d\Xi = 0$, which will imply Theorem 5.5.

The proof will proceed as follows. First, we can use (5.8)–(5.11) to write $d\Xi$ in terms of f, ψ^α, v^i, v'^i, μ^α, μ'^α, and the various structure functions. Second, we can collect terms that are of similar type with respect to the μ's and ψ's. We will see that each group of terms vanishes as a result of (5.4)–(5.7).

5.2.1. *Terms of type $\mu \wedge \mu$*

In $d\Xi$, the coefficient of $\mu^\alpha \wedge \mu'^\beta$ is $f^*(p_{i\alpha}q^i_\beta + p_{i\beta}q^i_\alpha) = f^*(\langle\Theta_\alpha, \Theta_\beta\rangle)$, which vanishes by the isotropy condition (5.4).

5.2.2. Terms of type $\psi \wedge \mu$

The coefficient of $\psi^\alpha \wedge \mu^\beta$ in $d\Xi$ is

$$f^*\left(\partial_j(p_{i\alpha})q^j_\beta - \partial_i(p_{j\alpha})q^j_\beta - q^j_\alpha \partial_j(p_{i\beta}) - p_{j\beta}\partial_i(q^j_\alpha) + p_{i\gamma}C^\gamma_{\alpha\beta}\right) v'^i$$
$$= \left(C^\gamma_{\alpha\beta}P_\gamma - L_{Q_\alpha}P_\beta + \iota_{Q_\beta}dP_\alpha\right)(v'),$$

which vanishes by the integrability conditions (5.6)–(5.7). Because of skew-symmetry, the coefficient of $\psi^\alpha \wedge \mu'^\beta$ will similarly vanish.

5.2.3. Terms of type $\psi \wedge \psi$

The coefficient of $\psi^\alpha \wedge \psi^\beta$ is

$$f^*\left(q^k_\alpha \partial_k\partial_j(p_{i\beta}) - q^k_\beta \partial_k\partial_j(p_{i\alpha}) - \partial_j(p_{i\gamma})C^\gamma_{\alpha\beta} + \partial_j(p_{k\beta})\partial_i(q^k_\alpha)\right.$$
$$\left. - \partial_j(p_{k\alpha})\partial_i(q^k_\beta) + \partial_k(p_{i\beta})\partial_j(q^k_\alpha) - \partial_k(p_{i\alpha})\partial_j(q^k_\beta) - p_{i\gamma}\partial_j(C^\gamma_{\alpha\beta})\right) v^i v'^j,$$

plus terms that are antisymmetric in i, j. This is equal to

$$f^*\partial_j\left(q^k_\alpha \partial_k(p_{i\beta}) - q^k_\beta \partial_k(p_{i\alpha}) + p_{k\beta}\partial_i(q^k_\alpha) + \partial_i(p_{k\alpha})q^k_\beta - p_{i\gamma}C^\gamma_{\alpha\beta}\right) v^i v'^j,$$

again plus terms that are antisymmetric in i, j. We can recognize this expression as

$$d\left(L_{Q_\alpha}P_\beta - \iota_{Q_\beta}dP_\alpha - C^\gamma_{\alpha\beta}P_\gamma\right)(v^i, v'^j),$$

which vanishes by the integrability conditions (5.6)–(5.7).

Remark 5.8. We observe that the proof of Theorem 5.5 does not rely on the assumption that $\mathcal{D}$ is *maximally* isotropic. Therefore, the result applies to any isotropic subbundle of $TM \oplus T^*M$ satisfying the integrability condition (5.5). The next section aims to address the question of what distinguishes the isotropic case from the maximally isotropic case.

6. Dirac structures and Lagrangian sub-2-groupoids

In Section 5, we considered the geometry of the inclusion $F_\bullet : \mathfrak{G}_\bullet(\mathcal{D}) \hookrightarrow \mathfrak{C}_\bullet(M)$, when $\mathcal{D}$ is a Dirac structure. Consider the composition of this map with the truncation map $\tau_{\le 2} : \mathfrak{C}(M) \to \mathrm{LWX}(M)$, and let $L_\mathcal{D}$ be the image of $\mathfrak{G}_2(\mathcal{D})$ in $\mathrm{LWX}_2(M)$. The image $L_\mathcal{D}$ determines a sub-2-groupoid $\mathfrak{L}_\mathcal{D}$ of $\mathrm{LWX}(M)$, and the 1-truncation of $\mathfrak{L}_\mathcal{D}$ can be identified with the 1-truncation of $\mathfrak{G}(\mathcal{D})$, which, as we noted in Section 5, is the Lie groupoid G integrating $\mathcal{D}$. As is explained in the proof of Lemma 5.1, $\mathfrak{G}_2(\mathcal{D})$ can be identified with the mapping space from Δ^2 to s-fibers of G such that $0 \in \Delta^2$ is mapped into the unit space of G. An argument similar (but easier) to the proof of Theorem 3.4 shows that $\tau_{\le 2}(\mathfrak{G}_\bullet(\mathcal{D}))_2$ is a smooth Banach manifold and therefore $\mathfrak{L}_\mathcal{D}$ is a Lie 2-groupoid.[3] We would now like to consider the geometry of the sub-2-groupoid $\mathfrak{L}_\mathcal{D} \subset \mathrm{LWX}(M)$.

For $x \in M$, the zero bundle map $\Psi^x : T\Delta \to \mathcal{D}$ over the constant map $f^x : \Delta \to M$, $s \mapsto x$, defines a point in $L_\mathcal{D}$. This defines an embedding $M \hookrightarrow L_\mathcal{D} \subset \mathrm{LWX}_2(M)$. We can think of this image of M as being the space of "units", since it is equal to the image of M under the "double degeneracy" maps. Recall that a subspace L of a (weak) symplectic Banach space (B, Ω) is *Lagrangian* if L is maximally isotropic with respect to Ω.

Proposition 6.1. *For all $x \in M$, $T_xL_\mathcal{D}$ is a Lagrangian subspace of $T_x\mathrm{LWX}_2(M)$.*

Proof. By Theorem 5.5, we already know that $L_\mathcal{D}$ is isotropic. It remains to show that $T_xL_\mathcal{D}$ is coisotropic, from which we can conclude that $T_xL_\mathcal{D}$ is a Lagrangian subspace.

We use the local description and notation from Section 5.2. Choose coordinates on M for which $x = 0$. Then, if we write $\Psi^x = (f^i, \psi^\alpha)$ as in Section 5.2, we have $f^i = 0$ and $\psi^\alpha = 0$.

Let (v^i, μ^α) be a tangent vector on $\mathfrak{G}_2(\mathcal{D})$ at Ψ^x. In this case, (5.10) and (5.11) reduce to

$$dv^i = q^i_\alpha(0)\mu^\alpha, \tag{6.1}$$
$$d\mu^\alpha = 0. \tag{6.2}$$

Any solution to (6.1)–(6.2) can be written in the form

$$v^i = q^i_\alpha(0)g^\alpha + c^i, \tag{6.3}$$
$$\mu^\alpha = dg^\alpha, \tag{6.4}$$

for some C^2 functions g^α on Δ^2 and constants c^i. From (5.12), we have that the induced tangent vector on $\mathfrak{C}_2(M)$ has

$$\chi_i = p_{i\alpha}(0)\mu^\alpha = p_{i\alpha}(0)dg^\alpha. \tag{6.5}$$

[3] In general, it is proved in [31, Theorem 1.2] that the truncation of $\mathfrak{G}_\bullet(\mathcal{D})$ at level 2 always gives a Lie 2-groupoid.

At the level of tangent vectors, the truncation map $\tau_{\leq 2}$ has the effect of pulling back to $\partial\Delta^2$. In what follows, let $j : \partial\Delta^2 \to \Delta^2$ be the natural inclusion map.

To prove that $T_x L_{\mathcal{D}}$ is coisotropic, we will show that, if any tangent vector $(v'^i, \chi_i') \in T_x\mathfrak{C}_2(M)$ is such that $\omega_2^H((v^i, \chi_i), (v'^i, \chi_i')) = 0$ for all (v^i, χ_i) of the form (6.3), (6.5), then $(j^* v'^i, j^*\chi_i')$ takes the same form.

We compute

$$\begin{aligned}\omega_2^H\left((v^i, \chi_i), (v'^i, \chi_i')\right) &= \int_{\Delta^2} d\left((q_\alpha^i(0)g^\alpha + c^i)\chi_i' - v'^i p_{i\alpha}(0)dg^\alpha\right) \\ &= \int_{\partial\Delta^2} c^i\chi_i' + \int_{\partial\Delta^2}\left(q_\alpha^i(0)\chi_i' + p_{i\alpha}(0)dv'^i\right)g^\alpha.\end{aligned}$$

The requirement that this vanishes for all g^α and c^i implies that $\int_{\partial\Delta^2}\chi_i'$ vanishes for all i, and that $j^*(q_\alpha^i(0)\chi_i' + p_{i\alpha}(0)dv'^i)$ vanishes for all α. From the first condition, we have that $j^*\chi_i'$ is exact, so let Λ_i be functions on $\partial\Delta^2$ such that $j^*\chi_i' = d\Lambda_i$. From the latter condition, we then have that

$$\begin{aligned}0 &= d(q_\alpha^i(0)\Lambda_i + p_{i\alpha}(0)j^*v'^i) \\ &= d\langle\Theta_\alpha(0), j^*v'^i\partial_i + \Lambda_i dx^i\rangle.\end{aligned} \tag{6.6}$$

Let 0 denote the 0th vertex of Δ^2, and let $e^i = v'^i(0)$, $\varepsilon_i = \Lambda_i(0)$. Then (6.6) implies that

$$(j^*v'^i - e^i)\partial_i + (\Lambda_i - \varepsilon_i)dx^i$$

annihilated $\mathcal{D}$. Using the fact that $\mathcal{D}$ is maximally isotropic, we deduce that there exist unique functions Φ^α on $\partial\Delta^2$ such that

$$(j^*v'^i - e^i)\partial_i + (\Lambda_i - \varepsilon_i)dx^i = \Phi^\alpha\Theta_\alpha(0),$$

implying that

$$j^*v'^i = q_\alpha^i(0)\Phi^\alpha + e^i, \tag{6.7}$$

$$\Lambda_i = p_{i\alpha}(0)\Phi^\alpha + \varepsilon_i. \tag{6.8}$$

Differentiating the latter equation, we have

$$j^*\chi_i' = p_{i\alpha}(0)d\Phi^\alpha. \tag{6.9}$$

From (6.7) and (6.9), we see that $j^*v'^i$ and $j^*\chi_i'$ indeed take the desired form of (6.3) and (6.5). □

Definition 6.2. An embedded submanifold L of a (weak) symplectic Banach manifold (B, Ω) is *Lagrangian* if the tangent space T_xL at each point $x \in L$ is a Lagrangian subspace of (T_xB, Ω_x), i.e. $T_xL \subset T_xB$ is a maximally isotropic subspace with respect to Ω_x.

A sub-2-groupoid $L_\bullet$ of a symplectic 2-groupoid $(G_\bullet, \Omega)$ is called *Lagrangian* if L_2 is a Lagrangian submanifold of G_2.

Proposition 6.1 suggests the following conjecture.

Conjecture 6.3. *$\mathfrak{L}_{\mathcal{D}}$ is a Lagrangian sub-2-groupoid of the symplectic 2-groupoid* LWX(M)*, i.e. L_D is a Lagrangian submanifold of* LWX$_2(M)$.

It is well-known that, if $B \in \Omega^2(M)$ is a closed 2-form on M, then the graph of $B^\flat : TM \to T^*M$ is a Dirac structure (with $H = 0$). We prove Conjecture 6.3 in this special case.

Proposition 6.4. *Let B be a closed 2-form on M, and let $\mathcal{D} \subset TM \oplus T^*M$ be the graph of $B^\flat$. Then $\mathfrak{L}_{\mathcal{D}}$ is a Lagrangian sub-2-groupoid of* LWX(M).

Proof. Because of Theorem 5.5, we only need to show that $L_{\mathcal{D}}$ is coisotropic. In the notation of Section 5.2, we can take the local trivialization of $\mathcal{D}$ to be given by $\Theta_i = \partial_i + B_{ij}dx^j$, where $B = \frac{1}{2}B_{ij}dx^i \wedge dx^j$. In this frame, the structure functions C_{ij}^k vanish.

Using (5.8), (5.9), and the above description of Θ_i, we have that a point $\Psi \in \mathfrak{G}_2(\mathcal{D})$ is given by (f^i, ψ^i), where $df^i = \psi^i$ and $\psi^i = 0$ (of course, the latter condition is redundant). Similarly, Eqs. (5.10) and (5.11), describing a tangent vector (v^i, μ^i) at Ψ, reduce to $dv^i = \mu^i$, $d\mu^i = 0$. From (5.12), we have that the induced tangent vector on $\mathfrak{C}_2(M)$ is of the form

$$\chi_i = v^k f^*(\partial_k B_{ij})\psi^j + f^*(B_{ij})\mu^j = v^k f^*(\partial_k B_{ij})df^j + f^*(B_{ij})dv^j. \tag{6.10}$$

As in the proof of Proposition 6.1, let $j : \partial\Delta^2 \to \Delta^2$ be the natural inclusion map. To prove that $L_{\mathcal{D}}$ is coisotropic, we will show that, if any tangent vector (v'^i, χ_i') is such that $\omega_2((v^i, \chi_i), (v'^i, \chi_i')) = 0$ for all v^i, with χ_i of the form (6.10), then $j^*\chi_i'$

takes the same form. We compute

$$\omega_2\left((v^i,\chi_i),(v'^i,\chi'_i)\right)=\int_{\partial\Delta^2} v^i\chi'_i - v'^i\left(v^k f^*(\partial_k B_{ij})df^j + f^*(B_{ij})dv^j\right)$$
$$=\int_{\partial\Delta^2} v^i\left(\chi'_i - v'^k f^*(\partial_i B_{kj})df^j - d(v'^j f^*(B_{ij}))\right).$$

The requirement that this vanishes for all v^i implies that

$$\begin{aligned} j^*\chi'_i &= j^*\left(v'^k f^*(\partial_i B_{kj})df^j + d(v'^j f^*(B_{ij}))\right)\\ &= j^*\left(v'^k f^*(\partial_i B_{kj})df^j + v'^j f^*(\partial_k B_{ij})df^k + f^*(B_{ij})dv'^j\right). \end{aligned} \tag{6.11}$$

Using the fact that B is closed, we may rewrite (6.11) as

$$j^*\chi'_i = j^*\left(v'^k f^*(\partial_k B_{ij})df^j + f^*(B_{ij})dv'^j\right),$$

showing that $j^*\chi'_i$ takes the desired form. □

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References

[1] T.J. Courant, A. Weinstein, Beyond Poisson structures, in: Action Hamiltoniennes de Groupes, Troisième Théorème de Lie, (Lyon, 1986), in: Travaux en Cours, vol. 27, Hermann, Paris, 1988, pp. 39–49.
[2] Z. Liu, A. Weinstein, P. Xu, Manin triples for Lie bialgebroids, J. Differential Geom. 45 (3) (1997) 547–574.
[3] H. Bursztyn, M. Crainic, A. Weinstein, C. Zhu, Integration of twisted Dirac brackets, Duke Math. J. 123 (3) (2004) 549-607.
[4] M. Crainic, R. Fernandes, Integrability of Lie brackets, Ann. of Math. (2) 157 (2) (2003) 575–620.
[5] P. Ševera, Some title containing the words "homotopy" and "symplectic", e.g. this one, in: Travaux Mathématiques. Fasc. XVI, in: Trav. Math., vol. XVI, Univ. Luxemb., Luxembourg, 2005, pp. 121–137.
[6] R.A. Mehta, X. Tang, From double Lie groupoids to local Lie 2-groupoids, Bull. Braz. Math. Soc. (N.S.) 42 (4) (2011) 651–681.
[7] D. Li-Bland, P. Ševera, Integration of exact Courant algebroids, Electron. Res. Announc. Math. Sci. 19 (2012) 58–76.
[8] Y. Sheng, C. Zhu, Higher extensions of Lie algebroids and application to Courant algebroids, 2011, arXiv:1103.5920.
[9] P. Ševera, L_∞-algebras as first approximations, in: XXVI Workshop on Geometrical Methods in Physics, AIP Conf. Proc., Vol. 956, Amer. Inst. Phys., Melville, NY, 2007, pp. 199–204.
[10] A. Cattaneo, G. Felder, Poisson sigma models and symplectic groupoids, in: Quantization of Singular Symplectic Quotients, in: Progr. Math., vol. 198, Birkhäuser, Basel, 2001, pp. 61–93.
[11] J. Villatoro, Poisson manifolds and their associated stacks, 2017. arXiv:1703.03542.
[12] D. Roytenberg, Courant Algebroids, Derived Brackets and Even Symplectic Supermanifolds (Ph.D. thesis), –University of California, ProQuest LLC, Ann Arbor, MI, Berkeley, 1999.
[13] D. Roytenberg, On the structure of graded symplectic supermanifolds and Courant algebroids, in: Quantization, Poisson Brackets and beyond, (Manchester, 2001), in: Contemp. Math., vol. 315, Amer. Math. Soc., Providence, RI, 2002, pp. 169–185.
[14] P. Ševera, A. Weinstein, Poisson geometry with a 3-form background, Progr. Theoret. Phys. Suppl. 144 (2001) 145–154, Noncommutative geometry and string theory (Yokohama, 2001).
[15] P. Ševera, Letters to Alan Weinstein, 1998–2000, http://sophia.dtp.fmph.uniba.sk/~severa/letters.
[16] T.J. Courant, Dirac manifolds, Trans. Amer. Math. Soc. 319 (2) (1990) 631–661.
[17] A. Henriques, Integrating L_∞-algebras, Compos. Math. 144 (4) (2008) 1017–1045.
[18] J. Duskin, Higher-dimensional torsors and the cohomology of topoi: the abelian theory, in: Applications of Sheaves (Proc. Res. Sympos. Appl. Sheaf Theory to Logic, Algebra and Anal., Univ. Durham, Durham, 1977), in: Lecture Notes in Math., vol. 753, Springer, Berlin, 1979, pp. 255–279.
[19] C. Zhu, Kan replacement of simplicial manifolds, Lett. Math. Phys. 90 (1–3) (2009) 383–405.
[20] B. Pym, P. Safronov, Shifted symplectic Lie algebroids, 2016, arXiv:1612.09446.
[21] R.A. Mehta, X. Tang, Constant symplectic 2-groupoids, Lett. Math. Phys. (2017) (in press).
[22] J.J. Duistermaat, J.A.C. Kolk, Lie Groups, in: Universitext, Springer-Verlag, Berlin, 2000.
[23] D. Sullivan, Infinitesimal computations in topology, Publ. Math. Inst. Hautes Études Sci. 1977 (47) (1978) 269–331.
[24] M. Alexandrov, M. Kontsevich, A. Schwarz, O. Zaboronsky, The geometry of the master equation and topological quantum field theory, Internat. J. Modern Phys. A 12 (7) (1997) 1405–1429.
[25] R. Bott, H. Shulman, J. Stasheff, On the de Rham theory of certain classifying spaces, Adv. Math. 20 (1) (1976) 43–56.
[26] D. Roytenberg, AKSZ-BV formalism and Courant algebroid-induced topological field theories, Lett. Math. Phys. 79 (2) (2007) 143–159.
[27] R. Fernandes, I. Marcut, Private communication.
[28] P. Ševera, Private communication.
[29] P. Ševera, M. Širaň, Integration of differential graded manifolds, 2015. arXiv:1506.04898.
[30] A. Cattaneo, On the integration of Poisson manifolds, Lie algebroids, and coisotropic submanifolds, Lett. Math. Phys. 67 (1) (2004) 33–48.
[31] C. Zhu, Lie ii theorem for lie algebroids via stacky lie groupoids, Trends Math. (2008) 115–139.