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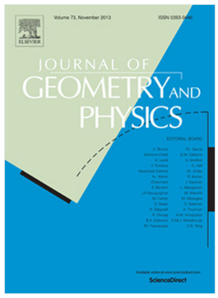

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Application of the moving frame method to deformed Willmore surfaces in space forms

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Abstract

The main goal of this paper is to use the theory of exterior differential forms in deriving variations of the deformed Willmore energy in space forms and study the minimizers of the deformed Willmore energy in space forms. We derive both first and second order variations of deformed Willmore energy in space forms explicitly using moving frame method. We prove that the second order variation of deformed Willmore energy depends on the intrinsic Laplace Beltrami operator, the sectional curvature and some special operators along with mean and Gauss curvatures of the surface embedded in space forms, while the first order variation depends on the extrinsic Laplace Beltrami operator.

Keywords: Deformed Willmore energy, Space forms, moving frame

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1. Introduction

The Willmore energy of a surface $M \subset \mathbb{R}^3$ is defined as

$$W(M) = \int_M H^2 \, dS, \tag{1}$$

where H is the mean curvature of the surface M and dS is the area element. This is also called classical bending energy and its applications appear in various fields of science and technology such as cell biology, optical design, nonlinear plate theory, nano tubes etc (see, e.g., [1], [2], [3],[4] and references therein).

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From the theoretical view point Willmore conjecture was one of the prominent results in the study of Willmore energies which was proved by F. C. Marques and A. A. Neves using min-max theory of minimal surfaces in [5]. This energy captures how much the surface has deviated from a round sphere. The minimizers of (1) are called Willmore surfaces which are the solutions of the Euler-Lagrange equation corresponding to (1). The Willmore energy is important in the context of conformal geometry as it is known to be invariant under the conformal transformations of $\mathbb{R}^3$; more details about this can be found in [6],[7],[8], [9], [10] and [11].

The generalization of Willmore energy connects to the elasticity of membranes which was studied by many authors in the literature (see [12],[13] and [14]). In [1], the elasticity bending energy of a membrane is defined by

$$E(M) = \int_M \epsilon + \beta(H - c_0)^2 - \gamma K \, dS,$$

where ϵ is the surface tension which describes the interaction between the membrane material and the ambient fluid material and β, γ, c_0 are the elastic constants which determines the inner interaction of the membrane and the spontaneous curvature of the membrane respectively. When $\epsilon, \gamma, c_0 = 0$ and $\beta = 1$ this reduces to (1). The [15] studies the behaviour of the Willmore energy under infinitesimal bending of a surface. In addition, the Willmore energy of curves under second order infinitesimal bending energy was studied in [16].

We define our generalized Willmore energy functional associated to a surface M immersed in $\mathbb{R}^3$ as

$$W(M) = \int_M (H^2 + \epsilon) \, dS. \tag{2}$$

The Euler-Lagrange equation corresponding to the functional (2) was first obtained in [17] as a particular case of a most general variational problem. By considering surface tension and neglecting other rigidities, the Euler-Lagrange equation,

$$\Delta H + 2H(H^2 - K - \epsilon) = 0 \tag{3}$$

corresponding to (2) was deduced in [18] and [19], where Δ is the Laplace Beltrami operator induced by the metric and K is the Gauss curvature of the surface . Its biological applications are demonstrated in [20]. The solutions of (3) are called generalized Willmore surfaces and in the case of $H = 0$, they are minimal surfaces. Further the studies in [21], [22], [23] and [24] motivated us to study Generalized Willmore flow of graphs and its numerical applications using automatic differentiation tools which was a novel approach in the study of Willmore flow (see [25]).

In [18] and [26] we studied Willmore-type energies and Willmore-type surfaces in space forms and deduced the Euler-Lagrange equation of the deformed Willmore energy in a space form using an extrinsic Laplace-Beltrami operator which depends on the metric of the surface and sectional curvature of the space form. Papers [14] and [15] motivated us to derive both first and second order variations of our deformed Willmore energy in space forms. In [26], we considered only normal variation of a surface.

In this report, we consider the variation of a surface embedded in $\mathbb{M}^3(k_0)$ along with a moving frame. Using exterior differential forms we derive the variations of the deformed Willmore energy in $\mathbb{M}^3(k_0)$. The first order variation of the deformed Willmore energy in space forms depends on the extrinsic Laplace Beltrami operator and the sectional curvature of ambient space form and the second order variation depends on the intrinsic Laplace Beltrami operator, the sectional curvature and some special operators ∇_K,$\nabla \cdot \nabla_K$ which will be defined in the next section. Then we study the minimizers of the deformed Willmore energy in space forms.

2. Background

Let p be a point of a Riemannian manifold M and let $\sigma \subset T_pM$ be a two dimensional subspace of the tangent space T_pM of M at p. The real number $k(x, y) = k(\sigma)$, where $\{x, y\}$ is an arbitrary basis of σ, is called the sectional curvature of σ at p. It measures the curvature of a Riemannian manifold. In other words, it is defined by

$$k(u,v) = \frac{\langle R(u,v)v, u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2},$$

where R is the Riemann curvature tensor which is given by

$$R(u,v)v, u = \nabla_v \nabla_u v - \nabla_u \nabla_v v + \nabla_{[u,v]} v,$$

and u, v are linearly independent tangent vectors at a point on the Riemannian manifold. Here, ∇ is the Riemann connection of M and $[u, v]$ denotes Lie brackets of u and v. If u and v are orthonormal, then $k(u,v) = \langle R(u,v)v, u\rangle$. If $M = \mathbb{R}^n$, then $\langle R(u,v)v, u\rangle = 0$ for all tangent vectors u and v, which implies $k(u,v) = 0$. In our work, the sectional curvature, $k(u,v) = k_0$ is a constant. The complete Riemannian manifolds with constant sectional curvature k_0 are said to be space forms. We consider a three-dimensional space form and it is denoted by $\mathbb{M}^3(k_0)$. When $k_0 = 1, k_0 = 0$ and $k_0 = -1$, the space form becomes $\mathbb{S}^3$, $\mathbb{R}^3$ and $\mathbb{H}^3$ spaces respectively.

The deformed Willmore energy in space form $\mathbb{M}^3(k_0)$ is defined by

$$\tilde{W}(M; k_1) = \int_M (H^2 + k_1)\, dS \tag{4}$$

where k_1 is an arbitrary constant. When $k_0, k_1 = 0$ this reduces to the classical Willmore energy in $\mathbb{R}^3$. When $k_0 = 0$ and $k_1 \neq 0$, this deformed Willmore energy reduces to (2). In that case, if we add a constant k_1 to the integrand of (4) then we subtract the same constant from the quantity $H^2 - K$ in the corresponding Euler-Lagrange equation as in [18] and [26].

Now, we review some basic facts about differential forms in connections with concepts in differential geometry. More details can be found in [27]. Let us consider $\mathbf{r} : M \to \mathbb{M}^3(k_0)$ to be an immersion of a smooth orientable surface M embedded in $\mathbb{M}^3(k_0)$. We start by recalling few basic facts about local geometry of M at a point $p \in M$. Let $O \subset M$ be a neighbourhood of p such that the restriction $\mathbf{r}|O$ is an embedding. Let $P \subset \mathbb{M}^3(k_0)$ be a neighbourhood of p such that $P \cap \mathbf{r}(M) = \mathbf{r}(O)$. Now it is possible to construct a moving frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ such that $\mathbf{e}_1, \mathbf{e}_2$ are tangent vectors to $\mathbf{r}(O)$ and $\mathbf{e}_3$ is normal to

$\mathbf{r}(O)$. Note that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ $(i, j = 1, 2, 3)$. Also, $d\mathbf{r} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2$, where $\{\omega_i\}$ is the coframe forms associated to the frame $\{e_i\}$. The connection forms are given by $\omega_{ij} = -\omega_{ji}$, $i, j = 1, 2, 3$ which satisfy the following structure equations of M with the additional relation $\omega_3 = 0$.

$$d\omega_1 = \omega_2 \wedge \omega_{21}, \tag{5}$$

$$d\omega_2 = \omega_1 \wedge \omega_{12}, \tag{6}$$

$$\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0, \tag{7}$$

$$d\omega_{12} = \omega_{13} \wedge \omega_{32}, \tag{8}$$

$$d\omega_{13} = \omega_{12} \wedge \omega_{23}, \tag{9}$$

and

$$d\omega_{23} = \omega_{21} \wedge \omega_{13}. \tag{10}$$

Using (7) and Cartan's lemma, we have

$$\omega_{13} = h_{11}\omega_1 + h_{12}\omega_2 \tag{11}$$

and

$$\omega_{23} = h_{21}\omega_1 + h_{22}\omega_2, \tag{12}$$

where $h_{ij} = h_{ji}$ are differentiable functions.

The area element is defined as $dS = \omega_1 \wedge \omega_2$. The first fundamental form is given by

$$I := d\mathbf{r} \cdot d\mathbf{r} = \omega_1^2 + \omega_2^2. \tag{13}$$

The second fundamental form is

$$II := -d\mathbf{r} \cdot d\mathbf{e}_3 = h_{ij}\omega_i\omega_j. \tag{14}$$

The mean curvature H and the Gauss curvature K of M are given by

$$H = \frac{h_{11} + h_{22}}{2} \quad \text{and} \quad K = h_{11}h_{22} - h_{12}^2 \tag{15}$$

respectively. It can be easily obtained that

$$d\omega_{12} = -K\omega_1 \wedge \omega_2 \quad \text{and} \quad \omega_{13} \wedge \omega_2 + \omega_1 \wedge \omega_{23} = 2H\omega_1 \wedge \omega_2. \tag{16}$$

The fundamental properties of the Hodge star operator ($*$) are given by the following formulae

$$*\xi = \xi\, \omega_1 \wedge \omega_2, \tag{17}$$

$$*\omega_1 = \omega_2, *\omega_2 = -\omega_1, \tag{18}$$

and

$$d * d\phi = \Delta\phi\, \omega_1 \wedge \omega_2, \tag{19}$$

where ϕ, ξ are functions defined on M and Δ is the intrinsic Laplace Beltrami operator on M (see[14] and [27]).

The extrinsic Laplace Beltrami operator, $\tilde{\Delta}$, in $\mathbb{M}^3(k_0)$ is defined by

$$\tilde{\Delta} = \Delta + 2k_0. \tag{20}$$

The following relations and operators can be easily proved using orthogonal local coordinates (u^1, u^2) at a point in a surface. Then we have

$$d\phi(u^1, u^2) = \phi_1\omega_1 + \phi_2\omega_2. \tag{21}$$

If we consider (13) in Einstein notation then we have

$$I = g_{11}(du^1)^2 + g_{22}(du^2)^2, \tag{22}$$

where $g_{ij} = \langle \frac{\partial \mathbf{r}}{\partial u^1}, \frac{\partial \mathbf{r}}{\partial u^2} \rangle$. Since $\omega_1 = \sqrt{g_{11}}du^1$ and $\omega_2 = \sqrt{g_{22}}du^2$, (14) implies

$$II = h_{11}g_{11}(du^1)^2 + 2h_{12}\sqrt{g_{11}}\sqrt{g_{22}}du^1du^2 + h_{22}g_{22}(du^2)^2. \tag{23}$$

By taking $h_{11}g_{11} = l_{11}$, $h_{12}\sqrt{g_{11}}\sqrt{g_{22}} = l_{12}$ and $h_{22}g_{22} = l_{22}$,

$$II = l_{ij}du^idu^j,$$

which is the second fundamental form in Einstein notation. Then we can define

$$\nabla_K = Kl^{ij}\mathbf{r}_i \frac{\partial}{\partial u^j}, \tag{24}$$

where $l^{ij} = (l_{ij})^{-1}$. Then we have

$$\nabla \cdot \nabla_K \phi = \frac{1}{\sqrt{g_{11}g_{22}}} \frac{\partial}{\partial u^i} \left(\sqrt{g_{11}g_{22}} K l^{ij} \frac{\partial \phi}{\partial u^j} \right) \tag{25}$$

and

$$d\tilde{*}\tilde{d}\phi = \nabla \cdot \nabla_K \phi \omega_1 \wedge \omega_2, \tag{26}$$

where $\tilde{*}\tilde{d}\phi = -\phi_2\omega_{13} + \phi_1\omega_{23}$.

3. Variations of deformed Willmore functional in space forms

In this section, we derive both first and second order variations of (4) of a surface embedded in $\mathbb{M}^3(k_0)$. We consider the variation of a surface in $\mathbb{M}^3(k_0)$ along an orthonormal moving frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and show that the variations of deformed Willmore energy along the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ vanish for a surface without a boundary and with a boundary in $\mathbb{M}^3(k_0)$ and then we obtain the variation formulae in the normal direction $\mathbf{e}_3$ to the surface.

3.1. Variation of a surface in $\mathbb{M}^3(k_0)$

Let M be a surface with mean curvature H and Gauss curvature K in space forms $\mathbb{M}^3(k_0)$ of sectional curvature k_0. Then $M' := \{\mathbf{r}' | \mathbf{r}' = \mathbf{r} + \delta\mathbf{r}\}$ is the deformed surface, where $\delta\mathbf{r}$ is the variation of the surface M and it is denoted by

$$\delta\mathbf{r} = \delta_1\mathbf{r} + \delta_2\mathbf{r} + \delta_3\mathbf{r}. \tag{27}$$

where

$$\delta_1\mathbf{r} = \Omega_1\mathbf{e}_1, \quad \delta_2\mathbf{r} = \Omega_2\mathbf{e}_2, \quad \text{and} \quad \delta_3\mathbf{r} = \Omega_3\mathbf{e}_3, \tag{28}$$

Ω_1, Ω_2 and Ω_3 are smooth functions. The moving frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ should be changed because of the deformation of M. Therefore, we have

$$\delta_k\mathbf{e}_i = \Omega_{kij}\mathbf{e}_j, \quad i, j, k = 1, 2, 3. \tag{29}$$

Since $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ $(i, j = 1, 2, 3)$, $\Omega_{kij} = -\Omega_{kji}$. Since the operators δ and d(exterior differential operator) are commutative we have $d\delta_k\mathbf{r} = \delta_k d\mathbf{r}$ and $d\delta_k\mathbf{e}_j = \delta_k d\mathbf{e}_j$.

3.2. First order variation of deformed Willmore energy

Here we derive the first order variation of (4) using exterior differential forms. We consider two cases a surface without a boundary and a surface with a boundary separately.

Theorem 1. *Let $\mathbf{r} : M \to \mathbb{M}^3(k_0)$ be an immersion of a smooth orientable surface M embedded in $\mathbb{M}^3(k_0)$. Let $M' := \{\mathbf{r}' | \mathbf{r}' = \mathbf{r} + \delta\mathbf{r}\}$ be a deformed surface of $M \subset \mathbb{M}^3(k_0)$, where $\delta\mathbf{r}$ is given by* (27)*. Then the first variation of* (4) *is given by*

$$\delta\tilde{W}(M; k_1) = \int_M \left(\tilde{\Delta}H + 2H(H^2 - K + k_0 - k_1)\right)\Omega_3 \, dS.$$

where $\tilde{\Delta}$ is the extrinsic Laplace Beltrami operator on M.

Proof. Case I: M is a surface without a boundary.

Let us consider

$$\delta\tilde{W}(M; k_1) = \delta_1\tilde{W}(M; k_1) + \delta_2\tilde{W}(M; k_1) + \delta_3\tilde{W}(M; k_1). \tag{30}$$

We calculate $\delta_1\tilde{W}(M; k_1), \delta_2\tilde{W}(M; k_1)$, and $\delta_3\tilde{W}(M; k_1)$. First we show that $\delta_1\tilde{W}(M; k_1) \equiv 0$ and $\delta_2\tilde{W}(M; k_1) \equiv 0$. By taking the variation δ_1 of (4), we have

$$\begin{aligned} \delta_1\tilde{W}(M; k_1) &= \delta_1\left(\int_M (H^2 + k_1) \, dS\right) \\ &= \int_M \left(2H\delta_1(H)\omega_1 \wedge \omega_2 + H^2\delta_1(\omega_1 \wedge \omega_2)\right) \\ &\quad + k_1 \int_M \delta_1(\omega_1 \wedge \omega_2)). \end{aligned}$$

Then we evaluate $\delta_1 H$ and $\delta_1(\omega_1 \wedge \omega_2)$.

$$\delta_1(\omega_1 \wedge \omega_2) = \delta_1\omega_1 \wedge +\omega_1 \wedge \delta_1\omega_2. \tag{31}$$

The following fundamental formulae can be found in [14].

$$\delta_1\omega_1 = d\Omega_1 - \omega_2\Omega_{121}, \tag{32}$$

$$\delta_2\omega_2 = \Omega_1\omega_{12} - \omega_1\Omega_{112}, \tag{33}$$

$$\Omega_{113} = h_{11}\Omega_1, \quad \text{and} \quad \Omega_{123} = h_{12}\Omega_1, \tag{34}$$

and

$$\delta_k \omega_{ij} = d\Omega_{kij} + \Omega_{kim}\omega_{mj} - \omega_{im}\Omega_{kmj}. \tag{35}$$

Using (32) and (33), (31) becomes

$$\delta_1(\omega_1 \wedge \omega_2) = d\Omega_1 \wedge \omega_2 + \Omega_1\omega_1 \wedge \omega_{12} = d(\Omega_1\omega_2). \tag{36}$$

Taking wedge product of (11) with ω_{12}, we obtain

$$d\omega_{23} = h_{11}d\omega_2 - h_{12}d\omega_1. \tag{37}$$

On the other hand, from (12) we have

$$d\omega_{23} = d(h_{21}) \wedge \omega_1 + h_{21}d\omega_1 + d(h_{22}) \wedge \omega_2 + h_{22}d\omega_2. \tag{38}$$

By equating (37) and (38), we obtain

$$d(h_{12}) \wedge \omega_1 + 2h_{12}d\omega_1 = (h_{11} - h_{22})d\omega_2 - d(h_{22}) \wedge \omega_2. \tag{39}$$

Using (39), (34) and (35) we have

$$\delta_1 H\omega_1 \wedge \omega_2 = dH \wedge \omega_2\Omega_1. \tag{40}$$

Therefore,

$$\delta_1\tilde{W}(M;k_1) = \int_M (2HdH) \wedge \omega_2\Omega_1 + (H^2 + k_1)d(\Omega_1\omega_2)) \tag{41}$$

$$= \int_M d((H^2 + k)\Omega_1\omega_2) = 0 \tag{42}$$

since M is a closed surface. Similarly, it is easily showed that

$$\delta_2\tilde{W}(M;k_1) = 0.$$

Now, we consider

$$\delta_3\tilde{W}(M;k_1) = \delta_3\Big(\int_M (H^2 + k_1)\,dS\Big) = \int_M \delta_3(H^2)\,dS + \int_M H^2\,\delta_3(dS) + k_1\int_M \delta_3(dS). \tag{43}$$

$$\delta_3\,(dS) = \delta_3(\omega_1 \wedge \omega_2) = \delta_3\omega_1 \wedge \omega_2 + \omega_1 \wedge \delta_3\omega_2. \tag{44}$$

Since $\delta_3\omega_1 = \Omega_3\omega_{31} - \omega_2\Omega_{321}$ and $\delta_3\omega_2 = \Omega_3\omega_{32} - \omega_2\Omega_{312}$, (44) becomes

$$\delta_3\omega_1 \wedge \omega_2 + \omega_1 \wedge \delta_3\omega_2 = \Omega_3(\omega_{31} \wedge \omega_2 + \omega_1 \wedge \omega_{32}) \tag{45}$$

Substituting (16) in (45), we obtain

$$\delta_3\,(dS) = -(2H)\Omega_3\,dS. \tag{46}$$

Now, we show that

$$\delta_3(H^2)\,dS = (4H^2 - 2K + 4k_0)H\Omega_3\,dS + H\Delta\Omega_3\,dS. \tag{47}$$

Using (15), we have

$$\delta_3(H^2)\,dS \quad = \quad H(\delta_3(h_{11}) + \delta_3(h_{22}))\omega_1 \wedge \omega_2. \tag{48}$$

By taking variation δ_3 of (11) and (12) we obtain

$$\delta_3\omega_{13} \quad = \quad \delta_3(h_{11})\omega_1 + h_{11}\delta_3\omega_1 + \delta_3(h_{12})\omega_2 + h_{12}\delta_3\omega_2 \tag{49}$$

and

$$\delta_3\omega_{23} \quad = \quad \delta_3(h_{12})\omega_1 + h_{12}\delta_3\omega_1 + \delta_3(h_{22})\omega_2 + h_{22}\delta_3\omega_2. \tag{50}$$

On the other hand, using (35) we have

$$\delta_3\omega_{13} = d\Omega_{313} + \Omega_{311}\omega_{13} - \omega_{12}\Omega_{323}, \tag{51}$$

$$\delta_3\omega_{23} = d\Omega_{323} + \Omega_{321}\omega_{13} - \omega_{22}\Omega_{323}, \tag{52}$$

and

$$d\Omega_3 = \Omega_{313}\omega_1 + \Omega_{323}\omega_2. \tag{53}$$

Using (15) and (18), we end up with the following formulae;

$$\delta_3(h_{11}) = (4H^2 - 2K + 4k_0)\Omega_3, \tag{54}$$

$$\delta_3(h_{22})\omega_1 \wedge \omega_2 = d * d\Omega_3. \tag{55}$$

Substituting (54) and (55) into (48), we obtain

$$\delta_3(H^2)\,dS = (4H^2 - 2K + 4k_0)H\Omega_3\,\omega_1 \wedge \omega_2 + H d * d\Omega_3. \tag{56}$$

Since $d * d\Omega_3 = \Delta\Omega_3\omega_1 \wedge \omega_2 = \Delta\Omega_3\,dS$, (56) implies (47). The Green's second identity implies that

$$\int_M H\Delta\Omega_3\,dS = \int_M \Omega_3\Delta H\,dS, \tag{57}$$

since M is a closed surface.

Using (46) and (47), (43) becomes

$$\begin{aligned}\delta_3\tilde{W}(M;k_1) &= \int_M ((\Delta + 4H^2 - 2K + 4k_0)H)\Omega_3\,dS \\ &+ \int_M H^2(-(2H)\Omega_3)\,dS + \int_M -2k_1 H\Omega_3\,dS \\ &= \int_M \Big((\Delta H + 2H(H^2 - K + 2k_0 - k_1)\Big)\Omega_3\,dS. \end{aligned}\tag{58}$$

Substituting (20) in (58), we obtain

$$\delta\tilde{W}(M;k_1) = \int_M \Big(\tilde{\Delta}H + 2H(H^2 - K + k_0 - k_1)\Big)\Omega_3\,dS. \tag{59}$$

□

Case II: M is a surface in $\mathbb{M}^3(k_0)$ with a boundary

Let

$$\tilde{W}(M;k_1) = \int_M (H^2 + k_1)\,dS + \int_{\partial M} (H^2 + k_1)\,dS. \tag{60}$$

Since $\Omega_1, \Omega_2, \Omega_3 = 0$, we have $d\Omega_1, d\Omega_2, d\Omega_3 = 0$ on the boundary, and taking the variation δ_1 of (60) we obtain

$$\delta_1\tilde{W}(M;k_1) = \int_M d((H^2 + k_1)\Omega_1\omega_2) = \int_{\partial M} (H^2 + k_1)\Omega_1\omega_2 = 0. \tag{61}$$

Similarly, we can easily show that

$$\delta_2\tilde{W}(M;k_1) = 0.$$

In case of a surface with a boundary, the Green's second identity gives

$$\int_M H\Delta\Omega_3\,dS - \Omega_3\Delta H\,dS = \int_{\partial M} (H * d\Omega_3 - \Omega_3 * dH). \tag{62}$$

Since $\Omega_3 = 0$ and $d\Omega_3 = 0$ on the boundary, we have (62) becomes

$$\int_M H\Delta\Omega_3 \, dS - \Omega_3 \Delta H \, dS = 0.$$

In case of taking variation δ_3 of (60), we still obtain (59).

Observe that if M is a surface with a boundary embedded in the space form $\mathbb{M}^3(k_0)$, then the variations δ_1 and δ_2 of (60) are zero and the presence of the boundary does not lead to additional relations. It was earlier shown in [28] and [3] that the variations δ_1 and δ_2 of (60) are zero when the ambient space is $\mathbb{R}^3$.

3.3. Second order variation of deformed Willmore energy in space forms

Theorem 2. *Let* $\mathbf{r} : M \to \mathbb{M}^3(k_0)$ *be an immersion of a smooth orientable surface* M *embedded in* $\mathbb{M}^3(k_0)$. *Let* $M' := \{\mathbf{r}' | \mathbf{r}' = \mathbf{r} + \delta\mathbf{r}\}$ *be a deformed surface of* $M \subset \mathbb{M}^3(k_0)$, *where* $\delta\mathbf{r}$ *is given by* (27). *Then the second order variation of* (4) *is given by*

$$\begin{aligned}
\delta^2 \tilde{W}(M; k_1) = \int_M \Omega_3^2 \bigg(& 8H^4 - 10H^2K + 2K^2 - 8Kk_0 + 12H^2k_0 + 8k_0^2 \\
& + 2Kk_1 - 4k_0k_1 \bigg) \, dS \\
& + \int_M \Omega_3 \Delta\Omega_3 \left(7H^2 - 2K + 4k_0 - k_1 \right) dS \\
& - \int_M 2H\Omega_3 \nabla \cdot \nabla_K \Omega_3 \, dS + \frac{1}{2} \int_M (\Delta\Omega_3)^2 \, dS \\
& + \int_M 2H(\nabla(H\Omega_3) \cdot \nabla\Omega_3 - \nabla\Omega_3 \cdot \nabla_K \Omega_3) \, dS. \qquad (63)
\end{aligned}$$

Proof. We consider

$$\delta^2 \tilde{W}(M; k_1) = \delta_3^2 \int_M H^2 + k_1 \, dS. \qquad (64)$$

Substituting (56) and (46) into (64), we have

$$\begin{aligned}
\delta^2\tilde{W}(M;k_1) &= \delta_3\int_M \Big((4H^2-2K+4k_0)H-2H^3\Big)\Omega_3\,dS \\
&+\delta_3\int_M Hd*d\Omega_3+k_1\int_M \delta_3(-2H\Omega_3\,dS) \\
&=\int_M \delta_3\Big((4H^2-2K+4k_0)H-2H^3\Big)\Omega_3\,dS \\
&+\int_M \Big((4H^2-2K+4k_0)H-2H^3\Big)\Omega_3\delta_3(dS) \\
&+\int_M (\delta_3 Hd*d\Omega_3+H\delta_3(d*d\Omega_3)) \\
&-2k_1\Omega_3\Big(\int_M \delta_3 H\,dS+\int_M H\delta_3(dS)\Big).
\end{aligned}$$

Define

$$I_1 := \int_M \delta_3(2H^3-2KH+4k_0H)\Omega_3\,dS. \tag{65}$$

Now we show that

$$\delta_3 K\,dS = 2KH\Omega_3+\nabla\cdot\nabla_K\Omega_3. \tag{66}$$

From (16), we have

$$\delta_3 K\,dS = -\delta_3(d\omega_{12})-K\delta_3(dS). \tag{67}$$

Using (35) and (46) we obtain (66). Substituting (47) and (66) into (65) we have

$$\begin{aligned}
I_1 = \Omega_3\Big(\int_M 6H^2\Big((2H^2-K+2k_0)\Omega_3+\frac{1}{2}\Delta\Omega_3\Big) \\
-2H\Big(2KH\Omega_3+\nabla\cdot\nabla_K\Omega_3\Big) \\
-2K\Big((2H^2-K+2k_0)\Omega_3+\frac{1}{2}\Delta\Omega_3\Big) \\
+4k_0\Big((2H^2-K+2k_0)\Omega_3+\frac{1}{2}\Delta\Omega_3\Big)\,dS\Big).
\end{aligned} \tag{68}$$

Substituting (57) into (68), we obtain

$$\begin{aligned}
I_1 = \int_M \Omega_3^2\Big(12H^4-14H^2K+2K^2-8Kk_0+20H^2k_0+8k_0^2\Big)\,dS \\
+\Omega_3\int_M (3H^2-K+2k_0)\Delta\Omega_3\,dS-2\Omega_3\int_M H\nabla\cdot\nabla_K\Omega_3\,dS.
\end{aligned} \tag{69}$$

Define

$$I_2 := \int_M \Big((4H^2 - 2K + 4k_0)H - 2H^3 \Big) \Omega_3 \delta_3(dS). \tag{70}$$

Using (46),

$$\begin{aligned} I_2 &= \int_M (2H^3 - 2KH + 4k_0 H)\Omega_3(-2H\Omega_3)\, dS, \\ &= \Omega_3^2 \int_M (-4H^4 + 4KH^2 - 8k_0 H^2)\, dS. \end{aligned} \tag{71}$$

Now we consider

$$\begin{aligned} \delta_3(d * d\Omega_3) = \Big(&2\nabla(H\Omega_3)\cdot\nabla\Omega_3 + 2H\Omega_3\Delta\Omega_3 \\ &-2\nabla\Omega_3\cdot\nabla_K\Omega_3 - 2\Omega_3\nabla\cdot\nabla_K\Omega_3 \Big) dS. \end{aligned} \tag{72}$$

Define

$$I_3 := \int_M (\delta_3 H d * d\Omega_3 + H\delta_3(d * d\Omega_3)). \tag{73}$$

Substituting (72) into (73) we have

$$\begin{aligned} I_3 = \int_M \Big(&(2H^2 - K + 2k_0)\Omega_3 + \frac{1}{2}\Delta\Omega_3 \Big) \Delta\Omega_3\, dS \\ &+ \int_M \Big(2H\nabla(H\Omega_3)\cdot\nabla\Omega_3) + 2H^2\Omega_3\Delta\Omega_3 \\ &-2H\nabla\Omega_3\cdot\nabla_K\Omega_3 - 2H\Omega_3\nabla\cdot\nabla_K\Omega_3 \Big)\, dS, \end{aligned}$$

which simplifies to

$$\begin{aligned} I_3 = \Omega_3 &\int_M (4H^2 - K + 2k_0)\Delta\Omega_3\, dS \\ &+\frac{1}{2}\int_M (\Delta\Omega_3)^2\, dS - \int_M 2H\Omega_3\nabla\cdot\nabla_K\Omega_3\, dS \\ &+\int_M 2H(\nabla(H\Omega_3)\cdot\nabla\Omega_3 - \nabla\Omega_3\cdot\nabla_K\Omega_3)\, dS. \end{aligned} \tag{74}$$

Define

$$I_4 := -2k_1\Omega_3\Big(\int_M \delta_3 H\, dS + \int_M H\delta_3(dS) \Big). \tag{75}$$

Using (46) and (47), (75) becomes

$$I_4 = -2k_1\Omega_3\Big(\int_M \Big((2H^2 - K + 2k_0)\Omega_3 + \frac{1}{2}\Delta\Omega_3\Big)\,dS. \\ +\Omega_3^2\int_M 4k_1 H^2\,dS. \tag{76}$$

$$\delta^2\tilde{W}(M;k_1) = I_1 + I_2 + I_3 + I_4.$$

□

4. Conclusion

We studied deformed Willmore energy in space forms with the framework of exterior differential forms. We considered the variation of a surface embedded in $\mathbb{M}^3(k_0)$ along with the moving frame. In that case we considered the variation of the moving frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and showed that the variations of deformed Willmore energy along the directions $\mathbf{e}_1$ and $\mathbf{e}_2$ are zero and obtained the both first and second order variations of the deformed Willmore energy in $\mathbb{M}^3(k_0)$ in the direction of $\mathbf{e}_3$ which is the normal to the surface. The papers [26] and [18] studied first variation of deformed Willmore energy with framework of Einstein notation and considered only normal variation of the surface. In this paper we derived both first and second order variations using exterior differential forms. In the first variation of the deformed Willmore energy depends on the extrinsic Laplace Beltrami operator $\tilde{\Delta}$, as we described in [26] and [18], the sectional curvature k_0 of the ambient space form, mean and Gauss curvatures of the surface and arbitrary constant k_1. We considered a surface in $\mathbb{M}^3(k_0)$ with a boundary and without a boundary in deriving the variation using exterior differential forms which is a novel approach in the study of Willmore energy in different ambient spaces. Then the second variation of the deformed Willmore energy is expressed in terms of the intrinsic Laplace Beltrami operator Δ, k_0, k_1, mean and Gauss curvatures and some special operators Δ_K, $\Delta \cdot \Delta_K$ which depends on the Gauss curvature and the metric of the surface. When $k_1 = 0$ in $\mathbb{R}^3$, $k_1 = 2$ in $\mathbb{S}^3$ and $k_1 = -2$ in $\mathbb{H}^3$, the minimizers of (4) share the same

Euler-Lagrange equation which is the classical Willmore equation, eventually up to extrinsic shift $2k_0 = k_1$ in the Laplace Beltrami operator. The equation (63) is useful to discuss the stability of the minimizers of (4) in different ambient spaces.

The papers [29] and [30] studies Lawson's 1-1 correspondence which represents CMC surfaces in different ambient spaces and they are called Lawson's cousins. Likewise, there should be a 1-1- correspondence between Willmore surfaces embedded in different ambient space forms. In future, we will study this 1-1 correspondence between isometric families of Willmore surfaces in different ambient space forms together with the generalized harmonic maps and Lie group theory.

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