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On flux quantization in M-theory and the effective action

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Abstract

The quantization law for the antisymmetric tensor field of M-theory contains a gravitational contribution not known previously. When it is included, the low energy effective action of M-theory, including one-loop and Chern–Simons contributions, is well defined. The relation of M-theory to the $E_8 \times E_8$ heterotic string greatly facilitates the analysis.

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1. Introduction

This paper is devoted to explaining some topological points concerning 11-dimensional (11D) M-theory. Roughly speaking, we will explain one new physical effect and show how it enters in reinterpreting, or avoiding, several potential anomalies.

The new physical effect involves the three-form C of 11D supergravity and its field strength $G = \mathrm{d}C$. It has been believed that G is constrained precisely by a flux quantization law, which says that if G is correctly normalized its periods are integer multiples of 2π. As we will see, this is not quite correct.

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For the precise statement, recall first that in M-theory we can assume that the space–time manifold Y is a spin manifold since spinors exist in the theory.[2] For Y a spin manifold, the first Pontryagin class $p_1(Y)$ is divisible by 2 in a canonical way. We set

$$\lambda(Y) = \tfrac{1}{2} p_1(Y). \tag{1.1}$$

λ in turn may or may not be divisible by 2, depending on the topology of Y.

It turns out that the object that must have integral periods is not $G/2\pi$ but $G/2\pi - \frac{1}{2}\lambda$. We denote the cohomology class of $G/2\pi$ as $[G/2\pi]$ and we describe the situation by saying that

$$\left[\frac{G}{2\pi}\right] - \frac{\lambda}{2} \in H^4(Y, \mathbb{Z}). \tag{1.2}$$

(Perhaps a more precise way to say this is that it is not $[G/2\pi]$ but $[G/\pi]$ that is well defined as an integral cohomology class, and that this class is congruent to λ modulo 2.) This shift from the naive quantization law is obtained in Section 2 by a very brief argument using the relation of M-theory to the $E_8 \times E_8$ heterotic string [1]. We also give a second (and perhaps more precise) derivation of the shift using membrane world-volume anomalies. The rest of the paper is devoted to applications of this effect, beginning at the end of Section 2 with an application to M-theory on $(\mathrm{S}^1)^5/\mathbb{Z}_2$.

In Section 3, we reconsider an effect noted recently in compactification of M-theory to three dimensions [2]. It was shown that standard M-theory compactification on $Y = \mathbb{R}^3 \times X$, with X an eight-manifold, is consistent only if X obeys a certain topological condition. We will see that this condition follows from (1.2). Eq. (1.2) implies that G can be zero only if $\lambda(X)$ is divisible by 2 (in the integral cohomology of X). In [2] (as in most discussions of compactification), G was set to zero. An inconsistency therefore arises unless $\frac{1}{2}\lambda(X)$ is integral. As we will show in Section 3, when $\frac{1}{2}\lambda(X)$ is integral, the anomaly found in [2] vanishes.

What to do when $\frac{1}{2}\lambda(X)$ is not integral is now clear. One must endow X with a suitable non-zero G with half-integral periods of $G/2\pi$. Compactifications with non-zero G have not been much studied, but in the case of compactification on a Calabi–Yau four-fold, there is a very elegant framework [3] for incorporating G in a fashion that preserves supersymmetry. (See also [4] for an older example of a different sort.) Of course, one can turn on (integral) G even if $\frac{1}{2}\lambda$ is integral, but when $\frac{1}{2}\lambda$ is non-integral, one *must* take G non-zero.

Integrality of $\frac{1}{2}\lambda$ has an attractive interpretation that enters the argument of Section 3: it is equivalent to evenness of the intersection form on $H^4(X, \mathbb{Z})$.

In Section 4, we move on to a question of which the discussion in Section 3 is really a special case: defining the Chern–Simons interactions of the low energy limit of M-theory. Thus, the quantization of G means that the $C \wedge G \wedge G$ interaction of 11D supergravity is a kind of Chern–Simons coupling. One may wonder whether, given the value of the quantum of G, this coupling is correctly normalized so as to be single-valued modulo 2π.

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[2] In this paper, we generally assume that Y is orientable, although, as M-theory conserves parity (this is one of the points we will re-examine), one could relax this. If Y is unorientable, it should carry a "pin" structure rather than a spin structure. We do consider an unorientable example at the end of Section 2.

The answer is "no"; it is too small by a factor of 6. To make sense of the Chern–Simons interaction requires two ingredients: gravitational corrections to the CGG coupling; and some congruences.

The gravitational corrections come from two sources: a term first computed in [5] and further studied in [6]; and the $\frac{1}{2}\lambda$ term in (1.2). The necessary congruences are potentially baffling – unless one knows the relation of M-theory to the $E_8 \times E_8$ heterotic string, which gives them naturally. Putting everything together, we show that the Chern–Simons couplings are completely well-defined modulo a term that is independent of G.

Section 5 is devoted to the G-independent term, which turns out to be essential in resolving a longstanding though perhaps relatively little-known puzzle. In $8k+3$ dimensions, Dirac-like operators have eigenvalues which are real but not necessarily positive. There is sometimes [7] (see [8–11] for more detail) a $\mathbb{Z}_2$ anomaly affecting the sign of a Dirac determinant. A case in point is the Rarita–Schwinger operator D of 11D supergravity. It has infinitely many positive and negative eigenvalues, raising the question of whether the sign of the fermion path integral can be consistently defined. In uncompactified 11D Minkowski space M^{11}, there is no problem; this follows from the absence of exotic 12-spheres, which implies that the diffeomorphism group of M^{11} is connected, so that there is no "room" for an anomaly. It has not been clear under what circumstances the sign of the Rarita–Schwinger determinant can be consistently defined after compactification.

In field theory, when there is such a $\mathbb{Z}_2$ anomaly, it can generally be canceled by including a Chern–Simons coupling kL_{CS}, where L_{CS} is a suitable Chern–Simons coupling that is well-defined modulo 2π, and k is a *half*-integer (congruent to $\frac{1}{2}$ modulo $\mathbb{Z}$), so that $\mathrm{e}^{\mathrm{i}kL_{\mathrm{CS}}}$ has a sign problem that just cancels the anomaly of $\det D$. The choice of a non-zero k, however, violates parity, so this effect is often described as a "parity anomaly". This has led to speculations that 11D supergravity or M-theory might have a parity anomaly, but this would rule out its relation [1] to the $E_8 \times E_8$ heterotic string, which depends on dividing by an involution that reverses the parity. A parity anomaly of M-theory would also have strange implications for other duality statements in string theory, as orientation-reversing transformations in M-theory are often mapped to known symmetries of string theories.

In Section 5, we resolve these questions. Let $I_M = CGG +$ gravitational terms be the Chern–Simons couplings in the low energy limit of M-theory. The G-dependence of $\mathrm{e}^{\mathrm{i}I_M}$ is well defined, according to our results in Section 4. However, $\mathrm{e}^{\mathrm{i}I_M}$ has a G-independent sign ambiguity, which just equals that of the Rarita–Schwinger determinant $\det D$, so that $\mathrm{e}^{\mathrm{i}I_M} \cdot \det D$ is well defined. Since the other massless fields of M-theory are bosons with manifestly well-defined determinants, this completes the demonstration that the low energy effective action of M-theory is well defined.

Despite this crucial role of the Chern–Simons terms, there is no parity violation. The Chern–Simons coupling that cancels the potential anomaly of the Rarita–Schwinger field is completely parity-invariant, with C and G understood (as usual) to be odd under parity. Roughly speaking, G plays the role that is usually played by the Chern–Simons coupling k in corresponding $8k+3$-dimensional gauge theories. When λ is not divisible by 2, one must choose a non-zero G, with half-integral $G/2\pi$ (just as in the field theory one chooses a non-zero and half-integral k), and this violates parity. But because G is a field (while the usual k is a coupling constant) what one has in M-theory is spontaneous parity violation by the choice of a particular physical state, rather than explicit parity violation of the theory. Notice that, even though G and λ transform oppositely under parity, the relation (1.2) is completely parity-invariant; in fact, as λ is integral, it is equivalent to say that $G/2\pi - \frac{1}{2}\lambda$ is integral or that $G/2\pi + \frac{1}{2}\lambda$ is integral.

2. Flux quantization

2.1. Boundary of the universe

The shifted quantization condition on G that was described in Section 1 is easily motivated given the relation [1] of M-theory to the $E_8 \times E_8$ heterotic string.

Suppose that the space–time manifold Y has a boundary N. Then, according to [1], there are E_8 gauge fields propagating on N. An E_8 bundle V has a 4D characteristic class $w(V)$ which is associated with the differential form $\mathrm{tr} F \wedge F/16\pi^2$. The topology of E_8 is such that $w(V)$ is subject to absolutely no restriction except for being integral. The characteristic class $\frac{1}{2}\lambda = p_1$ of the tangent bundle of Y is associated with the differential form $\mathrm{tr} R \wedge R/16\pi^2$. In the second paper in [1], it was determined that the boundary values of G are restricted by

$$\left.\frac{G}{2\pi}\right|_N = \frac{1}{16\pi^2}\left(\mathrm{tr} F \wedge F - \frac{1}{2}\mathrm{tr} R \wedge R\right). \tag{2.1}$$

At the level of cohomology, this implies that

$$\left.\left[\frac{G}{2\pi}\right]\right|_N = w(V) - \frac{\lambda}{2} \tag{2.2}$$

and therefore, since $w(V)$ is integral, that $[G/2\pi] - \frac{1}{2}\lambda$ is integral when restricted to N. So the promised relation (1.2) holds, at least when restricted to N.

Now consider a physicist who is integrating $G/2\pi - \frac{1}{2}\lambda$ over a four-cycle S in space–time. If anywhere in the universe – maybe 10^{10} light years away – there is a boundary N that contains a four-cycle S' homologous to S, then from what has just been said, the integral over S of $G/2\pi - \frac{1}{2}\lambda$ must be integral. If $\int_S (G/2\pi - \frac{1}{2}\lambda)$ were to be non-integral, the physicist measuring it would obtain the amazing information that no N and S' exist, at any distance from S, no matter how great. The ability to obtain such information about what there is at arbitrarily big distances from the observer seems counterintuitive. So it is natural to suspect that the right requirement is that $\int_S (G/2\pi - \frac{1}{2}\lambda)$ is integral whether N and S' exist or not, or in other words (since this is true for all S) the cohomology class $[G/2\pi] - \frac{1}{2}\lambda$ is integral. In the rest of this paper it will, hopefully, become clear that this is the right interpretation.

For some additional insight, note that, since the homotopy groups $\pi_i(E_8)$ vanish for $i < 15$, except for π_3, and since dim $N < 16$, there is no restriction on the characteristic class $w(V)$ except integrality, and the E_8 bundle V over N is completely determined (topologically) by $w(V)$. The first statement means that (2.2) imposes no restriction on G except integrality of the restriction to N of $[G/2\pi] - \frac{1}{2}\lambda$; this is the basis for an assertion made above. The second statement means that, since $[G/2\pi]$ and λ are determined by their restrictions to the interior of space–time, a physicist who has made thorough measurements of the physics away from the boundary of space–time can uniquely predict what the E_8 bundle will have to be on the boundary.

2.2. Membrane anomalies

Now we will offer a second and to my belief a conclusive approach to the same result, based on considering world-volume anomalies of M-theory membranes. There is no need here to assume that M-theory has "elementary" membranes, whatever those may be. It is only necessary that M-theory admits macroscopic membranes. Given this, the path integral in the presence of the membrane must be well defined; this is the issue that we will examine.

Rather than aiming for generality, I will formulate the argument in a representative situation. Suppose the 11-manifold Y contains a submanifold $D = \mathbf{S}^3 \times \mathbf{S}^1$. We consider a membrane whose world-volume T is wrapped over $\mathbf{S}^3 \times q$, where q is a point in $\mathbf{S}^1$. We want to determine whether the membrane path integral is single-valued when q moves all the way around $\mathbf{S}^1$.

There are two potentially dangerous factors to consider. One is the "Chern–Simons" factor coming from the coupling of the membrane world-volume to C. This factor gives in the membrane path integrand a factor

$$\exp\left(\mathrm{i}\int_T C\right). \tag{2.3}$$

The change in this factor when q moves once around $\mathbf{S}^1$ is

$$\exp\left(\mathrm{i}\int_D G\right). \tag{2.4}$$

This factor is 1 if and only if the period of G is a multiple of 2π.

The second dangerous factor is the path integral over the world-volume fermions on T. At this point, therefore, we need some remarks on fermion path integrals. The path integral in $8k+3$ dimensions for massless fermions coupled to gauge fields and gravity is naturally real, but can be positive or negative. Potentially, as in [7–11], there can be an inconsistency in defining the sign of the fermion path integral. This will happen in our problem under some conditions on the embedding of D in Y.

In fact, the fermions on the membrane world-volume T are sections of the spin bundle of T tensored with the normal bundle N to T in space–time. One has $N = \mathcal{O} \oplus N'$, where $\mathcal{O}$ – a 1D trivial bundle – is the direction tangent to the $\mathbf{S}^1$ factor in D, and N' is the normal bundle to D in space–time. N' can be absolutely any Spin(7) bundle. Such a bundle has a 4D characteristic class $\lambda(N')$ (equal to $\frac{1}{2}p_1(N')$). $\lambda(N')$ is equal to the restriction of $\lambda(Y)$

to D (as the tangent bundle to D has vanishing characteristic classes). The considerations of [7–11] show that as q moves once around $\mathbf{S}^1$, the sign of the fermion path integral changes by a factor of $(-1)^{\int_D \lambda}$. What must be single-valued is therefore not really (2.4) but

$$(-1)^{\int_D \lambda} \exp\left(\mathrm{i} \int_D G\right). \tag{2.5}$$

But this amounts to our main claim: $G/2\pi$ has a half-integral period on just those cycles on which the value of λ is odd.

The fact that this computation gives the same result as the argument based on boundaries of space–time and $E_8 \times E_8$ gauge fields is a satisfying test of quantum M-theory.

2.3. Application to an orbifold

From what we have seen, it will sometimes happen that a half-integral quantum of $G/2\pi$ will be trapped on a four-cycle D in space–time. This is reminiscent of an example already considered in the literature, where in M-theory on $\mathbb{R}^5/\mathbb{Z}_2$, a half-integral flux of G appeared [12]. To be precise, if D is a four-cycle surrounding the origin in $\mathbb{R}^5/\mathbb{Z}_2$ – so D can naturally be taken to be a copy of $\mathbf{S}^4/\mathbb{Z}_2 = \mathbb{R}\mathbf{P}^4$ – then it was found from considerations of space–time anomaly cancelation that $\int_D (G/2\pi) = \frac{1}{2}$ modulo $\mathbb{Z}$.

To put this in the present context, we should first delete a neighborhood of the origin in $\mathbb{R}^5/\mathbb{Z}_2$ – as field theory considerations may not be applicable near the singularity. Let $\widetilde{\mathbb{R}}^5/\mathbb{Z}_2$ be $\mathbb{R}^5/\mathbb{Z}_2$ with such a neighborhood omitted. Now, $\widetilde{\mathbb{R}}^5/\mathbb{Z}_2$ is not orientable – this is the only place in the present paper where we consider an unorientable space–time. The differential form λ can be taken to vanish for $\widetilde{\mathbb{R}}^5/\mathbb{Z}_2$ – as that manifold admits a flat metric. So any statement we make will have to involve torsion.

In any event, for unorientable manifolds, it is unnatural to state a relation between G and λ, as the two transform oppositely under reversal of orientation (being respectively odd and even). Note, though,[3] that in the orientable case, λ is congruent modulo 2 to the Stieffel–Whitney class w_4, so that our statement in the orientable case could equivalently have been

$$\int_D \frac{G}{2\pi} = \frac{1}{2} \int_D w_4 \,\text{modulo}\, \mathbb{Z}. \tag{2.6}$$

This is a way of expressing our result in the orientable case that may carry over better to the unorientable case. Indeed, if as above D is a copy of $\mathbb{R}\mathbf{P}^4$ wrapping around the "interior" of $\widetilde{\mathbb{R}}^5/\mathbb{Z}_2$, then a standard computation [4] shows that

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[3] The following formulation was improved after suggestions by D. Freed.

[4] The tangent bundle of $\widetilde{\mathbb{R}}^5/\mathbb{Z}_2$ is a sum of five copies of an unorientable real line bundle ϵ. If $x = w_1(\epsilon)$, then the total Stieffel–Whitney class of $\widetilde{\mathbb{R}}^5/\mathbb{Z}_2$ is $(1+x)^5 = 1 + x + x^4 + x^5$. In particular, $w_4(\widetilde{\mathbb{R}}^5/\mathbb{Z}_2) = x^4$. Now restrict this to a copy of $D = \mathbb{R}\mathbf{P}^4 \subset \widetilde{\mathbb{R}}^5/\mathbb{Z}_2$. The mod 2 cohomology of $\mathbb{R}\mathbf{P}^4$ is the polynomial ring

$$\int_D w_4 = 1 \text{ modulo } 2. \tag{2.7}$$

(Of course, as Stieffel–Whitney classes are mod 2 classes, the integral only makes sense modulo 2.) Therefore, the result of [12] that an $\mathbb{R}^5/\mathbb{Z}_2$ singularity is the source of a half-integral flux of $G/2\pi$ is actually a consequence of (2.6). If one considers other somewhat analogous $\mathbb{Z}_2$ orbifold singularities studied in M-theory, such as $\mathbb{R}^8/\mathbb{Z}_2$ and $\mathbb{R}^9/\mathbb{Z}_2$ considered in [13], a computation as in footnote 4 shows that $w_4 = 0$, so that no half-integral G flux is expected – or found.

3. Integrality of the number of branes

In [2], compactification of M-theory on an eight-manifold X was considered. It was tacitly assumed that the vacuum expectation value of G was zero. The effects of the interaction [5,6] $C \wedge I_8(R)$ (where $I_8(R)$ is a certain quartic polynomial in the curvature tensor) were considered. If $J = -\int_X I_8(R)$ is non-zero, one obtains a "tadpole" for the C field, which must be canceled by including in the vacuum J two-branes. Since the number of two-branes must be an integer, the theory is inconsistent unless J is integral.

It was pointed out in [2] that for X a Calabi–Yau four-fold, $J = c_4/24$ (or equivalently $J = \chi(X)/24$, with χ the topological Euler characteristic). Index theorems were used to show that c_4 is always divisible by 6, but explicit examples show that c_4 need not be divisible by 12 or 24, so that one gets a non-trivial restriction on these compactifications.

For a more general eight-manifold, the formula for J, if written in terms of $\lambda = \frac{1}{2}p_1$ and p_2, is

$$J = \frac{p_2 - \lambda^2}{48}. \tag{3.1}$$

Now let us assess the integrality of J, using index theorems together with our previous considerations. The index of the Dirac operator on X, written in terms of λ and p_2, is

$$I = \frac{1}{1440}(7\lambda^2 - p_2). \tag{3.2}$$

Since I is an integer, it follows that

$$p_2 - \lambda^2 \cong 6\lambda^2 \text{ modulo } 1440. \tag{3.3}$$

Now, if G is to vanish, then according to our shifted flux condition, λ must be divisible by 2, say $\lambda = 2x$ with x an integral class. So we can write

$$p_2 - \lambda^2 \cong 24x^2 \text{ modulo } 1440. \tag{3.4}$$

So $p_2 - \lambda^2$ is divisible by 24. Since we need divisibility by 48, we must probe more deeply.

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in x with relation $x^5 = 0$, and in particular x^4 is the mod 2 fundamental class of $\mathbb{RP}^4$, which integrates to 1 (modulo 2).

An extra factor of two arises as follows. Let x be any element of $H^4(X,\mathbb{Z})$. Then by a special case of the Wu formula,

$$x^2 \cong x\cdot\lambda \text{ modulo } 2. \tag{3.5}$$

If, therefore, λ is divisible by 2, then x^2 is even, and (3.4) implies that $(p_2-\lambda^2)/48$ is integral, showing that the number of branes is integral, as promised. Note that formula (3.5) implies that the intersection form on $H^4(X,\mathbb{Z})$ is even when λ is divisible by 2. Since this intersection form is in any case unimodular (by Poincaré duality), this gives another occurrence of even unimodular lattices in string theory.

Formula (3.5) can be put in the following theoretical context. Let x and y be elements of $H^4(X,\mathbb{Z})$. Because $(x+y)^2$ is congruent to x^2+y^2 modulo 2, the function $f(x)=x^2$ is a linear function of x modulo 2. By Poincaré duality, there is therefore an element $\alpha\in H^4(X,\mathbb{Z}_2)$ such that for all $x\in H^4(X,\mathbb{Z}_2)$, $x^2\cong x\cdot\alpha$ modulo 2. α is called the Wu class and in general can be written as a polynomial in Stieffel–Whitney classes. For X a spin manifold, so that $w_i=0$ for i <4, one has simply $\alpha=w_4$, and this in turn equals the mod 2 reduction of the class $\lambda=\frac{1}{2}p_1\in H^4(X,\mathbb{Z})$, leading to (3.5). We will not attempt here an account of these matters, but instead give a direct proof of (3.5), using E_8 index theory, in the next section.

In a similar way, because the function $f(x)=x^3$ is linear in x modulo 3 (that is, $(x+y)^3\cong x^3+y^3$ modulo 3), it is natural, if x is for example a 4D class in a 12D spin manifold W, to have a formula $x^3\cong x\cdot\beta$ modulo 3, with β some 8D class. In the next section, we will explain how the existence of such a formula is relevant to M-theory, and we will determine β using E_8 index theory.

The computation we did in this section was really a special case of the general question of whether the Chern–Simons couplings – the classical CGG coupling and the quantum correction $CI_8(R)$ – are single-valued. Indeed, non-integrality of the tadpole would mean that the $CI_8(R)$ coupling is not well-defined modulo 2π upon adding to C a closed form with properly normalized periods. In the rest of this paper, we study more systematically the well-definedness of the Chern–Simons couplings.

4. Chern–Simons couplings

Since G can have non-zero periods, the definition $G=\mathrm{d}C$ is not really valid globally, and C is not well defined as a differential form. There is therefore some subtlety in defining the classical Chern–Simons interaction $I=\int_Y C\wedge G\wedge G$. This is most naturally accomplished by realizing Y as the boundary of a 12D spin manifold Z, extending the closed four-form G over Z, [5] and setting $I=\int_Z G\wedge G\wedge G$. To check whether this is well-defined –

[5] The existence of a 12D spin manifold Z over which G extends is ensured by a computation by Stong [14]. For later purposes, we want to choose the extension of G to obey the shifted quantization condition. Z can be chosen to make this possible. For instance, the shifted quantization condition says that $[G/\pi]=\lambda+2\alpha$ where α is some integral class. λ, being a characteristic class, automatically extends over Z, and Stong's

that is, independent of the choice of Z and of the extension of G – one takes two different choices, involving 12-manifolds Z and Z', each with boundary Y, and one tries to show that the corresponding I's – call them I_Z and $I_{Z'}$ – are equal modulo 2π. If Q is the closed 12-manifold obtained by gluing Z' to Z along their common boundary (one takes Z' and Z with "opposite" orientations so that the orientations fit together to an orientation of Q), then

$$I_Z - I_{Z'} = \int_Q G \wedge G \wedge G. \tag{4.1}$$

For the Chern–Simons interaction to be well-defined modulo 2π is thus equivalent to the requirement that for a closed spin manifold Q, $I_Q = \int_Q G \wedge G \wedge G$ is a multiple of 2π.

One might expect that the quantization law of G would be such as to ensure this, but in fact it is not. The actual relation is that, if α is the cohomology class of $[G/2\pi]$, then

$$\frac{I_Q}{2\pi} = -\frac{1}{6}\int_Q \alpha^3. \tag{4.2}$$

We will see below where this crucial factor of $-\frac{1}{6}$ comes from in terms of the relation of M-theory to the heterotic string. It can also be verified using a recent careful analysis of numerical factors in M-theory [15].

Since $\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$, in analyzing integrality of the right-hand side of (4.2), mod 2 and mod 3 congruences are helpful. The mod 2 congruences one needs are generalizations of (3.5), and the mod 3 congruences are possible for a reason suggested toward the end of Section 3. In addition to certain congruences, restoring integrality depends on gravitational corrections to the CGG coupling, which are of two kinds. (1) It is not α, but $\alpha - \frac{1}{2}\lambda$, that is integral. (2) One must also include the $CI_8(R)$ quantum correction.

Including the gravitational corrections and working out all of the congruences that are relevant in understanding the integrality of $I_Q/2\pi$ would be extremely complicated in the absence of some insight about the structure that is appearing here. But the analysis is actually made extremely easy by the recognition in [1] that the Chern–Simons couplings in 11 dimensions are related to anomalies in E_8 supergravity in 10 dimensions. They are therefore also related to index theory in 12 dimensions. To be more exact, the 11D Chern–Simons couplings were related in [1] to the anomalies of gluinos of *one* E_8 gauge group (rather than $E_8 \times E_8$) plus *half* of the anomaly of the dilatino and gravitino. Moreover, the gluinos, gravitino, and dilatino are all Majorana–Weyl, so their anomalies are half of what one would have for "ordinary" Weyl fermions.

Introduce on the 12-manifold Q an E_8 bundle V whose characteristic class $w(V)$ obeys $w = \alpha + \frac{1}{2}\lambda$. ($V$ exists and is unique as explained at the end of Section 2.1.) The shifted flux condition permits w to be an arbitrary 4D class. Let $\mathrm{i}(E_8)$ be the index of the Dirac

theorem means that Z can be chosen so that α extends; one can then take $G/\pi = \lambda + 2\alpha$ as the definition of the extension of G.

operator on Q for fermions with values in V (taken in the adjoint representation of E_8), and let $\mathrm{i(RS)}$ be the index of the Rarita–Schwinger operator on Q. The analysis of the Chern–Simons terms in Section 3.1 of [1], as subsequently extended [15,16], is equivalent to the assertion that after including the gravitational corrections

$$\frac{I_Q}{2\pi} = \frac{\mathrm{i}(E_8)}{2} + \frac{\mathrm{i(RS)}}{4}. \tag{4.3}$$

Here $\mathrm{i}(E_8)$ appears with a factor of $\frac{1}{2}$ because of the Majorana–Weyl condition, while $\mathrm{i(RS)}$ has a factor of $\frac{1}{4}$ because of the Majorana–Weyl condition plus the fact that we want the characteristic class that is related to $\frac{1}{2}$ of the gravitino–dilatino anomaly.

Now, in $8k + 4$ dimensions, because of charge conjugation symmetry, the index of the Dirac operator with values in a real vector bundle is even. So $\frac{1}{2}\mathrm{i}(E_8)$ is an integer, but $\frac{1}{4}\mathrm{i(RS)}$ is in general a half-integer. Since the G or w dependence of $I_Q/2\pi$ is entirely in $\mathrm{i}(E_8)$ (the other term is not sensitive to the choice of E_8 bundle!), $I_Q/2\pi$ changes by an integer under any change of G or w. Thus, we have verified integrality of $I_Q/2\pi$ up to a term that is G-independent. Moreover, in general (4.3) makes it clear that $I_Q/2\pi$ takes values in $\frac{1}{2}\mathbb{Z}$, with integrality depending only on the value of $\mathrm{i(RS)}$ modulo 4. In the next section, we will interpret the meaning of this last, G-independent, potential failure of integrality.

To exhibit the congruences that are implicit here, and also to make clear just how much trouble one might have had with this analysis if one did not know the relation of M-theory to E_8 gauge theory, we will now expand (4.3) explicitly using the index theorem (or equivalently, the detailed knowledge of the various Chern–Simons terms) and make explicit *some* of the relevant congruences. We get

$$\frac{I_Q}{2\pi} = -\frac{1}{6}\int_Q \left(w - \tfrac{1}{2}\lambda\right)\left(\left(w - \tfrac{1}{2}\lambda\right)^2 - \frac{1}{8}(p_2 - \lambda^2)\right). \tag{4.4}$$

(Notice that $I_Q/2\pi = -(\frac{1}{6})\int_Q w^3$ modulo gravitational corrections; as w equals α modulo a gravitational correction, this is the basis for the assertion that $I_Q/2\pi$ is as given in (4.2) modulo gravitational corrections.) Let us see what congruences can be extracted from the knowledge that the w-dependent terms are integral. First we consider mod 3 congruences.

After expanding out the right-hand side of (4.4) and dropping terms that do not have a 3 in the denominator, the fact that the w-dependent part of $I_Q/2\pi$ is a 3-integer (it can be written as a rational number with denominator not divisible by 3) turns out to be equivalent to the assertion that

$$w^3 \cong -(p_2 - \lambda^2)w \text{ modulo } 3. \tag{4.5}$$

Such a congruence was promised at the end of Section 3, and enables one to re-express $\frac{1}{6}w^3$ (which appears in (4.2)) in terms of $\frac{1}{2}w^3$ and a gravitational correction, linear in w. If (4.5) were known independently, this would be a step in the direction of analyzing integrality of (4.4) without using E_8 gauge theory.

For an interesting mod 2 congruence, take $Q = \mathbf{S}^1 \times Y$, with Y an 11-manifold, and take $w = \mathrm{d}\theta \cdot u + v$, where $\mathrm{d}\theta$ is a closed one-form on $\mathbf{S}^1$ that integrates to 1, $u \in H^3(Y, \mathbb{Z})$, and $v \in H^4(Y, \mathbb{Z})$. Think of $I_Q/2\pi$ as a function of u and v. Since this function changes by an integer when w is changed, $T(u, v) = (I_Q(u, v) - I_Q(u, 0) - I_Q(0, v) + I_Q(0, 0))/2\pi$ is an integer. But concretely

$$T(u, v) = -\frac{1}{2}\int_Y (uv^2 - uv\lambda). \tag{4.6}$$

Integrality of this expression for arbitrary u implies that

$$v^2 \cong v\lambda \text{ modulo } 2 \tag{4.7}$$

for any 4D class v on an 11-manifold Y. If we specialize to the case that $Y = \mathbf{S}^3 \times X$, with X an 8D spin manifold and $v \in H^4(X, \mathbb{Z})$, then (4.7) reduces to (3.5), giving the promised proof of (3.5) based on E_8 index theory.

For one final comment, suppose that $\lambda(Q)$ is even. Then we can choose an E_8 bundle on Q with $w = \frac{1}{2}\lambda(Q)$. For such a bundle, I_Q is clearly zero according to (4.4). Hence, using (4.3), i_{RS} is divisible by 4 when λ is even. Since that is not so in general, we go on in the next section to analyze the significance of the fact that $I_Q/2\pi$ may be half-integral.

Note that formula (4.4) is completely invariant under reversal of orientation of Q (which changes the sign of the index) together with $w \to \lambda - w$. Of course $w \to \lambda - w$ corresponds to $G \to -G$, which customarily accompanies orientation reversal in M-theory.

5. Rarita–Schwinger path integral

We still must interpret the fact that $I_Q/2\pi$ is half-integral in general. On the other hand, there is one more potentially anomalous factor in the low energy effective action of M-theory. This is the determinant of the Rarita–Schwinger operator D. We will see that the two problems cancel each other.

As we have already noted in Section 2.2, in $8k + 3$ dimensions a massless fermion path integral is naturally real. This is essentially because the massless Dirac operator is hermitean and has real eigenvalues. But (for Majorana fermions) the path integral has no natural sign, because there is no natural way to fix the sign of the fermion measure.

One way, as in [7], to try to define the determinant of the Rarita–Schwinger operator D is to compare det D to the determinant of a massive Rarita–Schwinger operator $D_m = D + \mathrm{i}m$. The constant m is odd under parity, as a result of which det D_m is complex. Since the mass is a soft perturbation and one can define det D and det D_m using the same fermion measure, the ratio det D/detD_m is completely well defined. Then by taking the limit as $m \to \infty$, one comes, in a sense, as close as one can to getting a natural definition of det D that preserves all the formal properties. However, one gets different limits by taking $m \to +\infty$ or $m \to -\infty$, so that the parity violation which follows from a choice of sign of m survives as $|m| \to \infty$.

The limits are in fact essentially (up to a sign that depends only on the manifold and not on the metric)[6]

$$\det D \exp(\pm\tfrac{1}{2}\mathrm{i} I_{\mathrm{RS}}). \tag{5.1}$$

Here the sign of the exponent depends on the sign of m. Also I_{RS} is a properly normalized Chern–Simons functional, associated with the characteristic class $\frac{1}{2}\mathrm{i}(\mathrm{RS})$. (We recall that $\mathrm{i}(\mathrm{RS})$, the Rarita–Schwinger index in $8k+4$ dimensions, is even.) Because of the $\frac{1}{2}$ in the exponent, the exponential factor in (5.1) may change sign under some orientation-preserving diffeomorphism, but any such sign changes precisely cancel sign changes in $\det D$, as one can verify using index theory.

This has led some physicists to suspect over the years that there might be a quantum correction to the low energy effective action of M-theory of the form $\frac{1}{2}\mathrm{i} I_{\mathrm{RS}}$, or perhaps $-\frac{1}{2}\mathrm{i} I_{\mathrm{RS}}$, or some other expression that differs from these by properly normalized Chern–Simons terms, to cancel the potential sign anomaly of $\det D$.[7] At first sight, it seems that any such functional would violate the parity-invariance of M-theory, even if one allows the freedom to add terms that depend on G as well as the metric. Parity allows only terms odd in C and G (namely our friends CGG and $CI_8(R)$); how can these terms, which vanish at $G=0$, help with a problem – the sign of $\det D$ – that persists at $G=0$? But the shift in the quantization law of G that we have uncovered makes it just possible to find a parity-invariant Chern–Simons-like interaction whose difference from $\frac{1}{2}I_{\mathrm{RS}}$ is properly normalized. The interaction with this property is our friend, the Chern–Simons interaction of M-theory, schematically $I_M = CGG + CI_8(R)$. This is clear from (4.3): $I_Q/2\pi$ differs from $\frac{1}{4}\mathrm{i}(\mathrm{RS})$ by an integer, so I_M, which is the Chern–Simons coupling derived from $I_Q/2\pi$, differs from $\frac{1}{2}I_{\mathrm{RS}}$ by a properly normalized Chern–Simons coupling (derived from $\frac{1}{2}\mathrm{i}(E_8)$). The ability to "have our cake and eat it too," to cancel the sign ambiguity of $\det D$ with a Chern–Simons interaction and still maintain parity conservation, depends on the fact that, as G must be congruent to $\frac{1}{2}\lambda$ mod 2, one cannot set G to zero (unless λ is even, in which case, as we saw at the end of Section 4, $\mathrm{i}(\mathrm{RS})$ is divisible by 4 and the problem vanishes). Related to this, symmetry under reversal of orientation is rather hidden in (4.3); it acts in the peculiar fashion $w \to \lambda - w$ (which is manifest when (4.3) is expanded to get (4.4)). Of course that transformation law comes from the classical parity transformation law $G \to -G$ of M-theory, together with the shift in the flux quantization law of G.

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[6] A more precise result in which also this overall manifold-dependent sign is fixed can be obtained by using the eta invariant instead of Chern–Simons, as in [10,11]. This refinement is not important in studying M-theory on any given manifold, but would be important in comparing M-theory on different manifolds, for instance in order to analyze gravitational instantons.

[7] For example, see a brief comment in [17].

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