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Some non-abelian phase spaces in low dimensions

Dongping Hou, Chengming Bai *

Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, PR China

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ABSTRACT

A non-abelian phase space, or a phase space of a Lie algebra, is a generalization of the usual (abelian) phase space of a vector space. It corresponds to a para-Kähler structure in geometry. Its structure can be interpreted in terms of left-symmetric algebras. In particular, a solution of an algebraic equation in a left-symmetric algebra which is an analogue of classical Yang–Baxter equation in a Lie algebra can induce a phase space. In this paper, we find that such phase spaces have a symplectically isomorphic property. We also give all such phase spaces in dimension 4 and some examples in dimension 6. These examples can be a guide for a further development.

1. Introduction

It is known that the phase space T^*V of a vector space V over a field $\mathbf{F}$ can be defined as the direct sum of V and its dual space $V^* = \mathrm{Hom}(V, \mathbf{F})$ endowed with the symplectic form

$$\omega_p(x + a^*, y + b^*) = \langle a^*, y\rangle - \langle b^*, x\rangle, \quad \forall x, y \in V, a^*, b^* \in V^*, \tag{1.1}$$

where $\langle , \rangle$ is the ordinary pairing between V and V^*. In [30], Kupershmidt generalized the above definition to the non-abelian case in the sense of replacing V by a Lie algebra (in particular for a non-abelian Lie algebra). Let $\mathcal{G}$ be a Lie algebra and $\mathcal{G}^*$ be its dual space. A phase space of $\mathcal{G}$ is the vector space $T^*(\mathcal{G}) = \mathcal{G} \oplus \mathcal{G}^*$ as the direct sum of vector spaces such that $T^*(\mathcal{G})$ is a Lie algebra, $\mathcal{G}$ is its subalgebra and the symplectic form ω_p given by Eq. (1.1) is a 2-cocycle on $T^*(\mathcal{G})$, that is,

$$\omega_p([x_1 + a_1^*, x_2 + a_2^*], x_3 + a_3^*) + \mathrm{CP} = 0, \quad \forall x_i \in \mathcal{G},\ a_i^* \in \mathcal{G}^*, \tag{1.2}$$

where "CP" stands for "cyclic permutation". A further study of non-abelian phase spaces was given in [4]. In particular, the module structure on $\mathcal{G}^*$ in [30] (it is equivalent to the condition that $\mathcal{G}^*$ is an ideal of $T^*(\mathcal{G})$) can be generalized to be a subalgebra. However, it is not easy to get some concrete examples under this sense.

On the other hand, the (non-abelian) phases spaces are just the para-Kähler structures on Lie algebras. In geometry, a para-Kähler manifold is a symplectic manifold with a pair of transversal Lagrangian foliations [33]. The Lie algebra $\mathcal{G}$ of a Lie group G with a G-invariant para-Kähler structure is a para-Kähler structure on $\mathcal{G}$ [12,7,39]. It is a symplectic Lie algebra ([15, 34,35], etc.) which is a direct sum of the underlying vector spaces of two Lagrangian subalgebras [27].

In fact, both of these two structures can be interpreted in terms of a class of nonassociative algebras, namely, left-symmetric algebras (or Koszul–Vinberg algebras). Left-symmetric algebras arose from the study of convex homogeneous

* Corresponding author. Tel.: +86 13920400299.
E-mail address: baicm@nankai.edu.cn (C. Bai).

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cones, affine manifolds and deformation of algebras [43,23,2,29,36] and appeared in many fields in mathematics and mathematical physics, such as complex and symplectic structures on Lie groups and Lie algebras [25,15,17,18,28,1], integrable systems [42], classical and quantum Yang–Baxter equations [11,21,31,32,24,19], Poisson brackets and infinite-dimensional Lie algebras [22,9,44], vertex algebras [8], quantum field theory [16], operads [14] and so on.

In [5], we established that a phase space is isomorphic to a bialgebra structure, namely, a left-symmetric bialgebra. It has many similar properties to a Lie bialgebra [20]. In particular, such a structure (and hence the phase space) can be obtained through solving an algebraic equation (S-equation) in left-symmetric algebras which is an analogue of the classical Yang–Baxter equation in Lie algebras [40,10,32].

In this paper, we give a further study of the S-equation. We find a symplectically isomorphic property of the phase spaces constructed through the S-equation. We also give such phases in low dimensions and we hope that these examples can be a guide for a further development. This paper is organized as follows. In Section 2, for self-containment, we give a brief introduction to phase spaces and left-symmetric algebras. In Section 3, we recall the construction of phase spaces through the S-equation in left-symmetric algebras given in [5] and we prove a symplectically isomorphic property of such phase spaces. In Section 4, we give the four-dimensional phase spaces obtained from solving the S-equation in two-dimensional complex left-symmetric algebras. In Section 5, we give some examples in dimension 6 through giving all solutions of the S-equation in three-dimensional complex simple left-symmetric algebras. In Section 6, we give some conclusions and discussion.

Throughout this paper, all algebras are finite dimensional and over the complex field $\mathbf{C}$ and the parameters belong to the complex field $\mathbf{C}$, too. And $\langle | \rangle$ stands for a Lie or left-symmetric algebra with a basis and non-zero products on each side of $|$.

2. Preliminaries and some basic results

Definition 2.1. A para-Kähler Lie algebra $\mathcal{G}$ or a para-Kähler structure on a Lie algebra $\mathcal{G}$ is a triple $\{\mathcal{G}^+, \mathcal{G}^-, \omega\}$, where $\mathcal{G}^+, \mathcal{G}^-$ are two subalgebras of $\mathcal{G}$ and $\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{G}^-$ as vector spaces, ω (the symplectic form) is a nondegenerate skew-symmetric 2-cocycle on $\mathcal{G}$ such that

$$\omega([x,y],z) + \omega([y,z],x) + \omega([z,x],y) = 0, \quad \forall x,y,z \in \mathcal{G}, \tag{2.1}$$

and $\omega(\mathcal{G}^+, \mathcal{G}^+) = \omega(\mathcal{G}^-, \mathcal{G}^-) = 0$.

Definition 2.2. Two para-Kähler Lie algebras $(\mathcal{G}_1^+, {\mathcal{G}_1}^-, \omega_1)$ and $(\mathcal{G}_2^+, {\mathcal{G}_2}^-, \omega_2)$ are isomorphic if there exists a Lie algebra isomorphism $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ such that

$$\varphi(\mathcal{G}_1^+) = \mathcal{G}_2^+, \qquad \varphi(\mathcal{G}_1^-) = \mathcal{G}_2^-; \qquad \omega_1(x,y) = \varphi^*\omega_2(x,y) = \omega_2(\varphi(x),\varphi(y)), \quad \forall x,y \in \mathcal{G}_1. \tag{2.2}$$

Proposition 2.3 ([5]). *Any Lie algebra $\mathcal{G}$'s phase space $T^*(\mathcal{G})$ admits a para-Kähler structure $(\mathcal{G}, \mathcal{G}^*, \omega_p)$, where ω_p is given by Eq. (1.1). Conversely, every para-Kähler Lie algebra $(\mathcal{G}^+, \mathcal{G}^-, \omega)$ is isomorphic to a phase space of $\mathcal{G}^+$.*

Definition 2.4. Let A be a vector space over a field $\mathbf{F}$ with a bilinear product $(x,y) \to xy$. A is called a left-symmetric algebra if for any $x,y,z \in A$,

$$(xy)z - x(yz) = (yx)z - y(xz). \tag{2.3}$$

Proposition 2.5 ([36]). *Let A be a left-symmetric algebra. For any $x \in A$, let L_x denote the left multiplication operator, that is, $L_x(y) = xy$, $\forall\, y \in A$. Then we have:*

(i) *The commutator*

$$[x,y] = xy - yx, \quad \forall\, x,y \in A, \tag{2.4}$$

defines a Lie algebra $\mathcal{G} = \mathcal{G}(A)$, which is called the sub-adjacent Lie algebra of A, and A is called the compatible left-symmetric algebra structure on the Lie algebra $\mathcal{G}(A)$.

(ii) *$L : \mathcal{G}(A) \to gl(\mathcal{G}(A))$ with $x \to L_x$ gives a (regular) representation of the sub-adjacent Lie algebra $\mathcal{G}(A)$.*

Theorem 2.6 ([15,4]). *Let $\{\mathcal{G}^+, \mathcal{G}^-, \omega\}$ be a para-Kähler structure on a Lie algebra $\mathcal{G}$. Then there exists a compatible left-symmetric algebra structure "$*$" on $\mathcal{G}$ defined by*

$$\omega(x * y, z) = -\omega(y, [x,z]), \quad \forall x,y,z \in \mathcal{G}. \tag{2.5}$$

Moreover, $\mathcal{G}^\pm$ are two (left-symmetric) subalgebras of $\mathcal{G}$ with the above product.

Let $\mathcal{G}$ be a Lie algebra and $\rho : \mathcal{G} \to gl(V)$ be a representation. On a direct sum $\mathcal{G} \oplus$ of the underlying vector spaces $\mathcal{G}$ and V, there is a natural Lie algebra structure (denoted by $\mathcal{G} \ltimes_\rho V$) given as follows [26]:

$$[x_1 + v_1, x_2 + v_2] = [x_1,x_2] + \rho(x_1)v_2 - \rho(x_2)v_1, \quad \forall x_1,x_2 \in \mathcal{G}, v_1,v_2 \in V. \tag{2.6}$$

Example 2.7. Let A be a left-symmetric algebra. Then $T^*(\mathcal{G}(A)) = \mathcal{G}(A) \ltimes_{L^*} \mathcal{G}(A)^*$ is a phase space of the sub-adjacent Lie algebra $\mathcal{G}(A)$ with the symplectic form ω_p given by Eq. (1.1), where $\mathcal{G}(A)^*$ is the dual space of $\mathcal{G}(A)$ and L^* is the dual representation of the regular representation L of $\mathcal{G}(A)$. Such a construction was given by Medina and Revoy [37,38] and Kupershmidt [30] respectively.

Definition 2.8. Let A be a vector space. A left-symmetric bialgebra structure on A is a pair of linear maps (α, β) such that $\alpha : A \to A \otimes A$, $\beta : A^* \to A^* \otimes A^*$ and

(a) $\alpha^* : A^* \otimes A^* \to A^*$ is a left-symmetric algebra structure on A^*;
(b) $\beta^* : A \otimes A \to A$ is a left-symmetric algebra structure on A;
(c) α is a 1-cocycle of $\mathcal{G}(A)$ associated with $L \otimes 1 + 1 \otimes \mathrm{ad}$ with values in $A \otimes A$;
(d) β is a 1-cocycle of $\mathcal{G}(A^*)$ associated with $L \otimes 1 + 1 \otimes \mathrm{ad}$ with values in $A^* \otimes A^*$;

where L is the regular representation of $\mathcal{G}(A)$ and ad is the adjoint representation of $\mathcal{G}(A)$ satisfying $\mathrm{ad}x(y) = [x, y] = xy - yx$ for any $x, y \in A$. We also denote this left-symmetric bialgebra by (A, A^*, α, β) or simply (A, A^*). Two left-symmetric bialgebras $(A, A^*, \alpha_A, \beta_A)$ and $(B, B^*, \alpha_B, \beta_B)$ are isomorphic if there is a left-symmetric algebra isomorphism $\varphi : A \to B$ such that $\varphi^* : B^* \to A^*$ is also an isomorphism of left-symmetric algebras, that is, φ satisfies

$$(\varphi \otimes \varphi)\alpha_A(x) = \alpha_B(\varphi(x)), \qquad (\varphi^* \otimes \varphi^*)\beta_B(a^*) = \beta_A(\varphi^*(a^*)), \quad \forall x \in A, a^* \in B^*. \tag{2.7}$$

Theorem 2.9 ([5]). *Let $(A, \cdot)$ be a left-symmetric algebra and $(A^*, \circ)$ be a left-symmetric algebra structure on its dual space A^*. Then $(\mathcal{G}(A), \mathcal{G}(A)^*, \omega_p)$ (where ω_p is given by Eq. (1.1)) is a phase space if and only if (A, A^*) is a left-symmetric bialgebra. Moreover, two phase spaces are isomorphic if and only if their corresponding left-symmetric bialgebras are isomorphic.*

3. The S-equation and the symplectic isomorphism

Definition 3.1. A left-symmetric bialgebra (A, A^*, α, β) is called a coboundary if α is a 1-coboundary of $\mathcal{G}(A)$ associated with $L \otimes 1 + 1 \otimes \mathrm{ad}$, that is, there exists a $r \in A \otimes A$ such that

$$\alpha(x) = (L_x \otimes 1 + 1 \otimes \mathrm{ad}x)r, \quad \forall x \in A. \tag{3.1}$$

Notation 3.2. Let $(A, \cdot)$ be a left-symmetric algebra and $r = \sum_i a_i \otimes b_i \in A \otimes A$. Set

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \qquad r_{13} = \sum_i a_i \otimes 1 \otimes b_i; \qquad r_{23} = \sum_i 1 \otimes a_i \otimes b_i \in U(\mathcal{G}(A)), \tag{3.2}$$

where $U(\mathcal{G}(A))$ is the universal enveloping algebra of the sub-adjacent Lie algebra $\mathcal{G}(A)$. Set

$$r_{12} \cdot r_{13} = \sum_{i,j} a_i \cdot a_j \otimes b_i \otimes b_j; \qquad r_{12} \cdot r_{23} = \sum_{i,j} a_i \otimes b_i \cdot a_j \otimes b_j; \tag{3.3}$$

$$[r_{13}, r_{23}] = r_{13} \cdot r_{23} - r_{23} \cdot r_{13} = \sum_{ij} a_i \otimes a_j \otimes [b_i, b_j]. \tag{3.4}$$

Proposition 3.3 ([5]). *Let A be a left-symmetric algebra and $r \in A \otimes A$. Suppose r is symmetric. Then the map α defined by Eq. (3.1) induces a left-symmetric algebra structure on A^* such that (A, A^*) is a left-symmetric bialgebra if*

$$[[r, r]] = -r_{12} \cdot r_{13} + r_{12} \cdot r_{23} + [r_{13}, r_{23}] = 0. \tag{3.5}$$

Let A be a vector space. For any $r \in A \otimes A$, r can be regarded as a map from $A^* \to A$ in the following way:

$$\langle u^* \otimes v^*, r \rangle = \langle u^*, r(v^*) \rangle, \quad \forall u^*, v^* \in A^*. \tag{3.6}$$

r is symmetric if and only if $\langle u^* \otimes v^*, r \rangle = \langle v^* \otimes u^*, r \rangle$ for any $u^*, v^* \in A^*$.

Remark 3.4. Eq. (3.5) is called an S-equation in a left-symmetric algebra. This algebraic equation has a geometric meaning as follows. Let $(A, \cdot)$ be a left-symmetric algebra. A bilinear form $\mathcal{B} : A \otimes A \to \mathbf{F}$ is called a 2-cocycle of A if

$$\mathcal{B}(x \cdot y, z) - \mathcal{B}(x, y \cdot z) = \mathcal{B}(y \cdot x, z) - \mathcal{B}(y, x \cdot z), \quad \forall x, y, z \in A. \tag{3.7}$$

Suppose that $r \in A \otimes A$ is symmetric and nondegenerate. Then r is a solution of the S-equation in A if and only if the inverse of the isomorphism $A^* \to A$ induced by r, regarded as a bilinear form $\mathcal{B}$ on A, is a 2-cocycle of A. That is, $\mathcal{B}(x, y) = \langle r^{-1}x, y \rangle$ for any $x, y \in A$.

On the other hand, a left-symmetric algebra over the real field $\mathbf{R}$ is called Hessian if there exists a symmetric and positive definite 2-cocycle $\mathcal{B}$ of A. In geometry, a Hessian manifold M is a flat affine manifold provided with a Hessian metric g, that is, g is a Riemannian metric such that for any point $p \in M$ there exists a C^∞-function φ defined on a neighborhood of

p such that $g_{ij} = \frac{\partial^2\varphi}{\partial x^i \partial x^j}$. A Hessian left-symmetric algebra corresponds to an affine Lie group G with a G-invariant Hessian metric [41]. $\square$

Proposition 3.5 ([5]). *Let $(A, \cdot)$ be a left-symmetric algebra and $r \in A \otimes A$ be a symmetric solution of the S-equation in A. Then the left-symmetric algebra and its sub-adjacent Lie algebra structure in the phase space $T^*(\mathcal{G}(A))$ can be given from the products in A as follows:*

$$\text{(a) } a^* * b^* = a^* \circ b^* = -R_\cdot^*(r(b^*))a^* + \mathrm{ad}_\cdot^*(r(a^*))b^*, \quad \text{for any } a^*, b^* \in A^*; \tag{3.8}$$

$$\text{(b) } [a^*, b^*] = a^* \circ b^* - b^* \circ a^* = L_\cdot^*(r(a^*))b^* - L_\cdot^*(r(b^*))a^*, \quad \text{for any } a^*, b^* \in A^*; \tag{3.9}$$

$$\text{(c) } x * a^* = x \cdot r(a^*) - r(\mathrm{ad}_\cdot^*(x)a^*) + \mathrm{ad}_\cdot^*(x)a^*, \quad \text{for any } x \in A, a^* \in A^*; \tag{3.10}$$

$$\text{(d) } a^* * x = r(a^*) \cdot x + r(R_\cdot^*(x)a^*) - R_\cdot^*(x)a^*, \quad \text{for any } x \in A, a^* \in A^*; \tag{3.11}$$

$$\text{(e) } [x, a^*] = [x, r(a^*)] - r(L_\cdot^*(x)a^*) + L_\cdot^*(x)a^*, \quad \text{for any } x \in A, a^* \in A^*. \tag{3.12}$$

where $R_\cdot^* : A \to gl(A^*)$ satisfies $\langle R_\cdot^*(x)a^*, y\rangle = -\langle a^*, y \cdot x\rangle$ for any $x, y \in A$ and $a^* \in A^*$.

Example 3.6. Obviously, $r = 0$ is a symmetric solution of the S-equation in any left-symmetric algebra A. Moreover, it is obvious that the phase space constructed from $r = 0$ just corresponds to the phase space in Example 2.7 as the semi-direct sum $\mathcal{G}(A) \ltimes_{L^*} \mathcal{G}(A)^*$. $\square$

Definition 3.7. A symplectic Lie algebra or a symplectic structure on a Lie algebra is a pair $(\mathcal{G}, \omega)$, where $\mathcal{G}$ is a Lie algebra and ω is a nondegenerate skew-symmetric 2-cocycle on $\mathcal{G}$ satisfying Eq. (2.1). Two symplectic Lie algebras $(\mathcal{G}_1, \omega_1)$ and $(\mathcal{G}_2, \omega_2)$ are isomorphic if there exists a Lie algebra isomorphism $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ such that

$$\omega_1(x, y) = \varphi^*\omega_2(x, y) = \omega_2(\varphi(x), \varphi(y)), \quad \forall x, y \in \mathcal{G}_1. \tag{3.13}$$

Lemma 3.8 ([5]). *Let $(A, \cdot)$ be a left-symmetric algebra and $r \in A \otimes A$ be symmetric. Then r is a solution of the S-equation in A if and only if r satisfies*

$$[r(a^*), r(b^*)] = r(L_\cdot^*(r(a^*))b^* - L_\cdot^*(r(b^*))a^*), \quad \forall a^*, b^* \in A^*. \tag{3.14}$$

Theorem 3.9. *Let A be a left-symmetric algebra. Then as symplectic Lie algebras, the phase space $T^*(\mathcal{G}(A))$ given by any symmetric solution of the S-equation in A is isomorphic to the phase space given in Example 2.7, that is, the semi-direct sum $\mathcal{G}(A) \ltimes_{L^*} \mathcal{G}(A)^*$ which corresponds to the solution $r = 0$.*

Proof. Let r be a symmetric solution of the S-equation in A. Define a linear map $\varphi : \mathcal{G}(A) \ltimes_{L^*} \mathcal{G}(A)^* \to A \oplus A^*$ satisfying

$$\varphi(x) = x, \qquad \varphi(a^*) = -r(a^*) + a^*, \quad \forall x \in A, a^* \in A^*.$$

Obviously, φ is a linear isomorphism. Since r is symmetric and by Lemma 3.8, we know that

$$\begin{aligned}
&[\varphi(x), \varphi(y)] = [x, y] = \varphi([x, y]);\\
&[\varphi(x), \varphi(a^*)] = [x, -r(a^*) + a^*] = -[x, r(a^*)] + [x, r(a^*)] - r(L^*(x)a^*) + L^*(x)a^*\\
&\qquad = -r(L^*(x)a^*) + L^*(x)a^* = \varphi(L^*(x)a^*) = \varphi([x, a^*]);\\
&[\varphi(a^*), \varphi(b^*)] = [r(a^*), r(b^*)] - \{[r(a^*), r(b^*)] - r(L^*(r(a^*))b^*) + L^*(r(a^*))b^*\}\\
&\qquad + \{[r(b^*), r(a^*)] - r(L^*(r(b^*))a^*) + L^*(r(b^*))a^*\} + L^*(r(a^*))b^* - L^*(r(b^*))a^*\\
&\qquad = [r(b^*), r(a^*)] + r(L^*(r(a^*))b^*) - r(L^*(r(b^*))a^*) = 0 = \varphi([a^*, b^*]);\\
&\varphi^*\omega_p(x + a^*, y + b^*) = \langle a^*, y - r(b^*)\rangle - \langle x - r(a^*), b^*\rangle\\
&\qquad = \langle a^*, y\rangle - \langle x, b^*\rangle - \langle a^*, r(b^*)\rangle + \langle r(a^*), b^*\rangle = \omega_p(x + a^*, y + b^*),
\end{aligned}$$

for any $x, y \in A$ and $a^*, b^* \in A^*$. Therefore φ is an isomorphism of symplectic Lie algebras. $\square$

Remark 3.10. In general, the above two phase spaces are not isomorphic as para-Kähler Lie algebras. In particular, for the non-zero solution $r \neq 0$ of the S-equation in A, the linear isomorphism φ given in the above proof is not an isomorphism of para-Kähler Lie algebras since $\varphi(A^*) = r(A^*) + A^* \neq A^*$ when $r \neq 0$. $\square$

4. Four-dimensional phase spaces

Let $\{e_1, \ldots, e_n\}$ be a basis of a left-symmetric algebra $(A, \cdot)$ and $\{e_1^*, \ldots, e_n^*\}$ be its dual basis in A^*. Let $r \in A \otimes A$. Set $r = \sum_{i,j}^n r_{ij} e_i \otimes e_j$, $\; e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k$. Then $r(e_i^*) = \sum_{j=1}^n r_{ij} e_j$ and r is a symmetric solution of the S-equation in A if and only if r_{ij} satisfies

$$r_{ij} = r_{ji}, \qquad \sum_{t,l}^{n}\{-c_{tl}^{i} r_{tj} r_{lk} + c_{tl}^{j} r_{it} r_{lk} + (c_{tl}^{k} - c_{lt}^{k}) r_{it} r_{lj}\} = 0, \quad \forall i, j, k = 1, 2, \ldots, n. \tag{4.1}$$

We let SE(A) denote the set of the symmetric solutions of the S-equation in A. By Proposition 3.5, we obtain the structures of phase spaces obtained from solving the S-equation as follows:

$$[e_i, e_j] = \sum_{k=1}^{n}(c_{ij}^{k} - c_{ji}^{k})e_k;$$

$$[e_i, e_j^*] = \sum_{k=1}^{n}[r_{jl}(c_{il}^{k} - c_{li}^{k}) + \sum_{l=1}^{n} r_{lk} c_{il}^{j}]e_k - \sum_{k=1}^{n} c_{ik}^{j} e_k^*;$$

$$[e_i^*, e_j^*] = \sum_{l,k}^{n}(-r_{il} c_{lk}^{j} + r_{jl} c_{lk}^{i})e_k^*.$$

Now we consider the case $n = 2$. The classifications of two-dimensional complex left-symmetric algebras (with non-zero products) are given as follows [13]:

$(\mathrm{AI}) = \langle e_1, e_2 | e_1 \cdot e_1 = e_1, e_2 \cdot e_2 = e_2 \rangle;$
$(\mathrm{AII}) = \langle e_1, e_2 | e_2 \cdot e_2 = e_2, e_1 \cdot e_2 = e_2 \cdot e_1 = e_1 \rangle;$
$(\mathrm{AIII}) = \langle e_1, e_2 | e_1 \cdot e_1 = e_1 \rangle;$
$(\mathrm{AV}) = \langle e_1, e_2 | e_1 \cdot e_1 = e_2 \rangle :$
$(\mathrm{AIV}) = \langle e_1, e_2 | e_i \cdot e_j = 0, i, j = 1, 2 \rangle;$
$(\mathrm{NI}) = \langle e_1, e_2 | e_2 \cdot e_1 = -e_1, e_2 \cdot e_2 = -e_2 \rangle;$
$(\mathrm{NII})_{-1} = \langle e_1, e_2 | e_2 \cdot e_1 = -e_1, e_2 \cdot e_2 = e_1 - e_2 \rangle;$
$(\mathrm{NII})_k = \langle e_1, e_2 | e_2 \cdot e_1 = -e_1, e_2 \cdot e_2 = ke_2, k \neq -1 \rangle;$
$(\mathrm{NIII}) = \langle e_1, e_2 | e_1 \cdot e_2 = e_1, e_2 \cdot e_2 = e_2 \rangle;$
$(\mathrm{NIV})_k = \langle e_1, e_2 | e_1 \cdot e_2 = ke_1, e_2 \cdot e_1 = (k-1)e_2, e_2 \cdot e_2 = e_1 + ke_2, k \in \mathbf{C} \rangle;$
$(\mathrm{NV}) = \langle e_1, e_2 | e_1 \cdot e_1 = 2e_1, e_1 \cdot e_2 = e_2, e_2 \cdot e_2 = e_1 \rangle.$

By a direct computation, we give the solutions of the S-equation in the above (two-dimensional) left-symmetric algebras and their corresponding (four-dimensional) phase spaces (only the non-zero products are given) as follows:

$$\begin{aligned}
\mathrm{SE(AI)} &= \left\{\begin{pmatrix} r_{11} & 0 \\ 0 & r_{22} \end{pmatrix}\right\} \Longrightarrow \begin{cases} [e_1, e_1^*] = r_{11}e_1 - e_1^*; \\ [e_2, e_2^*] = r_{22}e_2 - e_2^*. \end{cases} \\
&\quad \bigcup \left\{\begin{pmatrix} r_{11} & r_{11} \\ r_{11} & r_{11} \end{pmatrix} \middle| r_{11} \neq 0\right\} \Longrightarrow \begin{cases} [e_1, e_1^*] = r_{11}e_1 + r_{11}e_2 - e_1^*; \\ [e_2, e_2^*] = r_{11}e_1 + r_{11}e_2 - e_2^*; \\ [e_1^*, e_2^*] = r_{11}e_1^* - r_{11}e_2^*. \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathrm{SE(AII)} &= \left\{\begin{pmatrix} 0 & r_{12} \\ r_{12} & r_{22} \end{pmatrix}\right\} \Longrightarrow \begin{cases} [e_1, e_1^*] = r_{12}e_2 - e_1^*; \\ [e_2, e_2^*] = r_{12}e_2 - e_2^*; \\ [e_1, e_2^*] = r_{12}e_1 + r_{22}e_2 - e_2^*. \end{cases} \\
&\quad \bigcup \left\{\begin{pmatrix} r_{11} & 0 \\ 0 & 0 \end{pmatrix} \middle| r_{11} \neq 0\right\} \Longrightarrow \begin{cases} [e_1, e_1^*] = r_{11}e_1 - e_1^*; \qquad [e_1, e_2^*] = -e_2^*; \\ [e_2, e_2^*] = r_{11}e_1 - e_1^*; \qquad [e_1^* e_2^*] = -r_{11}e_2^*. \end{cases}
\end{aligned}$$

$$\mathrm{SE(AIII)} = \left\{\begin{pmatrix} r_{11} & 0 \\ 0 & r_{22} \end{pmatrix}\right\} \Longrightarrow \{[e_1, e_1^*] = r_{11}e_1 - e_1^*.$$

$$\mathrm{SE(AIV)} = \left\{\begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix}\right\} \Longrightarrow \begin{cases} [e_1, e_2] = [e_1, e_1^*] = [e_2, e_2^*] = 0; \\ [e_1, e_2^*] = [e_2, e_1^*] = [e_1^*, e_2^*] = 0. \end{cases}$$

$$\mathrm{SE(AV)} = \left\{\begin{pmatrix} 0 & r_{12} \\ r_{12} & r_{22} \end{pmatrix}\right\} \Longrightarrow \{[e_1, e_1^*] = r_{12}e_2 - e_1^*.$$

$$\mathrm{SE(NI)} = \left\{\begin{pmatrix} r_{11} & \pm\sqrt{r_{11}r_{22}} \\ \pm\sqrt{r_{11}r_{22}} & r_{22} \end{pmatrix}\right\} \Longrightarrow \begin{cases} [e_1, e_2] = e_2; \\ [e_1, e_2^*] = r_{22}e_1; \\ [e_1, e_1^*] = \pm\sqrt{r_{11}r_{22}}e_1; \\ [e_2, e_1^*] = -2r_{11}e_1 \mp \sqrt{r_{11}r_{22}}e_2 + e_1^*; \\ [e_2, e_2^*] = \mp\sqrt{r_{11}r_{22}}e_1 - r_{22}e_2 + e_2^*; \\ [e_1^*, e_2^*] = \mp\sqrt{r_{11}r_{22}}e_2^* - r_{22}e_1^*. \end{cases}$$

$$\mathrm{SE}(\mathrm{NII}_k, k \neq \pm 1) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & r_{22} \end{pmatrix} \right\} \Longrightarrow \begin{cases} [e_1, e_2] = e_1; & [e_1, e_2^*] = r_{22} e_1 \\ [e_2, e_2^*] = e_1^*; & [e_1^*, e_2^*] = -r_{22} e_1^*; \\ [e_2, e_2^*] = k r_{22} e_2 - k e_2^*. \end{cases}$$

$$\mathrm{SE}(\mathrm{NII}_1) = \left\{ \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & 0 \end{pmatrix} \right\} \Longrightarrow \begin{cases} [e_1, e_2] = e_1; & [e_1, e_1^*] = r_{12} e_1; \\ [e_2, e_1^*] = -2r_{11} e_1 - r_{12} e_2 + e_1^*; \\ [e_2, e_2^*] = -e_2^*; & [e_1^*, e_2^*] = -r12 e_2^*. \end{cases}$$

$$\bigcup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & r_{22} \end{pmatrix} \middle| r_{22} \neq 0 \right\} \Longrightarrow \begin{cases} [e_1, e_2] = e_1; & [e_1, e_2^*] = r_{22} e_1; \\ [e_2, e_1^*] = e_1^*; & [e_1^*, e_2^*] = -r_{22} e_1^*; \\ [e_2, e_2^*] = r_{22} e_2 - e_2^*. \end{cases}$$

$$\mathrm{SE}(\mathrm{NII}_{-1}) = \left\{ \begin{pmatrix} r_{11} & 0 \\ 0 & 0 \end{pmatrix} \right\} \Longrightarrow \begin{cases} [e_1, e_2] = e_1; & [e_2, e_2^*] = e_2^*; \\ [e_2, e_1^*] = -2r_{11} e_1 + e_1^* - e_2^*. \end{cases}$$

$$\mathrm{SE}(\mathrm{NIII}) = \left\{ \begin{pmatrix} r_{11} & \pm\sqrt{r_{11} r_{22}} \\ \pm\sqrt{r_{11} r_{22}} & r_{22} \end{pmatrix} \right\} \Longrightarrow \begin{cases} [e_1, e_2] = e_1; & [e_1, e_2^*] = r_{22} e_1; \\ [e_1, e_1^*] = \pm 2\sqrt{r_{11} r_{22}} e_1 + r_{22} e_2 - e_2^*; \\ [e_2, e_1^*] = -r_{11} e_1; & [e_2, e_2^*] = r_{22} e_2 - e_2^*. \end{cases}$$

$$\mathrm{SE}(\mathrm{NIV}_k, k \neq 0, 2) = \left\{ \begin{pmatrix} r_{11} & (1-k) r_{11} \\ (1-k) r_{11} & (1-k)^2 r_{11} \end{pmatrix} \right\}$$
$$\Longrightarrow \begin{cases} [e_1, e_2] = e_1; & [e_1, e_2^*] = (1-k)^2 r_{11} e_1; \\ [e_1, e_1^*] = (1-k)^2 r_{11} e_1 + k(1-k)^2 r_{11} e_2 - k e_2^*; \\ [e_2, e_1^*] = (1-k) e_1^* - r_{11} e_1 - e_2^*; \\ [e_2, e_2^*] = -(1-k) r_{11} e_1 + k(1-k) r_{11} e_2 - k e_2^*; \\ [e_1^*, e_2^*] = -(1-k)^3 r_{11} e_1^* + k(1-k)^2 e_2^*. \end{cases}$$

$$\mathrm{SE}(\mathrm{NIV})_2 = \left\{ \begin{pmatrix} r_{11} & -r_{12} \\ -r_{12} & r_{22} \end{pmatrix} \right\} \Longrightarrow \begin{cases} [e_1, e_2] = e_1; & [e_1, e_2^*] = r_{22} e_1; \\ [e_1, e_1^*] = -3r_{22} e_1 + 2r_{22} e_2 - 2e_2^*; \\ [e_2, e_1^*] = -r_{22} e_1 - e_1^* - e_2^*; \\ [e_1^*, e_2^*] = r_{22}(e_1^* + e_2^*); \\ [e_2, e_2^*] = -r_{22} e_1 + 2r_{22} e_2 - 2e_2^*. \end{cases}$$

$$\mathrm{SE}(\mathrm{NV}) = \left\{ \begin{pmatrix} r_{11} & 0 \\ 0 & 0 \end{pmatrix} \right\} \Longrightarrow \begin{cases} [e_1, e_2] = e_2; & [e_1, e_2^*] = -e_2^*; \quad [e_1^*, e_2^*] = -r_{11} e_2^*; \\ [e_1, e_1^*] = 2r_{11} e_1 - 2e_1^*; & [e_2, e_1^*] = -r_{11} e_2 - e_2^*. \end{cases}$$

$$\bigcup \left\{ \begin{pmatrix} r_{11} & 0 \\ 0 & 2r_{11} \end{pmatrix} \middle| r_{11} \neq 0 \right\} \Longrightarrow \begin{cases} [e_1, e_2] = e_2; & [e_2, e_1^*] = r_{11} e_2 - e_2^*; \\ [e_1^*, e_2^*] = -r_{11} e_2^*; & [e_1, e_1^*] = 2r_{11} - 2e_1^*; \\ [e_1, e_2^*] = 4r_{11} e_2 - e_2^*. \end{cases}$$

$$\bigcup \left\{ \begin{pmatrix} r_{11} & -\mathrm{i} r_{11} \\ -\mathrm{i} r_{11} & -r_{11} \end{pmatrix} \middle| r_{11} \neq 0, \mathrm{i}^2 = -1 \right\} \Longrightarrow \begin{cases} [e_1, e_2] = e_2; & [e_2, e_2^*] = \mathrm{i} r_{11} e_2; \\ [e_1, e_1^*] = 2r_{11} e_1 - 3\mathrm{i} r_{11} e_2 - 2e_1^*; \\ [e_1, e_2^*] = -\mathrm{i} r_{11} e_1 - 2r_{11} e_2 - e_2^*; \\ [e_2, e_1^*] = -\mathrm{i} r_{11} e_1 - 2r_{11} e_2 - e_2^*; \\ [e_1^*, e_2^*] = -2\mathrm{i} r_{11} e_1^* - 2r_{11} e_2^*. \end{cases}$$

Corollary 4.1. *There are invertible solutions of the S-equation in the following two-dimensional left-symmetric algebras:* (AI)–(AV), $(\mathrm{NII})_1$, $(\mathrm{NIV})_2$, (NV).

Remark 4.2. In fact, the four-dimensional symplectic Lie algebras were described in [37].

5. Some six-dimensional phase spaces

The complete classification of three-dimensional complex left-symmetric algebras is very complicated [6]. On the other hand, it is interesting to consider the solutions of the S-equation in simple left-symmetric algebras (without any ideal besides zero and itself), like considering the solution of the classical Yang–Baxter equation in semisimple Lie algebras [10]. Although there is not a complete classification of simple left-symmetric algebras, either, we know that (NV) is the only two-dimensional simplex left-symmetric algebra and the classification of three-dimensional complex simplex left-symmetric algebras is given as follows [13,3,6]:

$$\mathrm{T}_1^{\lambda} = \langle e_1, e_2, e_3 | e_1 \cdot e_1 = (\lambda + 1) e_1, e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = \lambda e_3, e_2 \cdot e_3 = e_3 \cdot e_2 = e_1 \rangle;$$
$$0 \langle |\lambda| \langle 1, \text{ or } \lambda = \mathrm{e}^{\mathrm{i}\theta}, 0 \leq \theta \leq \pi;$$
$$\mathrm{T}_2 = \left\langle e_1, e_2, e_3 | e_1 \cdot e_1 = \frac{3}{2} e_1, e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = \frac{1}{2} e_3, e_2 \cdot e_3 = e_3 \cdot e_2 = e_1, e_3 \cdot e_3 = -e_2 \right\rangle.$$

Similarly to in the discussion in the above section, by a direct computation, we give the solutions of the S-equation in the above (three-dimensional) simple left-symmetric algebras and their corresponding (six-dimensional) phase spaces (only the non-zero products are given) as follows:

$$
\begin{aligned}
\mathrm{SE}(\mathrm{T}_1^{\lambda}, \lambda \neq 1) \;=\; & \left\{ \begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \Longrightarrow \left\{ \begin{array}{l} [e_1, e_2] = e_2; \qquad [e_1, e_3] = \lambda e_3; \\ [e_1, e_1^*] = (\lambda+1) r_{11} e_1 - (\lambda+1) e_1^*; \\ [e_1, e_2^*] = -e_2^*; \qquad [e_1, e_3^*] = -\lambda e_3^*; \\ [e_2, e_1^*] = -r_{11} e_2 - e_3^*; \qquad [e_1^*, e_2^*] = -r_{11} e_2^*; \\ [e_1^*, e_3^*] = -\lambda r_{11} e_3^*; \qquad [e_3, e_1^*] = -\lambda r_{11} e_3 - e_2^*. \end{array} \right. \\
& \bigcup \left\{ \left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & r_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right| r_{22} \neq 0 \right\} \Longrightarrow \left\{ \begin{array}{l} [e_1, e_2] = e_2; \qquad [e_1, e_3] = \lambda e_3; \\ [e_1, e_1^*] = -(\lambda+1) e_1^*; \qquad [e_1, e_2^*] = 2 r_{22} e_2 - e_2^*; \\ [e_1, e_3^*] = -\lambda e_3^*; \qquad [e_2, e_1^*] = -e_3^*; \\ [e_1^*, e_2^*] = r_{22} e_3^*; \qquad [e_3, e_1^*] = r_{22} e_2 - e_2^*. \end{array} \right. \\
& \bigcup \left\{ \left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r_{33} \end{pmatrix} \right| r_{33} \neq 0 \right\} \Longrightarrow \left\{ \begin{array}{l} [e_1, e_2] = e_2; \qquad [e_1, e_3] = \lambda e_3; \\ [e_1, e_1^*] = -(\lambda+1) e_1^*; \qquad [e_1, e_2^*] = -e_2^*; \\ [e_1, e_3^*] = 2\lambda r_{33} e_3 - \lambda e_3^*; \; [e_3, e_1^*] = -e_2^*; \\ [e_2, e_1^*] = r_{33} e_3 - e_3^*; \qquad [e_1^*, e_3^*] = r_{33} e_3^*. \end{array} \right. \\
& \bigcup \left\{ \left. \begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 0 & (\lambda+1) r_{11} \\ 0 & (\lambda+1) r_{11} & 0 \end{pmatrix} \right| (\lambda+1) r_{11} \neq 0 \right\} \\
& \Longrightarrow \left\{ \begin{array}{l} [e_1, e_2] = e_2; \qquad [e_1, e_3] = \lambda e_3; \\ [e_1, e_1^*] = (\lambda+1) r_{11} e_1 - (\lambda+1) e_1^*; \\ [e_1, e_2^*] = (\lambda+1)^2 r_{11} e_3 - e_2^*; \\ [e_1, e_3^*] = (\lambda+1)^2 r_{11} e_2 - \lambda e_3^*; \\ [e_1^*, e_2^*] = \lambda r_{11} e_2^*; \qquad [e_3, e_1^*] = r_{11} e_3 - e_2^*; \\ [e_2, e_1^*] = \lambda r_{11} e_2 - e_3^*; \qquad [e_1^*, e_3^*] = r_{11} e_3^*. \end{array} \right.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{SE}(\mathrm{T}_1^{1}) \;=\; & \mathrm{SE}(\mathrm{T}_1^{\lambda}, \text{ set } \lambda = 1) \\
& \bigcup \left\{ \left. \begin{pmatrix} \sqrt{r_{22} r_{33}} & 0 & 0 \\ 0 & r_{22} & \sqrt{r_{22} r_{33}} \\ 0 & \sqrt{r_{22} r_{33}} & r_{33} \end{pmatrix} \right| r_{22} r_{33} \neq 0 \right\} \\
& \Longrightarrow \left\{ \begin{array}{l} [e_1, e_2] = e_2; \qquad [e_1, e_3] = e_3; \qquad [e_2, e_1^*] = r_{33} e_3 - e_3^*; \\ [e_1, e_1^*] = 2\sqrt{r_{22} r_{33}} e_1 - 2 e_1^*; \\ [e_1, e_2^*] = 2 r_{22} e_2 + 2\sqrt{r_{22} r_{33}} e_3 - e_2^*; \\ [e_1^*, e_2^*] = r_{22} e_3^*; \qquad [e_1^*, e_3^*] = r_{33} e_2^*; \qquad [e_3, e_1^*] = r_{22} e_2 - e_2^*; \\ [e_1 e_3^*] = 2\sqrt{r_{22} r_{33}} e_2 + 2 r_{33} e_3 - e_3^*. \end{array} \right. \\
& \bigcup \left\{ \left. \begin{pmatrix} -\mathrm{i}\sqrt{2 r_{12} r_{13}} & r_{12} & r_{13} \\ r_{12} & \mathrm{i} r_{12} \sqrt{\dfrac{r_{12}}{2 r_{13}}} & \mathrm{i}\sqrt{\dfrac{r_{12} r_{13}}{2}} \\ r_{13} & \mathrm{i}\sqrt{\dfrac{r_{12} r_{13}}{2}} & \mathrm{i} r_{13} \sqrt{\dfrac{r_{13}}{2 r_{12}}} \end{pmatrix} \right| r_{12} r_{13} \neq 0 \right\} \\
& \Longrightarrow \left\{ \begin{array}{l} [e_1, e_2] = e_3; \qquad [e_1, e_3] = e_3; \\ [e_1, e_1^*] = -2\mathrm{i}\sqrt{2 r_{12} r_{13}} e_1 + 3 r_{12} e_2 + 3 r_{12} e_2 + 3 r_{13} e_3 - 2 e_1^*; \\ [e_1, e_2^*] = r_{12} e_1 + 2\mathrm{i} r_{12} \sqrt{\dfrac{r_{12}}{2 r_{13}}} e_2 + \mathrm{i}\sqrt{2 r_{12} r_{13}} e_3 - e_2^*; \\ [e_1, e_3^*] = r_{13} e_1 + \mathrm{i}\sqrt{2 r_{12} r_{13}} e_2 + 2\mathrm{i} r_{13} \sqrt{\dfrac{r_{13}}{2 r_{12}}} e_3 - e_3^*; \\ [e_2, e_1^*] = r_{13} e_1 + \dfrac{3\mathrm{i}}{2} \sqrt{r_{12} r_{13}} + \mathrm{i} r_{13} \sqrt{\dfrac{r_{13}}{2 r_{12}}} e_3 - e_3^*; \\ [e_2, e_2^*] = -r_{12} e_2; \qquad [e_2, e_3^*] = -r_{13} e_2; \qquad [e_3, e_2^*] = -r_{12} e_3; \\ [e_3, e_1^*] = r_{12} e_1 + \mathrm{i} r_{12} \sqrt{\dfrac{r_{12}}{2 r_{13}}} e_2 + \dfrac{3\mathrm{i}}{2} \sqrt{2 r_{12} r_{13}} e_3 - e_2^*; \\ [e_3, e_3^*] = -r_{13} e_3; \qquad [e_2^*, e_3^*] = r_{13} e_2^* - r_{12} e_3^*; \\ [e_1^*, e_2^*] = 2 r_{12} e_1^* + \dfrac{3\mathrm{i}}{2} \sqrt{2 r_{12} r_{13}} e_2^* + \mathrm{i} r_{12} \sqrt{\dfrac{r_{12}}{2 r_{13}}} e_3^*; \\ [e_1^*, e_3^*] = 2 r_{13} e_1^* + \mathrm{i} r_{13} \sqrt{\dfrac{r_{13}}{2 r_{12}}} e_2^* + \dfrac{3\mathrm{i}}{2} \sqrt{2 r_{12} r_{13}} e_3^*. \end{array} \right. \\
& \bigcup \left\{ \left. \begin{pmatrix} -\sqrt{r_{22} r_{33}} & 0 & 0 \\ 0 & r_{22} & -\sqrt{r_{22} r_{33}} \\ 0 & -\sqrt{r_{22} r_{33}} & r_{33} \end{pmatrix} \right| r_{22} r_{33} \neq 0 \right\}
\end{aligned}
$$

$$\Longrightarrow \begin{cases} [e_1,e_2]=e_2; \quad [e_1,e_3]=e_3; \quad [e_1,e_1^*]=-2\sqrt{r_{22}r_{33}}e_1-2e_1^*;\\ [e_2,e_1^*]=r_{33}e_3-e_3^*; \quad [e_1,e_2^*]=2r_{22}e_2-2\sqrt{r_{22}r_{33}}e_3-e_2^*;\\ [e_1^*,e_2^*]=r_{22}e_3^*; \quad [e_1^*,e_3^*]=r_{33}e_2^*; \quad [e_3,e_1^*]=r_{22}e_2-e_2^*;\\ [e_1,e_3^*]=-2\sqrt{r_{22}r_{33}}e_2+2r_{33}e_3-e_3^*. \end{cases}$$

$$\bigcup \left\{ \begin{pmatrix} \mathrm{i}\sqrt{2r_{12}r_{13}} & r_{12} & r_{13}\\ r_{12} & -\mathrm{i}r_{12}\sqrt{\dfrac{r_{12}}{2r_{13}}} & -\mathrm{i}\sqrt{\dfrac{r_{12}r_{13}}{2}}\\ r_{13} & -\mathrm{i}\sqrt{\dfrac{r_{12}r_{13}}{2}} & -\mathrm{i}r_{13}\sqrt{\dfrac{r_{13}}{2r_{12}}} \end{pmatrix} \right\}$$

$$\Longrightarrow \begin{cases} [e_1,e_2]=e_3; \quad [e_1,e_3]=e_3;\\ [e_1,e_1^*]=2\mathrm{i}\sqrt{2r_{12}r_{13}}e_1+3r_{12}e_2+3r_{13}e_3-2e_1^*;\\ [e_1,e_2^*]=r_{12}e_1-2\mathrm{i}r_{12}\sqrt{\dfrac{r_{12}}{2r_{13}}}e_2-\mathrm{i}\sqrt{2r_{12}r_{13}}e_3-e_2^*;\\ [e_1,e_3^*]=r_{13}e_1-\mathrm{i}\sqrt{2r_{12}r_{13}}e_2-2\mathrm{i}r_{13}\sqrt{\dfrac{r_{13}}{2r_{12}}}e_3-e_3^*;\\ [e_2,e_1^*]=r_{13}e_1-\dfrac{3\mathrm{i}}{2}\sqrt{r_{12}r_{13}}-\mathrm{i}r_{13}\sqrt{\dfrac{r_{13}}{2r_{12}}}e_3-e_3^*;\\ [e_2,e_2^*]=-r_{12}e_2; \quad [e_2,e_3^*]=-r_{13}e_2; \quad [e_3,e_2^*]=-r_{12}e_3;\\ [e_3,e_1^*]=r_{12}e_1-\mathrm{i}r_{12}\sqrt{\dfrac{r_{12}}{2r_{13}}}e_2-\dfrac{3\mathrm{i}}{2}\sqrt{2r_{12}r_{13}}e_3-e_2^*;\\ [e_3,e_3^*]=-r_{13}e_3; \quad [e_2^*,e_3^*]=r_{13}e_2^*-r_{12}e_3^*;\\ [e_1^*,e_2^*]=2r_{12}e_1^*-\dfrac{3\mathrm{i}}{2}\sqrt{2r_{12}r_{13}}e_2^*-\mathrm{i}r_{12}\sqrt{\dfrac{r_{12}}{2r_{13}}}e_3^*;\\ [e_1^*,e_3^*]=2r_{13}e_1^*-\mathrm{i}r_{13}\sqrt{\dfrac{r_{13}}{2r_{12}}}e_2^*-\dfrac{3\mathrm{i}}{2}\sqrt{2r_{12}r_{13}}e_3^*. \end{cases}$$

$$\mathrm{SE}(\mathrm{T}_2) = \left\{ \begin{pmatrix} r_{11} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix} \right\} \Longrightarrow \begin{cases} [e_1,e_2]=e_2; \quad [e_1,e_3]=\dfrac{1}{2}e_3;\\ [e_1,e_1^*]=\dfrac{3}{2}r_{11}e_1-\dfrac{3}{2}e_1^*; \quad [e_1,e_2^*]=-e_2^*;\\ [e_1,e_3^*]=-\dfrac{1}{2}e_3^*; \quad [e_2,e_1^*]=-r_{11}e_2-e_3^*;\\ [e_3,e_2^*]=e_3^*; \quad [e_1^*,e_2^*]=-r_{11}e_2^*;\\ [e_3^*,e_1^*]=\dfrac{r_{11}}{2}e_3^*; \quad [e_1^*,e_3]=\dfrac{r_{11}}{2}e_3-e_2^*. \end{cases}$$

$$\bigcup \left\{ \begin{pmatrix} 0 & 0 & 0\\ 0 & r_{22} & 0\\ 0 & 0 & 0 \end{pmatrix} \middle| r_{22}\neq 0 \right\} \Longrightarrow \begin{cases} [e_1,e_2]=e_2; \quad [e_1,e_3]=\dfrac{1}{2}e_3;\\ [e_1,e_1^*]=-\dfrac{3}{2}e_1^*; \quad [e_1^*,e_2^*]=-r_{22}e_3^*\\ [e_1,e_3^*]=-\dfrac{1}{2}e_3^*; \quad [e_2,e_1^*]=-e_3^*;\\ [e_3,e_1^*]=r_{22}e_2-e_2^*; \quad [e_3,e_2^*]=e_3^*;\\ [e_1,e_2^*]=2r_{22}e_2-e_2^*. \end{cases}$$

$$\bigcup \left\{ \begin{pmatrix} r_{11} & 0 & 0\\ 0 & 0 & \dfrac{3}{2}r_{11}\\ 0 & \dfrac{3}{2}r_{11} & 0 \end{pmatrix} \middle| r_{11}\neq 0 \right\}$$

$$\Longrightarrow \begin{cases} [e_1,e_2]=e_2; \quad [e_1,e_3]=\dfrac{1}{2}e_3; \quad [e_1,e_1^*]=\dfrac{3}{2}r_{11}e_1-\dfrac{3}{2}e_1^*;\\ [e_1,e_2^*]=\dfrac{9}{4}r_{11}e_3-e_2^*; \quad [e_1,e_3^*]=\dfrac{9}{4}r_{11}e_1-\dfrac{1}{2}e_3^*;\\ [e_2,e_1^*]=\dfrac{1}{2}r_{11}e_2-e_3^*; \quad [e_3,e_1^*]=r_{11}e_3-e_2^*;\\ [e_1^*,e_2^*]=\dfrac{1}{2}e_2^*; \quad [e_1^*,e_3^*]=r_{11}e_3^*; \quad [e_3,e_2^*]=-\dfrac{3}{2}r_{11}e_2+e_3^*. \end{cases}$$

Corollary 5.1. *There are invertible solutions of the S-equation in three-dimensional simple left-symmetric algebras besides* (T_1^{-1}).

6. Conclusions and discussion

From the study in the previous sections, we would like to give the following conclusions and discussion.

(1) The phase space of a Lie algebra obtained from a non-zero symmetric solution of the S-equation in a left-symmetric algebra is symplectically isomorphic to the one obtained from the zero solution. Therefore, it is invalid to obtain the new symplectic Lie algebras in this way. Moreover, there is a quite similar property of the classical Yang–Baxter equation which was given in [19]. It is interesting to know that both of them can be interpreted in terms of left-symmetric algebras.

(2) We have obtained the phase spaces from all solutions of the S-equation in two-dimensional complex left-symmetric algebras and three-dimensional complex simple left-symmetric algebras. Comparing with the construction in [30], these phase spaces (para-Kähler Lie algebras) are new and different. Furthermore, we would like to point out that it is hard and less practicable to extend what we have done in Sections 4 and 5 to other cases in higher dimensions since the S-equation in left-symmetric algebras involves the (nonlinear) quadratic Equations (4.1).

(3) We have not proved yet whether the phase spaces obtained from the S-equation in left-symmetric algebras are not isomorphic for any two different parameters as para-Kähler Lie algebras. In fact, they are closely related to $\mathrm{SE}(A)$ for a left-symmetric algebra A which relies on the choice of a basis of A and its corresponding structural constants. It is natural to consider whether there are meaningful "classification rules" such that the classification of the solutions of the S-equation in left-symmetric algebras can be more "interesting".

(4) Our study on phase spaces in low dimensions can provide some good examples for studying certain related geometric structures, like complex and Kähler structures [4]. It is natural to consider the possible applications in physics. Furthermore, we hope that they can be a guide for the cases in higher dimensions, even in infinite dimensions. For example, since left-symmetric algebras are the underlying spaces of vertex algebras which are the algebraic structures of conformal field theory [8] and play a crucial role in the Hopf algebraic approach of Connes and Kreimer to renormalization theory of perturbative quantum field theory [16], it would be interesting to consider the roles of phase spaces and the S-equation related to the left-symmetric algebras there.

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