



There exist no 4-dimensional geodesically equivalent metrics with the same stress–energy tensor



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ABSTRACT

We show that if two 4-dimensional metrics of arbitrary signature on one manifold are geodesically equivalent (i.e., have the same geodesics considered as unparameterized curves) and are solutions of the Einstein field equation with the same stress–energy tensor, then they are affinely equivalent or flat. If we additionally assume that the metrics are complete or that the manifold is closed, the result remains valid in all dimensions ≥ 3 .

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1. Definitions and results

Let (M^n, g) be a connected pseudo-Riemannian manifold of arbitrary signature of dimension $n \geq 3$.

We say that a metric \bar{g} on M^n is *geodesically equivalent* to g , if every geodesic of g is a (possibly, reparametrized) geodesic of \bar{g} . We say that \bar{g} is *affinely equivalent* to g , if the Levi-Civita connections of g and \bar{g} coincide.

In this paper we study the question whether two geodesically equivalent metrics g and \bar{g} can satisfy the Einstein field equation with the same stress–energy tensor:

$$R_{ij} - \frac{R}{2} \cdot g_{ij} = \bar{R}_{ij} - \frac{\bar{R}}{2} \cdot \bar{g}_{ij}, \quad (1)$$

where R_{ij} (\bar{R}_{ij} , respectively) is the Ricci tensor of the metric g (\bar{g} , respectively), and $R := R_{ij}g^{ij}$ ($\bar{R} := \bar{R}_{ij}\bar{g}^{ij}$, respectively), $\bar{g}^{k\ell}$ is the tensor dual to \bar{g}_{ij} : $\bar{g}^{sl}\bar{g}_{sj} = \delta^i_j$) is the scalar curvature.

There exist the following trivial examples of such a situation:

1. If geodesically equivalent metrics g and \bar{g} are flat, then their stress–energy tensors vanish identically and therefore coincide. Examples of geodesically equivalent flat metrics are classically known and can be constructed as follows: take the classical projective transformation p of $(U \subseteq \mathbb{R}^n, g_{\text{standard}})$ (i.e., a local diffeomorphism that takes straight lines to straight lines, there is a $(n^2 + 2n)$ -dimensional group of it) and consider the pullback of the standard euclidean metric g_{standard} ; $\bar{g} = p^*g_{\text{standard}}$. It is clearly flat and geodesically equivalent to the initial metric g_{standard} . If p is not a classical affine transformation (the subgroup of affine transformations is $n^2 + n$ -dimensional), \bar{g} is not affinely equivalent to g_{standard} .
2. If g and \bar{g} are affinely equivalent metrics with vanishing scalar curvature, then their stress–energy tensors coincide with the Ricci tensors and therefore coincide (since even Riemannian curvature tensors coincide). There are many examples of such a situation, a possibly simplest one is as follows: Take an arbitrary metric $h = h_{ij}$, $i, j = 2, \dots, n$ of zero scalar curvature on $\mathbb{R}^{n-1}(x_2, \dots, x_n)$ and consider the direct product metric $g = dx_1^2 + \sum_{i,j=2}^n h_{ij}dx_i dx_j$ on

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$\mathbb{R}^n = \mathbb{R}(x_1) \times \mathbb{R}^{n-1}(x_2, \dots, x_n)$. Then, for this metric, and also for the (affinely equivalent) metric $g = dx_1^2 + 2 \sum_{i,j=2}^n h_{ij} dx_i dx_j$, the scalar curvature is zero.

3. The metric $\bar{g} := \text{const} \cdot g$ has the same stress–energy tensor as g . Indeed, $R_{ij} = \bar{R}_{ij}$, and $\bar{R} := \bar{g}^{ij} R_{ij} = \frac{1}{\text{const}} R$ so $R g_{ij} = \frac{1}{\text{const}} R \cdot \text{const} g_{ij} = \bar{R} \bar{g}_{ij}$.

In the present paper we show that in dimensions 3 and 4 this list of trivial examples contains all possibilities:

Theorem 1. *If two geodesically equivalent metrics g and \bar{g} on a connected manifold M of dimension 3 or 4 satisfy (1), then at least one of the following possibilities takes place:*

1. g and \bar{g} are affinely equivalent metrics with zero scalar curvature, or
2. g and \bar{g} are flat, or
3. $\bar{g} = \text{const } g$ for a certain $\text{const} \in \mathbb{R}$.

By this theorem, unparameterized geodesics determine the Levi-Civita connection of a 3 or 4-dimensional metric uniquely within the solutions of the Einstein field equation with the same stress–energy tensor provided the metric is not flat.

The motivation to study this question came from physics. It is known that geodesics of a space–time metric correspond to the trajectories of the free falling uncharged particles, and that certain astronomical observations give the trajectories of free falling uncharged particles as unparameterized curves; moreover, unparameterized geodesics and how and whether they determine the metric were actively studied by theoretical physicists (cf. [1–4]) in the context of general relativity. The space–time metric is a solution of the Einstein equation (there of course could be many solutions of the Einstein equation with the same stress–energy tensor) and our theorem implies that if we know the (unparameterized) trajectories of free falling uncharged particles and the stress–energy tensor, then we know (i.e., can in theory reconstruct) the metric or at least the Levi-Civita connection of the metric.

The dimension 4 is probably the dimension that could be interesting for physics, since space–time metrics are naturally 4-dimensional. The result for dimension 3 is essentially easier; that is why we put it here. In dimension two, the stress–energy tensor of every metric is identically zero and (the analog of) Theorem 1 is evidently wrong. It is also wrong in higher dimensions, we show an example in dimensions ≥ 5 . The metrics g and \bar{g} in this example both have zero scalar curvature and their Riemannian curvature tensors coincide. We do not know whether all geodesically equivalent not affinely equivalent metrics with the same stress–energy tensors have zero scalar curvature, but can show that the scalar curvature must be constant.

Theorem 2. *Suppose two nonproportional geodesically equivalent metrics g and \bar{g} on a connected manifold M^n of dimension $n \geq 5$ satisfy (1). Then, the scalar curvatures of the metrics are constant.*

Combining this theorem with [5,6], we obtain that in the global setting, when the manifold is closed (= compact without boundary), or when both metrics are complete, the analog of Theorem 1 is still true in all dimensions.

We say that a (complete in both directions) g -geodesic $\gamma : \mathbb{R} \rightarrow M$ is \bar{g} -complete, if there exists a diffeomorphism $\tau : \mathbb{R} \rightarrow \mathbb{R}$ such that the curve $\bar{\gamma} := \gamma \circ \tau$ is a \bar{g} -geodesic.

Corollary 1. *Let M^n be a connected manifold of dimension $n \geq 5$. Suppose g and \bar{g} on M^n are geodesically equivalent and satisfy (1). Assume in addition that g has indefinite signature and that every light-like g -geodesic γ is complete in both directions and is \bar{g} -complete.*

Then, the metrics are affinely equivalent.

Corollary 2. *Suppose two geodesically complete geodesically equivalent metrics g and \bar{g} on a connected M^n , $n \geq 5$, such that g is positively definite or negatively definite, satisfy (1). Then, the metrics are affinely equivalent.*

Corollary 3. *Suppose two geodesically equivalent metrics g and \bar{g} on a closed connected M^n , $n \geq 5$, satisfy (1). Then, the metrics are affinely equivalent.*

Probably the most famous special case of Theorem 1 that was known before is due to A. Z. Petrov [2] (see also [7,8]): he showed that 4-dimensional Ricci-flat nonflat metrics of Lorentz signature cannot be geodesically equivalent, unless they are affinely equivalent. It is one of the results for which Petrov obtained in 1972 the Lenin prize, the most important scientific award of the Soviet Union.

Remark 1. We observe the same effect that was observed in [8]: as in [8], in dimension 4 we have a rigidity (in [8] it was proved that 4-dimensional Einstein metrics of nonconstant curvature do not allow nontrivial geodesic equivalence. In every dimension $n \geq 5$ there exist Einstein metrics of nonconstant curvature that are geodesically equivalent but not affinely equivalent). We do not have any conceptual explanation for this effect, and in our proof the dimension is used in many arguments.

2. Proof of Theorems 1 and 2

2.1. Plan of the proof

We start with recalling in Section 2.2 certain known facts from the theory of geodesically equivalent metrics that will be used in the proof. In Section 2.3 we prove an important technical statement: we show that if the minimal polynomial of the tensor a_j^i defined by (6) has degree 2, then geodesically equivalent metrics that were used to construct a_j^i are warped product metrics provided they are not affinely equivalent. In Section 2.4 we prove Theorem 1 for geodesically equivalent warped product metrics.

In Section 2.5 we use the connections between the Ricci tensors of geodesically equivalent metrics to derive the formula (32) which will play an important role in the proof.

The proof depends on the behavior of the scalar curvature of a metric: the following three cases use different ideas:

- Case 1: $R = \text{const} \neq 0$.
- Case 2: $dR \neq 0$.
- Case 3: $R = 0$.

Clearly, almost every point of M^n belongs to one of the cases 1, 2, 3. We prove Theorem 1 under the assumption of these cases in Sections 2.6–2.8 respectively. The first and second cases will be reduced to the warped product case solved in Section 2.4, but in each case the reduction will be different. In the second case, and also in the “warped product part” (i.e., in Section 2.4) we will work in arbitrary dimensions $n \geq 3$, so we simultaneously prove Theorem 2.

Let us explain why and how the results of Sections 2.6–2.8 imply Theorems 1 and 2.

In Section 2.7 we actually show that, under the assumptions of Theorem 2, if $dR \neq 0$ at a point (which of course imply that $dR \neq 0$ at every point of a neighborhood), then the metrics are proportional with constant coefficients in this neighborhood. It is known (see for example [9, Proposition 1] or [10]) that if two geodesically equivalent metrics on a connected manifold are proportional in some neighborhood, then they are proportional on the whole manifold. Thus, the assumption $dR \neq 0$ at a point implies that the metrics are proportional on the whole manifold as we claim in Theorem 1. This also implies that, under the assumptions of Theorem 2, the scalar curvature of g is constant as we claim in Theorem 2.

Now, in Section 2.6 we show that if, under the assumptions of Theorem 1, $R = \text{const} \neq 0$ in a neighborhood again implies that the metrics are proportional in the neighborhood which again in view of [9, Proposition 1] implies that the metrics are proportional on the whole manifold as we claim in Theorem 1.

Finally, if the scalar curvatures of both metrics vanish, we show in Section 2.8 that (under the assumptions of Theorem 1) the metrics are affinely equivalent on the whole manifold, or flat on the whole manifold, as we claim.

2.2. Standard formulas we will use

We work in tensor notations with the background metric g . That means, we sum with respect to repeating indices, use g for raising and lowering indices, and use the Levi-Civita connection of g for covariant differentiation which we denote by comma.

As was known already to Levi-Civita [11], two connections $\Gamma = \Gamma_{jk}^i$ and $\bar{\Gamma} = \bar{\Gamma}_{jk}^i$ have the same unparameterized geodesics, if and only if their difference is a pure trace: there exists a $(0, 1)$ -tensor ϕ_i such that

$$\bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + \delta_{jk}^i \phi_l + \delta_j^l \phi_k. \quad (2)$$

If Γ and $\bar{\Gamma}$ related by (2) are the Levi-Civita connections of metrics g and \bar{g} , then one can find explicitly (following Levi-Civita [11]) a function ϕ on the manifold such that its differential $\phi_{,i}$ coincides with the $(0, 1)$ -tensor ϕ_i : indeed, contracting (2) with respect to i and j , we obtain $\bar{\Gamma}_{si}^s = \Gamma_{si}^s + (n+1)\phi_i$. From the other side, for the Levi-Civita connection Γ of a metric g we have $\Gamma_{sk}^s = \frac{1}{2} \frac{\partial \log(|\det(g)|)}{\partial x_k}$. Thus,

$$\phi_i = \frac{1}{2(n+1)} \frac{\partial}{\partial x_i} \log \left(\left| \frac{\det(\bar{g})}{\det(g)} \right| \right) = \phi_{,i} \quad (3)$$

for the function $\phi : M \rightarrow \mathbb{R}$ given by

$$\phi := \frac{1}{2(n+1)} \log \left(\left| \frac{\det(\bar{g})}{\det(g)} \right| \right). \quad (4)$$

In particular, the derivative of ϕ_i is symmetric, i.e., $\phi_{i,j} = \phi_{j,i}$.

The formula (2) implies that two metrics g and \bar{g} are geodesically equivalent if and only if for a certain ϕ_i (which is, as we explained above, the differential of ϕ given by (4)) we have

$$\bar{g}_{ij,k} - 2\bar{g}_{ij}\phi_k - \bar{g}_{ik}\phi_j - \bar{g}_{jk}\phi_i = 0, \quad (5)$$

where “comma” denotes the covariant derivative with respect to the connection Γ . Indeed, the left-hand side of this equation is the covariant derivative with respect to $\bar{\Gamma}$, and vanishes if and only if $\bar{\Gamma}$ is the Levi-Civita connection for \bar{g} . Clearly, the metrics g and \bar{g} are affinely equivalent, if $\phi_i \equiv 0$, or, which is the same, if $\phi = \text{const}$.

Eq. (5) should be viewed a system of PDE on the unknowns \bar{g}_{ij} and ϕ_i . It can be linearized by a clever substitution (which was already known to R. Liouville [12] and Dini [13] in dimension 2 and is due to Sinjukov [14] in other dimensions, see also [15,10]): consider a_{ij} and λ_i given by

$$a_{ij} = e^{2\phi} \bar{g}^{sq} g_{si} g_{qj}, \quad (6)$$

$$\lambda_i = -e^{2\phi} \phi_s \bar{g}^{sp} g_{pi}, \quad (7)$$

where \bar{g}^{sp} is the tensor dual to \bar{g}_{ij} : $\bar{g}^{si} \bar{g}_{sj} = \delta^i_j$. It is an easy exercise to show that the following linear equation on the symmetric (0, 2)-tensor a_{ij} and (0, 1)-tensor λ_i is equivalent to (5)

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}. \quad (8)$$

Note that there exists a function λ such that its differential is precisely the (0, 1)-tensor λ_i : indeed, multiplying (8) by g^{ij} and summing with respect to repeating indices i, j we obtain $(g^{ij} a_{ij})_{,k} = 2\lambda_{,k}$. Thus, λ_i is the differential of the function

$$\lambda := \frac{1}{2} g^{qp} a_{qp}. \quad (9)$$

In particular, the covariant derivative of λ_i is symmetric: $\lambda_{i,j} = \lambda_{j,i}$. Clearly, the metrics g and \bar{g} are affinely equivalent, if $\lambda_i \equiv 0$, or, which is the same, if $\lambda = \text{const}$.

Remark 2. In this paper an important role plays the tensor $A := a^i_j$ which we will view as a field of endomorphisms of TM ; combining the formulas (6) and (4) we see that it is given by the formula

$$A = a^i_j := \left(\left| \frac{\det(\bar{g})}{\det(g)} \right| \right)^{\frac{1}{n+1}} \bar{g}^{is} g_{sj}. \quad (10)$$

One can reconstruct the metric \bar{g} (considered as a bilinear form) by A and g by the formula

$$\bar{g}(\cdot, \cdot) = \frac{1}{|\det(A)|} g(A^{-1} \cdot, \cdot). \quad (11)$$

Integrability conditions for Eq. (8) (we substitute the derivatives of a_{ij} given by (8) in the formula $a_{ij,\ell k} - a_{ij,k\ell} = a_{is} R^s_{j k \ell} + a_{sj} R^s_{i k \ell}$, which is true for every (0, 2)-tensor a_{ij}) were first obtained by Solodovnikov [16] and are

$$a_{is} R^s_{j k \ell} + a_{sj} R^s_{i k \ell} = \lambda_{\ell,i} g_{jk} + \lambda_{\ell,j} g_{ik} - \lambda_{k,i} g_{j\ell} - \lambda_{k,j} g_{i\ell}. \quad (12)$$

For further use let us recall the following well-known fact which can also be obtained by simple calculations (the straightforward way is to replace Γ by $\bar{\Gamma}$ given by (2) in the formula for the Riemannian curvature and then for the Ricci tensor): the Ricci-tensors of connections related by (2) are connected by the formula

$$\bar{R}_{ij} = R_{ij} - (n-1)(\phi_{i,j} - \phi_j \phi_i), \quad (13)$$

where R_{ij} is the Ricci-tensor of Γ and \bar{R}_{ij} is the Ricci-tensor of $\bar{\Gamma}$.

Important special case of the metrics we will consider in our proof will be the metrics such that they admit a solution (a_{ij}, λ_i) of (8) with $a_{ij} \neq \text{const} \cdot g_{ij}$, such that the derivative of λ_i satisfies, for a certain constant B and for a certain function μ , the equation

$$\lambda_{i,j} = \mu g_{ij} + B a_{ij}. \quad (14)$$

This condition may look artificial at first glance, but it is not, since it naturally appears in many situations in the theory of geodesically equivalent metrics. For example, if g is Einstein, then every solution (a_{ij}, λ_i) satisfies this condition (with $B = -\frac{R}{n(n-1)}$), see [8, Eq. (24)]. Moreover, if the dimension of the space of solutions of (8) is at least three, then there exists a constant B such that every solution of (8) satisfies (14) (the constant B is the same for all solutions but the function μ depends on the solution), see [5, Lemma 3]. Moreover, the constant B is unique for all solutions and is the same on the whole (connected) manifold [5, Sections 2.3.4, 2.3.5]. In our setting, under the assumption that the scalar curvature R is a constant, Eq. (1) implies Eq. (14) for the constant $B = -\frac{R}{2(n-1)}$, see Section 2.5.

Moreover, if (14) is satisfied, then the function μ necessary satisfies the equation $\mu_{,i} = 2B\lambda_{,i}$ (see [5, Rem. 10]), so the triple (a, λ, μ) satisfies the following Frobenius-type system:

$$\begin{cases} a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} \\ \lambda_{i,j} = \mu g_{ij} + B a_{ij} \\ \mu_{,i} = 2B\lambda_{,i}. \end{cases} \quad (15)$$

For further use we need the following.

Lemma 1 (cf. Lemma 9 of [5]). Let g, \bar{g} be geodesically equivalent metrics on a connected $M^{n \geq 3}$. Assume that the metric g admits a solution (a_{ij}, λ_i) with $\lambda_i \neq 0$ of (8) such that (14) holds. Assume also that the metric \bar{g} admits a solution $(\bar{a}_{ij}, \bar{\lambda}_i)$ of the natural analog of (8) with $\bar{\lambda}_i \neq 0$ such that the natural analog of (14) holds; we denote the natural analog of B by \bar{B} .

Then, the following formula holds:

$$\phi_{i,j} - \phi_i \phi_j = -B g_{ij} + \bar{B} \bar{g}_{ij}. \quad (16)$$

Proof. We covariantly differentiate (7) (the index of differentiation is “j”); then we substitute the expression (5) for $\bar{g}_{ij,k}$ to obtain

$$\begin{aligned} \lambda_{i,j} &= -2e^{2\phi} \phi_j \phi_p \bar{g}^{pq} g_{qi} - e^{2\phi} \phi_{p,j} \bar{g}^{pq} g_{qi} + e^{2\phi} \phi_p \bar{g}^{ps} \bar{g}_{sl,j} \bar{g}^{lq} g_{qi} \\ &= -e^{2\phi} \phi_{p,j} \bar{g}^{pq} g_{qi} + e^{2\phi} \phi_p \phi_s \bar{g}^{ps} g_{ij} + e^{2\phi} \phi_j \phi_l \bar{g}^{lq} g_{qi}, \end{aligned} \quad (17)$$

where \bar{g}^{pq} is the tensor dual to \bar{g}_{pq} , i.e., $\bar{g}^{pi} \bar{g}_{pj} = \delta_j^i$. We now substitute $\lambda_{i,j}$ from (14), use that a_{ij} is given by (6), and divide by $e^{2\phi}$ for cosmetic reasons to obtain

$$e^{-2\phi} \mu g_{ij} + B \bar{g}^{pq} g_{pj} g_{qi} = -\phi_{p,j} \bar{g}^{pq} g_{qi} + \phi_p \phi_s \bar{g}^{ps} g_{ij} + \phi_j \phi_l \bar{g}^{lq} g_{qi}. \quad (18)$$

Multiplying with $g^{im} \bar{g}_{mk}$, we obtain

$$\phi_{k,j} - \phi_k \phi_j = \underbrace{(\phi_p \phi_q \bar{g}^{pq} - e^{-2\phi} \mu)}_{\bar{b}} \bar{g}_{kj} - B g_{kj}. \quad (19)$$

The same holds with the roles of g and \bar{g} exchanged (the function (4) constructed by the interchanged pair \bar{g}, g is evidently equal to $-\phi$). We obtain

$$-\phi_{k,j} - \phi_k \phi_j = \underbrace{(\phi_p \phi_q g^{pq} - e^{2\phi} \bar{\mu})}_b g_{kj} - \bar{B} \bar{g}_{kj}, \quad (20)$$

where $\phi_{i,j}$ denotes the covariant derivative of ϕ_i with respect to the Levi-Civita connection of the metric \bar{g} . Since the Levi-Civita connections of g and of \bar{g} are related by the formula (2), we have

$$-\phi_{k,j} - \phi_k \phi_j = \underbrace{-\phi_{k,j} + 2\phi_k \phi_j - \phi_k \phi_j}_{-\phi_{k,j}} = -(\phi_{k,j} - \phi_k \phi_j).$$

We see that the left hand side of (19) is equal to minus the left hand side of (20). Thus, $b \cdot g_{ij} - \bar{B} \cdot \bar{g}_{ij} = B \cdot g_{ij} - \bar{b} \cdot \bar{g}_{ij}$ holds. Since the metrics g and \bar{g} are not proportional by assumption, $\bar{b} = \bar{B}$ as we explained above, and the formula (19) coincides with (16). Lemma is proved.

Remark 3. We see that under the assumptions of Lemma 1 the constant B is given in view of (20) by

$$B = \phi_p \phi_q g^{pq} - e^{2\phi} \bar{\mu}.$$

2.3. Geodesically equivalent metrics such that the minimal polynomial of $A = a_j^i$ has degree 2

Assume that (a_{ij}, λ_i) is a nontrivial (i.e., $\lambda_i \neq 0$) solution of (8). We assume $n = \dim(M) \geq 3$. We will discuss the situation when the minimal polynomial of the $(1, 1)$ -tensor $A = a_j^i$ (viewed as an endomorphism of TM) has degree at most 2 (in every point of some nonempty neighborhood $U \subseteq M$), i.e., when there exist functions c_1 and c_2 such that

$$A^2 + c_1 A + c_2 \text{Id} = a_k^i a_j^k + c_1 a_j^i + c_2 \delta_j^i = 0. \quad (21)$$

In other words, we assume that A has the following real Jordan normal form at every point of the neighborhood; in all matrices below we assume that zeros stay on the empty spaces and all diagonal blocks are square matrices

$$\left(\begin{array}{c|c} \rho_1 \text{Id}_{k \times k} & \\ \hline & \rho_2 \text{Id}_{(n-k) \times (n-k)} \end{array} \right), \quad (22)$$

$$\left(\begin{array}{cc|cc|cc} \rho & 1 & & & & \\ & \rho & & & & \\ \hline & & \ddots & & & \\ \hline & & & \rho & 1 & \\ & & & & \rho & \\ \hline & & & & & \rho \text{Id}_{(n-2k) \times (n-2k)} \end{array} \right), \quad \left(\begin{array}{cc|cc|cc} \alpha & \beta & & & & \\ -\beta & \alpha & & & & \\ \hline & & \ddots & & & \\ \hline & & & \alpha & \beta & \\ & & & -\beta & \alpha & \end{array} \right), \quad (23)$$

where $\text{Id}_{k \times k} = \text{diag}(1, \dots, 1)$ denotes the matrix of the identity endomorphism of \mathbb{R}^k . In (23), the number of the first 2×2 -blocks of the first matrix is k and we assume $1 \leq k \leq \frac{n}{2}$. In (22) we assume $0 < k < n$. Note that the minimal polynomial of the matrices (22), (23) can have degree ≤ 1 ; this corresponds to the case when $\rho_1 = \rho_2$ or when $\beta = 0$.

Our goal is to prove the following.

Lemma 2. Let (M^n, g) be a pseudo-Riemannian connected manifold of dimension $n \geq 3$ and (a_{ij}, λ_i) be a solution of (8) such that $\lambda_i \neq 0$ at least at one point and such that there exist functions c_1, c_2 on an open nonempty neighborhood $U \subseteq M$ such that (21) holds at every point of the neighborhood.

Then, at the generic point of M , the Jordan normal form of $A = a_j^i$ is as in (22) with $\rho_1 \neq \rho_2$ and $k = 1$.

In other words, at the generic point the Jordan normal form of $A = a_j^i$ has no Jordan blocks of dimension ≥ 2 , has precisely two different eigenvalues both of which are real, and one of the eigenvalues (without loss of generality, ρ_1) has algebraic multiplicity one.

Moreover, the other eigenvalue ρ_2 of A (considered as a function on M) is constant.

Moreover, in a neighborhood of every generic point there exists a coordinate system (x_1, \dots, x_n) such that the matrices of g and of a_j^i are given by

$$g_{ij} = \left(\begin{array}{c|c} \pm 1 & \\ \hline & \sigma(x_1)h \end{array} \right), \quad a_j^i = \left(\begin{array}{c|c} \sigma(x_1) + C & \\ \hline & C \text{Id} \end{array} \right), \quad (24)$$

where σ is a function of x_1 , C is a constant, and h is a symmetric nondegenerate $(n-1) \times (n-1)$ -matrix whose entries depend on x_2, \dots, x_n .

Proof. The proof is based on the Splitting and Gluing Lemmas from [17] and on [9, Proposition 1]; it could be useful for a reader to have [17,9] at hand while reading the present paper.

If the minimal polynomial of A has degree 1 or 0 in a neighborhood of a point, then at each point of this neighborhood A has one eigenvalue of the geometric multiplicity n . Then, by [9, Proposition 1], the eigenvalue is constant on the whole manifold, and its geometric multiplicity is n on the whole manifold. Then, $A = \text{const} \cdot \text{Id}$, which implies that $\lambda_i \equiv 0$ which contradicts the assumptions.

Assume the Jordan form of $A = a_j^i$ is as the first one in (23) in a neighborhood $U \subseteq M$. Then, the geometric multiplicity of the eigenvalue ρ is ≥ 2 implying by [9, Proposition 1] that (the function) ρ is constant on the whole manifold so $\lambda = \frac{1}{2}a_s^s = \frac{n}{2}\rho$ is also constant so $\lambda_i \equiv 0$ which contradicts the assumptions.

Similarly, if the Jordan form of a_j^i is as the second one in (23) in U , the geometric multiplicity of the eigenvalues $\alpha \pm i\beta$ is ≥ 2 implying by [9, Proposition 1] that (both functions) α and β are constant on the whole manifold so $\lambda = \frac{1}{2}a_s^s = \frac{n}{2}\alpha$ is also constant so $\lambda_i \equiv 0$ which contradicts the assumptions.

Thus, at every point of the manifold the Jordan normal form of $A = a_j^i$ is as in (22); the eigenvalues ρ_1 and ρ_2 may of course depend on the point and there may exist points such that $\rho_1 = \rho_2$. If $k \neq 1$ and $k \neq n-1$, then the geometric multiplicities of both eigenvalues are ≥ 2 so we again obtain that ρ_1, ρ_2 are constant on the whole manifold and therefore $\lambda = \frac{1}{2}a_s^s = \frac{k}{2}\rho_1 + \frac{n-k}{2}\rho_2$ is constant so $\lambda_i \equiv 0$ which contradicts the assumptions. Thus, $k = 1$ or $n-k = 1$; without loss of generality we assume $k = 1$; then on the whole manifold $\rho_2 = \text{const}$ and its multiplicity is at least $n-1$ at all points of the manifold implying the Jordan form of a_j^i is as in (22) with $k = 1$ on the whole manifold.

The characteristic polynomial of $A = a_j^i$ is $\chi = (t - \rho_1)(t - \rho_2)^{n-1}$; we denote $(t - \rho_1)$ by χ_1 and $(t - \rho_2)^{n-1}$ by χ_2 . At the points such that $\rho_1 \neq \rho_2$ the factorization $\chi = \chi_1 \cdot \chi_2$ is admissible in the terminology of [17, Section 1.1]. Then, by the Splitting [17, Theorem 3] and Gluing Lemmas [17, Theorem 4], there exists a coordinate system (x_1, \dots, x_n) such that in these coordinates the eigenvalue ρ_1 is a function of x_1 and the matrices of a_j^i and of g are as follows:

$$a_j^i = \left(\begin{array}{c|c} \rho_1 & \\ \hline & \rho_2 \text{Id} \end{array} \right), \quad (25)$$

$$g_{ij} = \left(\begin{array}{c|c} f(x_1)\chi_2(\rho_1) & \\ \hline & \chi_1(\rho_2)h \end{array} \right) = \left(\begin{array}{c|c} f(x_1)(\rho_1 - \rho_2)^{n-1} & \\ \hline & (\rho_2 - \rho_1)h \end{array} \right), \quad (26)$$

where h is a symmetric nondegenerate $(n-1) \times (n-1)$ matrix whose components depend on the variables x_2, \dots, x_n only and f is a function of x_1 . Replacing the first coordinate by an appropriate function X_1 of it (such that $dX_1 = \sqrt{|f(x_1)(\rho_1 - \rho_2)^{n-1}|}dx_1$) we can make the $(1, 1)$ -component of g to be ± 1 . Finally we see that g and a_j^i are given by the formulas (24) with $\sigma = \rho_1$ and $C = \rho_2$. Lemma is proved.

Remark 4. Assume in addition that the solution a_{ij} came from a metric \bar{g}_{ij} by (6), i.e., assume that a_{ij} is nondegenerate, i.e., assume that $\rho_2 \neq 0$ and $\rho_1(x_1) \neq 0$ at every point of the neighborhood we are working in. Then, by (11), the matrix of \bar{g} is given by

$$\bar{g}_{ij} = \frac{1}{C^{n-1}} \left(\begin{array}{c|c} \frac{\pm 1}{(\sigma+C)^2} & \\ \hline & \frac{\sigma}{(\sigma+C)C} h \end{array} \right). \quad (27)$$

Without loss of generality we can assume later that the sign \pm of the $(1, 1)$ -entry of g and of \bar{g} is “+”, since multiplication of g or of \bar{g} by -1 does not affect Eq. (1).

2.4. Proof for geodesically equivalent warped product metrics

We will now prove Theorem 1 under the additional assumption that the geodesically equivalent metrics g and \bar{g} satisfying (1) are given by the formulas (24) and (27). We prove the following.

Lemma 3. Assume the metrics g and \bar{g} given by (24) and (27) satisfy (1). Then, the function σ is a constant, so the metrics are affinely equivalent.

Proof. We prove the Lemma by direct calculations: a straightforward way to do it (at least in the 3- and 4-dimensional cases which will be used in the proof of Theorem 1) is to use any computer algebra program, for example Maple, to calculate the difference between the left- and the right hand sides of (1). One immediately sees that the i, j -component of the difference with $i \geq 2, j \geq 2$ is proportional to the corresponding entry of h_{ij} with the same coefficient of the proportionality which is proportional to $(\sigma')^2$. Since it is zero by assumptions, σ is a constant and the metrics are affinely equivalent.

As a part of the proof of Theorem 2 we need this calculation in arbitrary dimension; let us explain a small trick that helps to calculate the difference between the left- and the right hand sides of (1) ‘by hands’ and in any dimension.

We will use that the conformally equivalent metric $\frac{1}{\sigma}g$ is the direct product metric so its Ricci tensor has the form

$$\left(\begin{array}{c|c} 0 & \\ \hline & H_{ij} \end{array} \right) \quad (28)$$

where H is the Ricci-tensor of the $(n-1)$ -dimensional metric h_{ij} (viewed as a metric on $U \subseteq \mathbb{R}^{n-1}(x_2, \dots, x_n)$), and its scalar curvature is simply the scalar curvature of h_{ij} . Now, it is well known that the Ricci-tensors and the scalar curvatures of any conformally equivalent metrics g and $\hat{g} := e^{-2\psi}g$ are related by

$$\begin{aligned} \hat{R}_{ij} &= R_{ij} - (n-2)(\psi_{i,j} - \psi_i\psi_j) - (\Delta_2 + (n-2)\Delta_1)g_{ij}, \\ \hat{R} &= -e^{-2\psi}(R + 2(n-1)\Delta_2 + (n-1)(n-2)\Delta_1), \end{aligned} \quad (29)$$

where Δ_2 is the Laplacian of ψ , $\Delta_2 = \psi_{i,j}g^{ij}$, and Δ_1 is the square of the length of ψ_i in g , $\Delta_1 := g^{ij}\psi_{i,j}$. In our case the role of the metric g in (29) plays the direct product metric $\frac{1}{\sigma}g$ and $\phi = -\frac{1}{2}\log|\sigma|$. After some relatively simple calculations we obtain R_{ij} as an algebraic expression in $H_{ij}, h_{ij}, \sigma, \sigma'$ and σ'' , and also R as an algebraic expression in $H = H_{ij}h^{ij}, \sigma, \sigma'$ and σ'' .

Similarly, the metric $\frac{C(\sigma+C)}{\sigma}\bar{g}$ which is conformally equivalent to the metric \bar{g} is also the direct product metric so its Ricci curvature also is as in (28). We again combine it with (29) and calculate the scalar and Ricci curvatures of \bar{g} . Substituting the result of the calculation on the left hand side of (1) minus the right-hand side of (1), and considering the components of the result for $i, j \geq 2$, we see that H and H_{ij} disappear and we obtain the following condition on the function σ only

$$\frac{(n-2)(n-1)(\sigma')^2}{6\sigma(\sigma+C)} = 0.$$

Then, $\sigma' = 0$, which implies that σ is a constant and the metrics are affinely equivalent.

2.5. $\lambda_{i,j}$ is a linear combination of g_{ij} and a_{ij} (with functional coefficients)

Assume geodesically equivalent g and \bar{g} on M^n satisfy (1). Rearranging the terms in (1), we obtain $R_{ij} - \bar{R}_{ij} = \frac{R}{2}g_{ij} - \frac{\bar{R}}{2}\bar{g}_{ij}$. Substituting (13) inside, we obtain

$$\phi_{i,j} - \phi_i\phi_j = \frac{R}{2(n-1)}g_{ij} - \frac{\bar{R}}{2(n-1)}\bar{g}_{ij}. \quad (30)$$

Now we covariantly differentiate (7) (the index of differentiation is “j”); then we substitute the expression (5) for $\bar{g}_{ij,k}$, and finally we substitute (30) to obtain

$$\begin{aligned}\lambda_{i,j} &= -2e^{2\phi}\phi_j\bar{g}^{sp}g_{pi} - e^{2\phi}\phi_{s,j}\bar{g}^{sp}g_{pi} + e^{2\phi}\phi_s\bar{g}^{sq}\bar{g}_{q\ell,j}\bar{g}^{\ell p}g_{pi} \\ &\stackrel{(5)}{=} -e^{2\phi}\bar{g}^{sp}g_{pi}(\phi_{s,j} - \phi_s\phi_j) + e^{2\phi}\phi_s\phi_p\bar{g}^{sp}g_{ij} \\ &\stackrel{(16)}{=} -e^{2\phi}\bar{g}^{sp}g_{pi}\left(\frac{R}{2(n-1)}g_{sj} - \frac{\bar{R}}{2(n-1)}\bar{g}_{sj}\right) + e^{2\phi}\phi_s\phi_p\bar{g}^{sp}g_{ij},\end{aligned}\quad (31)$$

where \bar{g}^{sp} is the tensor dual to \bar{g}_{ij} . We combine this with (6) and see that

$$\lambda_{i,j} = \mu g_{ij} + Ba_{ij}, \quad (32)$$

where $B := -\frac{R}{2(n-1)}$ and $\mu := \frac{\bar{R}}{2(n-1)}e^{2\phi} + e^{2\phi}\phi_s\phi_q\bar{g}^{sq}$.

Note that $B = -\frac{R}{2(n-1)}$ is constant if and only if the scalar curvature R is a constant.

For further use let us also consider the (1, 3)-tensor

$$X^i_{jkl} := R^i_{jkl} + \frac{R}{2(n-1)}(\delta^i_{\ell}g_{jk} - \delta^i_k g_{j\ell}). \quad (33)$$

This tensor clearly satisfies the same algebraic symmetries w.r.t. g as the curvature tensor; by construction of X the contraction $X^s_{jsk} = R_{jk} - \frac{R}{2}g_{jk}$ is the stress-energy tensor of g . Let us observe that X^i_{jkl} satisfies

$$a_{si}X^s_{jkl} + a_{sj}X^s_{ikl} = 0 : \quad (34)$$

indeed, we substitute $\lambda_{i,j}$ given by (32) with $B = -\frac{R}{2(n-1)}$ in (12) and obtain (33).

2.6. Proof of Theorem 1 under the assumption of Case 1: $R = \text{const} \neq 0$

Without loss of generality we can assume $\frac{R}{2(n-1)} = 1$, since it always can be achieved by the rescaling of the metric. In this setting the system (15) reads

$$\begin{cases} a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} \\ \lambda_{i,j} = \mu g_{ij} - a_{ij} \\ \mu_{,i} = -2\lambda_i. \end{cases} \quad (35)$$

By the metric cone over (M, g) we understand the product manifold $\hat{M} = \mathbb{R}_{>0}(r) \times M(x)$ equipped by the metric \hat{g} such that in the coordinates (r, x) its matrix has the form

$$\hat{g}(r, x) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 g(x) \end{pmatrix}. \quad (36)$$

Let us recall the following relation between the parallel symmetric (0, 2)-tensors on the cones and the solutions of (35).

Theorem 3 (Proposition 3.1 and Corollary 3.3 of [6]). *If a symmetric tensor field a_{ij} on (M, g) satisfies (35), then the (0, 2)-tensor field A on (\hat{M}, \hat{g}) defined in the local coordinates (r, x) by the following (symmetric) matrix:*

$$A = \left(\begin{array}{c|ccc} \mu(x) & -r\lambda_1(x) & \dots & -r\lambda_n(x) \\ \hline -r\lambda_1(x) & & & \\ \vdots & & r^2 a(x) & \\ -r\lambda_n(x) & & & \end{array} \right), \quad (37)$$

is parallel with respect to the Levi-Civita connection of \hat{g} .

Moreover, if a symmetric (0, 2)-tensor A_{ij} on \hat{M} is parallel, then in the cone coordinates it has the form (37), where (a_{ij}, λ_i, μ) satisfy (35).

Remark 5. Since Proposition 3.1 and Corollary 3.3 of [6] are written in different mathematical language, let us note that the proof of Theorem 3 is actually an easy exercise. A straightforward way to do this exercise is to calculate the Levi-Civita connection of the metric \hat{g} (was done many times before), to write down the condition that a symmetric (0, 2)-tensor field on the cone is parallel, and to compare it with (35).

Our next goal is to show (using the results of [18]) that the existence of a parallel symmetric (0, 2)-tensor field on \hat{M} that is not proportional to the metric implies the existence of a nontrivial parallel 1-form. We will essentially use that $n = 3$ or 4 ; there are counterexamples to this claim in all higher dimensions (see e.g. [18, Section 3.3.3] for a counterexample in dimension $n + 1 = 6$).

Since $n = \dim(M) \leq 4$, the dimension of \hat{M} is $n + 1 \leq 5$. Then, the signature of \hat{g} is as required in the assumptions of [18, Theorems 5, 6]. Then, by [18, Theorems 5, 6], the dimension of the space of symmetric (0, 2)-tensor fields is $\frac{k(k+1)}{2} + \ell$,

where $\ell = 1, \dots, \lfloor \frac{n+1-k}{3} \rfloor$, where k is the dimension of the space of parallel vector fields and the brackets " $\lfloor \cdot \rfloor$ " mean the integer part. Now, for $n = 3, 4$ we evidently have $\lfloor \frac{n+1-k}{3} \rfloor \leq 1$. Then, the existence of a parallel symmetric $(0, 2)$ -tensor field that is not proportional to the metric implies that $k \geq 1$, i.e., the existence of a nontrivial parallel vector field, which implies the existence of a nontrivial parallel 1-form.

We will call this parallel 1-form by V_α ($\alpha = 0, \dots, n$); we work in the cone coordinates ($x_0 := r, x_1, \dots, x_n$); we will denote the 0-component of V by v so the 1-form V has entries (v, V_1, \dots, V_n) .

Since V_α is \hat{g} -parallel, the $(0, 2)$ -tensor field $V_\alpha V_\beta$ is also parallel (on \bar{M}); in the cone coordinates $x_0 := r, x_1, \dots, x_n$ it is given by the matrix

$$A = \left(\begin{array}{c|ccc} (v)^2 & vV_1 & \dots & vV_n \\ \hline vV_1 & & & \\ \vdots & & V_i V_j & \\ vV_n & & & \end{array} \right). \quad (38)$$

Comparing this with (37), we see that by Theorem 3 v does not depend on x_0 (so it is essentially a function on M); V_i for $i \geq 1$ have the form $V_i = rv_i$, where $v_i = (v_1, \dots, v_n)$ is a 1-form on M . Moreover, the triple $(a_{ij} = v_i v_j, \lambda_i = -vv_i, \mu = (v)^2)$ is a solution of (35). Note that (v_1, \dots, v_n) is not zero at a generic point of M since otherwise the 'cone' vector field $\frac{\partial}{\partial x_0}$ will be proportional to V^α (with a possible functional coefficient of the proportionality) which is impossible by [18, Lemma 4]. Note that the last equation of the system (35) for this solution looks $(v^2)_{,i} = 2vv_i$ implying $v_i = v_{,i}$. Combining this with $(a_{ij} = v_i v_j, \lambda_i = -vv_i, \mu = v^2)$ and with $\lambda_{i,j} = \mu g_{ij} - a_{ij}$, we obtain

$$v_{i,j} = -vg_{ij}. \quad (39)$$

Since the matrix of $a_{ij} = v_i v_j$ has rank two, its minimal polynomial has degree two. Then, by Lemma 2, in a neighborhood of a generic point there exists a coordinate system such that the metric g and the $(1, 1)$ -tensor a^i_j are given by (24) (cf. [19, Lemma 2.1]). Then, the constant C in (24) is 0 and the 1-form v_i is given by $v_i = (\sqrt{\pm\sigma}, 0, \dots, 0)$. We see that in this coordinate system v^i and λ^i are proportional to $\frac{\partial}{\partial x_1}$ and the coefficient of the proportionality depends on x_0 .

Next, let us observe that the vector field λ^i is a projective vector field of g , that is, the pullback of g w.r.t. the (local) flow of λ^i is geodesically equivalent to g . Indeed, it is known (see e.g. [20,16]), that a vector field v is projective if and only if for the Lie derivative $\ell_{ij} := \mathcal{L}_v g_{ij}$ the $(0, 2)$ tensor $a^v := \ell_{ij} - \frac{1}{n+1} \ell^s_{sgij}$ satisfies (8). For $v^i = \lambda^i$, the tensors $\frac{1}{2} \ell_{ij}$ and $\frac{1}{2} (\ell_{ij} - \frac{1}{n+1} \ell^s_{sgij})$ are given by

$$\frac{1}{2} \ell_{ij} = \lambda_{i,j} = \mu g_{ij} - a_{ij}, \quad \frac{1}{2} \left(\ell_{ij} - \frac{1}{n+1} \ell^s_{sgij} \right) = \mu g_{ij} - a_{ij} - \frac{1}{n+1} (n\mu - 2\lambda) g_{ij} = -a_{ij} + \text{Const} g_{ij};$$

in the last equality in the formula above we use that $\mu_i = 2vv_i = -2\lambda_i$, so $\mu = -2\lambda + \text{const}$. Thus, $\ell_{ij} - \frac{1}{n+1} \ell^s_{sgij}$ satisfies (8) and λ^i is a projective vector field.

Take a small time t and denote by \bar{g} the pullback of g with respect to the time- t -flow of λ_i . Since as we explained above in the coordinate system (x_1, \dots, x_n) constructed above the vector field λ^i has the form $(\lambda^1(x_1), 0, \dots, 0)$, the pullback of the metric g has also the warped product form $\alpha(x_1)dx_1^2 + \beta(x_1)h(x_2, \dots, x_n)$, where $h = h_{ij}, i, j = 2, \dots, n$, is a metric on $U \subset \mathbb{R}^{n-1}(x_2, \dots, x_n)$. Then, the minimal polynomial of the correspondent A has degree 2 and by Lemma 2 in a certain coordinate system the metrics g and \bar{g} have the form (24), (27).

Now, since the metric \bar{g} is isometric to the metric g (since it is its pull-back), every solution of Eq. (8) satisfies (14) with $B = -\frac{R}{2(n-1)}$. Indeed, both R and B are invariants in the sense they do not depend on the coordinate system; moreover, as we explained above, from the result of [5] it follows that the constant B is the same for all solutions. Then, by Lemma 1, it satisfies (16), which in view of $B = \bar{B} = -\frac{R}{2(n-1)} = -\frac{\bar{R}}{2(n-1)}$ reads

$$\phi_{i,j} - \phi_i \phi_j = \frac{R}{2(n-1)} g_{ij} - \frac{R}{2(n-1)} \bar{g}_{ij} = \frac{R}{2(n-1)} g_{ij} - \frac{\bar{R}}{2(n-1)} \bar{g}_{ij}. \quad (40)$$

Substituting this in (13), we obtain (30), which is equivalent to (1). Now, by Lemma 2, geodesically equivalent metrics of the form (24), (27) are affinely equivalent. Then, $R = 0$ which contradicts the assumptions.

2.7. Proof of Theorem 1 under the assumption $dR \neq 0$

We assume as usual that g and \bar{g} on M^n with $n \geq 3$ are geodesically equivalent and satisfy (1); we show that the assumption that the differential $R_{,i}$ of the scalar curvature R is not zero at a certain point leads to a contradiction.

As we have shown above, the solution (a_{ij}, λ_i) of (8) constructed by (6), (7) satisfies (14). From the results of [5] it follows then that the minimal polynomial of a^i_j has degree (at most) two.

More precisely, by [5, Section 2.3.3], under the assumption that a solution (a_{ij}, λ_i) of (8) satisfies (14) (cf. [5, Eq. (38)]); in our setting B from [5, Eq. (38)] equals $-\frac{R}{2(n-1)}$, the formula [5, Eq. (45)] holds. If we take a vector field ξ^i such that $\frac{1}{(n-1)2}\xi^i R_{,i} = 1$ and contract it with [5, Eq. (45)], we obtain (21).

Then, by Lemma 2, in a certain coordinate system (in a neighborhood of almost every point) the metrics g and \bar{g} are as in (24), (27). By Lemma 3, the metrics are affinely equivalent which implies that $R = 0$ which contradicts the assumptions.

2.8. Proof in the case $R = 0$

We assume that g and \bar{g} on a connected M^n of dimension $n = 4$ are geodesically equivalent, are not affinely equivalent, and satisfy (1). The proof in dimension $n = 3$ is similar, is much easier, and will be left to the reader. We assume $R = 0$. If $\bar{R} \neq 0$, then we swap g and \bar{g} and come to the situation considered in Sections 2.6 and 2.7. We can therefore assume without loss of generality that $\bar{R} = 0$.

First let us show the (local) existence of a nontrivial parallel 1-form proportional to ϕ_i (the coefficient of proportionality is a function). In view of $R = \bar{R} = 0$, Eq. (1) reads $R_{ij} = \bar{R}_{ij}$. Then, (13) implies $\phi_{i,j} - \phi_{i,j} = 0$. Recall that $\phi_i = \phi_{,i}$ for the function ϕ given by (4). Then, for the 1-form $e^{-\phi}\phi_i$ we have

$$(e^{-\phi}\phi_i)_{,j} = -e^{-\phi}\phi_i\phi_{j,i} + e^{-\phi}\phi_{i,j} = 0.$$

Now let us show that $A = a^i_j$ has precisely one nonconstant eigenvalue, moreover, the algebraic multiplicity of this eigenvalue is one (at a generic point). In order to prove this claim, let us observe that the tensor X^i_{jkl} given by (33) coincides in view of $R = 0$ with R^i_{jkl} so (34) implies

$$a_{is}R^s_{jkl} + a_{js}R^s_{ikl} = 0. \quad (41)$$

Now, Eq. (41) can be equivalently reformulated as follows: for every $X = X^k$, $Y = Y^\ell$ the endomorphism

$$Z := R(X, Y) = R^i_{jkl}X^kY^\ell$$

commutes with A , i.e., $AZ = ZA$.

Since the metrics are not affinely equivalent, at least one of the eigenvalues of A (considered as a function on M) is not constant. Assume now there exist two nonconstant eigenvalues of A , or the algebraic multiplicity of a nonconstant eigenvalue is greater than one. Recall that by [9, Proposition 1], the geometric multiplicity of every nonconstant eigenvalue is one (at the generic point). Now, it is a standard exercise in linear algebra to check that if g -selfadjoint A commutes with g -scwselfadjoint Z , then every vector from the generalized eigenspace of A corresponding to an eigenvalue of geometric multiplicity one lies in the kernel of Z . Thus, if A has two nonconstant eigenvalues, or if the algebraic multiplicity of a nonconstant eigenvalue is greater than one, then there exist two linearly independent vectors $u = u^i$ and $v = v^i$ such that the restriction of g to $\text{span}(v, u)$ is nondegenerate and both vectors lie in the kernel of each $Z = R(X, Y)$ implying

$$u^s R^i_{skl} = v^s R^i_{skl} = 0. \quad (42)$$

Take the basic such that the first two vectors of this basis are v and u and the last two vectors are orthogonal to the first one. In this basis, the components of curvature tensor with lowered indices R_{ijkl} are zero, if one of the indices i, j, k, ℓ is 1 or 2. In view of the algebraic symmetries of the curvature tensor, we obtain that the only nonzero components of R_{ijkl} are

$$R_{4343} = R_{3434} = -R_{4334} = -R_{3443}.$$

Calculating the scalar curvature, we obtain

$$R = g^{ik}g^{j\ell}R_{ijkl} = 2(g^{33}g^{44} - g^{34}g^{43})R_{4343} = 2 \det \begin{pmatrix} g^{33} & g^{34} \\ g^{43} & g^{44} \end{pmatrix} R_{4343}.$$

Since the restriction of g to $\text{span}(v, u)$ is nondegenerate and the first two vectors of our basis are orthogonal to the second two vectors, then the determinate in the formula above is not zero so our assumption that the scalar curvature is zero implies that the curvature tensor R^i_{jkl} vanishes and the metric is flat which contradicts the assumptions.

Thus, only one eigenvalue of A is not constant, and the geometric multiplicity of this eigenvalue is one. Then, by (9), this eigenvalue is equal to $\frac{1}{2}\lambda + \text{const}$ so λ_i is proportional to this eigenvalue. By the Splitting Lemma (see [17, Theorem 3]), λ^i is an eigenvector corresponding to this eigenvalue. Combining this with (7), we see that ϕ_i is proportional to λ_i .

As we explained in Section 2.2, from (14) and the assumption $R = 0$ it follows the existence of a function μ such that (15) with $B = 0$ holds. The third equation of (15) implies that $\mu = \text{const}$. By [18], $\mu = 0$: indeed, the second equation of (15) in view $B = 0$ reads $\lambda_{i,j} = \mu g_{ij}$. By scaling the metric we can achieve $\lambda_{i,j} = g_{ij}$. Now, in [18, Lemma 2 and Remark 2] it was shown that a one-form v_i such that $v_{i,j} = g_{ij}$ cannot be proportional to a parallel one-form at points where the curvature is not zero. Thus, $\mu = 0$.

Since $\mu = 0$, λ_i is parallel and therefore is proportional to $e^{-\phi}\phi_i$ with a (nonzero) coefficient. Swamping the metrics g and \bar{g} , we also see that the analog of the constant μ (which we, as in Lemma 1, denote by $\bar{\mu}$, is also zero).

Our metrics satisfy the assumptions of Lemma 1 with $B = \bar{B} = 0$. Then, by Remark 3, in view of $B = \bar{\mu} = 0$, we have $g^{pq}\phi_p\phi_q = 0$. But ϕ^i is an eigenvector of g -selfadjoint A such that the corresponding eigenvalue has algebraic multiplicity 1, so $g^{pq}\phi_p\phi_q = 0$ implies $\phi_i = 0$. Finally, the metrics are affinely equivalent.

3. Counterexample in dimensions >4 and the proof of Corollaries 1–3

3.1. Counterexample

Consider any metric $h = h_{ij}$ on $U \subseteq \mathbb{R}^{n-2}(x_3, \dots, x_n)$ of zero scalar curvature. Now, consider a metric g_{ij} and a $(1, 1)$ -tensor $A = a^i_j$ given by

$$g_{ij} = \left(\begin{array}{c|c} x_1 & \\ \hline x_1 & (x_2 - C)^2 h \end{array} \right), \quad A = a^i_j = \left(\begin{array}{c|c} x_2 & x_1 \\ \hline x_2 & C \\ & \ddots \\ & C \end{array} \right), \quad (43)$$

where C is a constant. By direct calculations one can check that g is geodesically equivalent to \bar{g} given by (11). Moreover, the scalar curvatures of g and of \bar{g} are equal to zero, and the Riemannian curvatures of the metrics g and \bar{g} coincide. Then, the metrics satisfy (1).

Note that in dimension $n = 4$ the metric h is the flat metric and therefore the metric g is also a flat metric.

3.2. Corollaries 1–3 follow from [5,6]

Corollaries 1 and 2 easily follow from the results of our paper combined with that of [5]. There, it was proved (under the assumption that the degree of mobility of g is ≥ 3), that any solution of (8) satisfies (14). Then, it was proved (see [5, Sections 2.4, 2.5]) that the metrics are affinely equivalent provided g is complete and \bar{g} is light-line complete (in the indefinite signature) or both metrics are complete (in the definite signature). In the proof the assumption that the degree of mobility is ≥ 3 was not used (only (14) is necessary for the proof).

Now, Corollary 3 follows from [5,6]. More precisely, (15) implies that the function μ satisfies the Gallot–Tanno equation

$$\lambda_{,ijk} = B(\lambda_{,i}g_{jk} + \lambda_{,j}g_{ik} + 2\lambda_{,k}g_{ij}),$$

see [5, Corollary 4]. In the case the metrics are not affinely equivalent, λ is not constant. Now, by [6, Theorem 1], the existence of a nonconstant solution of this equation implies that the metric has constant nonzero sectional curvature. Then, (1) implies that the metrics are proportional.

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