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Locally homogeneous pp-waves☆

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ABSTRACT

We show that every n-dimensional locally homogeneous pp-wave is a plane wave, provided it is indecomposable and its curvature operator, when acting on 2-forms, has rank greater than one. As a consequence we obtain that indecomposable, Ricci-flat locally homogeneous pp-waves are plane waves. This generalises a classical result by Jordan, Ehlers and Kundt in dimension 4. Several examples show that our assumptions on indecomposability and the rank of the curvature are essential.

1. Background and main results

A semi-Riemannian manifold $(\mathcal{M}, \mathrm{g})$ is *homogeneous* if it admits a transitive action by a group of isometries, i.e., for each pair of points p and q in $\mathcal{M}$ there is an isometry of $(\mathcal{M}, \mathrm{g})$ that maps p to q. In the spirit of Felix Klein's *Erlanger Programm* to characterise geometries by their symmetry group, homogeneous manifolds are fundamental building blocks in geometry. Homogeneity is strongly tied to the geometry and the curvature of a manifold. For example, homogeneous Riemannian manifolds are geodesically complete, and, as an example for the link to curvature, any Ricci-flat homogeneous Riemannian manifold is flat [1]. A weaker version of homogeneity is local homogeneity: a semi-Riemannian manifold $(\mathcal{M}, \mathrm{g})$ is *locally homogeneous* if for each point $p \in \mathcal{M}$ there is a neighbourhood $\mathcal{U}$ such that the Killing vector fields of $(\mathcal{U}, \mathrm{g}|_{\mathcal{U}})$ span $T_p\mathcal{M}$ when evaluated at p.

Here we will study local homogeneity for a certain class of Lorentzian manifolds, the so-called *pp-waves* and the *plane waves*. A *pp-wave* is a Lorentzian manifold $(\mathcal{M}, \mathrm{g})$ that admits a parallel null vector field V, i.e., $V \neq 0$, $\mathrm{g}(V, V) = 0$ and $\nabla V = 0$, and curvature endomorphism $\mathrm{R} : \Lambda^2 T\mathcal{M} \to \Lambda^2 T\mathcal{M}$ is non-zero and satisfies

$$\mathrm{R}|_{V^\perp \wedge V^\perp} = 0, \qquad (1.1)$$

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where $V^\perp := \{X \in T\mathcal{M} \mid \mathrm{g}(X, V) = 0\}$. A *plane wave* is a pp-wave with

$$\nabla_U \mathrm{R} = 0 \quad \text{for all } U \in V^\perp. \tag{1.2}$$

Locally, an $(n+2)$-dimensional *pp-wave* has coordinates $(x^-, x^1, \dots, x^n, x^+)$ such that

$$\mathrm{g} := 2\mathrm{d}x^+(\mathrm{d}x^- + H\mathrm{d}x^+) + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{1.3}$$

where $H = H(x^1, \dots, x^n, x^+)$ is a smooth function not depending on x^-. For a *plane wave*, this function is required to be quadratic in the x^i's with x^+-dependent coefficients. In general, they are not homogeneous.

Four-dimensional pp-waves were discovered in a mathematical context by Brinkmann [2] as one class of Einstein spaces that can be mapped conformally onto each other. In physics, plane waves and pp-waves appeared in general relativity (see for example [3,4] for references), where they continue to play an important role as metrics for which the Einstein equations become linear. Solutions of this type describe the propagation of gravitational waves with flat surfaces as wave fronts. Later Penrose discovered that when "zooming in on null geodesics" every space–times has a plane wave as limit [5]. More recently, the conditions under which the homogeneity of a Lorentzian manifold is inherited by its Penrose limit were studied extensively by Figueroa-O'Farrill, Meessen and Philip [6,7]. Moreover, having a large number of parallel spinor fields, higher-dimensional pp-waves constitute supergravity backgrounds, e.g. in [8], and there is now a vast amount of literature on them. For more recent results on homogeneity see the work by Figueroa-O'Farrill et al. in [9–11].

A systematic study of 4-dimensional pp-waves was undertaken by Jordan, Ehlers and Kundt in [4] (see the English republication [12] and also [13], where the name *pp-wave* for *plane fronted with parallel rays* was introduced). Among other aspects, in [4] the isometries of 4-dimensional, gravitational (i.e. Ricci-flat) pp-waves are considered and the Killing equation is solved completely. As a consequence, the possible dimensions of the space of Killing vector fields are given and in each case the form of the metric is determined explicitly. This rather satisfying result allows [4] to conclude:

(A) If a 4-dimensional Ricci-flat pp-wave $(\mathcal{M}^4, \mathrm{g})$ is locally $V^\perp$-homogeneous, then it is a plane wave. In particular, if $(\mathcal{M}^4, \mathrm{g})$ is Ricci-flat and locally homogeneous, then it is a plane wave.

Here, *local $V^\perp$-homogeneity* is a generalisation of local homogeneity taking into account the vector distribution $V^\perp$ that is given on a pp-wave: A semi-Riemannian manifold $(\mathcal{M}, \mathrm{g})$ with a vector distribution $\mathcal{D}$ is *locally $\mathcal{D}$-homogeneous* if for each point $p \in \mathcal{M}$ there is a neighbourhood $\mathcal{U}$ in $\mathcal{M}$ such that the Killing vector fields of $(\mathcal{U}, \mathrm{g}|_{\mathcal{U}})$ span $\mathcal{D}_p$ when evaluated at p.

Proving (A) amounts to showing that local homogeneity (in $V^\perp$-directions) forces H to be at most quadratic in the x^i coordinates. The methods used in [4] in order to solve the Killing equation are restricted to dimension 4 and also use that the function H is harmonic, as a consequence of Ricci-flatness.

Statement (A) is no longer true without the assumption of Ricci-flatness: Sippel and Goenner in [14] determined all solutions to the Killing equation for a 4-dimensional pp-wave $(\mathcal{M}^4, \mathrm{g})$ without assuming $\mathrm{Ric} = 0$ and gave an example of a homogeneous pp-wave that is not a plane wave (see our Example 4.3). However, it turns out that the metric in this example decomposes into a product of a 3-dimensional pp-wave and $\mathbb{R}$. Note that in [4] such a decomposition was implicitly excluded by the Ricci-flatness: if a 4-dimensional Ricci-flat manifold splits as a Riemannian product, then it is flat. Hence, the results in [14, Table II, p. 1234] establish the following result:

(B) If a 4-dimensional indecomposable pp-wave $(\mathcal{M}^4, \mathrm{g})$ is locally $V^\perp$-homogeneous, then it is a plane wave. In particular, if $(\mathcal{M}^4, \mathrm{g})$ is indecomposable and locally homogeneous, then it is a plane wave.

Here, the manifold is *indecomposable*, if the holonomy algebra acts indecomposably. Therefore, when looking for a generalisation of (A) or (B) to arbitrary dimensions the notion of indecomposability is relevant. We say that a semi-Riemannian manifold $(\mathcal{M}, \mathrm{g})$ is *strongly indecomposable* if $(\mathcal{M}, \mathrm{g})$ does not split as a local semi-Riemannian product anywhere, i.e, there is no point in $\mathcal{M}$ that has a neighbourhood on which g is a product metric. Clearly, by the local version of the de Rham–Wu splitting theorem, the holonomy algebra of a strongly indecomposable manifold acts indecomposably (i.e. without non-degenerate invariant subspace), but the converse in general is not true. In addition to strong indecomposability we will need another condition on the curvature tensor R of a pp-wave. From the very definition of a pp-wave it follows that the rank of R when acting on 2-forms does not exceed $\dim(\mathcal{M}) - 2$. For a generalisation of statement (B), we assume that generically the rank of R is larger than one:

Theorem 1. *Let $(\mathcal{M}, \mathrm{g})$ be a pp-wave of arbitrary dimension with parallel null vector field V. Assume that $(\mathcal{M}, \mathrm{g})$ is strongly indecomposable and in addition that almost everywhere the rank of its curvature endomorphism acting on $\Lambda^2 T\mathcal{M}$ is larger than one. Then $(\mathcal{M}, \mathrm{g})$ is a plane wave if it is locally $V^\perp$-homogeneous.*

Here by "almost everywhere" we mean that there is no open set on which the rank of the curvature endomorphism is ≤ 1. Note that the assumption on the rank of the curvature prevents us from applying Theorem 1 to 3-dimensional pp-waves. Indeed, in Example 4.1 we exhibit a 3-dimensional, locally homogeneous pp-wave that is *not* a plane wave. Three-dimensional homogeneous pp-waves were recently classified by Garcia-Rio et al. [15] and our example belongs to their class $\mathcal{N}_b$.

As Ricci-flat pp-waves always satisfy the assumption on the curvature (Lemma 3.3), we obtain a generalisation of statement (A) to arbitrary dimensions:

Corollary 1. *A strongly indecomposable, Ricci-flat and locally $V^\perp$-homogeneous pp-wave is a plane wave.*

It is well known [16, Section 7.3] that in locally homogeneous manifolds any two points have neighbourhoods that are isometric to each other. Hence, a locally homogeneous manifold is strongly indecomposable whenever it is indecomposable, and the rank of the curvature endomorphism is constant. This yields

Corollary 2. *An indecomposable, locally homogeneous pp-wave is a plane wave if, at one point, the rank of the curvature endomorphism is greater than one.*

Locally homogeneous plane waves are reductive (see Section 4.3.3) and have been classified by Blau and O'Loughlin [17] (see our Section 4.3.2). As a consequence, with the exception of the curvature rank one case, our reduction to the plane waves yields a classification of indecomposable locally homogeneous pp-waves in terms of possible functions H (see Section 4.3.2). In appropriate coordinates, we have either

$$H(\boldsymbol{x},x^+)=\boldsymbol{x}^\top \mathrm{e}^{x^+F}S\mathrm{e}^{-x^+F}\boldsymbol{x} \quad \text{or} \quad H(\boldsymbol{x},x^+)=\frac{\boldsymbol{x}^\top \mathrm{e}^{\log(x^++b)F}S\mathrm{e}^{\log(-x^+-b)F}\boldsymbol{x}}{(x^++b)^2}, \tag{1.4}$$

for $b\in\mathbb{R}$, a skew symmetric matrix F and a symmetric matrix S.

Corollary 3. *Indecomposable, Ricci-flat and locally homogeneous pp-waves are plane waves whose metric is locally given by H in formulae (1.4) with a trace-free matrix S.*

Corollary 3 is an instance of the phenomenon that Ricci-flat pp-waves with some additional conditions have to be plane waves. Another instance of this phenomenon is given in [18], where it is shown that compact Ricci-flat pp-waves are plane waves.

If the local Killing vector fields spanning $V_p^\perp$ are actually sections of $V^\perp$, we can drop the assumption on the rank of the curvature:

Theorem 2. *Let $(\mathcal{M},\mathrm{g})$ be a strongly indecomposable pp-wave such that every point p admits a neighbourhood $\mathcal{U}$ with Killing vector fields defined on $\mathcal{U}$ that are sections of $V^\perp|_{\mathcal{U}}$ and which span $V^\perp|_p$. Then $(\mathcal{M},\mathrm{g})$ is a plane wave.*

This is a version of a result for *commuting* Killing vector fields tangent to $V^\perp$:

Theorem 3. *Let $(\mathcal{M},\mathrm{g})$ be a semi-Riemannian manifold of dimension m and assume that there are commuting Killing vector fields that span a null distribution (i.e., a distribution on which g degenerates) of rank $m-1$. Then $(\mathcal{M},\mathrm{g})$ admits a parallel null vector field V and its curvature satisfies*

$$\mathrm{R}(X,Y)Z=0 \quad \textit{and} \quad \nabla_X\mathrm{R}=0,$$

for all $X,Y,Z\in V^\perp$. In particular, if $(\mathcal{M},\mathrm{g})$ is Lorentzian, then it is a plane wave.

Jordan, Ehlers and Kundt [4, Theorem 4.5.2] proved Theorem 3 for 4-dimensional Lorentzian manifolds, but their proof works in any dimension and signature (see our Section 3). In contrast, our proofs of Theorems 1 and 2 use completely different methods than those in [4]. In fact, our proof of Theorem 1 does not require a full solution of the Killing equation (which we derive in Section 4) but a detailed analysis of its consequences (in Section 5). Moreover, we use algebraic results such as the classification of subalgebras of the Lie algebra of similarity transformations $\mathfrak{sim}(n)=(\mathbb{R}\oplus\mathfrak{so}(n))\ltimes\mathbb{R}^n$ of $\mathbb{R}^n$ that act indecomposably on $\mathbb{R}^{1,n+1}$ via $\mathfrak{sim}(n)\subset\mathfrak{so}(1,n+1)$. This classification is due to Bérard-Bergery and Ikemakhen [19], and plays an important role in the classification of indecomposable Lorentzian holonomy algebras in [20].

As we have pointed out above, Example 4.1 shows that, at least in dimension 3, the condition on the rank of the curvature cannot be dropped. However, obvious generalisations of Example 4.1 lead either to *non-homogeneous* pp-waves (as in [13], see our Example 4.2) or to *decomposable* homogeneous pp-waves (as in [14], our Example 4.3). Hence, we are tempted to conjecture (see Section 4.2 for more details):

Conjecture. *Any indecomposable locally homogeneous pp-wave of dimension larger than* 3 *is a plane wave.*

In relation to this we should point out that the rank assumption is independent from the assumption of strong indecomposability: in Example 3.2 we present a 4-dimensional, strongly indecomposable plane wave metric whose curvature has rank 1. The curvature rank one case remains open for further study. Also we believe that our methods employed in Section 5 are useful in a wider context and will give a better understanding of the more general class of indecomposable locally homogeneous Lorentzian manifolds.

2. Killing vector fields and locally homogeneous spaces

Let $(\mathcal{M}, \mathrm{g})$ be a semi-Riemannian manifold with Levi-Civita connection ∇. A *Killing vector field* $K \in \Gamma(T\mathcal{M})$ is a vector field whose flow ϕ_t consists of local isometries of g, i.e. $\phi_t : (\mathcal{U}, \mathrm{g}) \to (\phi_t(\mathcal{U}), \mathrm{g})$ is an isometry, where $\mathcal{U}$ is a neighbourhood of p on which ϕ_t is defined. If K is complete, then all ϕ_t's are global isometries. Clearly, K is a Killing vector field if and only if the $(2, 0)$-tensor $\mathrm{g}(\nabla K, \cdot)$ is skew-symmetric, i.e.

$$\mathrm{g}(\nabla_X K, Y) + \mathrm{g}(X, \nabla_Y K) = 0 \quad \text{for all } X, Y \in T\mathcal{M}. \tag{2.1}$$

Let us denote the real vector space of Killing vector fields of $(\mathcal{M}, \mathrm{g})$ by $\mathfrak{k}$. The Lie bracket of vector fields equips $\mathfrak{k}$ with a Lie algebra structure.

In order to derive the integrability conditions for the Killing equation (2.1), we recall the classical approach by Kostant [21]. Let us denote by $\mathfrak{so}(T\mathcal{M}, \mathrm{g}) := \{\phi \in \mathrm{End}(T\mathcal{M}) \mid \mathrm{g}(\phi(X), Y) + \mathrm{g}(\phi(Y), X) = 0\}$ the bundle of skew-symmetric endomorphisms. For a Killing vector field K, we define the section $\phi^K := \nabla K$ of $\mathfrak{so}(T\mathcal{M}, \mathrm{g})$. A straightforward computation shows that the Killing equation (2.1) implies that

$$\nabla_X \phi^K = -\mathrm{R}(K, X), \tag{2.2}$$

where R denotes the curvature tensor of g defined by $\mathrm{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. Hence, if we define the vector bundle

$$\mathcal{K} := T\mathcal{M} \oplus \mathfrak{so}(T\mathcal{M}, \mathrm{g}) \longrightarrow \mathcal{M}$$

and furnish it with the covariant derivative

$$\nabla^{\mathcal{K}}_X \begin{pmatrix} K \\ \phi \end{pmatrix} := \begin{pmatrix} \nabla_X K - \phi(X) \\ \nabla_X \phi + \mathrm{R}(K, X) \end{pmatrix},$$

we get the vector space isomorphism

$$\mathfrak{k} \simeq \{\text{parallel sections of } (\mathcal{K}, \nabla^{\mathcal{K}})\}.$$

This shows that $\dim(\mathfrak{k}) \le \mathrm{rk}(\mathcal{K}) = \frac{1}{2}m(m+1)$, where $m = \dim(\mathcal{M})$, and that a Killing vector field K is uniquely determined by the values $K|_p \in T_p\mathcal{M}$ and $\nabla K|_p \in \mathfrak{so}(T_p\mathcal{M}, \mathrm{g}_p)$ at a point $p \in \mathcal{M}$. This yields an injection

$$\kappa \ : \ \mathfrak{k} \hookrightarrow \mathfrak{so}(T_p\mathcal{M}, \mathrm{g}_p) \ltimes T_p\mathcal{M}, \qquad K \mapsto -\left(\nabla K|_p, K_p\right). \tag{2.3}$$

By definition, $(\mathcal{M}, \mathrm{g})$ is locally homogeneous if and only if for each point the evaluation map combined with the projection on $\mathbb{R}^{r,s}$ is surjective.

By fixing an orthonormal basis $\mathbf{e}_i$ with $\mathrm{g}(\mathbf{e}_i, \mathbf{e}_j) = \varepsilon_i \delta_{ij}$, we can identify $\mathfrak{so}(T_p\mathcal{M}, \mathrm{g}_p) \ltimes T_p\mathcal{M}$ with the Lie algebra of semi-Euclidean motions $\mathfrak{so}(r, s) \ltimes \mathbb{R}^{r,s}$, where (r, s) is the signature of g. The minus sign in front of the image ensures that for the flat metric on $\mathbb{R}^{r,s}$ this map is a Lie algebra isomorphism (instead of an *anti*-isomorphism) between the Killing vector fields and the Lie algebra of Euclidean motions. In general, this map is *not* a Lie algebra homomorphism. For example, the Killing vector fields of the m-sphere are isomorphic to $\mathfrak{so}(m + 1)$ rather than $\mathfrak{so}(m) \ltimes \mathbb{R}^n$. In fact, a computation reveals

$$\nabla[K, \hat{K}] = [\nabla K, \nabla \hat{K}] - \mathrm{R}(K, \hat{K}),$$

where the right-hand side bracket is the commutator of linear maps, which yields

$$\kappa([K, \hat{K}]) - [\kappa(K), \kappa(\hat{K})] = -(\mathrm{R}_p(K_p, \hat{K}_p), 0), \tag{2.4}$$

again with the second Lie bracket the one in $\mathfrak{so}(T_p\mathcal{M}, \mathrm{g}_p) \ltimes T_p\mathcal{M}$.

Returning to the integrability condition for the Killing equation, we compute the curvature $\mathrm{R}^{\mathcal{K}}$ of $\nabla^{\mathcal{K}}$ and we get

$$\mathrm{R}^{\mathcal{K}}(X, Y) \begin{pmatrix} K \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ (\nabla_K \mathrm{R})(X, Y) - (\phi \cdot \mathrm{R})(X, Y) \end{pmatrix},$$

where $\phi \cdot \mathrm{R}$ denotes the canonical action of an endomorphism on $(3, 1)$-tensors. Hence the existence of a parallel section (K, ϕ) of $\mathcal{K}$ gives the integrability condition

$$\nabla_K \mathrm{R} = \phi \cdot \mathrm{R}, \tag{2.5}$$

where $\phi = \nabla K$ and R is the curvature of g.

Now, assume that $(\mathcal{M}, \mathrm{g})$ enjoys the existence of a parallel vector field V. We define two vector spaces

$$\mathfrak{k}(V) := \{K \in \mathfrak{k} \mid \mathrm{g}(K, V) = 0\}, \qquad \mathfrak{k}'(V) := \{K \in \mathfrak{k} \mid \nabla_V K = 0\},$$

and observe

Lemma 2.1. *If V is a parallel vector field, then we have the following inclusion of subalgebras*

$$\mathfrak{k}(V) \subset \mathfrak{k}'(V) \subset \mathfrak{k}.$$

Proof. First we check the inclusion $\mathfrak{k}(V) \subset \mathfrak{k}'(V)$. Indeed, for a Killing vector field $K \in \mathfrak{k}$, the derivative of the function $\mathrm{g}(V, K)$ satisfies

$$X(\mathrm{g}(K, V)) = \mathrm{g}(\nabla_X K, V) = -\mathrm{g}(\nabla_V K, X). \tag{2.6}$$

First, this implies that if $K \in \mathfrak{k}(V)$ then we also have $\nabla_V K = 0$, i.e., $K \in \mathfrak{k}'(V)$.

Next we note that both $\mathfrak{k}(V)$ and $\mathfrak{k}'(V)$ are subalgebras: Clearly, if V is parallel, $V^\perp$ is involutive and hence $\mathfrak{k}(V)$ is closed under the bracket. Moreover, for $K, \hat{K} \in \mathfrak{k}'(V)$ we have that

$$\nabla_V[K, \hat{K}] = \nabla_V \nabla_K \hat{K} - \nabla_V \nabla_{\hat{K}} K = \nabla_{[K,V]} \hat{K} - \nabla_{[\hat{K},V]} K = 0,$$

since $V \lrcorner \mathrm{R} = 0$ and $[K, V] = -\nabla_V K = 0$. Hence, also $\mathfrak{k}'(V)$ is a subalgebra. $\square$

This lemma implies the following: Let $\mathfrak{stab}(V|_p)$ be the stabiliser of $V|_p$ in $\mathfrak{so}(T_p\mathcal{M}, \mathrm{g}_p)$. Then

$$\kappa : \mathfrak{k}(V) \hookrightarrow \mathfrak{stab}(V|_p) \ltimes V^\perp|_p.$$

When proving the main results, we will work with a different vector space of Killing vector fields, namely with

$$\mathfrak{k}_p(V) := \{K \in \mathfrak{k} \mid \mathrm{g}(K, V)|_p = 0\},$$

for a fixed point $p \in \mathcal{M}$. In general, this is not a Lie algebra, however, for pp-waves it is, as we will see in Corollary 5.3. This fact turns out to be very useful for our approach.

Finally, note that a locally homogeneous manifold is strongly indecomposable (as defined in Section 1) whenever it is indecomposable (i.e., the holonomy algebra acts indecomposably, that is without non-degenerate invariant subspace): if a locally homogeneous manifold is a local product somewhere, it is a local product everywhere and hence the holonomy algebra has a non-degenerate invariant subspace.

This does not hold in the case of local $V^\perp$-homogeneity for a parallel null vector field V. This can be seen for pp-waves as in (1.3) on $\mathbb{R}^{n+2}$: here $V = \partial_- = \frac{\partial}{\partial x^-}$ and the leaves of $V^\perp$ are given as $x^+ = c$ constant. If $H(x^1, \dots, x^n, x^+) \equiv 0$ for $x^+ \in (a, b)$ but $\det(\partial_i\partial_j(H)|_{(x^1,\dots,x^n,x^+)}) \neq 0$ for some other x^+, then the holonomy algebra acts indecomposably. However, near a point with $x^+ \in (a, b)$ the metric is flat.

3. pp-waves and plane waves

Here we recall some basic properties of pp-waves and plane waves as defined in Section 1. First note that the defining Eq. (1.1) is equivalent to

$$\mathrm{R}(X, Y)U \in \mathbb{R}V \quad \text{for all } U \in V^\perp \text{ and } X, Y \in T\mathcal{M}. \tag{3.1}$$

A general pp-wave has an abelian holonomy algebra contained in $\mathbb{R}^n$, where $\mathbb{R}^n$ is an abelian ideal in the stabiliser $\mathfrak{so}(n) \ltimes \mathbb{R}^n$ in $\mathfrak{so}(1, n+1)$ of a null vector. The holonomy algebra is indecomposable if and only if it is equal to $\mathbb{R}^n$. A pp-wave has the following coordinate description:

Lemma 3.1. *Let $(\mathcal{M}, \mathrm{g})$ be a pp-wave and let $p \in \mathcal{M}$. Then there are local coordinates $\varphi = (x^-, \boldsymbol{x} = (x^1, \dots, x^n), x^+)$ on a neighbourhood $\mathcal{U}$ of p and a function $H \in C^\infty(\varphi(\mathcal{U}))$ such that $H = H(x^+, \boldsymbol{x})$ not depending on x^- and,*

$$\mathrm{g} = 2\mathrm{d}x^+(\mathrm{d}x^- + (H \circ \varphi)\mathrm{d}x^+) + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{3.2}$$

where δ_{ij} is the Kronecker symbol and where we use the summation convention. In these coordinates the parallel null vector field is given by $V|_{\mathcal{U}} = \partial_- = \frac{\partial}{\partial x^-}$. These coordinates are usually called Brinkmann coordinates after [2].

Moreover, these coordinates can be chosen such that $\varphi(p) = 0$ and

$$H(x^+, \mathbf{0}) = 0, \qquad \frac{\partial H}{\partial x^i}(x^+, \mathbf{0}) = 0, \tag{3.3}$$

for all x^+ from an interval around zero. We call these coordinates normal Brinkmann coordinates centred at p.

Proof. It is well known that a pp-wave admits local Brinkmann coordinates $\varphi = (x^+, x^-, \boldsymbol{x})$ as in (3.2). We have to show that the remaining freedom allows to chose these coordinates to be normal and centred at p. The general transformations preserving the form (3.2) are of the form

$$\widetilde{x}^- = \frac{1}{a}(x^- - \dot{\boldsymbol{c}}(x^+)^\top A\boldsymbol{x}) + \beta(x^+), \qquad \widetilde{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{c}(x^+), \qquad \widetilde{x}^+ = ax^+ + b, \tag{3.4}$$

where $\boldsymbol{c}$ is a vectorial function and β a real function both of x^+, $A \in \mathrm{O}(n)$ a constant matrix and $a \neq 0$ and b real numbers. After such a transformation the new function $\widetilde{H}$ is given as

$$\widetilde{H}(\widetilde{x}^+, \widetilde{\boldsymbol{x}}) = \tfrac{1}{a}(H(x^+, \boldsymbol{x}) + \ddot{\boldsymbol{c}}(x^+)^\top A\boldsymbol{x}) + \dot{\beta} - \tfrac{1}{2a}\dot{\boldsymbol{c}}(x^+)^\top\dot{\boldsymbol{c}}(x^+). \tag{3.5}$$

Clearly, by applying a translation we can centre these coordinates at p. Moreover, consider a transformation (3.4) with $A = \delta_{ij}$, $b = 0$, $a = 1$ and let $\boldsymbol{c} = (c_1, \dots, c_n)$ be the solution to the ODE

$$\ddot{c}_i(t) = -\frac{\partial}{\partial x^i} H(t, -\boldsymbol{c}(t)),$$

for $i = 1, \dots, n$ with one initial condition $c_i(0) = 0$. Given such a solution $\boldsymbol{c} = (c_1, \dots, c_n)$, let β be the solution to the ODE

$$\dot{\beta}(t) = \tfrac{1}{2}\dot{\boldsymbol{c}}(t)^\top \dot{\boldsymbol{c}}(t) - H(t, -\boldsymbol{c}(t)),$$

with the initial condition $\beta(0) = 0$. Eq. (3.5) shows that after such a transformation we have

$$\tilde{H}|_{(\widetilde{x}^+, \mathbf{0})} = 0, \qquad \frac{\partial \tilde{H}}{\partial \tilde{x}^i}|_{(\widetilde{x}^+, \mathbf{0})} = 0,$$

for all $\widetilde{x}^+$, and hence the new coordinates satisfy Eqs. (3.3) for all $\widetilde{x}^+$. $\square$

In Brinkmann coordinates the non-vanishing components of ∇ are

$$\nabla \partial_i = \partial_i(H) \mathrm{d}x^+ \otimes \partial_-, \qquad \nabla \partial_+ = \mathrm{d}H \otimes \partial_- - \mathrm{d}x^+ \otimes \mathrm{grad}(H), \tag{3.6}$$

where $\mathrm{grad}(H) = \delta^{ij}\partial_i(H)\partial_j$ denotes the gradient of H with respect to the flat metric $\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$ on $\mathbb{R}^n$. This property justifies the term "normal" in Lemma 3.1: the covariant derivatives vanish at $\boldsymbol{x} = \mathbf{0}$. The covariant derivatives of the corresponding 1-forms $\mathrm{d}x^i = \mathrm{g}(\partial_i, \cdot)$, $\mathrm{d}x^+ = \mathrm{g}(\partial_-, \cdot)$ and $\mathrm{d}x^- = \mathrm{g}(\partial_+ - 2H\partial_-, \cdot)$ are

$$\nabla \mathrm{d}x^+ = 0, \qquad \nabla \mathrm{d}x^i = \partial_i H \mathrm{d}x^+ \otimes \mathrm{d}x^+, \qquad \nabla \mathrm{d}x^- = -2\mathrm{d}H\mathrm{d}x^+.$$

For a pp-wave the parallel null vector field V defines a parallel null distribution $V^\perp$ of rank $n+1$ for which the connection induced by the Levi-Civita connection on the leaves of $V^\perp$ is flat. In Brinkmann coordinates, each leaf is defined by $x^+ = c$ constant and parametrised by the coordinates $x^-, x^1, \dots, x^n$. Moreover, Eqs. (3.6) imply that all the curvature components vanish apart from

$$\mathrm{R}(\partial_i, \partial_+, \partial_j, \partial_+) = \partial_i\partial_j H, \tag{3.7}$$

and the components that are determined by this term via the symmetries of R, i.e.,

$$\mathrm{R} = 4\partial_i\partial_j H(\mathrm{d}x^i \wedge \mathrm{d}x^+)(\mathrm{d}x^j \wedge \mathrm{d}x^+).$$

Here we use Einstein's summation convention, the alternating and the symmetric product of two tensors. Hence, the Ricci tensor of a pp-wave is given by

$$\mathrm{Ric} = -\Delta H(\mathrm{d}x^+)^2,$$

where $\Delta = \sum_{i=1}^n \partial_i^2$ is the flat Laplacian. Moreover, the covariant derivative of R is

$$\nabla R = 4\mathrm{d}H_{ij} \otimes (\mathrm{d}x^i \wedge \mathrm{d}x^+)(\mathrm{d}x^j \wedge \mathrm{d}x^+),$$

including the differentials of the functions $H_{ij} := \partial_i\partial_j H$. This shows that for a pp-wave to be a plane wave it requires $\partial_i\partial_j\partial_k H = 0$. Therefore, for a plane wave the function H is a quadratic polynomial in the x^i's, i.e., in normal Brinkmann coordinates we have

$$2H(x^+, \boldsymbol{x}) = \boldsymbol{x}^\top S(x^+)\boldsymbol{x}$$

where $\boldsymbol{x}$ denotes the column vector $(x^1, \dots, x^n)$ and $S(x^+)$ is a symmetric $n \times n$-matrix depending on x^+. Plane waves satisfy the vacuum Einstein equations, i.e., are Ricci-flat if and only if $S(x^+)$ is trace-free for all x^+.

A subclass of plane waves are the solvable Lorentzian symmetric spaces, the *Cahen–Wallach spaces* [22]. As symmetric spaces, they satisfy $\nabla \mathrm{R} = 0$ which forces the matrix S to be constant. The Cahen–Wallach spaces for which S is trace-free provide remarkable examples of Ricci-flat, non flat symmetric spaces, contrasting the Riemannian situation where Ricci-flat symmetric spaces are flat.

The relation (3.7) on a coordinate neighbourhood shows that the rank of R as an endomorphism of $\Lambda^2 T\mathcal{M}$ is equal to n if and only if $\det(\mathrm{Hess}(H)) \neq 0$. Indeed, the rank is smaller than n if and only if there is a vector $X = \xi^i\partial_i \in V^\perp$ such that $\mathrm{R}(X \wedge \partial_+) = 0$ which is equivalent to $\mathrm{R}(X, \partial_+, \partial_j, \partial_+) = 0$ for all j, i.e., $\xi^i\partial_i\partial_j H = 0$. The curvature of a pp-wave and its derivatives are mapped into its holonomy algebra at p as follows, where we work with normal Brinkmann coordinates centred at p:

$$(\nabla_{X_1} \dots \nabla_{X_k}\mathrm{R})(\partial_i, \partial_+) \mapsto \begin{pmatrix} 0 & \left(X_1 \dots (X_k(\partial_i(\partial_j H \dots)))|_0\right)_{j=1}^n & 0 \\ 0 & 0 & \vdots \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that if there is *one point* where the Hessian of H has determinant not zero, then the pp-wave is indecomposable. However, the following example shows that the converse not true, i.e., there are indecomposable pp-waves, for which the rank of the curvature endomorphism is smaller than n *on an open set*.

Example 3.2. We give an example of a strongly indecomposable 4-dimensional plane wave whose curvature has everywhere rank 1. Given two functions a_1 and a_2 on $\mathbb{R}$ with $a_1^2 + a_2^2 \neq 0$ we consider the matrix

$$S = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \end{pmatrix} = \begin{pmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{pmatrix}$$

which has constant rank one. Then S defines a plane wave metric

$$\mathrm{g} = 2\mathrm{d}x^+(\mathrm{d}x^- + \boldsymbol{x}^\top S(x^+)\boldsymbol{x}\mathrm{d}x^+) + \mathrm{d}\boldsymbol{x}^2.$$

Its curvature tensor is given by the matrix S and hence has everywhere rank 1. However the derivative of the curvature is given by the matrix

$$(\nabla_{\partial_+}\mathrm{R})(\partial_+, \partial_i, \partial_+, \partial_j) = \dot{a}_i a_j + a_i \dot{a}_j$$

with determinant $\det(\dot{S}) = -(\dot{a}_1 a_2 - a_1 \dot{a}_2)^2$, which in general is not zero. Therefore, as the first derivative of the curvature has no kernel, the holonomy of g is equal to $\mathbb{R}^2$ and hence g is strongly indecomposable.

We can even choose the matrix S in a way that the resulting plane wave is homogeneous. Indeed, if we set

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

then

$$S(x^+) = \mathrm{e}^{-x^+F} S_0 \mathrm{e}^{x^+F} = \begin{pmatrix} \cos(x^+)^2 & -\cos(x^+)\sin(x^+) \\ -\cos(x^+)\sin(x^+) & \sin(x^+)^2 \end{pmatrix},$$

has constant rank 1. According to [17] (see also Section 4.3.2), S defines a homogeneous plane wave metric on $\mathbb{R}^4$ and $\det(\dot{S}) = -1$ shows that it is indecomposable.

In order to deduce Corollaries 1 and 3 from Theorem 1 we observe

Lemma 3.3. *If a pp-wave is Ricci-flat and its curvature endomorphism has rank* ≤ 1*, then it is flat.*

Proof. If the curvature endomorphism has rank 1 and zero trace at a point p, then it has to vanish and so g is flat. □

When analysing the Killing equation, the following observation will be useful.

Lemma 3.4. *On a pp-wave* $(\mathcal{M}, \mathrm{g})$*, let* $\mathcal{U}$ *be simply connected patch of Brinkmann coordinates and let* $L = a^i\partial_i$ *be a non-zero vector field on* $\mathcal{M}$ *with constant coefficients* a^i *such that* $\mathrm{R}(X, Y)L = 0$ *for all* $X, Y \in T\mathcal{U}$*. Then the holonomy of* $(\mathcal{U}, \mathrm{g})$ *is properly contained in* $\mathbb{R}^n$*, i.e., it does not act indecomposably. Moreover,* g *is locally a product metric.*

Proof. Since $L = a^i\partial_i$ has constant coefficients and no ∂_+-component, it is easy to see that its parallel transport along a curve γ is given as $\mathrm{P}_\gamma(L|_{\gamma(0)}) = \lambda\partial_- + L|_{\gamma(1)}$ for some $\lambda \in \mathbb{R}$ depending on the curve. Since L as well as ∂_- are annihilated by the curvature tensor, we get that

$$\mathrm{R}(X, Y) \circ \mathrm{P}_\gamma(L) = \mathrm{R}(X, Y)(\lambda\partial_- + L) = 0.$$

Using the Ambrose–Singer holonomy theorem, this shows that not only the null vector ∂_- but also the space-like vector L is invariant under the holonomy algebra of $(\mathcal{U}, \mathrm{g}|_\mathcal{U})$, which, as a consequence is reduced from $\mathbb{R}^n$ to a decomposable subalgebra. The reminder of the statement follows from the local version of the de Rham–Wu decomposition theorem. □

We conclude this section with a proof Theorem 3. It generalises the proof in [4] but avoids the use of coordinates.

Proof of Theorem 3. Let $(\mathcal{M}, \mathrm{g})$ be a semi-Riemannian manifold of dimension $n+2$ and $K_-, K_1, \ldots, K_n$ *commuting* Killing vector fields such that K_- is null and the K_i are orthogonal to K_-. We will show that this implies that $V := K_-$ is parallel and that $\mathrm{R}(X, Y)Z = 0$ and $\nabla_X\mathrm{R} = 0$ whenever $X, Y, Z \in V^\perp$. Since the ∇K_i are skew and the K_i are mutually commuting, we get

$$\mathrm{g}(\nabla_{K_i}K_j, K_k) = 0 \tag{3.8}$$

for $i, j, k = -, 1, \ldots, n$. Set $\mathrm{g}_{ij} := \mathrm{g}(K_i, K_j)$. Clearly, $\mathrm{g}_{-i} = 0$ but also

$$\mathrm{d}\mathrm{g}_{jk}(K_i) = \mathrm{g}(\nabla_{K_i}K_j, K_k) + \mathrm{g}(\nabla_{K_k}K_i, K_j) = 0. \tag{3.9}$$

Now we show that $V = K_-$ is parallel. To this end fix a null vector field Z such that $\mathrm{g}(V, Z) = 1$ and $\mathrm{g}(Z, K_i) = 0$ for $i = 1, \ldots, n$. Clearly we have $\mathrm{g}(\nabla_Z K_i, Z) = 0$, and

$$0 = \mathrm{g}(\nabla_Z K_i, K_j) + \mathrm{g}(\nabla_{K_j}K_i, Z) = \mathrm{g}([Z, K_i], K_j).$$

This implies that

$$\mathrm{g}(\nabla_Z K_i, K_j) = -\mathrm{g}(\nabla_{K_j} K_i, Z) = \frac{1}{2} Z(\mathrm{g}_{ij}),$$

and in particular that $\nabla_Z V = 0$ and $\nabla_{K_i} V = 0$, i.e., that $V = K_-$ is parallel. Moreover, we obtain that

$$\nabla_{K_i} K_j = -\frac{1}{2} Z(\mathrm{g}_{ij}) V,$$

which yields

$$2\mathrm{R}(K_i, K_j)K_k = \big([K_j, Z](\mathrm{g}_{ik}) - [K_i, Z](\mathrm{g}_{jk})\big)\, V = 0, \tag{3.10}$$

because of (3.9) and since the equation $0 = \mathrm{g}([Z, K_i], V)$ from above shows that $[Z, K_i]$ has no Z-component. Hence, we have shown that $\mathrm{R}(K_i, K_j)K_k = 0$, i.e., that g is a pp-wave in the case when g is Lorentzian. In order to show that $\nabla_X \mathrm{R} = 0$ for all $X \in V^\perp$ we use the integrability condition (2.5). Denote by $\phi_i := \nabla K_i$. Obviously $\phi_- = 0$ and $\phi_i(K_j) = -\frac{1}{2} Z(\mathrm{g}_{ij})V$ and $\phi_i(Z) \in \mathrm{span}(K_i)_{i=0}^n$. This and (3.10) together with the integrability condition (2.5) gives us

$$\nabla_{K_i} \mathrm{R} = \phi_i \cdot \mathrm{R} = 0,$$

and hence the statement of Theorem 3 holds. □

4. The Killing equation for pp-waves

4.1. The Killing equation in normal Brinkmann coordinates

Here we derive the Killing equation in Brinkmann coordinates and then specialise this to normal Brinkmann coordinates found in Lemma 3.1. Mostly we follow [17] where the Killing equation for plane waves is solved. We fix Brinkmann coordinates $(x^-, \boldsymbol{x} = (x^1, \ldots, x^n), x^+)$ and, using (3.6), compute the Lie derivative $\mathcal{L}_K \mathrm{g}$ of the metric g in direction of a vector field

$$K = K^- \partial_- + K^i \partial_i + K^+ \partial_+,$$

as

$$\begin{aligned} \tfrac{1}{2}\mathcal{L}_K \mathrm{g} = {} & \partial_- K^+ (\mathrm{d}x^-)^2 + \delta_{ij} \partial_k K^i \mathrm{d}x^k \mathrm{d}x^j + \big(\dot{K}^- + K^i H_i + K^+ \dot{H} + 2H\dot{K}^+\big)\, (\mathrm{d}x^+)^2 \\ & + \big(\delta_{ij}\partial_- K^j + \partial_i K^+\big)\, \mathrm{d}x^- \mathrm{d}x^i + \big(\partial_i K^- + \dot{K}^i + 2H\partial_i K^+\big)\, \mathrm{d}x^i \mathrm{d}x^+ + \big(\partial_- K^- + 2H\partial_- K^+ + \dot{K}^+\big)\, \mathrm{d}x^- \mathrm{d}x^+, \end{aligned}$$

where we write $H_i := \partial_i H$, $H^i := \delta^{ij} H_j$, and a dot for ∂_+ derivatives. Following the arguments in [17, Sec. 2.3], one can show that $\mathcal{L}_K \mathrm{g} = 0$ if and only if the components of K are given as

$$K^+ = a_i x^i + \alpha^+, \qquad K^- = -\dot{\alpha}^+ x^- + A^-, \qquad K^i = -a_i x^- + A^i \tag{4.1}$$

for constants a_i, a function α^+ of x^+ and functions A^- and A^i of $(x^1, \ldots, x^n, x^+)$ subject to the equations

$$-(\ddot{\alpha}^+ + a_i H^i)x^- + \dot{A}^- + A^i H_i + (a_i x^i + \alpha^+)\dot{H} + 2H\dot{\alpha}^+ = 0 \tag{4.2}$$

$$\partial_i A^j + \partial_j A^i = 0 \tag{4.3}$$

$$\partial_i A^- + \dot{A}^i + 2H a_i = 0. \tag{4.4}$$

Differentiating (4.2) with respect to x^- and then with respect to x^j we obtain

$$a_i \partial_j \partial^i H = 0. \tag{4.5}$$

Recalling formula (3.7), this shows that the vector field $L = a^i \partial_i$ on $\mathcal{M}$, for $a^i := a_i$ constants, is annihilated by the curvature tensor R of g, i.e., $\mathrm{R}(X, Y)L = 0$ for all $X, Y \in T\mathcal{M}$.

From now on we will assume that $(\mathcal{M}, \mathrm{g})$ is strongly indecomposable, i.e., that the holonomy algebra of $(\mathcal{U}, \mathrm{g}|_{\mathcal{U}})$ acts indecomposably. Under this assumption, Lemma 3.4 implies by (4.5) that all the constants a_i vanish. With the a_i being zero, we can again proceed as in [17, Sec. 2.3] and obtain that $\alpha^+ = ax^+ + b$ is linear, that A^i is of the form

$$A^i(x^1, \ldots, x^n, x^+) = \psi^i(x^+) + f^i_j x^j,$$

with ψ^i functions of x^+, $f^i_j = -f^j_i$ a skew-symmetric matrix, and moreover that

$$A^-(x^+, \boldsymbol{x}) = -\boldsymbol{x}^\top \dot{\Psi}(x^+) + \varphi(x^+)$$

with a function φ of x^+, where we write $\Psi = (\psi^1, \dots, \psi^n)$. Plugging this back into Eqs. (4.1) we obtain that on an indecomposable pp-wave $(\mathcal{M}, \mathrm{g})$ in Brinkmann coordinates K is of the form

$$K = -\left(ax^- + \varphi(x^+) + \boldsymbol{x}^\top \dot{\Psi}(x^+)\right)\partial_- + \left(\Psi(x^+) + F\boldsymbol{x}\right)^i \partial_i + (ax^+ + b)\partial_+, \tag{4.6}$$

where a, b and $F = (f^i_j) \in \mathfrak{so}(n)$ are constant, and φ and $\Psi = (\psi^1, \dots, \psi^n)$ are functions of x^+ satisfying the equation

$$-\ddot{\Psi}^\top \boldsymbol{x} - \dot{\varphi} + \mathrm{grad}(H)^\top(\Psi + F\boldsymbol{x}) + (ax^+ + b)\dot{H} + 2aH = 0. \tag{4.7}$$

Now, in normal Brinkmann coordinates, we can simplify Eq. (4.7):

Theorem 4.1. *Let $(\mathcal{M}, \mathrm{g})$ be a strongly indecomposable pp-wave, $p \in \mathcal{M}$, and let $(\mathcal{U}, (x^+, x^-, \boldsymbol{x} = (x^1, \dots, x^n)))$ be normal Brinkmann coordinates centred at p with $2H := \mathrm{g}(\partial_+, \partial_+)$. Then K is a Killing vector field if and only if*

$$K = (c - ax^- - \dot{\Psi}^\top \boldsymbol{x})\partial_- + (\Psi + F\boldsymbol{x})^i \partial_i + (ax^+ + b)\partial_+, \tag{4.8}$$

where $a, b, c \in \mathbb{R}$, $F \in \mathfrak{so}(n)$ are constant and $\Psi \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ subject to the equation

$$\ddot{\Psi}^\top \boldsymbol{x} - \mathrm{grad}(H)^\top(\Psi + F\boldsymbol{x}) - (ax^+ + b)\dot{H} - 2aH = 0. \tag{4.9}$$

Moreover, for the commutator $K = [K_1, K_2]$ of two Killing vector fields K_1, K_2 the parameters are

$$\begin{aligned} a &= 0 \\ b &= a_2 b_1 - a_1 b_2 \\ c &= \dot{\Psi}_1^\top \Psi_2 - \Psi_1^\top \dot{\Psi}_2 - a_1 c_2 + a_2 c_1 \\ F &= -[F_1, F_2] \\ \Psi &= F_2 \cdot \Psi_1 - F_1 \cdot \Psi_2 + (a_1 x^+ + b_1)\dot{\Psi}_2 - (a_2 x^+ + b_2)\dot{\Psi}_1. \end{aligned} \tag{4.10}$$

Proof. Clearly, K in (4.8) is a Killing vector field as its components satisfy Eq. (4.7) with $\varphi(x^+) \equiv -c$.

On the other hand, we have seen that every Killing vector field in Brinkmann coordinates is of the form (4.6) with components satisfying Eq. (4.7). Choosing the Brinkmann coordinates to be normal at p, Eq. (4.7) when taken along $\boldsymbol{x} = \boldsymbol{0}$ becomes $\dot{\varphi} \equiv 0$, which we solve by $\varphi(x^+) \equiv -c$.

Finally, one checks that the Lie bracket is of the form (4.10). □

Theorem 4.1 allows us to compute the covariant derivatives of K explicitly as

$$\begin{aligned} \nabla_{\partial_-} K &= -a\partial_- \\ \nabla_{\partial_i} K &= -\left(\dot{\psi}^i - (ax^+ + b)\partial_i H\right)\partial_- + f_i^{\ k}\partial_k \\ \nabla_{E_+} K &= \left(\dot{\psi}^i - (ax^+ + b)\partial_i H\right)\partial_i + aE_+, \end{aligned} \tag{4.11}$$

where $E_+ = \partial_+ - H\partial_-$ and where we use Eq. (4.7) to obtain the last derivative. Hence, at zero, the Killing vector in (4.6) and its covariant derivative is given by

$$\begin{aligned} K|_0 &= c\partial_- + \psi^i(0)\partial_i + b\partial_+ \\ \nabla_{\partial_-} K|_0 &= -a\partial_- \\ \nabla_{\partial_i} K|_0 &= -\dot{\psi}^i(0)\partial_- + f_i^{\ k}\partial_k \\ \nabla_{\partial_+} K|_0 &= \dot{\psi}^i(0)\partial_i + a\partial_+. \end{aligned} \tag{4.12}$$

Moreover, differentiating equation (4.9) yields

$$\ddot{\Psi} + F\mathrm{grad}(H) - \mathrm{Hess}(H)(\Psi + F\boldsymbol{x}) - (ax^+ + b)\mathrm{grad}(\dot{H}) - 2a\,\mathrm{grad}(H) = 0. \tag{4.13}$$

By the properties of the normal Brinkmann coordinates from Lemma 3.1, this becomes a second order linear ODE system for $\Psi = (\psi^1, \dots, \psi^n)$ when taken along $\boldsymbol{x} = \boldsymbol{0}$:

$$\ddot{\Psi}(t) - \mathrm{Hess}(H)(t, \boldsymbol{0})\Psi(t) = 0. \tag{4.14}$$

Fixing initial conditions $\Psi(0)$ and $\dot{\Psi}(0)$ gives a unique solution to this system. This illustrates how K is completely determined by the initial conditions. Moreover it shows that the c in (4.10) is indeed a constant.

In the remainder of the section we will consider some special cases, known results, and examples.

4.2. Transversal Killing vector fields

We will see that a crucial issue of the Killing equation on pp-waves is the existence of Killing vector fields that are transversal to the parallel null distribution $V^\perp$ of rank $n+1$.

First note that, if $\dot{H}=0$, then there is always the transversal Killing vector field ∂_+, but in general transversal Killing vector fields are much harder to find and the situation is much more involved. For example, for certain pp-waves there exist Killing vector fields with $b=0$ but $a\neq 0$ being tangent to $V^\perp$ *only* along the leaf $x^+=0$ but transversal elsewhere, i.e., pp-waves for which $\mathfrak{k}(V)$ and $\mathfrak{k}_p(V)$, as defined in Section 2 are different. Note that Theorem 4.1 and formulae (4.11) show that $\mathfrak{k}'$ and its subalgebra $\mathfrak{k}(V)$ are actually ideals in the Lie algebra $\mathfrak{k}$ of Killing vector fields. In fact we have that $[\mathfrak{k},\mathfrak{k}]=\mathfrak{k}'$. Killing vector fields that are transversal *at some point* project onto non-zero elements in the quotient Lie algebra $\mathfrak{k}/\mathfrak{k}(V)$.

Corollary 4.2. *The Lie algebra $\mathfrak{k}/\mathfrak{k}(V)$ is isomorphic to a subalgebra of $\mathfrak{aff}(1)$, the Lie algebra of affine transformations of $\mathbb{R}$. In particular, if $\mathfrak{k}/\mathfrak{k}(V)$ is 2-dimensional, then there are two Killing vector fields K and $\hat{K}$ such that*

$$K=x^+\partial_+ \mod V^\perp,\qquad \hat{K}=\partial_+ \mod V^\perp.$$

Proof. Theorem 4.1 shows that there is a Lie algebra homomorphism

$$\mathfrak{k}\ni (ax^++b)\partial_+ + K^i\partial_i + K^-\partial_- \mapsto -(a,b)\in\mathfrak{aff}(1),$$

the kernel of which is $\mathfrak{k}(V)$. Hence $\mathfrak{k}/\mathfrak{k}(V)$ injects homomorphically into $\mathfrak{aff}(1)$. If $\mathfrak{k}/\mathfrak{k}(V)$ is 2-dimensional, we can invert this map obtaining two Killing vector fields of the required form. □

Example 4.1. Here we will give an example of a 3-dimensional pp-wave for which the Lie algebra $\mathfrak{k}/\mathfrak{k}(V)$ is indeed 2-dimensional, and more importantly, which is *locally homogeneous but not a plane wave*, showing that the assumption on the curvature in Theorem 1 is essential. Consider the pp-wave $(\mathcal{M},\mathrm{g})$ where $\mathcal{M}=\mathbb{R}^3$ and

$$\mathrm{g}=2\mathrm{d}x^+(\mathrm{d}x^-+\mathrm{e}^{2ax}\mathrm{d}x^+)+\mathrm{d}x^2,$$

where $a\in\mathbb{R}\setminus\{0\}$ is a constant and (x^+,x^-,x) are the standard coordinates in $\mathbb{R}^3$. In particular, $H(x^+,x)=H(x)=\mathrm{e}^{2ax}$. The algebra of Killing vector fields $\mathfrak{k}$ is 3-dimensional and spanned by

$$\partial_-,\ \partial_+,\quad ax^+\partial_+-ax^-\partial_- -\partial_x.$$

Since $\mathrm{g}(K,V)=ax^+$, we have $\mathfrak{k}(V)=\mathbb{R}\cdot\partial_-$ and thus $\dim(\mathfrak{k}/\mathfrak{k}(V))=2$. Moreover, the Killing vector fields span the tangent space $T_p\mathcal{M}$ at any point $p\in\mathcal{M}$, so $(\mathcal{M},\mathrm{g})$ is a locally homogeneous pp-wave. However, $(\mathcal{M},\mathrm{g})$ is strongly indecomposable since

$$\mathrm{R}(\partial_x,\partial_+)=\begin{pmatrix}0 & 2a\mathrm{e}^{2ax} & 0\\ 0 & 0 & -2a\mathrm{e}^{2ax}\\ 0 & 0 & 0\end{pmatrix}\neq 0,$$

for any $x\in\mathbb{R}$, but $(\mathcal{M},\mathrm{g})$ clearly is *not a plane wave* since

$$(\nabla_{\partial_x}\mathrm{R})(\partial_x,\partial_+)=2\,a\,\mathrm{R}(\partial_x,\partial_+)\neq 0.$$

This has recently been observed in [15] where 3-dimensional homogeneous pp-waves were classified.

Example 4.2 (*Ehlers & Kundt*)**.** Similar examples with $\dim(\mathfrak{k}/\mathfrak{k}(V))=2$ but in dimension 4 were given by Ehlers and Kundt in [13, Table 2-5.1] as a correction to [4]. For one class of examples H is given as the real part of the complex function e^{2az} of $z=x^1+\mathrm{i}x^2$ with $a>0$. Then ∂_- and ∂_+ and

$$-a(x^-\partial_-+x^+\partial_+)-\partial_1$$

span the Killing vector fields. For the other class, H is given as the real part of $\mathrm{e}^{2\mathrm{i}a\ln(z)}$, with $a\neq 0$. Here, the Killing vector fields are spanned by ∂_- and ∂_+ and

$$-a(x^-\partial_-+x^+\partial_+)+x^1\partial_2-x^2\partial_1.$$

Note that with $\dim(\mathfrak{k})=3$ and $\dim(\mathfrak{k}_p(V))=2$ both metrics are neither homogeneous nor $V^\perp$-homogeneous.

Example 4.3 (*Sippel & Goenner*)**.** Another example of this type with $\dim(\mathfrak{k}/\mathfrak{k}(V))=2$ in dimension 4 was given by Sippel and Goenner in [14, Table II, no. 9]. These examples are pp-wave metrics on $\mathbb{R}^4$ which are locally homogeneous but not plane waves. However, they turn out to be *decomposable*. The pp-wave metric is defined by

$$H(x^1,x^2):=c\,\mathrm{e}^{a_1x^1-a_2x^2},$$

with c, a_1, a_2 constants with $a_1^2 + a_2^2 \neq 0$. The Killing vector fields are given by ∂_-, ∂_+ and

$$K := x^+(a_2\partial_1 + a_1\partial_2) + (a_2x^1 + a_1x^2)\partial_- \in \mathfrak{k}(\partial_-),$$
$$K_i := \partial_i + a_i(x^+\partial_+ - x^-\partial_-),$$

for $i = 1, 2$, and span the tangent space. However, a coordinate transformation

$$x = a_1x^1 - a_2x^2, \qquad y = a_2x^1 + a_1x^2$$

reveals that this metric is decomposable.

For plane waves we can show

Proposition 4.3. *A strongly indecomposable plane wave satisfies* $\dim(\mathfrak{k}/\mathfrak{k}(V)) \leq 1$.

Proof. Assume there are two linearly independent Killing vector fields that are not tangent to $V^\perp$. They are of the form

$$K = x^+\partial_+ + (\psi + F\boldsymbol{x})^k\partial_k + K^-\partial_-$$
$$\hat{K} = \partial_+ + (\hat{\psi} + \hat{F}\boldsymbol{x})^k\partial_k + \hat{K}^-\partial_-.$$

Now, differentiating Eq. (4.13) again we obtain

$$(\psi + F\boldsymbol{x})^k\partial_k\mathrm{Hess}(H) + [F, \mathrm{Hess}(H)] + (ax^+ + b)\mathrm{Hess}(\dot{H}) + 2a\mathrm{Hess}(H) = 0. \tag{4.15}$$

For a plane wave in normal Brinkmann coordinates with $S = \mathrm{Hess}(H)$ we have that $\partial_k S = 0$. Thus, when taking Eq. (4.15) along $\boldsymbol{x} = \boldsymbol{0}$, we obtain for K and $\hat{K}$ that

$$[F, S] + x^+\dot{S} + 2S = 0, \qquad [\hat{F}, S] + \dot{S} = 0.$$

This implies that

$$[F - x^+\hat{F}, S] + 2S = 0,$$

for all x^+. Since the map $S \mapsto [F - x^+\hat{F}, S]$ when acting on symmetric matrices is skew-symmetric with respect to the trace form, which, on the other hand, is positive definite on symmetric matrices, we obtain that $S \equiv 0$, which is a contradiction. □

A fundamental question is whether, in dimensions greater than 3, ($V^\perp$-) homogeneity and indecomposability force $\mathfrak{k}/\mathfrak{k}(V)$ to have dimension 1. Because of the additional term $\partial_k\mathrm{Hess}(H)$, we are not able to prove Proposition 4.3 for arbitrary ($V^\perp$-) homogeneous pp-waves, but we conjecture that it is true:

Conjecture 4.4. *For an indecomposable locally homogeneous pp-wave of dimension greater than* 3*, the Lie algebra* $\mathfrak{k}/\mathfrak{k}(V)$ *is* 1*-dimensional.*

The proof of Theorem 1 in Section 5 will show that if this conjecture is true, then in dimensions >3 we can drop the assumption on the curvature in Corollary 2 (see Remark 5.7).

4.3. *Plane waves*

In this section we will recall how the Killing equation for plane waves is completely solved in [17].

4.3.1. *Plane waves and the Heisenberg algebra*

For a plane wave defined by a matrix $S(x^+)$ the Lie algebra $\mathfrak{k}(V)$ always contains the Heisenberg algebra $\mathfrak{he}(n)$. Indeed, for a plane wave we have

$$H = \tfrac{1}{2}\boldsymbol{x}^\top S(x^+)\boldsymbol{x}$$

for a symmetric x^+-dependent matrix S, and hence

$$\mathrm{grad}(H) = S\boldsymbol{x}, \qquad \mathrm{Hess}(H) = S.$$

For such H, multiplying the differentiated equation (4.13) by $\boldsymbol{x}$ implies the Killing equation (4.9), which therefore becomes equivalent to (4.13). On the other hand, when setting $F = 0$ and $a = b = 0$, Eq. (4.13) is equivalent to the linear ODE system (4.14) which, for a plane wave, becomes

$$\ddot{\Psi} - S\Psi = 0. \tag{4.16}$$

Hence, we have Killing vector fields

$$L_i := \phi_i^k \partial_k - \boldsymbol{x}^\top \dot{\Phi}_i \,\partial_-, \qquad K_i := \psi_i^k \partial_k - \boldsymbol{x}^\top \dot{\Psi}_i \,\partial_-, \tag{4.17}$$

where $\Phi_i = (\phi_i^k)_{k=1,\dots,n}$ and $\Psi_i = (\psi_i^k)_{k=1,\dots,n}$ are solutions to the linear ODE system (4.16) with initial conditions

$$\Phi_i(0) = \mathbf{0}, \qquad \dot{\Phi}_i(0) = \mathbf{e}_i$$
$$\Psi_i(0) = \mathbf{e}_i, \qquad \dot{\Psi}_i(0) = \mathbf{0},$$

which span $\mathfrak{he}(n)$. Clearly, ∂_- commutes with the K_i's and L_j's and we have

$$[L_i, K_j] = (\Phi_i^\top \dot{\Psi}_j - \Psi_j^\top \dot{\Phi}_i)\partial_- = -\delta_{ij}\partial_- \tag{4.18}$$

because the term $\Phi_i^\top \dot{\Psi}_j - \Psi_j^\top \dot{\Phi}_i$ is constant as a consequence of Eq. (4.14).

For plane waves, there are *commuting* Killing vector fields $X_1, \dots, X_n, \partial_-$ spanning the null distribution $V^\perp$. Theorem 3 shows that this can *only* happen for plane waves.

4.3.2. *Locally homogeneous plane waves*

For plane waves, Eq. (4.13) becomes the following ODE:

$$[S(x^+), F] + (ax^+ + b)\dot{S}(x^+) + 2a\,S(x^+) = 0. \tag{4.19}$$

Since $\mathfrak{k}$ contains a Heisenberg algebra, a plane wave is locally homogeneous if and only if at each point p there is a Killing vector field K transversal to $V^\perp|_p$. Hence, when working with normal Brinkmann coordinates centred at p, one has to find a solution of Eq. (4.19) with $b \neq 0$. Blau and O'Loughlin in [17] determined all such solutions and hence gave a classification of locally homogeneous plane waves as follows:

Let $(\mathcal{M}, \mathrm{g})$ be a locally homogeneous plane wave with parallel null vector field V and let $p \in \mathcal{M}$. Then, for the non-empty set of Killing vector fields that are transversal to $V^\perp|_p$ two cases can occur: it contains a Killing vector field K with $\nabla_V K|_p = 0$ or it does not. In the first case, using normal Brinkmann coordinates $\varphi = (x^-, \boldsymbol{x}, x^+)$ centred at p, there is a solution $K = K(a, b, c, \Psi, F)$ to Eq. (4.19) with the parameter $a = 0$, and we may assume that $b = 1$. From Eq. (4.19) we see that F satisfies

$$[S(x^+), F] + \dot{S}(x^+) = 0.$$

This is an ODE for $S(x^+)$ and its general solution is

$$S(x^+) = \mathrm{e}^{x^+F} S_0 \mathrm{e}^{-x^+F}$$

with a constant skew symmetric matrix F and a constant symmetric matrix S_0. Hence, the metrics in the first case are of the form

$$\mathrm{g} = 2\mathrm{d}x^+\mathrm{d}x^- + (\boldsymbol{x}^\top \mathrm{e}^{x^+F} S_0 \mathrm{e}^{-x^+F} \boldsymbol{x})(\mathrm{d}x^+)^2 + \mathrm{d}\boldsymbol{x}^2. \tag{4.20}$$

When defined on all of $\mathbb{R}^{n+2}$, g is geodesically complete (see for example results by Candela, Flores and Sánchez [23, Prop. 3.5]).

In the second case there is no solution $K = K(a, b, c, \Psi, F)$ to Eq. (4.19) that is transversal to $V^\perp|_p$ and with $a = 0$. We may assume that such a solution has $a = 1$. It satisfies Eq. (4.19), which becomes an ODE with singularity at $x^+ = -b$,

$$(x^+ + b)\dot{S}(x^+) + [S(x^+), F] + 2\,S(x^+) = 0.$$

Its general solution is

$$S(x^+) = \frac{1}{(x^+ + b)^2}(\mathrm{e}^{\log(x^+ + b)F} S_0 \mathrm{e}^{\log(-(x^+ + b))F}),$$

again for constant (skew) symmetric matrices F and S_0. Hence, locally homogeneous plane wave metrics in the second family are of the form

$$\mathrm{g} = 2\mathrm{d}x^+\mathrm{d}x^- + \frac{1}{(x^+ + b)^2}(\boldsymbol{x}^\top \mathrm{e}^{\log(x^+ + b)F} S_0 \mathrm{e}^{\log(-(x^+ + b))F} \boldsymbol{x})(\mathrm{d}x^+)^2 + \mathrm{d}\boldsymbol{x}^2, \tag{4.21}$$

for constants F, S_0 and b. They are only defined for $x^+ > -b$ and hence geodesically incomplete. Clearly, metrics for different b can be pulled back by a translation $x^+ \mapsto x^+ + b$ to the metric with $b = 0$ on $\{x^+ > 0\}$. Hence, for given F and S_0, metrics with different b are isometric to each other.

This provides a classification of locally homogeneous plane waves as it shows that the local form of the metric is either given by (4.20) or by (4.21). Note that non flat metrics in (4.20) and (4.21) cannot be locally isometric: the metric in (4.20) admits s local Killing vector field K with $\mathrm{g}(V, K) \neq 0$ and $\nabla_V K = 0$, whereas the metric in (4.21) does not, unless $S_0 = 0$. In both cases the metric g is Ricci-flat if and only if S_0 is trace-free.

4.3.3. Reductivity of homogeneous plane waves

Here we will show that homogeneous plane waves are always *reductive*. This means that for some subalgebra $\mathfrak{k}_0$ of $\mathfrak{k}$ generating a (locally) transitive group action, the stabiliser $\mathfrak{h} := \{K \in \mathfrak{k}_0 \mid K|_p = 0\}$ in $\mathfrak{k}_0$ of a point p has a vector space complement $\mathfrak{m}$ in $\mathfrak{k}_0$ with $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$.

Proposition 4.5. *Homogeneous plane waves are reductively homogeneous.*

Proof. For a homogeneous plane wave, we take $\mathfrak{k}_0$ to be the subalgebra generated by

$$K_+, \partial_-, K_1, \dots, K_n, L_1, \dots, L_n,$$

where K_i, L_j are defined in (4.17) and $K_+ = -ax^-\partial_- + (F\boldsymbol{x})^i\partial_i + (ax^+ + b)\partial_+$ for a certain $F = (f_i^j) \in \mathfrak{so}(n)$ is transversal to $V^\perp$, which exists for homogeneous plane waves according to [17, (2.42)]. Working at p with normal Brinkmann coordinates, we see that $\mathfrak{h}$ is spanned by the L_i's defined in (4.17). Then the $\mathfrak{h}$-invariant complement $\mathfrak{m}$ is spanned by ∂_-, K_+ and the n Killing vector fields

$$M_i := [K_+, L_i].$$

This implies that

$$M_i|_p = b\dot{\phi}_i^k(0)\partial_k|_p = b\partial_i|_p.$$

Hence, since also $K_+|_p = b\partial_+|_p$, the vector space $\mathfrak{m}$ defined in this way is indeed a complement to $\mathfrak{h}$. Moreover, since both M_i and L_j are tangent to $V^\perp$ and without rotational component we obtain from (4.10) that

$$[L_j, M_i] = c\partial_-$$

for a constant c. Therefore we have $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ and the plane wave is reductive. □

4.3.4. Cahen–Wallach spaces

For Cahen–Wallach spaces, the matrix $S(x^+)$ is constant and thus Eq. (4.19) always has a solution with $b = 1$, $F = 0$ and $a = 0$ yielding a Killing vector field transversal to $V^\perp$. Generically, $\mathfrak{k}$ contains the oscillator algebra $\mathbb{R} \ltimes \mathfrak{he}(n)$. The stabiliser algebra is equal to the holonomy algebra $\mathbb{R}^n$. A Cahen–Wallach space may have additional Killing vector fields in addition to $\mathbb{R} \ltimes \mathfrak{he}(n)$. In fact, the additional Killing vector fields are isomorphic to the centraliser in $\mathfrak{so}(n)$ of the constant matrix S. Hence, it might have at most $\frac{1}{2}n(n-1)$ additional symmetries.

4.4. Dimension four

In [4] the Killing equation (4.9) for 4-dimensional pp-waves is explicitly solved under the assumption that $(\mathcal{M}, g)$ is Ricci-flat, i.e., that H is harmonic, so that methods from complex analysis can be used. In particular, in [4, table on p. 79], the dimension of the space of Killing vector fields of a 4-dimensional, indecomposable, *Ricci-flat* pp-wave has been determined to be one of $\dim(\mathfrak{k}) = 1, 2, 3, 5, 6$, and the metrics are explicitly given for each case. Moreover, in [14] the assumption of Ricci-flatness was dropped and new algebras of dimension 5, 6 and 7 appeared, almost reaching the upper bound of 8 we will deduce from Theorem 4.1 in Corollary 5.2.

5. Proofs of the main results

In this section we will draw the conclusions from Theorem 4.1 that eventually will lead to a proof of Theorem 1. We assume that $(\mathcal{M}, g)$ is a strongly indecomposable pp-wave with parallel null vector field V. First we note:

Corollary 5.1. *A Killing vector field satisfies $\nabla_V K \in \mathbb{R}V$.*

Proof. Let $p \in \mathcal{M}$ be an arbitrary point and choose normal Brinkmann coordinates centred at p. In these a Killing vector field K is of the form (4.8) with its covariant derivative as in (4.11). Since $V = \partial_-$ on the coordinate patch, we get $\nabla_V K = -aV$. □

We describe the evaluation map κ at a point $p \in \mathcal{M}$, at which we choose normal Brinkmann coordinates, and in a basis of $T_p\mathcal{M}$,

$$E_- = \partial_-|_p, \qquad E_i = \partial_i|_p, \qquad E_+ = (\partial_+ - H\partial_-)|_p = \partial_+|_p \tag{5.1}$$

in which

$$g_p(E_-, E_+) = 1, \qquad g_p(E_i, E_j) = \delta_{ij},$$

where $i, j = 1, \ldots, n$, and all other $\mathrm{g}_p(E_\alpha, E_\beta) = 0$ for $\alpha, \beta \in \{-, +, 1, \ldots, n\}$. By Theorem 4.1, for each $K \in \mathfrak{k}$ there are real numbers $a, b, c, X^i, Y^i, F = (f_i{}^j) \in \mathfrak{so}(n)$ such that

$$\begin{aligned} K|_p &= cE_- + X^i E_i + bE_+ \\ \nabla_{E_-} K|_p &= -aE_- \\ \nabla_{E_i} K|_p &= -Y_i E_- + f_i{}^k E_k \\ \nabla_{E_+} K|_p &= Y^i E_i + aE_+. \end{aligned} \tag{5.2}$$

Furthermore, we write $Y = (Y_i), X^\top = (X_i)$ for the row vectors and $X = (X^i), Y^\top = (Y^i)$ for the column vectors.

If we denote by $v \in \mathbb{R}^{1,n+1}$ the null vector that is the image of V under the evaluation map κ, i.e. $\kappa(V) = (0, v) \in \mathfrak{so}(1, n+1) \ltimes \mathbb{R}^{1,n+1}$, by Corollary 5.1 for $\phi = \nabla K$ we have

$$\phi \in \mathfrak{stab}(\mathbb{R}v) \subset \mathfrak{so}(1, n+1),$$

which is equal to the Lie algebra of similarity transformations of $\mathbb{R}^n$,

$$\mathfrak{stab}(\mathbb{R}v) = \mathfrak{sim}(n) = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n = \left\{ \begin{pmatrix} a & u^\top & 0 \\ 0 & F & -u \\ 0 & 0 & -a \end{pmatrix} \;\middle|\; \begin{array}{l} a \in \mathbb{R} \\ F \in \mathfrak{so}(n) \\ u \in \mathbb{R}^n \end{array} \right\}.$$

This is the minimal parabolic subalgebra in $\mathfrak{so}(1, n+1)$. Hence we obtain

Corollary 5.2. *The evaluation map κ in (2.3) is an injective vector space homomorphism*

$$\begin{aligned} \kappa \;:\; \mathfrak{k} &\hookrightarrow \mathfrak{sim}(n) \ltimes \mathbb{R}^{1,n+1} \\ K &\mapsto \left(\begin{pmatrix} a & Y & 0 \\ 0 & -F & -Y^\top \\ 0 & 0 & -a \end{pmatrix}, \begin{pmatrix} -c \\ -X \\ -b \end{pmatrix} \right). \end{aligned} \tag{5.3}$$

In particular,

$$1 \leq \dim(\mathfrak{k}) \leq (2n+3) + \frac{1}{2} n(n-1).$$

The map in (5.3) is *not* a Lie algebra homomorphism. In fact, a direct computation using the bracket formula (4.10) confirms formula (2.4),

$$\begin{aligned} [\kappa(K), \kappa(\hat{K})] - \kappa([K, \hat{K}]) &= \left(\mathrm{R}(K, \hat{K})|_p, 0 \right) \\ &= \left(\begin{pmatrix} 0 & (bS\hat{X} - \hat{b}SX)^\top & 0 \\ 0 & 0 & \hat{b}SX - bS\hat{X} \\ 0 & 0 & 0 \end{pmatrix}, 0 \right), \end{aligned} \tag{5.4}$$

where $S = \mathrm{Hess}(H)|_p$ and the second equality uses the basis in (5.1). As a remedy, we consider the vector space

$$\mathfrak{k}_p(V) = \{K \in \mathfrak{k} \mid \mathrm{g}(K, V)|_p = 0\}.$$

According to Theorem 4.1, when using normal Brinkmann coordinates around p, elements in $\mathfrak{k}_p(V)$ are characterised by the condition $b = 0$. Hence, consulting formula (4.10) for the Lie bracket of two Killing vector fields, we make the following observation:

Corollary 5.3. *$\mathfrak{k}_p(V)$ is a Lie subalgebra of $\mathfrak{k}$. Moreover, the evaluation map at p, when restricted to $\mathfrak{k}_p(V)$, is an injective Lie algebra homomorphism,*

$$\begin{aligned} \kappa : \mathfrak{k}_p(V) &\hookrightarrow \mathfrak{co}(n) \ltimes \mathfrak{he}(n) \\ K &\mapsto \begin{pmatrix} a & Y & c \\ 0 & -F & X \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

where $\mathfrak{co}(n) = \mathbb{R} \oplus \mathfrak{so}(n)$ denotes the conformal Lie algebra and $\mathfrak{he}(n)$ the $(2n+1)$-dimensional Heisenberg algebra.

Proof. That the evaluation map κ at p becomes a Lie algebra monomorphism follows from (5.4) and the defining property of pp-waves, which ensures that $\mathrm{R}(K, \hat{K}, \cdot, \cdot)|_p = 0$ whenever $K, \hat{K} \in \mathfrak{k}_p(V)$. Moreover, if $b = 0$, the image of K_p lies in $v^\perp$, i.e., $\kappa(\mathfrak{k}_p(V)) \subset \mathfrak{sim}(n) \ltimes v^\perp$. A direct computation shows that

$$\left(\begin{pmatrix} a & Y & 0 \\ 0 & -F & -Y \\ 0 & 0 & -a \end{pmatrix}, \begin{pmatrix} -c \\ -X \\ 0 \end{pmatrix} \right) \mapsto \begin{pmatrix} a & Y & c \\ 0 & -F & X \\ 0 & 0 & 0 \end{pmatrix}$$

is indeed a Lie algebra isomorphism between $\mathfrak{sim}(n) \ltimes v^\perp$ and $\mathfrak{co}(n) \ltimes \mathfrak{he}(n)$. $\square$

Next, note that the Lie algebra $\mathfrak{co}(n) \ltimes \mathfrak{he}(n)$ contains an abelian ideal

$$\mathfrak{a} := \left\{ \begin{pmatrix} a & Y & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, Y \in \mathbb{R}^n,\ a \in \mathbb{R},\ c \in \mathbb{R} \right\} \subset \mathfrak{co}(n) \ltimes \mathfrak{he}(n).$$

Therefore, the quotient $(\mathfrak{co}(n) \ltimes \mathfrak{he}(n))/\mathfrak{a}$ is a Lie algebra which turns out to be isomorphic to the Lie algebra of Euclidean motions $\mathfrak{so}(n) \ltimes \mathbb{R}^n$ via

$$\begin{pmatrix} a & Y^\top & c \\ 0 & -F & X \\ 0 & 0 & 0 \end{pmatrix} + \mathfrak{a} \mapsto \begin{pmatrix} -F & X \\ 0 & 0 \end{pmatrix}.$$

Hence, we obtain

Corollary 5.4. *The evaluation map κ induces a Lie algebra homomorphism $\lambda : \mathfrak{k}_p(V) \to \mathfrak{so}(n) \ltimes \mathbb{R}^n$ given by*

$$\begin{aligned} \mathfrak{k}_p(V) &\xrightarrow{\lambda} \mathfrak{so}(n) \ltimes \mathbb{R}^n \\ K &\longmapsto \begin{pmatrix} -F & X \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Moreover, if $\mathfrak{k}_p(V)$ at p spans $V^\perp|_p$, then $\mathfrak{g} := \lambda(\mathfrak{k}_p(V)) \subset \mathfrak{so}(n) \ltimes \mathbb{R}^n$ is a subalgebra that acts indecomposably on $\mathbb{R}^{1,n+1}$ via

$$\begin{pmatrix} 0 & -X^\top & 0 \\ 0 & -F & X \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. Since there are Killing vector fields that span $V^\perp|_p$, by the definition of λ for the projection $\mathrm{pr}_{\mathbb{R}^n} : \mathfrak{so}(n) \ltimes \mathbb{R}^n \to \mathbb{R}^n$ onto the translations we have that

$$\mathrm{pr}_{\mathbb{R}^n}(\lambda(\mathfrak{k}_p(V))) = \mathbb{R}^n.$$

This implies that $\mathfrak{g} = \lambda(\mathfrak{k}_p(V))$ acts indecomposably on $\mathbb{R}^{1,n+1}$. □

We have seen in Lemma 2.1 that Killing vector fields $\mathfrak{k}(V) = \{K \in \mathfrak{k} \mid \mathrm{g}(K, V) = 0\}$ form a subalgebra of $\mathfrak{k}$. In analogy to Corollary 5.4, for $\mathfrak{k}(V)$ we have:

Corollary 5.5. *The evaluation map κ induces a Lie algebra homomorphism $\lambda : \mathfrak{k}(V) \to \mathfrak{so}(n) \ltimes \mathbb{R}^n$. Moreover, if $\mathfrak{k}(V)$ spans $V^\perp$, then $\mathfrak{h} := \lambda(\mathfrak{k}(V)) \subset \mathfrak{so}(n) \ltimes \mathbb{R}^n$ is a subalgebra that acts indecomposably on $\mathbb{R}^{1,n+1}$ as in Corollary 5.4.*

For the proof of Theorem 1 we will need a description of subalgebras of $\mathfrak{sim}(n)$ that act indecomposably on $\mathbb{R}^{1,n+1}$. Such a classification is due to Bérard-Bergery and Ikemakhen [19]:

Proposition 5.6. *Let $\mathfrak{g} \subset \mathfrak{sim}(n)$ act indecomposably on $\mathbb{R}^{1,n+1}$. Then either $\mathfrak{g}$ contains the translations $\mathbb{R}^n$, or $\mathfrak{g}$ contains $\mathbb{R}^N$ for $1 < N < n$, in which case there is a subalgebra $\mathfrak{h} \subset \mathfrak{so}(q)$ and a surjective linear map $\varphi : \mathfrak{h} \to \mathbb{R}^{n-N}$ such that*

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & X^\top & \varphi(F)^\top & 0 \\ 0 & F & 0 & -X \\ 0 & 0 & 0 & -\varphi(F) \\ 0 & 0 & 0 & 0 \end{pmatrix} \,\middle|\, F \in \mathfrak{h}, X \in \mathbb{R}^N \right\}. \tag{5.5}$$

The important property in this proposition is that the rotational part F of a transitively acting group of similarity transformations acts only on $\mathbb{R}^N$ and annihilates the corresponding translational part $\varphi(F)$. With this at hand we are ready to prove Theorem 1.

Proof of Theorem 1. By the defining property (1.2) of a plane wave, we have to show that at each point $p \in \mathcal{M}$ we have $\nabla_U \mathrm{R}|_p = 0$ for all $U \in V^\perp|_p$. Working with a basis of the form (5.1), it follows that the only possibly non-vanishing terms of $\nabla \mathrm{R}$ are $\nabla_{E_+}\mathrm{R}(E_+, E_i, E_+, E_j)$ and

$$\nabla_k \mathrm{R}_{ij} := \nabla_{E_k}\mathrm{R}(E_+, E_i, E_+, E_j) \tag{5.6}$$

for $i, j, k = 1, \ldots, n$ and, because of the Bianchi identity, being symmetric in its indices. We will now use the integrability condition (2.5) to show that this term also vanishes. Because of our assumption that the curvature has rank greater than 1 almost everywhere, it suffices to work at a $p \in \mathcal{M}$ at which the rank of R is greater than 1. Hence, the rank of the matrix

$$\mathrm{R}_{ij} := \mathrm{R}(E_+, E_i, E_+, E_j)$$

is greater than 1.

Since there are Killing vector fields that span $V^\perp|_p$, we can apply Corollary 5.4 and Proposition 5.6 to $\mathfrak{g} = \lambda(\mathfrak{k}_p(V))$ giving two possible cases for $\mathfrak{g}$. In the first case, $\mathfrak{g}$ contains the translations $\mathbb{R}^n$, i.e., there are Killing vector fields $K_1, \dots, K_n$ with

$$\lambda(K_k) = \begin{pmatrix} 0 & \mathbf{e}_k^\top & 0 \\ 0 & 0 & -\mathbf{e}_k \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sim}(n).$$

By the definition of λ and recalling (5.2), this implies $K_k|_p = E_k + c_k E_-$. Since E_- is a Killing vector field, we can assume that $c_k = 0$. Then, for $\phi_k = \nabla K_k|_p$ we have

$$\begin{aligned} \phi_k(E_j) &\in \mathbb{R}V_p, \quad \text{for } j = 1, \dots, n, \\ \phi_k(E_+) &= a_k \partial_+ \quad \bmod V^\perp|_p, \end{aligned} \tag{5.7}$$

for $k = 1, \dots, n$. Without loss of generality we may assume that all but one a_i are equal to zero, $a_1 = \dots = a_{n-1} = 0$. Then the integrability condition (2.5) becomes

$$\begin{aligned} -\nabla_k \mathrm{R}_{ij} &= \mathrm{R}(\phi_k(E_+), E_i, E_+, E_j) + \mathrm{R}(E_+, \phi_k(E_i), E_+, E_j) + \mathrm{R}(\phi_k(E_+), E_j, E_+, E_i) + \mathrm{R}(E_+, \phi_k(E_j), E_+, E_i) \\ &= 2a_k \mathrm{R}_{ij}, \end{aligned} \tag{5.8}$$

for $i, j, k = 1, \dots, n$. Therefore, we get

$$\nabla_k \mathrm{R}_{ij} = 0,$$

for $k = 1, \dots, n-1$ and $i, j = 1, \dots, n$, as well as

$$2a_n \mathrm{R}_{ki} = -\nabla_n \mathrm{R}_{ki} = 0,$$

for all $i = 1, \dots, n$ and $k = 1, \dots, n-1$. Hence, if a_n was not zero, R_{nn} would be the only non-vanishing component of R_{ij} which contradicts the assumption that its rank is greater than one. Hence, also $a_n = 0$ and therefore $\nabla_k \mathrm{R}_{ij} = 0$ for all i, j, k.

This gives us an idea how to proceed in the other case, in which $\mathfrak{g}$ does not contain $\mathbb{R}^n$, but only an $\mathbb{R}^N$, for $1 < N < n$. Here, according to Proposition 5.6, $\mathfrak{g}$ is of the form (5.5). In the following, we will use indices $A, B, C \dots \in \{1, \dots, N\}$ and $b, c, d, \dots \in \{N+1, \dots, n\}$ and $i, j, k \in \{1, \dots, n\}$. For such $\mathfrak{g}$'s we have N Killing vector fields such that

$$\lambda(K_A) = \begin{pmatrix} 0 & \mathbf{e}_A^\top & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{e}_A \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(n) \ltimes \mathbb{R}^n,$$

with

$$\begin{aligned} K_A|_p &= E_A \\ \phi_A(E_-) &= -a_A \partial_- \\ \phi_A(E_i) &\in \mathbb{R}V_p, \quad \text{for } i = 1, \dots, n \\ \phi_A(E_+) &= a_A \partial_+ \mod V^\perp|_p, \end{aligned} \tag{5.9}$$

and $n - N$ Killing vector fields K_b, with

$$\lambda(K_b) = \begin{pmatrix} 0 & 0 & \mathbf{e}_b^\top & 0 \\ 0 & \overset{(b)}{F} & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{e}_b \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(n) \ltimes \mathbb{R}^n.$$

Note that by Proposition 5.6 all the $\overset{(b)}{F} \in \mathfrak{so}(N)$ are non-zero. By the definition of λ, and looking at (5.2), this implies for $\phi_b = \nabla K_b|_p$ that

$$\begin{aligned} K_b|_p &= E_b \\ \phi_b(E_-) &= -a_b \partial_- \\ \phi_b(E_A) &= \overset{(b)}{f}{}_A{}^B E_B \mod \mathbb{R}V_p \\ \phi_b(E_c) &\in \mathbb{R}V_p \\ \phi_b(E_+) &= a_b \partial_+ \mod V^\perp|_p. \end{aligned} \tag{5.10}$$

As before, without loss of generality, we can assume that a_N and a_n are the only a_i's that are possibly non-zero. Then we have

$$-\nabla_A \mathrm{R}_{ij} = 2a_A \mathrm{R}_{ij} \tag{5.11}$$
$$-\nabla_b \mathrm{R}_{cd} = 2a_b \mathrm{R}_{cd} \tag{5.12}$$
$$-\nabla_b \mathrm{R}_{cA} = 2a_b \mathrm{R}_{cA} + \overset{(b)}{f}{}_A{}^B \mathrm{R}_{cB} \tag{5.13}$$
$$-\nabla_b \mathrm{R}_{AB} = 2a_b \mathrm{R}_{AB} + 2\,\overset{(b)}{f}{}_{(A}{}^C \mathrm{R}_{B)C}. \tag{5.14}$$

With our assumption $a_1 = \cdots a_{N-1} = a_{N+1} = \cdots = a_{n-1} = 0$ Eq. (5.11) gives

$$\nabla_A \mathrm{R}_{ij} = 0, \quad \text{for all } A \neq N \tag{5.15}$$

and thus

$$a_N \mathrm{R}_{Aj} = 0, \quad \text{for all } A \neq N. \tag{5.16}$$

Similarly, Eq. (5.12) yields

$$\nabla_b \mathrm{R}_{cd} = 0, \quad \text{for all } b \neq n \tag{5.17}$$

and hence

$$a_n \mathrm{R}_{bc} = 0, \quad \text{for all } (b, c) \neq (n, n). \tag{5.18}$$

Furthermore, using the total symmetry of $\nabla_i \mathrm{R}_{jk}$ we observe that Eq. (5.13) gives

$$2a_A \mathrm{R}_{bc} = 2a_b \mathrm{R}_{cA} + \overset{(b)}{f}{}_A{}^B \mathrm{R}_{cB} \tag{5.19}$$

and (5.14) yields

$$2a_A \mathrm{R}_{Bc} = 2a_c \mathrm{R}_{AB} + \overset{(c)}{f}{}_{(A}{}^D \mathrm{R}_{B)D}. \tag{5.20}$$

With all these relations, the total symmetry of $\nabla_i \mathrm{R}_{jk}$ implies that

$$\begin{aligned} -\nabla_N \mathrm{R}_{NN} &= 2a_N \mathrm{R}_{NN} \\ -\nabla_n \mathrm{R}_{nn} &= 2a_n \mathrm{R}_{nn} \\ -\nabla_N \mathrm{R}_{nN} &= a_N \mathrm{R}_{nN} = 2a_n \mathrm{R}_{NN} + \overset{(n)}{f}{}_N{}^C \mathrm{R}_{NC} \\ -\nabla_n \mathrm{R}_{nN} &= a_N \mathrm{R}_{nn} = 2a_n \mathrm{R}_{nN} + \overset{(n)}{f}{}_N{}^B \mathrm{R}_{nB}. \end{aligned} \tag{5.21}$$

Now we consider two cases: First assume that $a_N \neq 0$. In this case Eq. (5.16) implies that

$$\mathrm{R}_{Aj} = 0 \quad \text{for all } A \neq N. \tag{5.22}$$

Evaluating (5.19) for $A = N$ yields

$$2a_N \mathrm{R}_{bc} = 2a_b \mathrm{R}_{cN} + \overset{(b)}{f}{}_N{}^B \mathrm{R}_{cB} = 2a_b \mathrm{R}_{cN} = 2a_c \mathrm{R}_{bN} \tag{5.23}$$

since $\overset{(b)}{F}$ is skew and hence $\overset{(b)}{f}{}_N{}^N = 0$. Evaluating this for $b \neq n$ we get that

$$\mathrm{R}_{bc} = 0, \quad \text{for all } (b, c) \neq (n, n). \tag{5.24}$$

Moreover, Eq. (5.20) for $A = B = N$ for $c \neq n$ gives

$$2a_N \mathrm{R}_{Nc} = \overset{(c)}{f}{}_N{}^D \mathrm{R}_{ND} = 0$$

again because of (5.22) and the skew-symmetry of $\overset{(c)}{F}$. So we get

$$\mathrm{R}_{Nb} = 0 \quad \text{for } b \neq n. \tag{5.25}$$

Putting (5.22), (5.24) and (5.25) together we get that R_{NN}, R_{nn} and R_{Nn} are the only non vanishing components of R_{ij}. According to the last two equations of (5.21) they are related by

$$\begin{aligned} a_N \mathrm{R}_{nN} &= 2\,a_n \mathrm{R}_{NN} \\ a_N \mathrm{R}_{nn} &= 2\,a_n \mathrm{R}_{nN}. \end{aligned}$$

This implies that $a_n \neq 0$ because otherwise R_{NN} would be the only non-vanishing component of R_{ij}, which contradicts to the rank of R_{ij} being greater than one. But this implies

$$a_n a_N \det \begin{pmatrix} \mathrm{R}_{NN} & \mathrm{R}_{Nn} \\ \mathrm{R}_{nN} & \mathrm{R}_{nn} \end{pmatrix} = 0,$$

which finally leads a contradiction to the rank of R_{ij} being greater than one.

It remains to derive a contradiction in the case when $a_N = 0$. If also $a_n = 0$ we are done, so we assume $a_n \neq 0$. In this case (5.18) implies that

$$\mathrm{R}_{bc} = 0, \quad \text{for all } (b, c) \neq (n, n). \tag{5.26}$$

Moreover (5.19) for $b = n$ implies that each $(\mathrm{R}_{cB})_{B=1}^{N}$ is an eigenvector of $\overset{(n)}{F}$. Since $a_n \neq 0$ is real and $\overset{(n)}{F}$ skew, this implies that $\mathrm{R}_{cB} = 0$ for all c and B.

Moreover Eq. (5.20) for $c = n$ becomes

$$-2a_n \mathrm{R}_{AB} = \overset{(n)}{f}{}_A{}^D \mathrm{R}_{BD} + \overset{(n)}{f}{}_B{}^D \mathrm{R}_{AD} = \overset{(n)}{f}{}_A{}^D \mathrm{R}_{BD} - \overset{(n)}{f}{}_D{}^B \mathrm{R}_{AD}$$

which just means that the matrix (R_{AB}) is an eigenvector with eigenvalue $-2a_n$ for the adjoint action of $\overset{(n)}{F} \in \mathfrak{so}(n)$ on the symmetric matrices, i.e.,

$$-2a_n \mathrm{R} = [\overset{(n)}{F}, \mathrm{R}]. \tag{5.27}$$

Since $\overset{(n)}{F}$, when acting on symmetric matrices via the commutator, is skew-symmetric with respect to the trace form, which, on the other hand, is positive definite on symmetric matrices, (5.27) implies $\mathrm{R}_{AB} = 0$. Hence, again R_{nn} is the only non-vanishing component of R_{ij} which contradicts our assumption that the rank of the curvature endomorphism is larger than one. This concludes the proof of Theorem 1. $\square$

This proof and Corollary 5.5 immediately give us a proof of Theorem 2 when taking into account that Killing vector fields from $\mathfrak{k}(V)$ have $a_i = 0$ for $i = 1, \ldots, n$.

Remark 5.7. Note that our proof shows that for indecomposable homogeneous pp-waves with 1-dimensional Lie algebra $\mathfrak{k}/\mathfrak{k}(V)$, we could drop the assumption on the rank of the curvature in Corollary 2. Indeed, if $(\mathcal{M}, \mathrm{g})$ is homogeneous, at each point p we have, in addition to the Killing vector fields $V, K_1, \ldots, K_n$ spanning $V_p^{\perp}$, a Killing vector field $\hat{K}$ transversal to $V_p^{\perp}$. In normal Brinkmann coordinates this vector field would have $b = 1$ and hence, by the assumption $\dim(\mathfrak{k}/\mathfrak{k}(V)) = 1$, all the K_i's would have $a_i = 0$. The proof of Theorem 1 then shows that $(\mathcal{M}, \mathrm{g})$ is a plane wave.

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