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Long-time existence of a geometric flow on closed Hessian manifolds

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ABSTRACT

In this paper we study Hessian manifolds. We define a flow, which we call the Hessian flow, to study the existence of Einstein–Hessian metrics on Hessian manifolds. The flow will considered as a real Monge–Ampére equation and we prove the short-time existence, the global existence and the uniqueness of it.

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1. Introduction

A Riemannian metric on a flat affine manifold is called a Hessian metric if it is locally expressed by the Hessian of a smooth function. A triple of (M, D, g) with an affine manifold M, flat connection D, and Hessian metric g is called a Hessian manifold. Hessian structure was first studied by Koszul [1], then Vinberg [2] considered this metric on convex cone and S.Y. Cheng and S.T. Yau [3] used Hessian metrics to solve Monge–Ampére equations. Hessian geometry finds connection with various fields of sciences in both Pure mathematical fields such as affine differential geometry, homogeneous spaces, cohomology and applied sciences such as physics, statistics and information geometry. It is well-known that many important smooth families of probability distributions (e.g. exponential families) admit Hessian structures [4]. The geometry of Hessian manifolds is also deeply related to Kählerian manifolds. A Hessian structure is formally analogous to a Kählerian structure because a Kählerian metric is locally expressed as a complex Hessian of a function with respect to the holomorphic coordinate systems. The relation of these two kinds of manifolds was motivation of defining a flow on Hessian manifolds similar to Kähler–Ricci flow.

One of the most important questions in Riemannian geometry is that whether a manifold admits a canonical metric? Geometric flows, as a class of important geometric partial differential equations, are used to answer this question in some cases. In 1982 Richard Hamilton [5] introduced the so called Ricci flow, for solving Thurston's conjecture and classified compact 3-manifolds of positive Ricci curvature as spherical space forms. Later in 1985 Cao [6] studied the Ricci flow on complex Kählerian manifolds and showed that on closed Kähler manifolds this flow exists for all time and converges to

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an Einstein–Kähler metric under normalized Kähler–Ricci flow. Shi [7] generalized this result to complete noncompact Kählerian manifolds.

In Hessian manifolds we intend to deform Hessian structure and alter Hessian metric on M to a canonical metric with good behavior, for this reason we define a PDE, as a geometric flow that evolves the metric tensor on manifold, and we will show the global existence of the flow.

2. Preliminaries

This section includes some definitions and facts that we need in following sections.

2.1. Geometry of Hessian manifolds

Definition 2.1.1. Let (M^m, D) be a flat affine manifold, if a Riemannian metric g on M has a local expression $g = Ddu$ for $u \in C^\infty(M)$, g is called a Hessian metric, (D, g) a Hessian structure and (M, D, g) a Hessian manifold and u is said to be a potential function of (D, g), that is,

$$g_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j},$$

where $(x^1, x^2, \ldots, x^m)$ is an affine coordinate system with respect to D (i.e. $Ddx^i = 0$).

At the following, in our notation we use this coordinate system.

Proposition 2.1.1 ([8]). *Let (M, D) be a flat manifold, then (M, D, g) is a Hessian manifold if and only if*

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{kj}}{\partial x^i}, \qquad \forall i, j, k = 1, \ldots, m.$$

Let (M, D) be a flat manifold, g a Riemannian metric, and ∇ the Levi-Civita connection of g. We denote the difference tensor of D and ∇ by

$$\gamma := \nabla - D.$$

It should be remarked that the components of γ with respect to affine coordinate systems of D coincide with the Christoffel symbols of ∇.

Definition 2.1.2 ([8]). Let (D, g) be a Hessian structure. A tensor field Q of type(1,3) defined by covariant differential of γ with respect to D

$$Q = D\gamma,$$

is called the Hessian curvature tensor for (D, g). The components of Q with respect to an affine coordinate system are given by

$$Q^i_{jkl} = \frac{\partial \gamma^i_{jl}}{\partial x^k}.$$

Proposition 2.1.2 ([8]). *Let (M, D, g) be a Hessian manifold and $g_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}$, for some smooth function ϕ on M. Then we have*

(1) $Q_{ijkl} = \frac{1}{2} \frac{\partial^4 \phi}{\partial x^i \partial x^j \partial x^k \partial x^l} - \frac{1}{2} g^{rs} \frac{\partial^3 \phi}{\partial x^i \partial x^r \partial x^k} \frac{\partial^3 \phi}{\partial x^j \partial x^l \partial x^s}$,
(2) $Q_{ijkl} = Q_{kjil} = Q_{klij} = Q_{jilk}$.

Proposition 2.1.3 ([8]). *Let (M, D, g) be a Hessian manifold and R the Riemannian curvature tensor for g. Then*

$$R_{ijkl} = \frac{1}{2}(Q_{ijkl} - Q_{jikl}).$$

Definition 2.1.3 ([8]). Let (D, g) be a Hessian structure and ν the volume element of g. We define a closed 1-form α and a symmetric 2-form β by

$$D^\nu_X = \alpha(X)\nu,$$

$$\beta = D\alpha,$$

then α and β are called **the first Koszul form** and **the second Koszul form** for (D, g) respectively.

Proposition 2.1.4 ([8]). *We have*

(1) $\alpha_i = \frac{1}{2}\frac{\partial \log \det[g_{mn}]}{\partial x^i} = \gamma^k_{ki} = tr\gamma$,
(2) $\beta_{ij} = \frac{\partial \alpha_i}{\partial x^j} = \frac{1}{2}\frac{\partial^2 \log \det[g_{mn}]}{\partial x^i \partial x^j} = Q^r_{rij} = trQ$.

Definition 2.1.4 ([8]). If a Hessian structure (D, g) satisfies the condition

$$\beta = \lambda g, \qquad \lambda = \frac{\beta^i_i}{n},$$

for some constant λ, then the Hessian structure is said to be Einstein–Hessian.

Proposition 2.1.5 ([8]). *The first affine Chern class $C_1(M)$ of a Hessian manifold (M, D, g) is represented by -2β, where β is the second Koszul form of (D, g).*

2.2. Kählerian structure on the tangent bundle of a Hessian manifold

Let $(x^1, x^2, \ldots, x^m)$ be an affine coordinate system on an affine manifold (M, D), we set

$$q^i := x^i o\pi, \qquad q^{m+i} := dx^i,$$

then $(q^1, \ldots, \ldots q^{2m})$ forms local coordinates on TM, (π is the projection map of vector bundle TM).

Let $J : TTM \longrightarrow TTM$ be a linear endomorphism on the tangent bundle of TM characterized by

$$J\left(\frac{\partial}{\partial q^i}\right) = -\frac{\partial}{\partial q^{m+i}}, \qquad and \qquad J\left(\frac{\partial}{\partial q^{m+i}}\right) = \frac{\partial}{\partial q^i},$$

therefore the endomorphism J satisfies $J^2 = -Id_{TTM}$, and it is an almost complex structure on TM.

Let (M, D) be a flat manifold. For an affine coordinate system $(x^1, \ldots, x^m)$ on M, we set

$$z^i = q^i + \sqrt{-1}q^{m+i},$$

in which $q^i = x^i o\pi$ and $q^{m+i} = dx^i$. Then m-tuples of functions given by $(z^1, \ldots, z^m)$ yield a holomorphic coordinate system on tangent bundle TM. Since the manifold is flat with respect to D, the almost complex structure defined above is a complex structure on TM. For Riemannian metric g on M we put

$$g^T = \sum_{i,j=1}^{m} (g_{ij} o\pi) dz^i d\bar{z}^j.$$

Then the Sasaki metric g^T is a Hermitian metric on the complex manifold (TM, J).

Proposition 2.2.1 ([8]). *Let (M, D) be a flat manifold and g a Riemannian metric on M . Then the following conditions are equivalent.*

(1) *g is a Hessian metric on (M, D).*
(2) *g^T is a Kählerian metric on (TM, J).*

Proposition 2.2.2 ([8]). *Let $R^T_{i\bar{j}k\bar{l}}$ and $R^T_{i\bar{j}}$ be the Riemannian curvature tensor and the Ricci tensor on the Kählerian manifold (TM, J, g^T). Then we have*

$$R^T_{i\bar{j}k\bar{l}} = \frac{1}{2} Q_{ijkl} o\pi,$$
$$R^T_{i\bar{j}} = -\frac{1}{2}\beta_{ij} o\pi.$$

Proposition 2.2.3 ([8]). *Let (M, D, g) be a Hessian manifold and (J, g^T) the Kählerian structure on the tangent bundle TM. Then the following two conditions are equivalent.*

1. *(D, g) is Einstein–Hessian.*
2. *(J, g^T) is Einstein–Kählerian (that is $R^T_{i\bar{j}} = \lambda g^T_{i\bar{j}}$ for some constant λ).*

3. Introduction of a flow on Hessian manifolds

In this section we intend to introduce a geometric evolution equation on Hessian manifolds to deform the geometry of the manifolds by changing the Hessian metric via a second-order PDE on symmetric (0, 2) tensors.

3.1. Hessian flow

Let (M, D, g_0) be a Hessian manifold, then we consider an evolution equation on M by

$$\frac{\partial}{\partial t} g_{ij} = 2\beta_{ij}, \tag{1}$$

with boundary condition $g_{ij}(0) = g_0$, called Hessian flow, where β is the second Koszul form of g. It can be shown that the Hessian flow (1) is a second-order partial differential equation on symmetric $(0, 2)$ tensors and defines a geometric flow on M. A solution to this equation is a one-parameter family of metrics on Hessian manifold M, defined on a time interval, and satisfying Eq. (1).

At the following, we will show that the metrics along the flow remain Hessian.

Lemma 3.1.1. *On a Hessian manifold (M, D, g) the evolved metric by (1) remains Hessian along the flow.*

Proof. Let $(x^1, x^2, \ldots, x^m)$ be an affine coordinate system with respect to D, using Proposition 2.1.4(2) we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial g_{ij}}{\partial x^k}\right) &= \frac{\partial}{\partial x^k}\left(\frac{\partial g_{ij}}{\partial t}\right) = \frac{\partial}{\partial x^k}(2\beta_{ij}) = \frac{\partial}{\partial x^k}\left(\frac{\partial^2 \log\det[g_{mn}]}{\partial x^i \partial x^j}\right)\\ &= \frac{\partial}{\partial x^i}\left(\frac{\partial^2 \log\det[g_{mn}]}{\partial x^k \partial x^j}\right) = \frac{\partial}{\partial x^i}(2\beta_{kj}) = \frac{\partial}{\partial x^i}\left(\frac{\partial g_{kj}}{\partial t}\right)\\ &= \frac{\partial}{\partial t}\left(\frac{\partial g_{kj}}{\partial x^i}\right).\end{aligned}$$

Since at $t = 0$, g_0 is Hessian, Proposition 2.1.1 and the above formula complete the proof. □

Proposition 3.1.1. *The Hessian flow (1) on Hessian manifold (M, D, g), induces a Kähler–Ricci flow on the tangent bundle (TM, J, g^T).*

Proof. By composition of (1) with projection map $\pi : TM \longrightarrow M$, we have

$$\frac{\partial g_{ij}}{\partial t} o\pi = 2\beta_{ij} o\pi = -4R^T_{\bar{i}\bar{j}}.$$

The above equality is obtained by Proposition 2.2.2. So we conclude

$$\frac{\partial g^T_{\bar{i}\bar{j}}}{\partial t} = -4R^T_{\bar{i}\bar{j}}, \tag{2}$$

and since by Proposition 2.1.4, $g^T_{\bar{i}\bar{j}}$ is a Kähler metric on TM, (2) is Kähler–Ricci flow on TM. □

3.2. Short-time existence

An important step in study of any partial differential equation is to show that it enjoys short time existence and uniqueness. In this section we will prove the short-time existence and uniqueness of the Hessian flow (1) on Hessian manifolds.

The first task is to show that Hessian flow (1) is a second-order partial differential equation. As for any $n \times n$ matrix M, we have $\det M = \sum_j M_{ij}(AdjM)^T_{ij}$, where $AdjM$ is the adjugate of M, so

$$\begin{aligned}\frac{\partial \log\det[g_{mn}]}{\partial x^i} &= \frac{1}{det[g_{mn}]}\left(\frac{\partial \det[g_{mn}]}{\partial x^i}\right)\\ &= \frac{1}{det[g_{mn}]}\left(\frac{\partial \det[g_{mn}]}{\partial g_{rs}}\right)\left(\frac{\partial g_{rs}}{\partial x^i}\right)\\ &= \frac{1}{det[g_{mn}]}\left(\frac{\partial \sum_k g_{rk}(Adjg)^T_{rk}}{\partial g_{rs}}\right)\left(\frac{\partial g_{rs}}{\partial x^i}\right)\\ &= \frac{1}{det[g_{mn}]}\sum_k \frac{\partial (g_{rk}(Adjg)^T_{rk})}{\partial g_{rs}}\left(\frac{\partial g_{rs}}{\partial x^i}\right).\end{aligned}$$

By product rule and since $\frac{\partial (Adjg)^T_{rk}}{\partial g_{rs}} = 0$ and $\frac{\partial g_{rk}}{\partial g_{rs}} = \delta_{ks}$ we have

$$\begin{aligned}\frac{\partial \log \det[g_{mn}]}{\partial x^i} &= \frac{1}{det[g_{mn}]}(Adjg)^T_{rs}\left(\frac{\partial g_{rs}}{\partial x^i}\right)\\ &= \frac{1}{det[g_{mn}]}(det[g_{mn}]g^{rs})\left(\frac{\partial g_{rs}}{\partial x^i}\right)\\ &= g^{rs}\left(\frac{\partial g_{rs}}{\partial x^i}\right),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 \log \det[g_{mn}]}{\partial x^i \partial x^j} &= g^{rs}\left(\frac{\partial^2 g_{rs}}{\partial x^i \partial x^j}\right) + \left(\frac{\partial g_{rs}}{\partial x^i}\right)\left(\frac{\partial g^{rs}}{\partial x^j}\right)\\ &= g^{rs}\left(\frac{\partial^2 g_{rs}}{\partial x^i \partial x^j}\right) - g^{rm}g^{ns}\left(\frac{\partial g_{mn}}{\partial x^j}\right)\left(\frac{\partial g_{rs}}{\partial x^i}\right).\end{aligned}$$

Thus by Proposition 2.1.4, the second Koszul form β locally given by

$$\beta_{ij} = g^{rs}\left(\frac{\partial^2 g_{rs}}{\partial x^i \partial x^j}\right) - g^{rm}g^{ns}\left(\frac{\partial g_{mn}}{\partial x^j}\right)\left(\frac{\partial g_{rs}}{\partial x^i}\right),$$

is a second order operator from $C^\infty(S_2^+T^*M)$ of positive definite symmetric $(0,2)$-tensors into itself and the Hessian flow (1) is a second-order partial differential equation on the vector bundle $C^\infty(S_2^+T^*M)$.

Now we need the following lemma to define the elliptic partial differential operators.

Lemma 3.2.1 (*[9]*). *Let P be a differential operator of order at most m between vector bundles E and F on M. Then there exists a unique smooth map, $\sigma_m(P) : T^*M \to Hom(E,F)$, such that the following is valid. Given $(x,\xi) \in T^*M$ and $\phi \in C^\infty(M)$ with $d\phi(x) = \xi$, for every $s \in C^\infty(M,E)$,*

$$\sigma_m(P,x,\xi)s(x) = \lim_{t\to\infty} t^{-m}e^{-t\phi(x)}P[e^{t\phi}s](x),$$

where $Hom(E,F)$ denotes the vector bundle whose fiber $Hom(E,F)_p$ equals the space of homeomorphisms of linear maps $E_p \to F_p$.

Definition 3.2.1 (*[9]*). For an m-order partial differential operator P from $C^\infty(E)$ into $C^\infty(F)$, $\sigma_m(P,\xi)$ is called the principal symbol of P in the direction of the one-form ξ, and an m-order partial differential operator P is said to be elliptic if its principal symbol $\sigma_m(P,\xi)$ is an isomorphism whenever $\xi \neq 0$.

Remark 3.2.1. For an m-order differential operator $P : C^\infty(E) \to C^\infty(F)$, of the form $P(u) = \sum_{|\alpha|\leq m} P_\alpha \partial^\alpha u$ where $P_\alpha \in Hom(E,F)$, the principal symbol of P in direction $\xi \in T^*M$ is the bundle homomorphism $\sigma_m[P](\xi) = \sum_{|\alpha|=m} P_\alpha \xi^\alpha$.

Remark 3.2.2. Let P be a differential operator of order at most m between vector bundles E and F, and Q a differential operator of order at most n between vector bundles F and G on M. Then $Po\,Q$ is a differential operator of order at most m+n between vector bundles E and G, and the principal symbol is given by

$$\sigma_{m+n}(Po\,Q,x,\xi) = \sigma_m(P,x,\xi)o\,\sigma_n(Q,x,\xi), \qquad (x \in M, \xi \in T^*_xM).$$

Proposition 3.2.1. *On a Hessian manifold (M,D,g) the second Koszul form β is an elliptic second-order operator.*

Proof. We have already seen that β is a second order operator and we know ([8], Corollary 7.5) it is the Laplace operator of Hessian metric g defined by $\Delta_\partial = \partial\delta + \delta\partial$. Now we will show that Laplace operator is an elliptic operator. The second order principal symbol at point $x \in M$ in the direction of a one-form $\xi \in T^*_xM$ is given by

$$\sigma_2(\Delta_\partial,\xi) = \sigma_2(\partial\delta + \delta\partial,\xi) = \sigma_1(\partial,\xi)\sigma_1(\delta,\xi) + \sigma_1(\delta,\xi)\sigma_1(\partial,\xi).$$

Let $e(\omega)$ be an exterior product operator for a (p,q)-form ω, $i(X)$ an interior product operator for $X \in TM$, $\star$ the Hodge star operator and $\{E_1,\ldots,E_m\}$ be an orthonormal frame field with respect to the Riemannian metric g and set $\theta^j = i(E_j)g$, it is easy to show that for $x \in M$ and $\xi \in T^*_xM$, $\sigma_0(\star,\xi) = \star$ and $\sigma_1(\partial,\xi) = e(\xi)$. Without loss of generality we can consider $\xi = \theta^j$, so by [8], Proposition 7.2 and Lemma 7.1 we have

$$\begin{aligned}\sigma_1(\delta,\theta^j) &= \sigma_0((-1)^p\star^{-1},\theta^j)\sigma_1(\partial,\theta^j)\sigma_0(\star,\theta^j)\\ &= (-1)^p\star^{-1}e(\theta^j)\star\\ &= -i(E_j).\end{aligned}$$

So

$$\sigma_2(\Delta_\partial, \theta^j) = -i(E_j)e(\theta^j) - e(\theta^j)i(E_j) = -g(E_j, E_j)I,$$

so for every $\xi \neq 0$, $\sigma_2(\Delta_\partial, \xi)$ is an isomorphism and the proof is complete. □

Definition 3.2.2. An evolution equation on the vector bundle of positive (0, 2)-tensors h, of the form

$$\frac{\partial}{\partial t}h_{ij} = P(h_{ij}),$$

where P is a m-th order differential operator, is said to be parabolic if P is an elliptic operator.

Theorem 3.2.1. *If (M, D, g_0) is a closed Hessian manifold, then there exists a unique smooth solution $g(t)$ for the Hessian flow (1) defined on some positive time interval $[0, T]$, $T > 0$, with $g(0) = g_0$.*

Proof. Since the second Koszul form β is an elliptic operator, so the Hessian flow (1) is a parabolic equation. It is known from PDE theory, given a smooth initial metric g_0 for a parabolic equation, there exist $T > 0$ and a smooth family of metrics $g(t)$ for $t \in [0, T]$ that satisfies (1) and it is unique. □

4. Long time existence of the flow on a closed Hessian manifold

In this section we intend to show global existence of the Hessian flow (1). We will use the flow to prove a problem which is similar to the Calabi conjecture for Kählerian manifolds. The proof follows closely the methods given in [6].

Let (M, D, g) be a closed Hessian manifold of dimension m, as is well-known the cohomology class of $-2\beta_{ij}$, where β_{ij} is the second Koszul form of g, is equal to the first Chern class $C_1(M)$ of M. Given any 2-form T_{ij} which represents the first Chern class $C_1(M)$, we can show that there is another Hessian metric $\tilde{g}_{ij}$ on M such that T_{ij} is the second Koszul form of $\tilde{g}_{ij}$. Cheng–Yau and Delanoë proved independently this result analogous to the Calabi conjecture [3,10]. This problem can be reduced to solve a real Monge–Ampére equation and we will prove long time existence of this by using the continuity method.

We consider the following equation:

$$\frac{\partial \tilde{g}_{ij}}{\partial t} = 2(\tilde{\beta}_{ij} - T_{ij}), \qquad \tilde{g}_{ij}(0) = g_{ij}, \tag{3}$$

where $\tilde{\beta}_{ij}$ denotes the second Koszul form of the metric $\tilde{g}_{ij}$ and g_{ij} is a Hessian metric. If we can prove that solution of (3) exists for all time and $\tilde{g}_{ij}(t)$ converges as t goes to the infinity and that $\frac{\partial \tilde{g}_{ij}}{\partial t}$ converges uniformly in t to a constant, then $\tilde{g}_{ij}(\infty)$ is a metric as we want. By assumption, T_{ij} and β_{ij} belong to the same cohomology class $C_1(M)$, so there exists a smooth function h on M that

$$\beta_{ij} - T_{ij} = \frac{\partial^2 h}{\partial x^i \partial x^j},$$

therefore, if we choose the smooth function u on $M \times [0, T)$, and $0 < T \leq \infty$, with $u(0) = 0$, such that

$$\tilde{g}_{ij} = g_{ij} + \frac{\partial^2 u}{\partial x^i \partial x^j},$$

then Eq. (3) becomes

$$\frac{\partial^2}{\partial x^i \partial x^j}\left(\frac{\partial u}{\partial t}\right) = 2\left(\tilde{\beta}_{ij} - \beta_{ij} + \frac{\partial^2 h}{\partial x^i \partial x^j}\right), \tag{4}$$

and according to Proposition 2.1.4 (2)

$$\begin{aligned}\frac{\partial^2}{\partial x^i \partial x^j}\left(\frac{\partial u}{\partial t}\right) &= 2\frac{\partial^2}{\partial x^i \partial x^j}\left(\frac{1}{2}\log\det(\tilde{g_{ij}}) - \frac{1}{2}\log\det(g_{ij})\right) + 2\frac{\partial^2 h}{\partial x^i \partial x^j} \\ &= \frac{\partial^2}{\partial x^i \partial x^j}\left(\log\det\left(g_{ij} + \frac{\partial^2 u}{\partial x^i \partial x^j}\right) - \log\det(g_{ij})\right) + 2\frac{\partial^2 h}{\partial x^i \partial x^j}.\end{aligned}$$

The maximum principle for compact manifolds, implies that the function u satisfies the equation

$$\frac{\partial u}{\partial t} = \log\det\left(g_{ij} + \frac{\partial^2 u}{\partial x^i \partial x^j}\right) - \log\det(g_{ij}) + f + \phi(t), \tag{5}$$

where $f := 2h$ and $\phi(t)$ is a smooth function in t. Then it results the compatibility condition

$$\int_M \exp\Big(\frac{\partial u}{\partial t} - f\Big)dV = \exp(\phi(t))Vol(M), \tag{6}$$

where dV is the volume element of the metric g_{ij}.

Eq. (5) is a nonlinear parabolic equation, so standard theory of PDE provides short time solution existence. In other words (5) is a real Monge–Ampére equation. At the following, we will try to get a prior estimates up to third order which will imply the global existence.

We consider that u satisfies the initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \log\det\Big(g_{ij} + \frac{\partial^2 u}{\partial x^i \partial x^j}\Big) - \log\det(g_{ij}) + f, \\ &u(x, 0) = 0, \end{aligned} \tag{7}$$

on the maximal time interval $[0, T)$, such that $\tilde{g}_{ij}(t) = g_{ij}(t) + \frac{\partial^2 u}{\partial x^i \partial x^j}$ is a Hessian metric on M for any time $t \in [0, T)$.

4.1. Zero order estimate

Lemma 4.1.1. *There exists a positive constant C_0 such that*

$$\Big|\frac{\partial u}{\partial t}\Big| \le C_0. \tag{8}$$

Proof. Differentiating Eq. (7) with respect to t, we have

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t}\Big) = \tilde{g}^{ij}\frac{\partial^2}{\partial x^i \partial x^j}\Big(\frac{\partial u}{\partial t}\Big) = \tilde{\Delta}\Big(\frac{\partial u}{\partial t}\Big), \tag{9}$$

where $\tilde{\Delta}$ is the Laplace operator with respect to $\tilde{g}_{ij}$. So $\frac{\partial u}{\partial t}$ satisfies the heat equation and applying the maximum principle for the parabolic heat equation (9) we have the result. In a more precise way, if we assume that $\frac{\partial u}{\partial t}$ achieves its maximum at point (x_0, t_0) then $\tilde{\Delta}(\frac{\partial u}{\partial t})(x_0, t_0) \le 0$. On the other hand, if $t_0 \neq 0$ then $\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t}\Big)(x_0, t_0) \ge 0$, so by (9) $t_0 = 0$ and

$$\max_M \Big|\frac{\partial u}{\partial t}\Big| \le \max_M |f|, \tag{10}$$

and the proof is complete. □

Lemma 4.1.2. *There exist positive constants C and C_1 such that*

$$0 < m + \Delta u \le C_1 \exp\Big(C(u - \inf_M u)\Big) \quad \forall t \in [0, T). \tag{11}$$

Proof. We have $\tilde{g}_{ij} = g_{ij} + \frac{\partial^2 u}{\partial x^i \partial x^j}$, so $tr\tilde{g} = m + \Delta u$ with respect to g_{ij}, and positivity follows from the fact that $\tilde{g}_{ij}$ is positive definite.

For the second inequality, we set $\frac{\partial u}{\partial t} - f = F$, then by (7) we have

$$\det(\tilde{g}_{ij})\Big[\det(g_{ij})\Big]^{-1} = \exp(F), \tag{12}$$

differentiating (12), with respect to x^k we have

$$\exp(F)\Big[\tilde{g}^{ij}\Big(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k}\Big) - g^{ij}\frac{\partial g_{ij}}{\partial x^k}\Big] = \exp(F)\frac{\partial F}{\partial x^k}.$$

Again differentiating the above equation with respect to x^l we obtain

$$-\tilde{g}^{mi}\tilde{g}^{nj}\frac{\partial \tilde{g}_{mn}}{\partial x^l}\Big(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k}\Big) + \tilde{g}^{ij}\Big(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^4 u}{\partial x^i \partial x^j \partial x^k \partial x^l}\Big) + g^{mi}g^{nj}\frac{\partial g_{mn}}{\partial x^l}\frac{\partial g_{ij}}{\partial x^k} - g^{ij}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} = \frac{\partial^2 F}{\partial x^k \partial x^l}. \tag{13}$$

Let $\tilde{\Delta}$ be the Laplacian with respect to $\tilde{g}_{ij}$, then

$$\tilde{\Delta}(\Delta u) = \tilde{g}^{kl}\frac{\partial^2 g^{ij}}{\partial x^k \partial x^l}\frac{\partial^2 u}{\partial x^i \partial x^j} + \tilde{g}^{kl}\frac{\partial g^{ij}}{\partial x^l}\frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k} + \tilde{g}^{kl}\frac{\partial g^{ij}}{\partial x^k}\frac{\partial^3 u}{\partial x^i \partial x^j \partial x^l} + \tilde{g}^{kl}g^{ij}\frac{\partial^4 u}{\partial x^i \partial x^j \partial x^k \partial x^l}. \tag{14}$$

Since the left hand side of (14) is independent of choice of coordinate system, we can compute the right hand side in a coordinate system so that $g_{ij} = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{ij}}{\partial x^l} = 0$. Then we have

$$\tilde{\Delta}(\Delta u) = \tilde{g}^{kl}\frac{\partial^2 g^{ij}}{\partial x^k \partial x^l}\frac{\partial^2 u}{\partial x^i \partial x^j} + \tilde{g}^{kl}\frac{\delta^{ij}\partial^4 u}{\partial x^i \partial x^j \partial x^k \partial x^l}. \tag{15}$$

The same strategy in (13) concludes

$$\Delta F = \sum_l \left[-\tilde{g}^{mi}\tilde{g}^{nj}\frac{\partial^3 u}{\partial x^i \partial x^j \partial x^l}\frac{\partial^3 u}{\partial x^m \partial x^n \partial x^l} + \tilde{g}^{ij}\frac{\partial^2 g_{ij}}{\partial x^l \partial x^l} + \tilde{g}^{ij}\frac{\partial^4 u}{\partial x^i \partial x^j \partial x^l \partial x^l} - \frac{\partial^2 g_{ii}}{\partial x^l \partial x^l}\right]. \tag{16}$$

Then by comparing (15) and (16) we have

$$\tilde{\Delta}(\Delta u) = \Delta F + \tilde{g}^{kl}\frac{\partial^2 g^{ij}}{\partial x^k \partial x^l}\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_l \left[\tilde{g}^{mi}\tilde{g}^{nj}\frac{\partial^3 u}{\partial x^i \partial x^j \partial x^l}\frac{\partial^3 u}{\partial x^m \partial x^n \partial x^l} + \frac{\partial^2 g_{ii}}{\partial x^l \partial x^l} - \tilde{g}^{ij}\frac{\partial^2 g_{ij}}{\partial x^l \partial x^l}\right]. \tag{17}$$

Also in this coordinate system Hessian curvature has the expression $Q_{ijkl} = \frac{1}{2}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}$, so inserting this tensor in (17) and the fact that $\frac{\partial^2 g_{ij}\delta^{ij}}{\partial x^k \partial x^l} = -\frac{\partial^2 g^{ij}\delta_{ij}}{\partial x^k \partial x^l}$ we have

$$\tilde{\Delta}(\Delta u) = \Delta F - 2\sum_{i,j}\tilde{g}^{kl}Q_{ijkl}\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_l \left[\tilde{g}^{mi}\tilde{g}^{nj}\frac{\partial^3 u}{\partial x^i \partial x^j \partial x^l}\frac{\partial^3 u}{\partial x^m \partial x^n \partial x^l} - 2\tilde{g}^{ij}Q_{ijll}\right] + \sum_{i,l} 2Q_{iill}. \tag{18}$$

If we choose another coordinate system so that $g_{ij} = \delta_{ij}$ and $u_{ij} = \delta_{ij}u_{ii}$ (u_{ij} denotes $\frac{\partial^2 u}{\partial x^i \partial x^j}$) then we have

$$\tilde{g}^{ij} = \delta^{ij}(1 + u_{ii})^{-1}.$$

In this coordinate system we have

$$\begin{aligned} -2\sum_{i,j}\tilde{g}^{kl}Q_{ijkl}u_{ij} - 2\sum_l \tilde{g}^{ij}Q_{ijll} + 2\sum_{i,l}Q_{iill} &= \sum_{i,j,l}[-2(1+u_{ll})^{-1}Q_{ijll}u_{ij}] \\ &\quad + \sum_{i,l}[-2(1+u_{ii})^{-1}Q_{iill} + 2Q_{iill}] \\ &= \sum_{i,l} 2Q_{iill}\Big(\frac{u_{ii}(u_{ll}-u_{ii})}{(1+u_{ii})(1+u_{ll})}\Big) \\ &= \sum_{i,l} Q_{iill}\Big(\frac{-(u_{ll}-u_{ii})^2}{(1+u_{ii})(1+u_{ll})}\Big). \end{aligned} \tag{19}$$

Using (19) we rewrite (18)

$$\tilde{\Delta}(\Delta u) \geq \Delta F + \sum_l \tilde{g}^{mi}\tilde{g}^{nj}u_{ijl}u_{mnl} + \sum_{i,l}(-\sup_{i\neq l} Q_{iill})\Big[\frac{1+u_{ii}}{1+u_{ll}} - m^2\Big]. \tag{20}$$

On the other hand,

$$\tilde{\Delta}u = \tilde{g}^{ij}u_{ij} = \delta^{ij}(1+u_{ii})^{-1}u_{ij} = \sum_{i=1}^m \frac{u_{ii}}{1+u_{ii}} = m - \sum_{i=1}^m \frac{1}{1+u_{ii}}. \tag{21}$$

Now let C be a positive constant, then

$$\begin{aligned} \tilde{\Delta}(\log(m+\Delta u) - Cu) &= \tilde{g}^{jk}\partial_j\Big[(m+\Delta u)^{-1}\partial_k(\Delta u) - Cu_k\Big] \\ &= -(m+\Delta u)^{-2}\tilde{g}^{jk}\partial_j(\Delta u)\partial_k(\Delta u) + (m+\Delta u)^{-1}\tilde{\Delta}(\Delta u) - C\tilde{\Delta}u, \end{aligned}$$

so

$$(m+\Delta u)\tilde{\Delta}(\log(m+\Delta u) - Cu) = -(m+\Delta u)^{-1}\tilde{g}^{jk}\partial_j(\Delta u)\partial_k(\Delta u) + \tilde{\Delta}(\Delta u) - C(m+\Delta u)\tilde{\Delta}u. \tag{22}$$

Since

$$\Delta u = \sum_{i=1}^m u_{ii} = \sum_{i=1}^m \tilde{g}_{ii} - m,$$

then

$$m + \Delta u \geq 0,$$

and

$$\partial_j(\Delta u) = \partial_j(g^{ab}u_{ab}) = \sum_a u_{aaj},$$

if we compute the first phrase on the right hand side of Eq. (22) we have

$$-(m+\Delta u)^{-1}\tilde{g}^{ij}\partial_j(\Delta u)\partial_j(\Delta u) = -(m+\Delta u)^{-1}\sum_j \tilde{g}^{jj}\left(\sum_a u_{aaj}\right)^2$$
$$= \frac{-\sum_{j=1}^m \tilde{g}^{jj}\sum_{a=1}^m \left(\sqrt{\tilde{g}^{aa}}u_{aaj}\sqrt{\tilde{g}_{aa}}\right)^2}{\sum_{a=1}^m \tilde{g}_{aa}},$$

by applying the Cauchy–Schwarz inequality we have

$$-(m+\Delta u)^{-1}\tilde{g}^{ij}\partial_j(\Delta u)\partial_j(\Delta u) \geq \frac{-\sum_{j=1}^m \tilde{g}^{jj}\left(\sum_{a=1}^m \left(\sqrt{\tilde{g}^{aa}}u_{aaj}\right)^2\right)\times\left(\sum_{a=1}^m\left(\sqrt{\tilde{g}_{aa}}\right)^2\right)}{\sum_{a=1}^m \tilde{g}_{aa}}$$
$$\geq -\sum_{b,a,j}\tilde{g}^{jj}\tilde{g}^{aa}(u_{abj})^2 \tag{23}$$

so (20), (21) and (23) imply

$$(m+\Delta u)\tilde{\Delta}(\log(m+\Delta u) - Cu) \geq (\Delta F - m^2T_2) + T_2\sum_{i,l}\left(\frac{1+u_{ii}}{1+u_{ll}}\right)$$
$$- Cm(m+\Delta u) + C(m+\Delta u)\left(\sum_i \frac{1}{1+u_{ii}}\right)$$
$$\geq (\Delta F - m^2T_2) - Cm(m+\Delta u)$$
$$+ (C+T_2)(m+\Delta u)\left(\sum_{l=1}^m \frac{1}{1+u_{ll}}\right),$$

in which $T_2 := -\sup_{i\neq l} Q_{iill}$, and since $\sum_{l=1}^m \frac{1}{1+u_{ll}} \geq (\frac{\sum_i(1+u_{ii})}{\prod_l(1+u_{ll})})^{\frac{1}{m-1}}$ then

$$(m+\Delta u)\tilde{\Delta}(\log(m+\Delta u) - Cu) \geq \Delta F - m^2T_2 - Cm(m+\Delta u) + (C+T_2)(m+\Delta u)^{\frac{m}{m-1}}\exp\frac{-F}{m-1}.$$

The function T_2 is continuous, so we can choose C such that $C + T_2 > 1$ and D be a positive constant such that $T_2 < D$, moreover $\Delta F \geq F_0$ imply $\Delta F - m^2T_2 \geq F_0 - m^2D$, if we choose F_1 such that $F \leq F_1$ so $\exp\frac{-F}{m-1} \geq \exp\frac{-F_1}{m-1}$. Let $A := F_0 - m^2D$ and $C' := \exp\frac{-F_1}{m-1}$ then

$$(m+\Delta u)\tilde{\Delta}(\log(m+\Delta u) - Cu) \geq A - B(m+\Delta u) + C'(m+\Delta u)^{\frac{m}{m-1}},$$

in which, B depends on m and C. Now if we assume that the function $\log(m+\Delta u) - Cu$ achieves its maximum at point x_0 then in x_0

$$\tilde{\Delta}(\log(m+\Delta u) - Cu) \leq 0 \Longrightarrow A - B(m+\Delta u) + C'(m+\Delta u)^{\frac{m}{m-1}} \leq 0.$$

In addition the function $R : (0, +\infty) \longrightarrow \mathbb{R}, x \mapsto A - Bx + C'x^{\frac{m}{m-1}}$ is continuous and when x tends to $+\infty$ the function tends to $+\infty$, so there is a constant C_1 (depends on A, B, C') such that $R(x) > 0$ for all $x > C_1$. But $m + \Delta u > 0$ while $R(m+\Delta u) \leq 0$, thus $m + \Delta u(x_0) \leq C_1$, and

$$\log(m+\Delta u) - Cu \leq \log(m+\Delta u(x_0)) - Cu(x_0) \leq \log C_1 - Cu(x_0),$$

so by applying exponential function on both sides of last inequality, we get

$$m + \Delta u \leq C_1\exp(C\Big(u - u(x_0)\Big)\Big) \leq C_1\exp\Big(C(\,u - \inf_M u)\Big). \quad \square$$

Let v be normalization of u such that its mean value is zero with respect to g, i.e.

$$v = u - \frac{1}{Vol(M)} \int_M u dV_g. \tag{24}$$

Then, we have the following.

Lemma 4.1.3.

$$\sup_{M\times[0,T)} v \leq C_2, \tag{25}$$

$$\sup_{M\times[0,T)} \int_M |v| dV \leq C_3, \tag{26}$$

in which C_2 and C_3 are constants.

Proof. We have (25) and (4.1.3) based on the calculations in [11]. □

Cao proved the following Lemma in Kähler geometry in [12], here we will prove it for our case by a different technique.

Lemma 4.1.4 (C^0 *Estimates*)**.** *There exists a constant C_4 so that*

$$\sup_{M\times[0,T)} |v| \leq C_4. \tag{27}$$

Proof. Set for a constant A, $w = u - At$. Then by (7) we have

$$\frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} - A = \log\det\left(g_{ij} + \frac{\partial^2 u}{\partial x^i \partial x^j}\right) - \log\det(g_{ij}) + f - A,$$

if we assume that u achieves its minimum at point (x_0, t_0) then $\frac{\partial^2 u}{\partial x^i x^j}(x_0, t_0) \leq 0$ and then

$$\frac{\partial w}{\partial t} \leq f - A,$$

let $A := \sup_{x\in M} f(x) + 1$, since M is compact A is constant and

$$\frac{\partial w}{\partial t} \leq -1.$$

Hence we can deduce that the maximum of w in $[0, T]$, must occur at $t = 0$. Recalling $u|_{t=0} = 0$, we obtain that $w \leq 0$ on $M \times [0, T]$, so

$$u \leq At \leq AT \quad on \quad M \times [0, T],$$

and this gives the upper bound $C = AT$ for u. Similarly, the lower bound is obtained and proof is complete. □

Proposition 4.1.1. *There exists a constant C_5 such that*

$$0 < m + \Delta v < C_5. \tag{28}$$

Proof. A simple calculation shows that $\Delta v = \Delta u$, combining Lemmas 4.1.2 and 4.1.4 and this fact that $\inf(-u) = -\sup u$ we have

$$m + \Delta v = m + \Delta u \leq C_1(\exp(C_0(u - \inf_M u)) \leq C_1 \exp(C_0(v - \inf_M v)) \leq C_5,$$

and the proof is complete. □

4.2. First order estimate

For the first order estimate, we have the following result originally from Schauder theory.

Lemma 4.2.1. *There exists a constant C_6 such that*

$$\sup_{M\times[0,T)} |\nabla v| \leq C_6. \tag{29}$$

Proof. According to Schauder estimate we know

$$\sup_{M\times[0,T)} |\nabla v| \leq K\left(\sup_{M\times[0,T)} |\Delta v| + \sup_{M\times[0,T)} |v|\right), \tag{30}$$

where K depends on the bounds of the coefficients of Δ, so by applying (27), (28) and (30), we have

$$\sup_{M\times[0,T)} |\nabla v| \leq K(C_5 - m + C_4) =: C_6,$$

so the proof is complete. □

4.3. Higher order estimates

First we estimate all the second ordered derivatives.

Lemma 4.3.1 (C^2 *Estimates*). *Let the Hessian flow (7) exist for $t \in [0, T)$, $T < \infty$. Then there exists a constant C_7 such that*

$$\sup |u|_{C^2} \leq C_7. \tag{31}$$

Proof. Proposition 4.1.1 implies that $m + \Delta u$, and thus $\sum_i (1 + u_{ii})$ is bounded from above for all i. Also note that

$$\prod_i (1 + u_{ii}) = \frac{\det \tilde{g}_{ij}}{\det g_{ij}} = \exp\left(\frac{\partial u}{\partial t} - f\right),$$

which is bounded by (10). So there exist positive constants $A \leq B$ such that

$$A \leq \frac{\partial^2 u}{\partial x^i \partial x^j} \leq B,$$

so all the second order derivatives are bounded and the proof is complete. □

The above Lemma also shows that $\tilde{g}_{ij}$ stay as a metric along the flow.

Lemma 4.3.2. *Let Hessian flow (7) exist for $t \in [0, T)$, $T < \infty$. Then there exists a constant C_8 such that*

$$\sup_{t\in[0,T)} |v|_{C^3} \leq C_8. \tag{32}$$

Proof. We can consider the below quantity similar to Calabi and Yau,

$$S = g^{ir} g^{sj} g^{kt} v_{ijk} v_{rst}.$$

Similar calculation as in [11] shows that there exist positive constants A_1, A_2 and A_3 such that

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)(S + A_1 \Delta v) \geq A_2 S - A_3.$$

If $p(t)$ be the maximum point of $S + A_1 \Delta v$ at time t, we have $A_2 S - A_3 \leq 0$, hence at t

$$A_2(S + A_1 \Delta v) \leq A_3 + A_1 A_2 \Delta v,$$

on the other hand by Proposition 4.1.1, $\Delta v \leq C_5 - m$, so $\sup_{M\times[0,T)} S$ is bounded, i.e.

$$\sup_{t\in[0,T)} |v|_{C^3} \leq C_8.$$

And we estimate all the third order derivatives of v. □

Theorem 4.3.1. *The Hessian flow (7) has a global existence on closed manifolds.*

Proof. Assume the Hessian flow (7) exists for $t \in [0, T)$, where T is the maximal time. With the above estimates, one can see T cannot be finite. The higher order estimates for u follow from below argument.

Applying the differential operator $\frac{\partial}{\partial x^k}$ to Eq. (7) we have

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial x^k}\right) = g^{ij}\frac{\partial}{\partial x^k}(g_{ij}) - \tilde{g}^{ij}\frac{\partial}{\partial x^k}(g_{ij}) - \frac{\partial f}{\partial x^k}, \tag{33}$$

which is a linear parabolic equation in $\frac{\partial u}{\partial x^k}$. The coefficients of operator $(\tilde{\Delta} - \frac{\partial}{\partial t})$ and also the right hand side of (33) are bounded in $C^{0,\alpha}$ norm, for all $0 < \alpha < 1$. The standard parabolic estimates give a $C^{2,\alpha}$ bound for $\frac{\partial u}{\partial x^k}$. Since $\frac{\partial}{\partial x^k}$ was a first

order operator with constant coefficients we obtain a bound for u in $C^{3,\alpha}$. This gives higher regularity for the coefficients of the above parabolic equation. Using the Schauder theory again we conclude higher regularity for u. So the C^{∞}-norm of u is uniformly bounded for all $t \in [0, T)$. Since the bounds on u are independent of $t \in [0, T)$, we can apply Arzela–Ascoli theorem to select a time sequence t_n such that $v(x, t_n) \to v_{\infty}(x)$ as $t_n \to \infty$. In fact, since $\frac{\partial u}{\partial t}$ is uniformly bounded for all $t \in [0, T)$, v_{∞} is unique and the function u cannot blow up in finite time. So the solution u exists for all time and the proof is complete. □

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