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Bi-warped products and applications in locally product Riemannian manifolds

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ABSTRACT

In this paper, we consider M_θ, a pointwise slant submanifold and prove that every bi-warped product $M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ in a locally product Riemannian manifold satisfies a general inequality:

$$\|\sigma\|^2 \geq n_2\|\vec{\nabla}^T(\ln f_1)\|^2 + n_3\cos^2\theta\|\vec{\nabla}^\theta(\ln f_2)\|^2,$$

where $n_2 = \dim(M_T)$, $n_3 = \dim(M_\theta)$ and σ is the second fundamental form and $\nabla^T(\ln f_1)$ and $\nabla^\theta(\ln f_2)$ are the gradient components along M_T and M_θ, respectively. We also discuss the equality case of this inequality. Furthermore, we give some applications and non-trivial examples.

1. Introduction

In [12], B.-Y. Chen and F. Dillen introduced a generalized class of CR-warped products, called multiply CR-warped product submanifold $M = M_T \times_{f_i} N$ in an arbitrary Kaehler manifold $\tilde{M}$, where $N =_{f_1} M_\perp^1 \times_{f_2} M_\perp^2 \times \cdots \times_{f_k} M_\perp^k$ is a product of k-totally real submanifolds and M_T is a holomorphic submanifold of $\tilde{M}$. They have obtained the following sharp inequality for the squared norm of the second fundamental form $\|\sigma\|^2 \geq 2\sum_{i=1}^{k} n_i\|\nabla(\ln f_i)\|^2$ in terms of the warping functions, where $n_i = \dim M_\perp^i$, for each $i = 1, \ldots, k$. They also discussed the equality case and provided some examples to illustrate the obtained inequality. Recently, H.M. Tastan [22] studied bi-warped product submanifolds of the form $M = M_T \times_{f_1} M_\perp \times_{f_2} M_\theta$ in a Kaehler manifold $\tilde{M}$, where M_T, $M_\perp$ and M_θ are holomorphic, totally real and proper pointwise slant submanifolds of $\tilde{M}$, respectively. Notice that bi-warped product submanifolds are spacial case of multiply warped product submanifolds which were introduced by S. Nolker [19] and B.-Y. Chen and F. Dillen [12].

In our previous paper, we studied bi-warped product submanifolds in locally product Riemannian manifolds. We showed that only $M_\theta \times_{f_1} M_T \times_{f_2} M_\perp$ bi-warped products exist in a locally product Riemannian manifold $\tilde{M}$, where M_T, $M_\perp$ and M_θ are invariant, anti-invariant and proper slant submanifolds of $\tilde{M}$, respectively. On the other hand, we proved that

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the bi-warped products of the form $M = M_T \times_{f_1} M_\perp \times_{f_2} M_\theta$ are Riemannian product manifolds, i.e., both f_1 and f_2 are constant on M in a locally product Riemannian manifold $\tilde{M}$, while the bi-warped products in the form $M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ are single warped products [18]. In the third case, if we consider M_θ, a pointwise slant fibre instead of slant, then these kinds of bi-warped products exist, which is the case we have to discuss in the present paper.

The purpose of this paper is to investigate the geometric properties of bi-warped product submanifolds of the form $M = M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ of a locally product Riemannian manifold $\tilde{M}$, where M_T, $M_\perp$ and M_θ are invariant, anti-invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively. We prove that for any bi-warped product submanifold in a locally product Riemannian manifold $\tilde{M}$, the second fundamental form σ of $M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ in $\tilde{M}$ satisfies the following:

Theorem 1. *Let $M = M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ be a $\mathfrak{D}^\perp - \mathfrak{D}^\theta$ mixed totally geodesic bi-warped product submanifold in a locally product Riemannian manifold $\tilde{M}$, where M_T, $M_\perp$ and M_θ are invariant, anti-invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively. Then, we have*

(i) The second fundamental form σ and the warping functions f_1, f_2 satisfy

$$\|\sigma\|^2 \geq n_2 \|\vec{\nabla}^T(\ln f_1)\|^2 + n_3 \cos^2\theta \|\vec{\nabla}^\theta(\ln f_2)\|^2 \tag{1.1}$$

where $n_2 = \dim M_T$, $n_3 = \dim M_\theta$ and $\vec{\nabla}^T(\ln f_1)$ and $\vec{\nabla}^\theta(\ln f_2)$ are the gradient components of $\ln f_1$ and $\ln f_2$ along M_T and M_θ, respectively.

(ii) If the equality sign holds identically in (i), then $M_\perp$ is a totally geodesic submanifold of $\tilde{M}$ and M_T and M_θ are totally umbilical in $\tilde{M}$. Moreover, M is $\mathfrak{D}^\perp$-geodesic submanifold of $\tilde{M}$.

The paper is organized as follows. In Section 2 we provide some basic notations, formulas, definitions and results. Section 3 is devoted to the study of bi-warped product submanifolds of locally product Riemannian manifolds. In Section 4, we prove Theorem 1 and in Section 5, we give some applications. Section 6, we provide some non-trivial examples of bi-warped product submanifolds in Euclidean spaces.

2. Preliminaries

An m-dimensional Riemannian manifold $\tilde{M}$ is said to be an *almost product Riemannian manifold* (see, for instance, [1,25]) if there is a $(1, 1)$ tensor field F satisfying $F^2 = I$ and $F \neq \pm I$ and a Riemannian metric g such that

$$g(FX, FY) = g(X, Y), \tag{2.1}$$

for any vector fields X, Y on $\tilde{M}$. It is easy to see that for an almost product Riemannian manifold, we have $g(FX, Y) = g(X, FY)$, for any $X, Y \in \Gamma(T\tilde{M})$, where $\Gamma(T\tilde{M})$ is the Lie algebra of vector fields on $\tilde{M}$. In addition, if $(\tilde{\nabla}_X F)Y = 0$, where $\tilde{\nabla}$ is the Riemannian connection with respect to g, then $\tilde{M}$ is called a locally product Riemannian manifold [5,17].

Let M be a submanifold of a Riemannian manifold $\tilde{M}$ with induced metric g. Let $\Gamma(TM)$ be the Lie algebra of vector fields of M in $\tilde{M}$ and $\Gamma(T^\perp M)$, set of all vector fields normal to M. Then, the Gauss and Weingarten formulas are given respectively by (see, for instance, [10,11,25])

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.2}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.3}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where ∇ and $\nabla^\perp$ are the induced connections on the tangent and normal bundles of M, respectively, and σ denotes the second fundamental form, A the shape operator of the submanifold. The second fundamental form σ and the shape operator A are related by (see, [7,25])

$$g(\sigma(X, Y), N) = g(A_N X, Y). \tag{2.4}$$

Let M be an n-dimensional submanifold of a Riemannian m-manifold $\tilde{M}$. We choose a local frame field $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ in $\tilde{M}$ such that restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M and hence $e_{n+1}, \ldots, e_m$ are normal to M. Let $\{\sigma^r_{ij}\}$, $i, j = 1, \ldots, n$; $r = n+1, \ldots, m$ denote the coefficients of the second fundamental form σ with respect to the local frame field. Then, we have

$$\sigma^r_{ij} = g(\sigma(e_i, e_j), e_r) = g(A_{e_r} e_i, e_j), \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j)). \tag{2.5}$$

The mean curvature vector $\vec{H}$ is defined by $\vec{H} = \frac{1}{n}\operatorname{trace}\sigma = \frac{1}{n}\sum_{i=1}^n \sigma(e_i, e_i)$, where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of the tangent bundle TM of M. A submanifold M is called totally geodesic, if $\sigma(X, Y) = 0$; totally umbilical if $\sigma(X, Y) = g(X, Y)\vec{H}$ and minimal if $\vec{H} = 0$.

For any $X \in \Gamma(TM)$, we write

$$FX = TX + \omega X, \tag{2.6}$$

where TX is the tangential component of FX and ωX is the normal component of FX. Similarly, for any vector field N normal to M, we put

$$FN = BN + CN, \tag{2.7}$$

where BN and CN are the tangential and normal components of FN, respectively.

The invariant and anti-invariant submanifolds of an almost product Riemannian manifold $\tilde{M}$ depend on the behaviour the tangent spaces under the action of the almost product structure F. A submanifold M is said to be *invariant* (resp. *anti-invariant*) if $F(T_pM) \subseteq T_pM,\ \forall\, p \in M$ (resp. $F(T_pM) \subseteq T_p{}^{\perp}M,\ \forall\, p \in M$).

A submanifold M of an almost product Riemannian manifold $\tilde{M}$ is called slant (see [8,9,20]) if for each non-zero vector $X \in T_pM$, the angle $\theta(X)$ between FX and T_pM is constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_pM$.

First, we give the following non-trivial example of a slant submanifold of an almost product Riemannian manifold.

Example 1. Consider a 4-Euclidean space $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ with the cartesian coordinates (x_1, x_2, y_1, y_2) and the almost product structure

$$F\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial x_i}, \quad F\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial y_j}, \quad 1 \le i, j \le 2.$$

Let M be a submanifold of $\mathbb{R}^4$ defined by immersion

$$\psi(u, v) = (u,\ \frac{1}{\sqrt{3}}(u+v),\ v,\ \frac{1}{\sqrt{3}}(u-v)).$$

If we put

$$Z_1 = \frac{\partial}{\partial x_1} + \frac{1}{\sqrt{3}}\frac{\partial}{\partial x_2} + \frac{1}{\sqrt{3}}\frac{\partial}{\partial y_2}, \quad Z_2 = \frac{1}{\sqrt{3}}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1} - \frac{1}{\sqrt{3}}\frac{\partial}{\partial y_2},$$

then we find

$$FZ_1 = -\frac{\partial}{\partial x_1} - \frac{1}{\sqrt{3}}\frac{\partial}{\partial x_2} + \frac{1}{\sqrt{3}}\frac{\partial}{\partial y_2}, \quad FZ_2 = -\frac{1}{\sqrt{3}}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1} - \frac{1}{\sqrt{3}}\frac{\partial}{\partial y_2}.$$

Thus, we observe that M is a slant submanifold of $\mathbb{R}^4$ with slant angle $\theta = \cos^{-1}\left(\frac{3}{5}\right)$.

As an extension of slant submanifolds, F. Etayo [15] introduced the notion of pointwise slant submanifolds under the name of quasi-slant submanifolds. Later, these submanifolds of almost Hermitian manifolds were studied by B.-Y. Chen and O.J. Garay in [13]. On the similar line of B.-Y. Chen, we introduced pointwise slant and semi-slant submanifolds (for instance, see [2,23]).

A submanifold M of an almost product Riemannian manifold $\tilde{M}$ is said to be pointwise slant submanifold, if for each point $p \in M$, the Wirtinger angle $\theta(X)$ between FX and T_pM is independent of the choice of the non-vanishing vector field $X \in T_pM$. In this case, the Wirtinger angle gives rise to a real-valued function $\theta : TM - \{0\} \to \mathbb{R}$, which is called the slant function of M. Notice that a pointwise slant submanifold M is slant, if its slant function θ is globally constant on M. Moreover, invariant and anti-invariant submanifolds are pointwise slant submanifolds with slant functions $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A pointwise slant submanifold is proper if it is neither invariant nor anti-invariant.

Now, we give the following non-trivial examples of pointwise slant submanifolds of almost product Riemannian manifolds.

Example 2. Let $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ be a Euclidean space with the cartesian coordinates (x_1, x_2, y_1, y_2) and the almost product structure defined in Example 1. Consider a submanifold M of $\mathbb{R}^4$ defined by immersion

$$\psi(u, v) = (\cos(u-v),\ \frac{1}{\sqrt{2}}(u+v),\ \sin(u-v),\ -\frac{1}{\sqrt{2}}(u+v))$$

such that $u, v\,(u \neq v)$ are non-vanishing real valued functions on M. Then the tangent space of M is spanned by the following vector fields

$$Z_1 = -\sin(u-v)\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} + \cos(u-v)\frac{\partial}{\partial y_1} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_2},$$
$$Z_2 = \sin(u-v)\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} - \cos(u-v)\frac{\partial}{\partial y_1} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_2}.$$

Thus, clearly we obtain

$$FZ_1 = \sin(u-v)\frac{\partial}{\partial x_1} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} + \cos(u-v)\frac{\partial}{\partial y_1} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_2},$$

$$FZ_2 = -\sin(u-v)\frac{\partial}{\partial x_1} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} - \cos(u-v)\frac{\partial}{\partial y_1} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_2}.$$

Then, we find that the slant function $\theta = \cos^{-1}\left(\frac{\cos 2(u-v)}{2}\right)$. Since $u, v(u \neq v)$ are non-vanishing real valued functions on M, hence the slant function θ is not a constant. Thus M is a pointwise slant submanifold of $\mathbb{R}^4$.

Example 3. Consider a 6-Euclidean space $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ with the cartesian coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ and the almost product structure

$$F\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial x_i}, \quad F\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial y_j}, \quad 1 \leq i, j \leq 3.$$

If M is a submanifold $\mathbb{R}^6$ defined by the immersion ψ as follows

$$\psi(u, v) = (\cos u,\ \cos v,\ u,\ \sin u,\ \sin v,\ v)$$

for any non-vanishing functions u and v such that $u \neq v$, then, the tangent space of M is spanned by

$$Z_1 = -\sin u\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \cos u\frac{\partial}{\partial y_1}, \quad Z_2 = -\sin v\frac{\partial}{\partial x_2} + \cos v\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}.$$

Hence, we find

$$FZ_1 = \sin u\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3} + \cos u\frac{\partial}{\partial y_1}, \quad FZ_2 = \sin v\frac{\partial}{\partial x_2} + \cos v\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}.$$

Then, we find two slant functions $\theta_1 = \cos^{-1}\left(\frac{1+\cos 2u}{2}\right)$ and $\theta_2 = \cos^{-1}\left(\frac{1+\cos 2v}{2}\right)$. Since $u, v\,(u \neq v)$ are non-vanishing functions on M. Thus, M is a pointwise bi-slant submanifold with slant distributions $\mathfrak{D}_1 = \mathrm{Span}\{Z_1\}$ and $\mathfrak{D}_2 = \mathrm{Span}\{Z_2\}$ with slant functions θ_1 and θ_2, respectively. The idea of pointwise bi-slant submanifolds is introduced by B.-Y. Chen and the second author in [14].

In a similar way of B.-Y. Chen's result (Lemma 2.1) of [13], it was shown in [16,23] that a Riemannian submanifold M of an almost product Riemannian manifold $\tilde{M}$ is pointwise slant if and only if

$$T^2 = (\cos^2\theta)I, \tag{2.8}$$

for some real-valued function θ on M, where I is the identity transformation of the tangent bundle TM of M.

The following relations are straightforward consequences of the above relation

$$g(TX, TY) = \cos^2\theta g(X, Y), \tag{2.9}$$

$$g(\omega X, \omega Y) = \sin^2\theta g(X, Y), \tag{2.10}$$

for any vector fields X, Y tangent to M.

Also, for a pointwise slant submanifold of an almost product Riemannian manifold, we have the following useful relations.

$$(i)\quad B\omega X = (\sin^2\theta)X, \qquad (ii)\quad C\omega X = -\omega TX \tag{2.11}$$

for any $X \in \Gamma(TM)$.

3. Bi-warped product submanifolds of locally product Riemannian manifolds

Let M_1, M_2, M_3 be Riemannian manifolds and let $M = M_1 \times M_2 \times M_3$ be the Cartesian product of M_1, M_2, M_3. For each i, denote by $\pi_i : M \to M_i$ the canonical projection of M onto M_i, $i = 1, 2, 3$. Then, if $f_2, f_3 : M_1 \to \mathbb{R}^+$ are positive real valued functions, then

$$g(X, Y) = g(\pi_{1*}X, \pi_{1*}Y) + (f_2 \circ \pi_1)^2 g(\pi_{2*}X, \pi_{2*}Y) + (f_3 \circ \pi_1)^2 g(\pi_{3*}X, \pi_{3*}Y)$$

defines a Riemannian metric g on M, called a bi-warped product metric, for any X, Y tangent to M and $*$ denotes the symbol for tangent maps. The product manifold M endowed with this metric denoted by $(M_1 \times_{f_2} M_2 \times_{f_3} M_3, g)$ is called a *bi-warped product manifold*. In this case, f_2 , f_3 are non-constant functions, called warping functions on M. It is clear that if both f_2, f_3 are constant on M, then M is simply a Riemannian product manifold and if anyone of the functions is constant, then M is a single warped product manifold. Also, if neither f_2 nor f_3 is constant, then M is a proper bi-warped product manifold.

Remark 1. If $M = B \times_f F$ be a warped product manifold, then, B is totally geodesic in M and F is totally umbilical in M (for instance, see, [6,10]).

Let $M = M_1 \times_{f_2} M_2 \times \times_{f_3} M_3$ be a bi-warped product submanifold of a Riemannian manifold $\tilde{M}$. Then, we have

$$\nabla_X Z = \sum_{i=2}^{3} (X(\ln f_i))\, Z^i, \tag{3.1}$$

for any $X \in \mathfrak{D}_1$, the tangent space of M_1 and $Z \in \Gamma(TN)$, where $N =_{f_2} M_2 \times \times_{f_3} M_3$ and Z^i is M_i-component of Z, for each $i = 2, 3$, and ∇ is the Levi-Civita connection on M (for instance, see [24]).

In a previous paper, we proved that the bi-warped product submanifold $M = M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ of a locally product Riemannian manifold $\tilde{M}$ is a single warped product submanifold, where M_T, $M_\perp$ and M_θ are invariant, anti-invariant and proper slant submanifolds of $\tilde{M}$, respectively [18]. If M_θ is a pointwise slant submanifold, then such warped product exists and we provide some nontrivial examples in the last section.

In this section, we study bi-warped products of the form $M = M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ with pointwise slant factor M_θ. For the simplicity, we denote the tangent bundles of M_T, $M_\perp$ and M_θ by $\mathfrak{D}$, $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$, respectively.

First, we have the following useful results.

Lemma 1. *Let $M = M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ be a bi-warped product submanifold of a locally product Riemannian manifold $\tilde{M}$ such that M_T, $M_\perp$ and M_θ are invariant, anti-invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively. Then, we have*

(i) $g(\sigma(X, Z), FW) = 0$,
(ii) $g(\sigma(X, Y), FZ) = -Z(\ln f_1)g(X, FY)$,
(iii) $g(\sigma(Z, W), \omega V) = -g(\sigma(Z, V), FW)$,
(iv) $g(\sigma(U, V), FZ) + g(\sigma(Z, U), \omega V) = -Z(\ln f_2)g(U, TV)$,

for any $X, Y \in \Gamma(\mathfrak{D})$, $Z, W \in \Gamma(\mathfrak{D}^\perp)$ and $U, V \in \Gamma(\mathfrak{D}^\theta)$.

Proof. For any $X \in \Gamma(\mathfrak{D})$ and $Z, W \in \Gamma(\mathfrak{D}^\perp)$, we have

$$g(\sigma(X, Z), FW) = g(\tilde{\nabla}_Z X, FW) = g(\tilde{\nabla}_Z FX, W).$$

Using (3.1), we get

$$g(\sigma(X, Z), FW) = Z(\ln f_1)g(FX, W) = 0,$$

which is the first part of the lemma. Similarly, we have

$$g(\sigma(X, Y), FZ) = g(\tilde{\nabla}_X Y, FZ) = g(\tilde{\nabla}_X FY, Z) = -g(FY, \tilde{\nabla}_X Z).$$

Again, using (3.1), we derive

$$g(\sigma(X, Y), FZ) = -Z(\ln f_1)g(X, FY),$$

which is second part of the lemma. Now, for any $V \in \Gamma(\mathfrak{D}^\theta)$, we have

$$\begin{aligned} g(\sigma(Z, W), \omega V) &= g(\tilde{\nabla}_Z W - \nabla_Z W, FV - TV) \\ &= g(\tilde{\nabla}_Z FW, V) - g(\tilde{\nabla}_Z W, TV) - g(F\nabla_Z W, V) + g(\nabla_Z W, TV). \end{aligned}$$

Since $\nabla_Z W \in \Gamma(\mathfrak{D}^\perp)$, for any $Z, W \in \Gamma(\mathfrak{D}^\perp)$ (see, Remark 1), then the last two terms in the right hand side of the above equation are identically zero. Thus, by using (2.3), we derive

$$g(\sigma(Z, W), \omega V) = -g(A_{FW} Z, V) + g(\tilde{\nabla}_Z TV, W).$$

Then from (2.4) and (3.1), we obtain

$$g(\sigma(Z, W), \omega V) = -g(\sigma(Z, V), FW) + Z(\ln f_2)g(TV, W).$$

By orthogonality of vector fields, the second term in the right hand side of above relation is identically zero and hence we find the third relation. Now, we have

$$\begin{aligned} g(\sigma(Z, U), \omega V) &= g(\tilde{\nabla}_U Z - \nabla_U Z, FV - TV) \\ &= g(\tilde{\nabla}_U FZ, V) - g(\tilde{\nabla}_U Z, TV) - g(\nabla_U Z, FV) + g(\nabla_U Z, TV), \end{aligned}$$

for any $Z \in \Gamma(\mathfrak{D}^\perp)$ and $U, V \in \Gamma(\mathfrak{D}^\theta)$. Using (2.2), (2.3), (2.4) and (3.1), we derive

$$g(\sigma(Z, U), \omega V) = -g(\sigma(U, V), FZ) - Z(\ln f_2)g(U, TV),$$

which is the fourth part of the lemma. Hence, the proof is complete. ■

Theorem 2. *Let $M = M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ be a bi-warped product submanifold of a locally product Riemannian manifold $\tilde{M}$. Then, either M is a $\mathfrak{D}^\perp - \mathfrak{D}^\theta$ mixed totally geodesic bi-warped product in $\tilde{M}$ or $\omega\mathfrak{D}^\theta$ has no component of $\sigma(\mathfrak{D}^\perp, \mathfrak{D}^\theta)$ or both statements are true.*

Proof. For any $Z \in \Gamma(\mathfrak{D}^{\perp})$ and $U, V \in \Gamma(\mathfrak{D}^{\theta})$, we have

$$g(\sigma(Z,U),\omega V) = g(\tilde{\nabla}_Z U - \nabla_Z U, FV - TV) = g(\tilde{\nabla}_Z FU, V) - g(\tilde{\nabla}_Z U, TV).$$

Using (2.6), (2.2) and (3.1), we obtain

$$g(\sigma(Z,U),\omega V) = g(\tilde{\nabla}_Z TU, V) + g(\tilde{\nabla}_Z \omega U, V) - Z(\ln f_2) g(U, TV).$$

Again, using (2.2), (2.1) and (3.1), we derive

$$\begin{aligned} g(\sigma(Z,U),\omega V) &= Z(\ln f_2) g(U,TV) + g(F\tilde{\nabla}_Z \omega U, FV) - Z(\ln f_2) g(U,TV) \\ &= g(\tilde{\nabla}_Z B\omega U, FV) + g(\tilde{\nabla}_Z C\omega U, FV). \end{aligned}$$

Then, from (2.11), we find

$$g(\sigma(Z,U),\omega V) = \sin^2\theta\, g(\tilde{\nabla}_Z U, FV) + \sin 2\theta\, Z(\theta) g(U, FV) - g(\tilde{\nabla}_Z \omega TU, FV).$$

Using (2.6), (2.1) and (3.1), we get

$$\begin{aligned} g(\sigma(Z,U),\omega V) =& Z(\ln f_2)\sin^2\theta\, g(U,TV) + \sin^2\theta\, g(\sigma(Z,U),\omega V) \\ &+ \sin 2\theta\, Z(\theta) g(U,TV) - g(\tilde{\nabla}_Z F\omega TU, V). \end{aligned}$$

From (2.7), we deduce that

$$\begin{aligned} \cos^2\theta\, g(\sigma(Z,U),\omega V) =& Z(\ln f_2)\sin^2\theta\, g(U,TV) + \sin 2\theta\, Z(\theta) g(U,TV) \\ &- g(\tilde{\nabla}_Z B\omega TU, V) - g(\tilde{\nabla}_Z C\omega TU, V). \end{aligned}$$

Again, using (2.11) and (3.1), we derive

$$\begin{aligned} \cos^2\theta\, g(\sigma(Z,U),\omega V) =& Z(\ln f_2)\sin^2\theta\, g(U,TV) + \sin 2\theta\, Z(\theta) g(U,TV) - Z(\ln f_2)\sin^2\theta\, g(TU,V) \\ &- \sin 2\theta\, Z(\theta) g(TU,V) + g(\tilde{\nabla}_Z \omega T^2 U, V). \end{aligned}$$

Then, using (2.8), we find that

$$\cos^2\theta\, g(\sigma(Z,U),\omega V) = \cos^2\theta\, g(\tilde{\nabla}_Z \omega U, V) - \sin 2\theta\, Z(\theta) g(\omega U, V).$$

By orthogonality of vector fields and using (2.3), we obtain

$$g(\sigma(Z,U),\omega V) = -g(\sigma(Z,V),\omega U). \tag{3.2}$$

On the other hand, from Lemma 1(iv), we have

$$g(\sigma(U,V),FZ) + g(\sigma(Z,U),\omega V) = -Z(\ln f_2) g(U,TV). \tag{3.3}$$

By polarization identity, we derive

$$g(\sigma(U,V),FZ) + g(\sigma(Z,V),\omega U) = -Z(\ln f_2) g(TU,V). \tag{3.4}$$

From (3.3) and (3.4), we get

$$g(\sigma(Z,V),\omega U) = g(\sigma(Z,U),\omega V). \tag{3.5}$$

Then, from (3.2) and (3.5), we find that $g(\sigma(Z,V),\omega U) = 0$, which means that either M is $\mathfrak{D}^{\perp} - \mathfrak{D}^{\theta}$ mixed totally geodesic or $\sigma(\mathfrak{D}^{\perp}, \mathfrak{D}^{\theta}) \perp \omega\mathfrak{D}^{\theta}$. Hence, the theorem is proved completely. ∎

From Lemma 1, we have the following useful results.

Corollary 1. *A bi-warped product $M = M_{\perp} \times_{f_1} M_T \times_{f_2} M_{\theta}$ of a locally product Riemannian manifold $\tilde{M}$ is a single warped product submanifold of the form $M = M_{\perp} \times M_T \times_{f_2} M_{\theta}$, i.e., f_1 is constant on M if and only if $\sigma(\mathfrak{D}, \mathfrak{D}) \perp F\mathfrak{D}^{\perp}$.*

Proof. The proof follows from Lemma 1(ii). ∎

Corollary 2. *Let $M = M_{\perp} \times_{f_1} M_T \times_{f_2} M_{\theta}$ be a $\mathfrak{D}^{\perp} - \mathfrak{D}^{\theta}$ mixed totally geodesic bi-warped product submanifold of a locally product Riemannian manifold $\tilde{M}$. Then, M is a single warped product submanifold of the form $M = M_{\perp} \times_{f_1} M_T \times M_{\theta}$, i.e., f_2 is constant on M if and only if $\sigma(\mathfrak{D}^{\theta}, \mathfrak{D}^{\theta}) \perp F\mathfrak{D}^{\perp}$.*

Proof. The proof follows from Lemma 1(iv) by using mixed totally geodesic condition. ∎

4. Proof of Theorem 1

In this section, we prove Theorem 1. In order to prove our main theorem, first we state the following result.

Lemma 2. *Let $M = M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ be a bi-warped product submanifold of a locally product Riemannian manifold $\tilde{M}$. Then, we have*

(i) $g(\sigma(X, Z), \omega V) = 0$,
(ii) $g(\sigma(X, V), FZ) = 0$

for any $X \in \Gamma(\mathfrak{D})$, $Z \in \Gamma(\mathfrak{D}^\perp)$ and $V \in \Gamma(\mathfrak{D}^\theta)$.

Proof. For any $X \in \Gamma(\mathfrak{D})$, $Z \in \Gamma(\mathfrak{D}^\perp)$ and $V \in \Gamma(\mathfrak{D}^\theta)$, we have

$$\begin{aligned} g(\sigma(X, Z), \omega V) &= g(\tilde{\nabla}_Z X - \nabla_Z X, FV - TV) \\ &= g(\tilde{\nabla}_Z FX, V) - g(F\nabla_Z X, V). \end{aligned}$$

Using (3.1) and orthogonality of vector fields, the right hand side of above relation is identically zero and hence the first part of the lemma is proved. For the second relation, we find that

$$g(\sigma(X, V), FZ) = g(\tilde{\nabla}_V X, FZ) = -g(\tilde{\nabla}_U Z, FX),$$

for any $X \in \Gamma(\mathfrak{D})$, $Z \in \Gamma(\mathfrak{D}^\perp)$ and $V \in \Gamma(\mathfrak{D}^\theta)$. Using (3.1), we get

$$g(\sigma(X, V), FZ) = -Z(\ln f_2) g(FX, V) = 0,$$

which is the second relation. This ends the proof. ■

Now, we construct the following frame fields for a bi-warped product submanifold. Let $M = M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ be an n-dimensional bi-warped product submanifold of an m-dimensional locally product Riemannian manifold $\tilde{M}$. Then the tangent and normal bundles of M respectively are decomposed by

$$TM = \mathfrak{D} \oplus \mathfrak{D}^\perp \oplus \mathfrak{D}^\theta, \quad T^\perp M = F\mathfrak{D}^\perp \oplus \omega\mathfrak{D}^\theta \oplus \mu \tag{4.1}$$

where μ is the F-invariant normal subbundle of $T^\perp M$. Let us consider the dimensions of $\dim M_\perp = n_1$, $\dim M_T = n_2$ and $\dim M_\theta = n_3$ and their corresponding tangent spaces are denoted by $\mathfrak{D}^\perp$, $\mathfrak{D}$ and $\mathfrak{D}^\theta$, respectively. We set the orthonormal frame fields of $\mathfrak{D}^\perp$ as follows

$$\{e_1, e_2, \ldots, e_{n_1}\}$$

and the orthonormal frame fields of $\mathfrak{D}$ and $\mathfrak{D}^\theta$, respectively are

$$\begin{aligned} &\{e_{n_1+1} = \hat{e}_1 = F\hat{e}_1, \ldots, e_{n_1+k} = \hat{e}_k = F\hat{e}_k, e_{n_1+k+1} = \hat{e}_{k+1} = -F\hat{e}_{k+1} \\ &\quad \cdots, e_{n_1+n_2} = \hat{e}_{n_2} = -F\hat{e}_{n_2}\}, \\ &\{e_{n_1+n_2+1} = e_1^* = \sec\theta T e_1^*, \ldots, e_n = e_{n_3}^* = \sec\theta T e_{n_3}^*\}. \end{aligned}$$

Then the orthonormal frames of the normal subbundles $F\mathfrak{D}^\perp$, $\omega\mathfrak{D}^\theta$ and μ, respectively are

$$\begin{aligned} &\{e_{n+1} = \tilde{e}_1 = Fe_1, \ldots, e_{n+n_1} = \tilde{e}_{n_1} = Fe_{n_1}\}, \\ &\{e_{n+n_1+1} = \tilde{e}_{n_1+1} = \csc\theta\,\omega e_1^*, \ldots, e_{n+n_1+n_3} = \tilde{e}_{n_1+n_3} = \csc\theta\,\omega e_{n_3}^*\}; \\ &\{e_{n+n_1+n_3+1} = \tilde{e}_{n_1+n_3+1}, \ldots, e_m = \tilde{e}_{m-n-n_1-n_3}\}. \end{aligned}$$

Clearly $\dim \mu = m - n - n_1 - n_3$.

Proof of Theorem 1. Now, we are able to prove the main theorem of this paper.

Proof. From (2.5), we have

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j)) = \sum_{r=n+1}^{m} \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), e_r)^2.$$

Then, with the help of (4.1), we derive

$$\|\sigma\|^2 = \sum_{r=1}^{n_1} \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), Fe_r)^2 + \sum_{r=1}^{n_3} \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \csc\theta\,\omega e_r^*)^2 + \sum_{r=n_1+n_3+1}^{m-n-n_1-n_3} \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \tilde{e}_r)^2. \tag{4.2}$$

Leaving the positive third μ-components term on the right hand side of (4.2) and using the constructed frame fields, we find

$$
\begin{aligned}
\|\sigma\|^2 \geq & \sum_{r=1}^{n_1}\sum_{i,j=1}^{n_1} g(\sigma(e_i,e_j),Fe_r)^2 + \sum_{r=1}^{n_1}\sum_{i,j=1}^{n_2} g(\sigma(\hat{e}_i,\hat{e}_j),Fe_r)^2 + \sum_{r=1}^{n_1}\sum_{i,j=1}^{n_3} g(\sigma(e_i^*,e_j^*),Fe_r)^2 \\
& + 2\sum_{r=1}^{n_1}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} g(\sigma(e_i,\hat{e}_j),Fe_r)^2 + 2\sum_{r=1}^{n_1}\sum_{i=1}^{n_1}\sum_{j=1}^{n_3} g(\sigma(e_i,e_j^*),Fe_r)^2 + 2\sum_{r=1}^{n_1}\sum_{i=1}^{n_2}\sum_{j=1}^{n_3} g(\sigma(\hat{e}_i,e_j^*),Fe_r)^2 \\
& + \csc^2\theta\sum_{r=1}^{n_3}\sum_{i,j=1}^{n_1} g(\sigma(e_i,e_j),\omega e_r^*)^2 + \csc^2\theta\sum_{r=1}^{n_3}\sum_{i,j=1}^{n_2} g(\sigma(\hat{e}_i,\hat{e}_j),\omega e_r^*)^2 + \csc^2\theta\sum_{r=1}^{n_3}\sum_{i,j=1}^{n_3} g(\sigma(e_i^*,e_j^*),\omega e_r^*)^2 \\
& + 2\csc^2\theta\sum_{r=1}^{n_3}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} g(\sigma(e_i,\hat{e}_j),\omega e_r^*)^2 + 2\csc^2\theta\sum_{r=1}^{n_3}\sum_{i=1}^{n_1}\sum_{j=1}^{n_3} g(\sigma(e_i,e_j^*),\omega e_r^*)^2 \\
& + 2\csc^2\theta\sum_{r=1}^{n_3}\sum_{i=1}^{n_2}\sum_{j=1}^{n_3} g(\sigma(\hat{e}_i,e_j^*),\omega e_r^*)^2.
\end{aligned}
\tag{4.3}
$$

We have no relation for the first, eighth, ninth and last twelfth terms on the right hand side of the above equation. Therefore, we can leave these positive terms. On the other hand, by using Lemmas 1 and 2 with $\mathfrak{D}^\perp - \mathfrak{D}^\theta$ mixed totally geodesic condition, all the terms of above relations vanish identically except the second and third terms. Then, using Lemma 1(ii)–(iv) in the second and third terms, we obtain

$$
\|\sigma\|^2 \geq \sum_{r=1}^{n_1}\sum_{i,j=1}^{n_2}(-e_r\ln f_1)^2 g(\hat{e}_i,F\hat{e}_j)^2 + \sum_{r=1}^{n_1}\sum_{i,j=1}^{n_3}(-e_r\ln f_2)^2 g(e_i^*,Te_j^*)^2.
$$

Since $F\hat{e}_j=\hat{e}_j$, $\forall j=1,\dots,k$ or $F\hat{e}_j=-\hat{e}_j$, $\forall j=k+1,\dots,n_2$ and $e_j^*=\sec\theta Te_j^*$, $\forall j=1,\dots,n_3$, which means that $Te_j^*=\cos\theta e_j^*$. Thus, we get

$$
\begin{aligned}
\|\sigma\|^2 &\geq n_2\sum_{r=1}^{n_1}(e_r\ln f_1)^2 + n_3\cos^2\theta\sum_{r=1}^{n_1}(e_r\ln f_2)^2 \\
&= n_2\|\vec{\nabla}^T(\ln f_1)\|^2 + n_3\cos^2\theta\|\vec{\nabla}^\theta(\ln f_2)\|^2,
\end{aligned}
$$

which is the inequality (i) of Theorem 1. For the equality case, we have from the leaving third term in (4.2)

$$
\sigma(TM,TM)\perp\mu \tag{4.4}
$$

From the leaving first term and vanishing seventh term in (4.3), we find

$$
\sigma(\mathfrak{D}^\perp,\mathfrak{D}^\perp)\perp F\mathfrak{D}^\perp \ \text{ and } \sigma(\mathfrak{D}^\perp,\mathfrak{D}^\perp)\perp\omega\mathfrak{D}^\theta. \tag{4.5}
$$

Then from (4.4) and (4.5), we obtain

$$
\sigma(\mathfrak{D}^\perp,\mathfrak{D}^\perp)=0. \tag{4.6}
$$

Since M is $\mathfrak{D}^\perp - \mathfrak{D}^\theta$ mixed totally geodesic, then we have

$$
\sigma(\mathfrak{D}^\perp,\mathfrak{D}^\theta)=0. \tag{4.7}
$$

Also, from the vanishing fourth and tenth terms on the right hand side of (4.3), we get

$$
\sigma(\mathfrak{D},\mathfrak{D}^\perp)\perp F\mathfrak{D}^\perp \ \text{ and } \sigma(\mathfrak{D},\mathfrak{D}^\perp)\perp\omega\mathfrak{D}^\theta. \tag{4.8}
$$

Then from (4.4) and (4.8), we conclude that

$$
\sigma(\mathfrak{D},\mathfrak{D}^\perp)=0. \tag{4.9}
$$

On the other hand, from the leaving eighth term in (4.3) and (4.4), we find that

$$
\sigma(\mathfrak{D},\mathfrak{D})\subset F\mathfrak{D}^\perp. \tag{4.10}
$$

Similarly, from the leaving ninth term in (4.3) and (4.4), we obtain

$$
\sigma(\mathfrak{D}^\theta,\mathfrak{D}^\theta)\subset F\mathfrak{D}^\perp. \tag{4.11}
$$

And, from the vanishing sixth term and leaving twelfth term in (4.3) , we get

$$
\sigma(\mathfrak{D},\mathfrak{D}^\theta)\perp F\mathfrak{D}^\perp \ \text{ and } \sigma(\mathfrak{D},\mathfrak{D}^\theta)\perp\omega\mathfrak{D}^\theta. \tag{4.12}
$$

Then from (4.4) and (4.12), we find

$$\sigma(\mathfrak{D},\mathfrak{D}^{\theta})=0. \tag{4.13}$$

Thus, $M_{\perp}$ is totally geodesic in $\tilde{M}$ by using Remark 1 and (4.6), (4.7) and (4.9). Again, from (4.10), (4.9) and Remark 1, we conclude that M_T and M_{θ} are totally umbilical in $\tilde{M}$. Using all conditions (4.6)–(4.13), M is a $\mathfrak{D}^{\perp}$-geodesic submanifold of $\tilde{M}$. ■

5. Applications of Theorem 1

We have the following applications of Theorem 1.

In Theorem 1, if $n_3 = 0$, then M is a single warped product of the form $M = M_{\perp} \times_{f_1} M_T$, which has been studied in [3,4,21]. In this case, Theorem 1 implies:

Corollary 3 (*Theorem 4.1 of [4] and Theorem 4.2 of [21]*). *Let $M = M_{\perp} \times_{f_1} M_T$ be a warped product semi-invariant submanifold in a locally product Riemannian manifold $\tilde{M}$, where M_T and $M_{\perp}$ are invariant and anti-invariant submanifolds of $\tilde{M}$, respectively. Then, we have*

(i) The second fundamental form σ of M satisfies

$$\|\sigma\|^2 \geq n_2\|\vec{\nabla}^T(\ln f_1)\|^2 \tag{5.1}$$

where $n_2 = \dim M_T$ and $\vec{\nabla}^T(\ln f_1)$ is the gradient component of $\ln f_1$ along M_T.

(ii) If the equality sign holds identically in (i), then $M_{\perp}$ is a totally geodesic submanifold of $\tilde{M}$ and M_T is totally umbilical in $\tilde{M}$. Moreover, M is mixed totally geodesic submanifold of $\tilde{M}$.

On the other hand, if $n_2 = 0$, then warped product takes form $M = M_{\perp} \times_{f_2} M_{\theta}$, studied in [2]. In this case, Theorem 1 gives:

Corollary 4 (*Theorem 5.1 of [2]*). *Let $M = M_{\perp} \times_{f_2} M_{\theta}$ be a $\mathfrak{D}^{\perp} - \mathfrak{D}^{\theta}$ mixed totally geodesic warped product pointwise pseudo-slant submanifold in a locally product Riemannian manifold $\tilde{M}$, where $M_{\perp}$ and M_{θ} are anti-invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively. Then, we have*

(i) The second fundamental form σ of M satisfies

$$\|\sigma\|^2 \geq n_3\cos^2\theta\|\vec{\nabla}^{\theta}(\ln f_2)\|^2 \tag{5.2}$$

where $n_3 = \dim M_{\theta}$ and $\vec{\nabla}^{\theta}(\ln f_2)$ is the gradient component of $\ln f_2$ along M_{θ}.

(ii) If the equality sign holds identically in (i), then $M_{\perp}$ is a totally geodesic submanifold of $\tilde{M}$ and M_{θ} is totally umbilical in $\tilde{M}$. Moreover, M is a mixed totally geodesic submanifold of $\tilde{M}$.

Another application of Theorem 1 is to describe the Dirichlet energy of the warping functions f_1 and f_2, which is a useful tool in physics. The Dirichlet energy of a function f on a compact manifold M is defined as

$$E(f)=\frac{1}{2}\int_M \|\vec{\nabla}(f)\|^2\, dV \tag{5.3}$$

where $\vec{\nabla}(f)$ is the gradient of the function f and dV is the volume element.

Theorem 1 and (5.3) imply the following.

Theorem 3. *Let $M = M_{\perp} \times_{f_1} M_T \times_{f_2} M_{\theta}$ be a compact $\mathfrak{D}^{\perp} - \mathfrak{D}^{\theta}$ mixed totally geodesic bi-warped product submanifold in a locally product Riemannian $\tilde{M}$. Then*

$$n_2E(\ln f_1)+n_3\cos^2\theta E(\ln f_2) \leq \frac{1}{2}\int_M \|\sigma\|^2 dV$$

where dV is the volume element and $n_2 = \dim M_T$, $n_3 = \dim M_{\theta}$; while $E(\ln f_i)$ is gradient of $\ln f_i$, $i = 1, 2$.

Theorem 5 and Corollaries 3–4 imply:

Theorem 4. *Let $M = M_{\perp} \times_f M_T$ be a compact semi-invariant warped product submanifold in a locally product Riemannian $\tilde{M}$. Then*

$$E(\ln f) \leq \frac{1}{2n_2}\int_M \|\sigma\|^2 dV$$

where dV is the volume element and $n_2 = \dim M_T$; while $E(\ln f)$ is gradient of $\ln f$.

Theorem 5. *Let $M = M_\perp \times_f M_\theta$ be a compact $\mathfrak{D}^\perp - \mathfrak{D}^\theta$ mixed totally geodesic warped product pseudo-slant (hemi-slant) submanifold of a locally product Riemannian $\tilde{M}$. Then*

$$E(\ln f) \leq \frac{1}{2n_3} \sec^2\theta \int_M \|\sigma\|^2 dV$$

where dV is the volume element and $n_3 = \dim M_\theta$; while $E(\ln f)$ is gradient of $\ln f$.

6. Examples of bi-warped products

In this section, we construct the following non-trivial examples of bi-warped product submanifolds of the form $M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ in Euclidean spaces.

Example 4. Consider a submanifold of $\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^3 \times \mathbb{R}$ with the cartesian coordinates $(x_1, x_2, x_3, y_4 y_1, y_2, y_3, z)$ and the almost product structure

$$F\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial x_i}, \quad F\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial y_j}, \quad F\left(\frac{\partial}{\partial z}\right) = -\frac{\partial}{\partial z}, \quad 1 \leq i \leq 4,\ 1 \leq j \leq 3.$$

Let M be defined by the immersion ψ as follows

$$\psi(u, v, t) = (\cos u,\ \sin u,\ u\sin t,\ u\cos t,\ u\sin v,\ u\cos v,\ u,\ v),$$

for any non-vanishing function u on M. Then, the tangent space TM of M is spanned by the following vectors

$$Z_1 = -\sin u\frac{\partial}{\partial x_1} + \cos u\frac{\partial}{\partial x_2} + \sin t\frac{\partial}{\partial x_3} + \cos t\frac{\partial}{\partial x_4} + \sin v\frac{\partial}{\partial y_1} + \cos v\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3},$$
$$Z_2 = u\cos v\frac{\partial}{\partial y_1} - u\sin v\frac{\partial}{\partial y_2} + \frac{\partial}{\partial z}, \quad Z_3 = u\cos t\frac{\partial}{\partial x_3} - u\sin t\frac{\partial}{\partial x_4}.$$

Then, we find

$$FZ_1 = \sin u\frac{\partial}{\partial x_1} - \cos u\frac{\partial}{\partial x_2} - \sin t\frac{\partial}{\partial x_3} - \cos t\frac{\partial}{\partial x_4} + \sin v\frac{\partial}{\partial y_1} + \cos v\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3},$$
$$FZ_2 = u\cos v\frac{\partial}{\partial y_1} - u\sin v\frac{\partial}{\partial y_2} - \frac{\partial}{\partial z}, \quad FZ_3 = -u\cos t\frac{\partial}{\partial x_3} + u\sin t\frac{\partial}{\partial x_4}.$$

It is easy to see that $FZ_1 \perp TM = \mathrm{Span}\{Z_1, Z_2, Z_3\}$ and thus we consider $\mathfrak{D}^\perp = \mathrm{Span}\{Z_1\}$ is an anti-invariant distribution, $\mathfrak{D} = \mathrm{Span}\{Z_3\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{Z_2\}$ is a pointwise slant distribution with slant function $\theta = \arccos\left(\frac{u^2-1}{u^2+1}\right)$. It is easy to observe that $\mathfrak{D}, \mathfrak{D}^\theta$ and $\mathfrak{D}^\perp$ are integrable (each distribution is spanned by a single vector field). If we denote the integral manifolds of $\mathfrak{D}, \mathfrak{D}^\theta$ and $\mathfrak{D}^\perp$ by M_T, M_θ and $M_\perp$, respectively, then the metric tensor of M is given by

$$ds^2 = 4du^2 + (1+u^2)dv^2 + u^2dt^2.$$

Thus M is a bi-warped product submanifold of the form $M = M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ in $\mathbb{R}^8$ with the warping functions $f_1 = u$ and $f_2 = \sqrt{1+u^2}$.

Example 5. Let $\mathbb{R}^{17}$ be the 17-Euclidean space endowed with the cartesian coordinates $(x_1, \ldots, x_8, y_1, \ldots, y_8, z)$ and the usual Euclidean metric $\langle .\,, .\rangle$. We define the almost product structure $F : \mathbb{R}^{17} \to \mathbb{R}^8 \times \mathbb{R}^8 \times \mathbb{R}$ by:

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i},\ F\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial y_k},\ F\left(\frac{\partial}{\partial z}\right) = -\frac{\partial}{\partial z},\ 1 \leq i, k \leq 8$$

which verifies $F^2 = I (F \neq \pm I)$ and $\langle X, FY\rangle = \langle FX, Y\rangle$, for any $X, Y \in \mathbb{R}^{17}$. Let $\psi : M \to \mathbb{R}^{17}$ be an immersion defined by

$$\begin{aligned}\psi(u, v, w, r, t) =&(u\cos\theta,\ u\sin\theta,\ v\cos\theta,\ v\sin\theta,\ u\cos w,\ u\sin w,\ v\cos w,\ v\sin w,\\ &u\cos r,\ u\sin r,\ v\cos r,\ v\sin r,\ u\cos t,\ u\sin t,\ v\cos t,\ v\sin t,\ kw)\end{aligned}$$

for $k \neq 0$ and the non-vanishing functions u and v, where $M = \{(u, v, w, r, t) \mid v, u \neq 0;\ w, r, t \in \mathbb{R}\}$. We can find the local orthonormal frame on TM as follows:

$$Z_1 = \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2} + \cos w\frac{\partial}{\partial x_5} + \sin w\frac{\partial}{\partial x_6} + \cos r\frac{\partial}{\partial y_1} + \sin r\frac{\partial}{\partial y_2} + \cos t\frac{\partial}{\partial y_5} + \sin t\frac{\partial}{\partial y_6},$$
$$Z_2 = \cos\theta\frac{\partial}{\partial x_3} + \sin\theta\frac{\partial}{\partial x_4} + \cos w\frac{\partial}{\partial x_7} + \sin w\frac{\partial}{\partial x_8} + \cos r\frac{\partial}{\partial y_3} + \sin r\frac{\partial}{\partial y_4} + \cos t\frac{\partial}{\partial y_7} + \sin t\frac{\partial}{\partial y_8},$$

$$Z_3 = -u\sin w\frac{\partial}{\partial x_5} + u\cos w\frac{\partial}{\partial x_6} - v\sin w\frac{\partial}{\partial x_7} + v\cos w\frac{\partial}{\partial x_8} + k\frac{\partial}{\partial z},$$
$$Z_4 = -u\sin r\frac{\partial}{\partial y_1} + u\cos r\frac{\partial}{\partial y_2} - v\sin r\frac{\partial}{\partial y_3} + v\cos r\frac{\partial}{\partial y_4},$$
$$Z_5 = -u\sin t\frac{\partial}{\partial y_5} + u\cos t\frac{\partial}{\partial y_6} - v\sin t\frac{\partial}{\partial y_7} + v\cos t\frac{\partial}{\partial y_8}.$$

Clearly, we obtain

$$FZ_1 = \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2} + \cos w\frac{\partial}{\partial x_5} + \sin w\frac{\partial}{\partial x_6} - \cos r\frac{\partial}{\partial y_1} - \sin r\frac{\partial}{\partial y_2} - \cos t\frac{\partial}{\partial y_5} - \sin t\frac{\partial}{\partial y_6},$$
$$FZ_2 = \cos\theta\frac{\partial}{\partial x_3} + \sin\theta\frac{\partial}{\partial x_4} + \cos w\frac{\partial}{\partial x_7} + \sin w\frac{\partial}{\partial x_8} - \cos r\frac{\partial}{\partial y_3} - \sin r\frac{\partial}{\partial y_4} - \cos t\frac{\partial}{\partial y_7} - \sin t\frac{\partial}{\partial y_8},$$
$$FZ_3 = -u\sin w\frac{\partial}{\partial x_5} + u\cos w\frac{\partial}{\partial x_6} - v\sin w\frac{\partial}{\partial x_7} + v\cos w\frac{\partial}{\partial x_8} - k\frac{\partial}{\partial z},$$
$$FZ_4 = u\sin r\frac{\partial}{\partial y_1} - u\cos r\frac{\partial}{\partial y_2} + v\sin r\frac{\partial}{\partial y_3} - v\cos r\frac{\partial}{\partial y_4},$$
$$FZ_5 = u\sin t\frac{\partial}{\partial y_5} - u\cos t\frac{\partial}{\partial y_6} + v\sin t\frac{\partial}{\partial y_7} - v\cos t\frac{\partial}{\partial y_8}.$$

We note that FZ_1 and FZ_2 are perpendicular to TM. Then $\mathfrak{D}^\perp = \mathrm{Span}\{Z_1, Z_2\}$ is an anti-invariant distribution, $\mathfrak{D} = \mathrm{Span}\{Z_4, Z_5\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{Z_3\}$ is a pointwise slant distribution with slant function $\theta = \arccos\left(\frac{u^2-v^2-k^2}{u^2+v^2+k^2}\right)$. All the distributions $\mathfrak{D}$, $\mathfrak{D}^\theta$ and $\mathfrak{D}^\perp$ are completely integrable. Let M_T, M_θ and $M_\perp$ be the integral manifolds of $\mathfrak{D}$, $\mathfrak{D}^\theta$ and $\mathfrak{D}^\perp$, respectively. Then the induced Riemannian metric tensor of M is given by

$$ds^2 = 4(du^2 + dv^2) + (k^2 + u^2 + v^2)dw^2 + (u^2 + v^2)(dr^2 + dt^2).$$

Hence, $M = M_\perp \times_{f_1} M_T \times_{f_2} M_\theta$ is a bi-warped product submanifold of $\mathbb{R}^{17}$ with the warping functions $f_1 = \sqrt{u^2 + v^2}$ and $f_2 = \sqrt{k^2 + u^2 + v^2}$.

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