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A groupoid proof of the Lefschetz fixed point formula

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ABSTRACT

The purpose of this article is to present a "Groupoid proof" to the Lefschetz fixed point formula for elliptic complexes whose differentials are of order one. We shall define a "relative version" of tangent groupoid which resemble the appropriate localization phenomena, describe the corresponding groupoid algebra and explain the relation with the Lefschetz fixed point formula.

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1. Introduction

The notion of tangent groupoid

$$\mathbb{T}V = TV \times \{0\} \sqcup V \times V \times (0, 1]$$

is invented by Alain Connes [1] initially to simplify the proof of the Atiyah–Singer index theorem(see also [2]). It can be viewed as a smoothly varying family of manifolds over the interval $[0, 1]$ whose fiber over nonzero number is $V \times V$ and whose fiber over zero is the tangent bundle TV. Roughly speaking, the smooth structure of the tangent groupoid encodes a localization phenomena, that is, as $V \times V$ approaches the tangent bundle TV, geometric information is concentrating more and more on the infinitesimal neighborhood of the diagonal V inside $V \times V$. This is reminiscent to the concentration phenomena that the support of heat kernel $e^{-t\Delta}$ is concentrating on the diagonal as $t \to 0$. In fact, this analogy can be made precise by taking the family of heat kernel $\left\{e^{-t\Delta}\right\}_{t\in(0,1]}$ as a family of smoothly varying pseudodifferential operators acting on source fibers of $\mathbb{T}V$. The crucial point is that there is a natural completion at $t = 0$. Namely, we can add a pseudodifferential operator to the position $t = 0$, so that the new family is smoothly varying with respect to $t \in [0, 1]$.

On the other hand, the local index theorem and the heat kernel is connected by the famous McKean–Singer formula. Along this line, in [3], by also deforming the spinor bundle, the authors construct a "rescaled vector bundle" over the tangent groupoid of an even dimensional spin manifold. The rescaled vector bundle has the feature that its space of smooth sections has an algebra structure and supports a smooth family of supertraces $\{\mathrm{Str}_t\}_{t\in[0,1]}$. The existence of such a smooth family of supertraces can be thought as a version of the McKean–Singer formula and the process of sending the parameter t to 0 is equivalent to the Getzler rescaling.

As oppose to the Getzler rescaling in the vector bundle direction, the Lefschetz fixed point formula with zero dimensional fixed point set requires no rescaling in the vector bundle direction and more rescaling in the manifold direction. In this paper, instead of building a rescaled bundle, we shall modify the tangent groupoid construction to encode the appropriate "Lefschetz type rescaling".

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To fix the notations, let us quickly go through the basic setting of the Lefschetz fixed point formula [4]. Let V be a compact manifold, $E_0, E_1, \dots, E_N$ be a sequence of vector bundles over V. Let

$$0 \to C^\infty(V, E_0) \xrightarrow{d_0} C^\infty(V, E_1) \xrightarrow{d_1} \cdots \xrightarrow{d_{N-1}} C^\infty(V, E_N) \to 0 \tag{1.1}$$

be a sequence of differential operators such that $d_{i+1}d_i = 0$. By standard terminology, such a sequence is called a *complex*. An *Elliptic complex* over V is a complex such that its sequence of principal symbols

$$0 \to C^\infty(T^*V, \pi^*E_0) \xrightarrow{\sigma(d_0)} C^\infty(T^*V, \pi^*E_1) \xrightarrow{\sigma(d_1)} \cdots \xrightarrow{\sigma(d_{N-1})} C^\infty(T^*V, \pi^*E_N) \to 0$$

is exact outside the zero section of T^*V. Let

$$E = \bigoplus_i E_i \tag{1.2}$$

be the direct sum vector bundle and

$$d = \bigoplus_i d_i : C^\infty(V, E) \to C^\infty(V, E) \tag{1.3}$$

be the corresponding direct sum of differentials. It is convenient, for our purpose, to consider the $\mathbb{Z}/2\mathbb{Z}$-grading on $E = E^{\mathrm{ev}} \oplus E^{\mathrm{odd}}$ where $E^{\mathrm{ev}} = \bigoplus_i E_{2i}$ and $E^{\mathrm{odd}} = \bigoplus_i E_{2i+1}$. If $T : C^\infty(V, E) \to C^\infty(V, E)$ is an even smoothing operator, meaning that T preserves the decomposition $C^\infty(V, E) = C^\infty(V, E^{\mathrm{ev}}) \oplus C^\infty(V, E^{\mathrm{odd}})$ and its Schwartz kernel belongs to $C^\infty(V \times V, E \boxtimes E^*)$, then its supertrace $\mathrm{Str}(T)$ is defined as the difference

$$\mathrm{Str}(T) = \mathrm{Tr}\Big(T : C^\infty(V, E^{\mathrm{ev}}) \to C^\infty(V, E^{\mathrm{ev}})\Big) - \mathrm{Tr}\Big(T : C^\infty(V, E^{\mathrm{odd}}) \to C^\infty(V, E^{\mathrm{odd}})\Big) \tag{1.4}$$

where Tr is the usual trace on smoothing operators(trace class).

Let $\varphi : V \to V$ be a smooth map with only simple fixed points (meaning that $\left|\det(\varphi_{*,m} - 1)\right|$ is invertible for all its fixed point $m \in V$). Assume in addition that there is a bundle map $\zeta : \varphi^*E \to E$ which induces the smooth map $\zeta : C^\infty(V, \varphi^*E) \to C^\infty(V, E)$. If the composition

$$T : C^\infty(V, E) \xrightarrow{\varphi^*} C^\infty(V, \varphi^*E) \xrightarrow{\zeta} C^\infty(V, E).$$

commutes with the differential d defined in (1.3), it lifts to an endomorphism on the cohomology ring (which is a finite dimensional vector space by the ellipticity condition) $T : H^*(E, d) \to H^*(E, d)$. The Lefschetz fixed point formula is a way to calculate the alternating sum

$$\sum_{i=0}^{m} (-1)^i \,\mathrm{Tr}\left(T : H^i(E, d) \to H^i(E, d)\right). \tag{1.5}$$

The Hodge decomposition theorem(see [5, Chapter 1]) identifies the cohomology ring $H^*(E, d)$ with the kernel of Laplacian $\Delta = (d + d^*)^2$ as vector space. Under this light, the alternating sum (1.5) can be expressed as a supertrace

$$\mathrm{Str}\left(T.g(t\Delta) : C^\infty(V, E) \to C^\infty(V, E)\right). \tag{1.6}$$

where g is a rapidly decreasing function on $\mathbb{R}^+$ with $g(0) = 1$ and $g(t\Delta)$ is given by functional calculus (see [6, Chapter 5]). In fact, it can be shown that only zero eigenvalue of Δ contributes to (1.6). A standard choice in most literature is $g = e^{-x}$.

The role of groupoids is to help us justify and compute the limit of (1.6) as $t \to 0$. To make it more precise, we shall modify the construction of the tangent groupoid $\mathbb{T}V$ by the fixed point set M

$$\mathbb{T}_M V = \sqcup_{m \in M} T_m V \oplus T_m V \sqcup V \times V \times (0, 1]$$

such that the smooth structure of the new groupoid $\mathbb{T}_M V$ resembles the localization phenomena that the kernel of $T.g(t\Delta)$ concentrates more and more on M as $t \to 0$. By a finite propagation argument given in [7], when d is of first order, $T.g(t\Delta)$ can be treated as an element of the groupoid algebra $C_c^\infty(\mathbb{T}_M V)$. The value of (1.6) can then be computed by setting $t = 0$.

Theorem 1.1 (*The Lefschetz Fixed Point Formula*)**.** *If (E, d) is an elliptic complex and d is a first order differential operator then*

$$\mathrm{Str}\left(T.g(t\Delta) : C^\infty(V, E) \to C^\infty(V, E)\right) = \sum_{m \in M} \left|\det(\varphi_{*,m} - 1)\right|^{-1} \mathrm{str}(\zeta_m).$$

Basic staffs on groupoids shall be reviewed in Section 2. The modified groupoid that serves our purpose will be built in Sections 3 and 4. The final calculation of (1.6) will be performed in Section 5.

This paper grows out of an effort to apply [3] to Bismut's hypoelliptic Laplacian [8] where both Getzler rescaling and "Lefschetz type rescaling" are needed, the detail of which we plan to pursue elsewhere. This paper can also be taken as a complement to [9] where the author uses a similar rescaling to describe the Witten deformation.

2. Lie groupoids

A Lie groupoid, usually denoted as $G \rightrightarrows G^{(0)}$, consists of the following data:

1. Two smooth manifolds $G = G^{(1)}$ and $G^{(0)}$ with two submersions $s, r : G \to G^{(0)}$ called source and range maps of the groupoid.
   Roughly speaking, $G^{(0)}$ can be thought of as a set of points and $G^{(1)}$ as a set of arrows between those points. The source and target maps send an arrow to its initial and terminal points respectively;
2. An associative multiplication map $m : G^{(2)} \to G$ where $G^{(2)} = \{(\gamma, \eta) \in G \times G \mid s(\gamma) = r(\eta)\}$.
   Given two arrows, if the initial point of the first arrow coincides with the terminal point of the second one, they can be "connected" to produce a new arrow. This type of "connection" determines the multiplication map m; We shall write $\gamma \cdot \eta$ for $m(\gamma, \eta)$.
3. A smooth map $\varepsilon : G^{(0)} \to G^{(1)}$.
   Given a point $x \in G^{(0)}$, it can be viewed as an arrow(loop) which goes from x to itself;
4. A smooth map $\iota : G \to G$ such that

$$\iota(\gamma) \circ \gamma = \varepsilon(s(\gamma)), \quad \gamma \circ \iota(\gamma) = \varepsilon(r(\gamma)) \quad \forall \gamma \in G.$$

   Given an arrow, the map ι reverse its direction. We shall write γ^{-1} for $\iota(\gamma)$.

Let $G \rightrightarrows G^{(0)}$ be a Lie groupoid, $G_x = s^{-1}(x)$ and $G^y = r^{-1}(y)$ be the corresponding source and range fibers where $x, y \in G^{(0)}$.

3. Deformation space

In this section, we briefly review the deformation to the normal cone construction. For a more detailed account see [9–11].

To the embedding $M \subseteq V$, the associated deformation to the normal cone $\mathbb{N}_V M$ is a smooth manifold whose underlying set is

$$\mathbb{N}_V M = N_V M \sqcup V \times (0, 1].$$

Let $V \supseteq U \xrightarrow{\psi} \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ be a sliced local coordinate chart of V such that $M \cap U$ is contained in $\psi^{-1}(\mathbb{R}^p \times \{0\})$. Then $\mathbb{N}_V M \supseteq \mathbb{N}_U(M \cap U) \xrightarrow{\phi} \mathbb{R}^{n+1}$ is seen as a local coordinate chart of the deformation space with the diffeomorphism ϕ given by

$$\begin{aligned} &(v, t) \mapsto (\psi_p(v), \frac{1}{t}\psi_q(v), t); \\ &(X, m) \mapsto (\psi_p(m), \psi_{q,*}(X), 0), \end{aligned} \tag{3.1}$$

where ψ_p and ψ_q are the first p and last q components of ψ respectively and $\psi_{q,*}$ is the differential of ψ_q.

Equivalently, the smooth structure is determined by declaring the following functions to be smooth:

1. The assignment $\mathbb{N}_V M \to \mathbb{R}$

$$\begin{aligned} (v, t) &\mapsto f(v) \\ X_m &\mapsto f(m) \end{aligned}$$

   where f is a smooth function on V and X_m is a normal vector at $m \in M$.
2. The assignment $\mathbb{N}_V M \to \mathbb{R}$

$$\begin{aligned} (v, t) &\mapsto \frac{1}{t^r} f(v) \\ X_m &\mapsto \frac{1}{r!} X_m^r(f) \end{aligned}$$

   where f is a smooth function on V that vanishes to order r on M.

The deformation to the normal cone construction is functorial in the following sense. Given a commutative diagram

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \uparrow & & \uparrow \\ M' & \longrightarrow & M \end{array}$$

where the vertical maps are inclusion of submanifolds and the horizontal maps are any smooth maps, there is an induced map between deformation spaces $\mathbb{N}'_V M' \to \mathbb{N}_V M$. Moreover, if the horizontal maps in the above diagram are submersions, then the induces map between deformation spaces is also a submersion(see [9, Proposition 1.1]).

In the following discussion, fix on V a Riemannian metric g and we choose a metric on the normal bundle $TV|_M/TM$ such that for each $m \in M$, the decomposition $T_mV = T_mM \oplus T_mV/T_mM$ is orthogonal.

Definition 3.1. Let μ_V be the Riemannian density on V. There is an induced metric and induced Riemannian density μ_M on M. There is also an induced Riemannian density μ_m on each T_mV/T_mM such that $d\mu_{V,m} = d\mu_{M,m} \cdot d\mu_m$.

Proposition 3.2. *Let $f \in C_c^\infty(\mathbb{N}_V M)$ then $t^{\dim M - \dim V} \int_V f(v,t) d\mu_V(v)$ converges, as $t \to 0$, to $\int_M \int_{T_mV/T_mM} f(X_m) d\mu_m(X_m) d\mu_M(m) < \infty$.*

Proof. Let U be an open subset of V. According to the smooth structure (3.1) of the deformation space, $\mathbb{N}_U(M \cap U)$ is a local coordinate chart of $\mathbb{N}_V M$. Without loss of generality, we may assume that f is supported inside the coordinate chart $\mathbb{N}_U(M \cap U)$. Then the result follows from a local calculation:

$$\begin{aligned} t^{\dim M - \dim V} \int_V f(v,t) d\mu_V(v) &= t^{-q} \int_{\mathbb{R}^n} f(v_1, \dots, v_p, t^{-1}v_{p+1}, \dots, t^{-1}v_{p+q}, t) d\mu(v) \\ &= \int_{\mathbb{R}^n} f(v_1, \dots, v_{p+q}, t) d\mu(v) \\ &\to \int_{\mathbb{R}^n} f(v_1, \dots, v_{p+q}, 0) d\mu(v). \quad \square \end{aligned}$$

Consider the diagonal embedding $M \hookrightarrow V \times V$, which sends $m \in M$ to $(m,m) \in V \times V$, and its associated deformation to the normal cone $\mathbb{N}_{V^2} M$.

Proposition 3.3. *The deformation space $\mathbb{N}_{V^2} M$ is a Lie groupoid with the unit space being $\mathbb{N}_V M$:*

$$\mathbb{N}_{V^2} M \rightrightarrows \mathbb{N}_V M.$$

The range and source maps are induced from the commutative diagrams

$$\begin{array}{ccc} V \times V & \xrightarrow{\pi_1} & V \\ \uparrow & & \uparrow \\ M & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} V \times V & \xrightarrow{\pi_2} & V \\ \uparrow & & \uparrow \\ M & \longrightarrow & M \end{array}$$

where in both diagrams the left vertical maps are the diagonal embeddings, the right vertical maps are the embeddings of M into V, the lower horizontal maps are the identity map and π_1, π_2 are the projections onto the first and second factors respectively.

Proof. It is clear from the functoriality of the deformation to the normal cone construction, the source map and the target map are submersions. The set of composable pairs of $\mathbb{N}_{V^2} M$ is the deformation space $\mathbb{N}_{V^3} M$ and the multiplication map is induced from the diagram

$$\begin{array}{ccc} V \times V \times V & \xrightarrow{\pi_{1,3}} & V \times V \\ \uparrow & & \uparrow \\ M & \longrightarrow & M \end{array}$$

where $\pi_{1,3} : V^3 \to V^2$ is the projection onto the first and the third components. The unit map ε and the inverse map ι are induced from the following diagrams respectively

$$\begin{array}{ccc} V & \xrightarrow{\Delta} & V \times V \\ \uparrow & & \uparrow \\ M & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} V \times V & \xrightarrow{s} & V \times V \\ \uparrow & & \uparrow \\ M & \longrightarrow & M \end{array}$$

where Δ is the diagonal map which sends $v \in V$ to $(v,v) \in V \times V$ and s switch the order of two components, namely $s(v_1, v_2) = (v_2, v_1)$. $\square$

To emphasis the groupoid structure and hint the similarity to the tangent groupoid, we shall write $\mathbb{T}_M V$ for $\mathbb{N}_{V^2} M$ and call it *the relative tangent groupoid*. Remember we have chosen a Riemannian metric g on V. Let $\gamma \in \mathbb{N}_V M$, we shall assign the source fiber over $\gamma = (v,t)$ which is $V \times \{t\}$ metric $t^{-1} \cdot g$ and the source fiber over $\gamma = X_m$ which is T_mV metric g_m. In the following discussion, we shall use G and $\mathbb{T}_M V$ interchangeably.

4. The coefficient bundle

In this section, we shall build a vector bundle $\mathbb{E}$ over the relative tangent groupoid $\mathbb{T}_M V$ to account for vector bundles appeared in the elliptic complex (1.1). We shall also build a functional calculus for the groupoid by using an argument from [7].

For the deformation space $\mathbb{N}_V M$, there is a canonical map $\mathbb{N}_V M \to V$ which sends $(v, \lambda) \in V \times (0, 1]$ to v and $X_m \in T_m V / T_m M$ to $m \in M \subset V$. Let F be the pullback of the vector bundle E defined in (1.2) along this canonical map and let $\mathbb{E}$ be the tensor product $r^*F \otimes s^*F^* \to \mathbb{T}_M V$. It is easy to check that

$$\mathbb{E}_{(v_1, v_2, t)} \cong E_{v_1} \otimes E^*_{v_2}, \qquad \text{and} \qquad \mathbb{E}_{(X_m, Y_m)} \cong \mathrm{End}(E_m)$$

where $(v_1, v_2, t) \in V \times V \times (0, 1]$ and $(X_m, Y_m) \in T_m V \times T_m V$.

Let $\{\mathbb{D}_x\}_{x \in G^{(0)}}$ be a family of first order differential operators acting on source fibers $\mathbb{D}_x : C^\infty_c(G_x, r^*E) \to C^\infty_c(G_x, r^*E)$ defined by

$$\mathbb{D}_x = \begin{cases} t(d + d^*) & x = (v, t) \in V \times (0, 1] \\ d_m + d^*_m & x = X_m \in TV|_M / TM; \end{cases}$$

where $d_m + d^*_m$ is the principal part of $d + d^*$ at $m \in M$. It can be checked that $\mathbb{D}$ is a smooth family of differential operators in the sense that it maps $C^\infty_c(G, \mathbb{E})$ to itself. For any $x \in G^{(0)}$, the first order differential operator $\mathbb{D}_x : C^\infty_c(G_x, r^*E) \to C^\infty_c(G_x, r^*E)$ is elliptic, self-adjoint and its propagation speed is uniformly bounded by a constant $c > 0$. In the following, we shall write σ_x for the restriction of $\sigma \in C^\infty_c(G, \mathbb{E})$ to G_x.

Proposition 4.1. *If $\sigma \in C^\infty_c(G, \mathbb{E})$ then the solution of wave equation $e^{is\mathbb{D}}\sigma$ is still a smooth compactly support section of $\mathbb{E}$.*

Proof. By definition $\left(e^{is\mathbb{D}}\sigma\right)_x = e^{is\mathbb{D}_x}\sigma_x$. Since the propagation speed of $\mathbb{D}_x$ is uniformly bounded, $e^{it\mathbb{D}}\sigma$ is still a compactly supported section of $\mathbb{E}$. We need to show the smoothness.

For any open subset $U \subset V$ we denote by U_R the R-neighborhood of U in V. Choose a finite open cover $\{U_\alpha\}$ of V and a positive number $R > 0$ such that $\{U_{\alpha, R}\}$ is an open cover of V by local coordinate charts. The relative tangent groupoid $\mathbb{T}_M V$ is covered by $\{U_\alpha \times U_\beta \times (0, 1]\} \cup \{\mathbb{T}_{M \cap U_\alpha} U_\alpha\}$. By choosing a partition of unit according to this cover, we may assume that the section σ is supported within single open subset, say $\mathbb{T}_{M \cap U} U \subset \mathbb{T}_M V$. Then the solution $e^{is\mathbb{D}}\sigma$ is supported within the local coordinate chart $\mathbb{T}_{M \cap U_R} U_R$ for $s < R/c$. According to [12], the symmetric hyperbolic system

$$\begin{aligned} &\frac{\partial f}{\partial s} + iDf = 0 \\ &\quad f|_{s=0} = \sigma \end{aligned}$$

can be solved on the local coordinate chart, and has smooth solution for $s < R/c$. So, $e^{is\mathbb{D}}$ maps $C^\infty_c(G, \mathbb{E})$ to itself when $s < R/c$. The proposition then follows from the property $e^{is_1\mathbb{D}} \cdot e^{is_1\mathbb{D}} = e^{i(s_1 + s_2)\mathbb{D}}$. □

Proposition 4.2. *Let f be a smooth function whose Fourier transformation $\widehat{f}$ is smooth and supported within $[-r/c, r/c]$, then $f(\mathbb{D})$ determines an element k in $C^\infty(G, \mathbb{E})$. Moreover, when $t \neq 0$, the restriction of k to t-fiber $k(\cdot, \cdot, t)$ is supported within r-neighborhood of the diagonal in $V \times V$ and the restriction of k to $t = 0$ fiber $k(\cdot, \cdot, 0)$ is supported within r-neighborhood of the diagonal in $T_m V \times T_m V$.*

Proof. Let $x \in G^{(0)}$, $f(\mathbb{D})$ acts on each source fiber as $f(\mathbb{D}_x) : C^\infty_c(G_x, r^*E) \to C^\infty_c(G_x, r^*E)$ whose integral kernel is smooth and supported within r-neighborhood of the diagonal in $G_x \times G_x$ according to [6, Proposition 7.23]. Then $f(\mathbb{D})$ determines a section k of $\mathbb{E}$ by

$$\begin{cases} k(v_1, v_2, t) = f(t(d + d^*))(v_1, v_2) & t \neq 0 \\ k(X_m, Y_m) = f(d_m + d^*_m)(X_m, Y_m) \end{cases}$$

where $X_m, Y_m \in T_m V$. Notice that $k(X_m, Y_m)$ is well defined for $[X_m, Y_m] \in T_m(V \times V)/T_m M$ since $f(d_m + d^*_m)$ is translation invariant.

Let $U \subset G$ be an open subset which is diffeomorphic, through source fiber preserving map, to $W \times W \times [0, \varepsilon)$ where $W \subset \mathbb{R}^n$ is an open subset and $\varepsilon > 0$. Let $\phi \in C^\infty_c(W)$. We could write

$$\phi f(\mathbb{D}) \phi \sigma(w_1, t) = \int \phi(w_1) k(w_1, w_2, t) \phi(w_2) \sigma(w_2, t) dw_2$$

and $k(w_1, w_2, t)$ is smooth in $(w_1, w_2) \in W \times W$. We still need to show the smoothness in t-position. Write

$$f(\mathbb{D}) = \int \widehat{f}(s) e^{-is\mathbb{D}} ds,$$

under this light

$$f(\mathbb{D})\sigma = \int \widehat{f}(s)e^{-is\mathbb{D}}\sigma\, ds$$

for any $\sigma \in C_c^\infty(\mathbb{T}_M V, \mathbb{E})$. Since $e^{-is\mathbb{D}}\sigma$ is a smooth compactly supported section of $\mathbb{E}$, we see that $f(\mathbb{D})$ maps smooth compactly supported sections to smooth compactly supported sections therefore has smooth integral kernel. □

Definition 4.3. If k is a smooth section of $\mathbb{E} \to \mathbb{T}_M V$. We shall use $\mathrm{Str}_t(k)$ to denote the following quantity

$$\mathrm{Str}_t(k) = t^{\dim(M)-\dim(V)} \int_V \mathrm{str}\,(k(v, v, t))\, d\mu_V(v).$$

for all $t \neq 0$. And define

$$\mathrm{Str}_0(k) = \int_{m\in M} \int_{T_mV} \mathrm{str}\,(k(X_m, X_m))\, d\mu_m(X_m)$$

where $\mathrm{str} : \mathrm{End}(E_v) \to \mathbb{C}$ is the supertrace which reflects the $\mathbb{Z}/2\mathbb{Z}$-grading of E_v.

From now on we assume that $\varphi : V \to V$ is a smooth map with only simple fixed points and M is the fixed point set. It is straightforward to see that

$$\mathbb{T}_M V = \sqcup_{m\in M} T_mV \oplus T_mV \sqcup V \times V \times (0, 1].$$

The source map sends (v_1, v_2, t) to (v_2, t) when $t \neq 0$ and sends $(X_m, Y_m) \in T_mV \oplus T_mV$ to Y_m. The range map sends (v_1, v_2, t) to (v_1, t) when $t \neq 0$ and sends $(X_m, Y_m) \in T_mV \oplus T_mV$ to X_m. Let $k \in C^\infty(\mathbb{T}_M V, \mathbb{E})$ then $k_\varphi(x, y) = \zeta.k(\varphi(x), y)$ is still a smooth section on the bundle $\mathbb{E} \to \mathbb{T}_M V$. Its valued at $(X, Y) \in T_mV \oplus T_mV$ can be computed $k_\varphi(X, Y) = \zeta_m.k(\varphi_*(X), Y)$ where $\varphi_{*,m} : T_mV \to T_mV$ is the differential of φ at $m \in M$. Assume, in addition, that k satisfies

$$k(X_m, Y_m) = k(X_m - Y_m, 0) \tag{4.1}$$

which is saying that the pseudodifferential operator, of which k is the Schwartz kernel, is translation invariant. Then

$$\begin{aligned}\mathrm{Str}_0(k_\varphi) &= \sum_{m\in M} \int_{T_mV} \mathrm{str}\,(\zeta_m.k((\varphi_* - 1)X_m, 0))\, d\mu_m(X_m)\\ &= \sum_{m\in M} \int_{T_mV} \left|\det(\varphi_{*,m} - 1)\right|^{-1} \mathrm{str}\,(\zeta_m.k(X_m, 0))\, d\mu_m(X_m)\\ &= \sum_{m\in M} \left|\det(\varphi_{*,m} - 1)\right|^{-1} \mathrm{str}\left(\zeta_m.\int_{T_mV} k(X_m, 0) d\mu_m(X_m)\right).\end{aligned} \tag{4.2}$$

Proposition 4.4. *Let f be a smooth function whose Fourier transformation is compactly supported. Let k be the smooth section of $\mathbb{E}$ determined by $f(\mathbb{D})$, then $\mathrm{Str}_t(k_\varphi) \to \mathrm{Str}_0(k_\varphi)$ as $t \to 0$.*

Proof. Let $f(v, t) = \mathrm{str}(\zeta.k(\varphi(v), v, t))$ and $f(X_m, 0) = \mathrm{str}(\zeta_m.k(\varphi_* X_m, X_m, 0))$, this defines a smooth function on $\mathbb{N}_M V$. Indeed, let $V \supset U \xrightarrow{\psi} \mathbb{R}^n$ be a coordinate chart of V centered at $m \in M$ and contains only one fixed point $m \in U$, it induces coordinate charts $\mathbb{N}_V M \supset \mathbb{N}_U\{m\} \xrightarrow{\phi_1} \mathbb{R}^{n+1}$ and $\mathbb{T}_M V \supset \mathbb{T}_{\{m\}}U \xrightarrow{\phi_2} \mathbb{R}^{2n+1}$ by (3.1). Under these local coordinates f can be represented as

$$\begin{cases}\widetilde{f}(x, t) = \mathrm{str}(\zeta.\widetilde{k}(\widetilde{\varphi}(tx)/t, x, t)) & t \neq 0\\ \widetilde{f}(x, 0) = \mathrm{str}(\zeta.\widetilde{k}(\widetilde{\varphi}'(0)x, x, 0))\end{cases}$$

where $\widetilde{f}, \widetilde{k}, \widetilde{\varphi}$ denote the local coordinate representations $f \circ \phi_1^{-1}$, $k \circ \phi_2^{-1}$, $\psi \circ \varphi \circ \psi^{-1}$ respectively. So f defines a smooth function on $\mathbb{N}_V M$.

In fact, f is compactly supported. The result then follows from Proposition 3.2. It suffices to show the restriction of f to the coordinate chart $\mathbb{N}_U\{m\}$ is compactly supported. On one hand, according to Proposition 4.2, $k(\cdot, \cdot, t)$ is supported within finite neighborhood of the diagonal, on the other hand, φ is non-degenerated, $\widetilde{\varphi}'(0)$ has no eigenvalue 1. Given large x enough, $(\widetilde{\varphi}(tx)/t, x)$ falls out the finite neighborhood of the diagonal on which k is supported. □

Now, $\Delta = (d+d^*)^2 : C^\infty(V, E) \to C^\infty(V, E)$ is an elliptic differential operator of order 2. Write Δ_m for the principal part of Δ at $m \in M$. Choose a rapidly decreasing function g, such that $f(x) = g(x^2)$ has smooth compactly supported Fourier transformation and $f(0) = g(0) = 1$. According to Proposition 4.2, $f(\mathbb{D}) = g(\mathbb{D}^2)$ defines an element in $C_c^\infty(\mathbb{T}_M V, \mathbb{E})$ with

$$f(\mathbb{D})(\gamma) = \begin{cases} t^n g\left(t^2\Delta\right)(v_1, v_2) & \gamma = (v_1, v_2, t) \in V \times (0, 1]\\ g\left(\Delta_m\right)(X_m, Y_m) & \gamma = (X_m, Y_m) \in T_mV \oplus T_mV.\end{cases}$$

where $g(t^2\Delta)(\cdot,\cdot)$ and $g(\Delta_m)(\cdot,\cdot)$ stand for the integral kernels of corresponding smooth operators. Since Δ_m has constant coefficients, $g(\Delta_m)$ is translation invariant.

Remark 4.5. Notice here we do not require g to be smooth. Given any positive integer p we can choose g to be p-times differentiable. However, if we want f to be Schwartz, it is impossible to choose g smooth. This little defect would not affect our calculation.

5. Lefschetz fixed point formula

In this section we apply the gadget developed above to calculate the supertrace (1.6) as $t \to 0$. Let I be a smooth section of the bundle $\mathbb{E} \to \mathbb{T}_M V$ which is defined by

$$\begin{cases} I(v_1, v_2, t) = \zeta \cdot g(t^2\Delta)(\varphi(v_1), v_2, t)t^n \\ I(X_m, Y_m) = \zeta_m \cdot g(\Delta_m)(\varphi_* X_m, Y_m). \end{cases}$$

The Lefschetz fixed point formula is about to calculate

$$\mathrm{Str}\left(T.g(t^2\Delta) : C^\infty(V,E) \to C^\infty(V,E)\right) = \mathrm{Str}_t(I). \tag{5.1}$$

By Proposition 4.4 and (4.2), as $t \to 0$ (5.1) converges to

$$\mathrm{Str}_0(I) = \sum_{m\in M} \left|\det(\varphi_{*,m} - 1)\right|^{-1} \mathrm{str}\left(\zeta_m \cdot \int_{T_m V} g(\Delta_m)(X_m, 0)d\mu_m\right)$$

Proposition 5.1. *Recall that $g(x^2) = f(x)$ where f is a smooth function with compactly supported Fourier transformation, then*

$$\int_{T_m V} g(\Delta_m)(X,0)dX = g(0).$$

Proof. Let $\sigma(\cdot)$ be symbol of corresponding differential operators. $g(\Delta_m)$ is a pseudo-differential operator on $T_m V$ whose symbol is calculated by

$$\begin{aligned} a(x,\xi) &= e^{ix\xi} g(\Delta_m) e^{-ix\xi} = \int \widehat{f}(t) e^{ix\xi} e^{it(d_m + d_m^*)} e^{-ix\xi} dt \\ &= f(\sigma(d_m + d_m^*)(\xi)) = g(\sigma(\Delta_m)(\xi)). \end{aligned}$$

Notice that since Δ_m has constant coefficients, $a(x,\xi)$ is independent of x. Then the integral kernel of $g(\Delta_m)$ can be expressed in terms of $a(x,\xi)$

$$g(\Delta_m)(x,y) = \int a(x,\xi) e^{i(x-y)\cdot\xi} d\xi.$$

Finally,

$$\int_{T_m V} g(\Delta_m)(X,0)dX = \int a(x,\xi) e^{ix\xi} dx d\xi = a(x,0) = g(0). \quad \square$$

Overall we have

$$\mathrm{Str}_0(I) = \sum_m \left|\det(\varphi_{*,m} - 1)\right|^{-1} \mathrm{str}(\zeta_m).$$

which completes the proof of Theorem 1.1.

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